

# Generalized Lyapunov conditions for $k$ -contraction: analysis and feedback design

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**Abstract**—Recently, the concept of  $k$ -contraction has been introduced as a promising generalization of contraction for dynamical systems. However, the study of  $k$ -contraction properties has faced significant challenges due to the reliance on complex mathematical objects called matrix compounds. As a result, related control design methodologies have yet to appear in the literature. In this paper, we overcome existing limitations and propose new sufficient conditions for  $k$ -contraction which do not require matrix compounds computation. Notably, these conditions are also necessary in the linear time-invariant framework. Leveraging on these findings, we propose a feedback design methodology for both the linear and the nonlinear scenarios which can be used to enforce  $k$ -contractivity properties on the closed-loop dynamics.

**Index Terms**—Contraction analysis, Nonlinear systems, Linear matrix inequalities, Inertia Theorems, Compound matrices.

## I. INTRODUCTION

Contraction theory is an emerging topic that has been used in numerous applications, such as observer design [1], multi-agent system synchronization [2]–[4] and controller design [5]–[7]. Nonetheless, many systems cannot present classical contractivity properties, e.g. multi-stable systems or systems that admit a (non-trivial) periodic solution. This fact motivated the study of suitable generalizations, such as horizontal contraction [8, Section VII] and transversal exponential stability [9]. Motivated by the results of Muldowney [10], the recent work [11] presented the notion of  $k$ -contraction as the generalization to  $k$ -dimensional objects of the standard contraction concept for distances. As such,  $k$ -contraction includes classical contraction as the special case  $k = 1$ . For  $k > 1$ , this property can be used to analyze the asymptotic behavior of systems that are not contractive in the classical sense. For example, for 2-contractive time-invariant systems, every bounded solution converges to an equilibrium point (which may not be the same for every solution).

Existing sufficient conditions for  $k$ -contraction are given in terms of a particular matrix compound of the Jacobian of the vector field dynamics [10]–[12]. Although these conditions are adequate for system analysis, their application for feedback design is limited. First, matrix compounds rapidly explode in dimension for low values of  $k$  and systems of large dimension. This fact drastically increases the computational complexity of potential feedback design algorithms. Second, the use of matrix compounds hinders the derivation of

a tractable matrix inequality problem for feedback design. Consequently, a  $k$ -contractive design methodology has yet to be developed.

Considering these limitations, this work presents alternative design-oriented conditions for  $k$ -contraction that do not rely on matrix compounds, but rather on matrix inequalities on the given system dynamics. In particular, we build upon the generalized Lyapunov matrix inequalities studied for instance in [13], [14]. By exploiting these novel conditions, we devise a feedback design methodology in both the linear and the nonlinear framework. In the linear time-invariant framework, our design is based on a new generalization of the notion of stabilizability, which is, then, generalized to the nonlinear framework.

The remainder of this document is organized as follows. In Section II, we provide a refined definition of  $k$ -contraction which strongly focuses on its geometrical interpretation. Then, we recover the notion of infinitesimal  $k$ -contraction, which has been used in [15], and link it to the proposed definition of  $k$ -contraction. Subsequently, we recall matrix compound-based sufficient conditions for  $k$ -contraction and discuss their limitations. Section III focuses on linear systems. First, we derive necessary and sufficient conditions for  $k$ -contraction that do not require matrix compounds, but rather generalized Lyapunov matrix inequalities. Then, based on these results, we propose the notion of  $k$ -order stabilizability together with a new  $k$ -contractive feedback design. In doing so, we also collect and recall in a unified theorem a series of results on inertia theorems (see, e.g. [16, Lemma 1, Section 3] and [17, Theorem 2.5]) and we provide new inertia results on algebraic inequalities of the form  $WA^T + AW - BB^T \prec 0$  not requiring any controllability assumption (see for instance [18]). Section IV focuses on extending these results to nonlinear systems. Similarly, we first provide sufficient conditions for  $k$ -contraction in nonlinear systems and, then, propose a design methodology for  $k$ -contraction. In section IV-D, we discuss asymptotic behaviors of  $k$ -contractive dynamics. The results are validated in some numerical simulations in Section V. All the proofs are postponed in Sections VI–VIII and in the Appendix. Finally, some conclusions and future perspectives are drawn in Section IX.

*Contribution:* This work builds on the initial results published in the conference paper [19]. The main differences between [19] and the current paper are summarized in the following points:

- We refine the necessary and sufficient conditions for  $k$ -contraction in linear systems presented in [19] and provide a general result for state matrices with arbitrary eigenvalues.
- We present a state-feedback design methodology for linear systems. In doing so, we introduce the notion of  $k$ -order stabilizability and an associated generalized Lyapunov test.
- We extend the proposed design to nonlinear dynamics.
- We discuss the asymptotic behavior of  $k$ -contractive systems, with a specific focus on  $k \in \{1, 2, 3, n\}$ .
- We provide multiple numerical examples of  $k$ -contraction analysis and design, with a particular focus on the multi-stability properties of a grid-connected synchronverter [20], [21].

*Notation:*  $\mathbb{R}_{\geq 0} := [0, \infty)$  and  $\mathbb{N} := \{0, 1, 2, \dots\}$ .  $|\cdot|$  denotes the

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standard Euclidean norm. Given  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^m$ , we set  $(x, y) := (x^\top, y^\top)^\top$ . We denote  $\binom{n}{k} := \frac{n!}{k!(n-k)!}$  as the binomial coefficient, with  $n!$  denoting the factorial of  $n \in \mathbb{N}$ . The inertia of a matrix  $P$  is defined by the triplet of integers  $\text{In}(P) := (\pi_-(P), \pi_0(P), \pi_+(P))$ , where  $\pi_-(P)$ ,  $\pi_+(P)$  and  $\pi_0(P)$  denote the numbers of eigenvalues of  $P$  with negative, positive and zero real part, respectively, counting multiplicities. The cardinality of a set is denoted as  $\text{card}(\cdot)$ .  $A \succ 0$  (resp.  $A \succeq 0$ ) denotes  $A$  being a positive definite (resp. positive semidefinite) matrix.

## II. PRELIMINARIES ON $k$ -CONTRACTION

### A. Definition of $k$ -contraction

In this work, we consider nonlinear systems of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \quad (1)$$

where  $f$  is sufficiently smooth. The flow of  $f$  is denoted by  $\psi^t$ , and  $\psi^t(x_0)$  is the trajectory of (1) at time  $t$ . By definition,  $\psi^0(x_0) = x_0$ . We now formally define the property of  $k$ -contraction studied in this article. Our definition strongly focuses on a geometrical interpretation and it is related to the notion presented in the works [10], [11]. Moreover, when considering objects of dimension 1 ( $k = 1$ ), it matches the definition of contraction presented in [9], [22].

In [9], [22], 1-contraction expresses the fact that the length of any parametrized and sufficiently smooth curve exponentially decreases with time. To extend such a notion to any positive integer  $k \in [1, n]$ , we consider a set of sufficiently smooth functions  $\mathcal{I}_k$  defined on  $[0, 1]^k$ , namely

$$\mathcal{I}_k := \left\{ \Phi : [0, 1]^k \rightarrow \mathbb{R}^n \mid \Phi \text{ is a smooth immersion} \right\}. \quad (2)$$

Let  $P \in \mathbb{R}^{n \times n}$  be a positive definite symmetric matrix. For each  $\Phi$  in  $\mathcal{I}_k$ , we define the volume  $V^k(\Phi)$  of  $\Phi$  as

$$V^k(\Phi) := \int_{[0, 1]^k} \sqrt{\det \left\{ \frac{\partial \Phi}{\partial r}(r)^\top P \frac{\partial \Phi}{\partial r}(r) \right\}} dr. \quad (3)$$

Note that, since  $f$  in (1) is sufficiently smooth, for each  $t \in \mathbb{R}_{\geq 0}$  the corresponding flow  $\psi^t$  is also sufficiently smooth. Consequently, for each  $\Phi$  in  $\mathcal{I}_k$ ,  $\psi^t \circ \Phi$  belongs to  $\mathcal{I}_k$ .

**Remark 1** When  $\Phi$  is injective and  $P$  is the identity matrix, the volume  $V^k$  defined in (3) coincides with the standard Euclidean  $k$ -volume of the submanifold  $\Phi([0, 1]^k) \subseteq \mathbb{R}^n$ . Note that 1-volumes are lengths, 2-volumes are areas and 3-volumes are standard volumes.

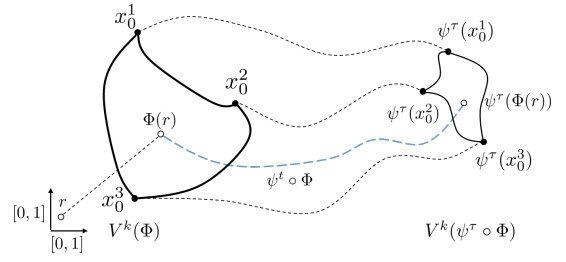
**Remark 2** The volume definition (3) can be generalized to the Riemannian framework by substituting the Euclidean metric  $P$  with a symmetric positive definite 2-tensor  $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ , see [23, Lemma 3.2]. However, in this paper, we will focus on the Euclidean scenario in order to obtain more tractable conditions.

From now on, we let  $k$  be a fixed integer between 1 and  $n$ . We now define  $k$ -contraction for nonlinear systems of the form (1).

**Definition 1 ( $k$ -contraction)** System (1) is said to be  $k$ -contractive on a forward invariant set  $\mathcal{S} \subseteq \mathbb{R}^n$  if there exist real numbers  $a, b > 0$  such that, for every  $\Phi \in \mathcal{I}_k$  satisfying  $\text{Im}(\Phi) \subseteq \mathcal{S}$ , the following holds

$$V^k(\psi^t \circ \Phi) \leq b e^{-at} V^k(\Phi), \quad \forall t \in \mathbb{R}_{\geq 0}. \quad (4)$$

In other words, a system is  $k$ -contractive if, for any parametrized  $k$ -dimensional submanifold of  $\mathbb{R}^n$  from which trajectories are complete, its volume exponentially shrinks along the system dynamics. An



**Fig. 1.** Flow of a 2-contractive system. The initial submanifold of initial conditions is described by  $\Phi$ . The volume of this submanifold decreases exponentially along the trajectories of the system.

intuitive representation of the required volume convergence condition is presented in Fig. 1. When  $k = 1$ , (4) implies that the length of any sufficiently smooth curve exponentially decreases, matching the definition in [9]. Moreover, this definition includes the ones in [10], and [11, Section 3.2]. We highlight the following property, which is directly obtained by using the definition of the volume  $V^k$ .

**Lemma 1** If system (1) is  $k$ -contractive for an integer  $1 \leq k < n$ , then it is also  $(k + 1)$ -contractive.

We remark that Definition 1 is invariant under globally Lipschitz diffeomorphism on  $\mathcal{S}$ , due to the presence of the overshoot term  $b$ .

**Lemma 2** Assume that the system (1) is  $k$ -contractive on a forward invariant set  $\mathcal{S} \subseteq \mathbb{R}^n$  for some positive constants  $a, b > 0$ . Moreover, let  $\varphi : \mathcal{S} \rightarrow \mathcal{S}$  be a diffeomorphism which satisfies for some positive constants  $\bar{c}, \underline{c} > 0$

$$\underline{c}I \preceq \frac{\partial \varphi}{\partial x}(x)^\top \frac{\partial \varphi}{\partial x}(x) \preceq \bar{c}I, \quad \forall x \in \mathcal{S}. \quad (5)$$

Then, there exists a positive constant  $\bar{b} > 0$  such that for every  $\Phi \in \mathcal{I}_k$  satisfying  $\text{Im}(\Phi) \subseteq \mathcal{S}$ , the following holds

$$V^k(\varphi \circ \psi^t \circ \Phi) \leq \bar{b} e^{-at} V^k(\varphi \circ \Phi),$$

The proof is postponed to Appendix A.

### B. Infinitesimal $k$ -contraction

Inspired by classical works on contraction theory [24], we now provide a result linking the exponential stability properties of the variational system of (1) to the  $k$ -contraction property proposed in Definition 1. We start by recalling the dynamics of the variational system, which describe the evolution of an infinitesimal displacement along the trajectories of the system. Specifically, the variational system of (1) along the trajectory  $\psi^t(x_0)$  is

$$\dot{v} = \frac{\partial f}{\partial x}(\psi^t(x_0)) v, \quad v \in \mathbb{R}^n. \quad (6)$$

Then,  $\frac{\partial \psi^t}{\partial x}(x_0)v_0$  is a trajectory of (6) at time  $t$  initialized at  $v_0$  at  $t = 0$ . From linearity, we have that  $\frac{\partial \psi^t}{\partial x}(x_0)$  is the state transition matrix of (6). Thus,  $\frac{\partial \psi^t}{\partial x}(x_0)v_0$  depicts the infinitesimal displacement with respect to the solution  $\psi^t(x_0)$  induced by the initial condition  $x_0 + v_0$ .

We recall that the trajectory  $\psi^t(x_0)$  is locally exponentially stable, that is, the trajectory generated from any initial condition close enough to  $x_0$  will exponentially converge to  $\psi^t(x_0)$ , if and only if the variational system (6) is exponentially stable [25, Theorem 3.13]. In classical contraction theory [24], this property is generalized by considering simultaneously all the trajectories in a set. That is, system (1) is contracting in a forward invariant set  $\mathcal{S} \subseteq \mathbb{R}^n$  if the variational system (6) is exponentially stable for all  $x_0 \in \mathcal{S}$ . Then,

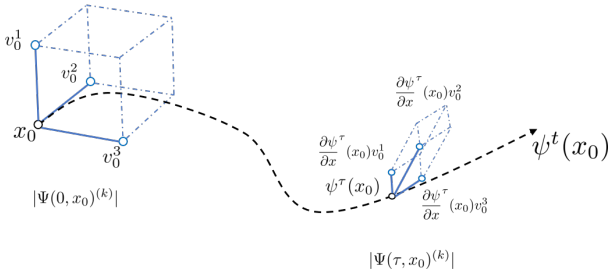


Fig. 2. Flow of an infinitesimally 3-contractive system.

contraction on  $\mathcal{S}$  implies that every solution in  $\mathcal{S}$  converges to the same trajectory [24], or equivalently, the distance between any pair of trajectories shrinks to zero. In a sense, contraction exemplifies how the local linearization along trajectories can be used to derive global incremental properties of the original system. In this section, we generalize this idea by considering  $k$ -contraction properties on the variational system (6).

Pick any  $x_0 \in \mathbb{R}^n$  and consider any  $k$  initial conditions  $v_0^1, \dots, v_0^k$  of the variational system (6). We define the following matrix

$$\Psi(t, x_0) := \begin{bmatrix} \frac{\partial \psi^t}{\partial x}(x_0)v_0^1 & \dots & \frac{\partial \psi^t}{\partial x}(x_0)v_0^k \end{bmatrix} \in \mathbb{R}^{n \times k}. \quad (7)$$

Note that  $\Psi(0, x_0) = \frac{\partial \Phi_{\text{loc}}}{\partial r}(r)$ , where  $\Phi_{\text{loc}}$  is an immersion parameterized by the variable  $r \in [0, 1]^k$  and whose image is an infinitesimal  $k$ -order parallelepiped with vertices at  $x_0$  and  $v_0^i + x_0$ , namely,

$$\Phi_{\text{loc}}(r) = \sum_{i=1}^k r_i (v_0^i + x_0) + \left(1 - \sum_{i=1}^k r_i\right) x_0,$$

with  $r_i \in [0, 1]$  for  $i \in \{1, \dots, k\}$  being the  $i$ -th component of  $r$ . For  $k = 1$ ,  $\Phi_{\text{loc}}(r)$  defines a straight line between  $x_0$  and  $x_0 + v_0^1$ . The volume of the infinitesimal parallelepiped can be computed by means of the multiplicative compound, which is defined as follows.

**Definition 2 (Multiplicative Compound [26])** Consider a matrix  $Q \in \mathbb{R}^{n \times m}$  and select an integer  $k \in [1, \min\{n, m\}]$ . Moreover, define a minor of order  $k$  of the matrix  $Q$  as the determinant of some  $k \times k$  submatrix of  $Q$ . The  $k$ -th multiplicative compound of  $Q$ , denoted as  $Q^{(k)}$ , is the  $\binom{n}{k} \times \binom{m}{k}$  matrix including all the minors of order  $k$  of  $Q$  in a lexicographic order.

As an example, consider a  $3 \times 3$  matrix  $Q$  with entries  $q_{ij}$  for  $i, j = 1, \dots, 3$ . The  $2^{\text{nd}}$  multiplicative compound  $Q^{(2)}$  is

$$Q^{(2)} = \begin{bmatrix} \det \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} & \det \begin{pmatrix} q_{11} & q_{13} \\ q_{21} & q_{23} \end{pmatrix} & \det \begin{pmatrix} q_{12} & q_{13} \\ q_{22} & q_{23} \end{pmatrix} \\ \det \begin{pmatrix} q_{11} & q_{12} \\ q_{31} & q_{32} \end{pmatrix} & \det \begin{pmatrix} q_{11} & q_{13} \\ q_{31} & q_{33} \end{pmatrix} & \det \begin{pmatrix} q_{12} & q_{13} \\ q_{32} & q_{33} \end{pmatrix} \\ \det \begin{pmatrix} q_{21} & q_{22} \\ q_{31} & q_{32} \end{pmatrix} & \det \begin{pmatrix} q_{21} & q_{23} \\ q_{31} & q_{33} \end{pmatrix} & \det \begin{pmatrix} q_{22} & q_{23} \\ q_{32} & q_{33} \end{pmatrix} \end{bmatrix}.$$

Note that for a  $Q \in \mathbb{R}^{n \times n}$ ,  $Q^{(n)} = \det(Q)$  and  $Q^{(1)} = Q$ .

Thanks to the previous definition, by considering  $P = I$ , we can compute the volume of  $\Phi_{\text{loc}}$  and  $\frac{\partial \psi^t}{\partial x} \circ \Phi_{\text{loc}}$  as follows [11]

$$V^k(\Phi_{\text{loc}}) = |\Psi(0, x_0)^{(k)}|, \quad V^k\left(\frac{\partial \psi^t}{\partial x} \circ \Phi_{\text{loc}}\right) = |\Psi(t, x_0)^{(k)}|.$$

The second equality is a consequence of the linearity of the dynamics of  $\Psi(t, x_0)^{(k)}$  and we postpone further details at the beginning of Appendix B. Given these notions, we have the next definition.

**Definition 3 (Infinitesimal  $k$ -contraction)** System (1) is said to be infinitesimally  $k$ -contractive on a forward invariant set  $\mathcal{S} \subseteq \mathbb{R}^n$  if there exist real numbers  $a, b > 0$  such that

$$\left| \Psi(t, x_0)^{(k)} \right| \leq e^{-at} \left| \Psi(0, x_0)^{(k)} \right|, \quad (8)$$

for all  $(t, x_0) \in \mathbb{R}_{\geq 0} \times \mathcal{S}$  and any  $\Psi(0, x_0)$  defined as in (7).

Roughly speaking, the bound in (8) implies that the volume of any infinitesimal parallelepiped connected to the trajectory  $\psi^t(x_0)$  and generated by the vectors  $\frac{\partial \psi^t}{\partial x}(x_0)v_0^1, \dots, \frac{\partial \psi^t}{\partial x}(x_0)v_0^k$  exponentially shrinks to zero. A depiction of this property is presented in Fig. 2. Notice that, for the case  $k = 1$ , Definition 3 boils down to (6) being exponentially stable for all  $x_0 \in \mathcal{S}$ , which is a sufficient condition for the classical notion of contraction [24]. In the next proposition, we link the notion of infinitesimal  $k$ -contraction to  $k$ -contraction.

**Theorem 1** Suppose system (1) is infinitesimally  $k$ -contractive on a forward invariant set  $\mathcal{S}$ . Then, it is also  $k$ -contractive on  $\mathcal{S}$  according to Definition 1.

The proof is postponed to Appendix B. We highlight that in [22] it has also been shown that for  $k = 1$ , that is, standard contraction, the converse is also true, namely contraction of the system implies infinitesimal contraction. For  $k > 1$  the problem is currently open.

### C. Sufficient conditions based on additive matrix compounds

Sufficient conditions for  $k$ -contraction were originally given in the seminal work by Muldowney [10] and were recently re-proposed in the works [11], [27]. These conditions strongly depend on the use of the additive matrix compound, which is defined below.

**Definition 4 (Additive Compound [26])** Consider a matrix  $Q \in \mathbb{R}^{n \times n}$  and select an integer  $k \in [1, n]$ . The  $k$ -th additive compound of  $Q$  is the  $\binom{n}{k} \times \binom{n}{k}$  matrix defined as

$$Q^{[k]} := \frac{d}{d\epsilon} \Big|_{\epsilon=0} (I + \epsilon Q)^{\binom{n}{k}}.$$

The additive compound can be explicitly computed in terms of the entries of  $Q$ . For example, for  $Q \in \mathbb{R}^{n \times n}$  we have  $Q^{[n]} = \text{tr}(Q)$  and  $Q^{[1]} = Q$ . More details on this operation can be found in [28].

Bearing this definition in mind, we now reframe the sufficient condition for  $k$ -contraction presented in [10], [11] in the framework of this paper, namely, we view them through the lenses of Definition 1.

**Theorem 2** Let  $\mathcal{S} \subseteq \mathbb{R}^n$  be a forward invariant set and suppose there exist a real number  $\eta > 0$  and a symmetric positive definite matrix  $Q \in \mathbb{R}^{\binom{n}{k} \times \binom{n}{k}}$  such that

$$Q \left( \frac{\partial f}{\partial x}(x)^{[k]} \right) + \left( \frac{\partial f}{\partial x}(x)^{[k]} \right)^\top Q \preceq -\eta I, \quad \forall x \in \mathcal{S}. \quad (9)$$

Then, system (1) is infinitesimally  $k$ -contractive on  $\mathcal{S}$  (therefore,  $k$ -contractive on  $\mathcal{S}$  according to Definition 1).

The proof is postponed to Appendix C. The extension of Theorem 2 to time-varying systems can be found in [19].

**Remark 3** Inequality (9) is equivalent to the condition in [11, Theorem 9] using the logarithmic norm induced by the weighted  $\ell_2$  norm (e.g. [29, Equation 2.56]). However, in our statement, the set  $\mathcal{S}$  is allowed to be non-convex. Furthermore, when  $k = 1$ , we recover the well-known Demidovich conditions (see [30]) and the proof in [9] for contraction of lengths in the context of Euclidean metrics. Nonetheless, it is worth noting that, similarly to the case of standard contraction (see, e.g. [29, Section 3]), also  $k$ -contraction can be studied through different logarithmic norms [11, Section 3.1].

**Remark 4** Theorem 2 can be generalized to the case of Riemannian volumes, see Remark 2. However, we omit these results to ease the reading of the document. It should be remarked that such generalization also expands on point IV in [31, Proposition 2.5], since we consider volume objects of dimension lower than  $n$ .

#### D. Limitations of matrix compound-based conditions

Although Theorem 2 provides a suitable condition for system analysis, we claim that the presence of matrix compounds hinders the process of devising  $k$ -contractive feedback designs. Indeed, consider a linear control system of the form

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (10)$$

where  $u$  is the control input. Assume we want to design a state-feedback controller of the form  $u = -Kx$ , with  $K \in \mathbb{R}^{n \times m}$ , such that the closed-loop system is  $k$ -contractive. Then, Theorem 2 reduces to designing  $K$  such that condition (9) is satisfied for the closed-loop system, namely,

$$Q \left( (A - BK)^{[k]} \right) + \left( (A - BK)^{[k]} \right)^\top Q \preceq -\eta I.$$

However, this is a non-convex matrix inequality, due to the strong coupling between the matrices  $B, K$  imposed by the additive matrix compound. Consequently, even for a simple linear case, a design methodology for the gain  $K$  cannot be straightforwardly derived. Another motivation for deriving alternative conditions is related to the computational complexity of the matrix inequality (9). Although for  $k = n$  interesting properties may arise (see below Lemma 8), most useful asymptotic behaviors (from a control perspective) require small values of  $k$ , specifically  $k \in \{1, 2, 3\}$  (see [11] and Section IV-D). Notice that matrix compounds rapidly grow in dimension when the order of the system is large and the  $k$  is low, since they involve matrices of dimensions  $\binom{n}{k} \times \binom{n}{k}$ . Consequently, as highlighted in previous works [32], compound-based conditions often explode in size.

With this in mind, a consistent portion of the following sections is dedicated the presentation of alternative design-oriented  $k$ -contraction conditions that do not require matrix compound computation. These conditions will be fundamental in the derivation of control laws guaranteeing  $k$ -contractivity of the closed-loop.

### III. $k$ -CONTRACTION FOR LINEAR SYSTEMS

We start our analysis by focusing on the linear scenario. This will provide fundamental intuitions on the notion of  $k$ -contraction that will be instrumental in the subsequent analysis of nonlinear dynamics.

#### A. Generalized Lyapunov necessary and sufficient conditions

Consider a linear system of the form

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n. \quad (11)$$

We now provide a set of sufficient and necessary conditions guaranteeing that (11) is  $k$ -contractive according to Definition 1. This result is based on the following two facts:

- a necessary and sufficient condition for system (11) to be  $k$ -contractive is that the sum of the real part of any combination of  $k$  eigenvalues of  $A$  is negative, see Lemma 9 below in Section VI;
- the generalized Lyapunov matrix inequality (see, e.g [13], [16])

$$PA + A^\top P \prec 2\mu P$$

admits a symmetric solution  $P$  of inertia  $\text{In}(P) = (p, 0, n - p)$  if and only if  $A$  has  $p$  eigenvalues with real part larger than  $\mu$  and  $n - p$  eigenvalues with real part smaller than  $\mu$ , see below Lemma 10 in Section VI.

Combining the two properties above, we state now the following main result.

**Theorem 3** System (11) is  $k$ -contractive if and only if there exist:

- a positive integer  $\ell \in \mathbb{N}$  satisfying  $1 \leq \ell \leq k$ ,
- $\ell$  real numbers  $\mu_i \in \mathbb{R}$ , with  $i \in \{0, \dots, \ell - 1\}$ ,
- $\ell$  positive integers  $d_i \in \mathbb{N}$ , with  $i \in \{0, \dots, \ell - 1\}$ , satisfying

$$0 =: d_0 < d_1 < \dots < d_{\ell-1} \leq k - 1,$$

- and  $\ell$  symmetric matrices  $P_i \in \mathbb{R}^{n \times n}$  of respective inertia  $(d_i, 0, n - d_i)$ , with  $i \in \{0, \dots, \ell - 1\}$ ,

such that

$$A^\top P_i + P_i A \prec 2\mu_i P_i, \quad \forall i \in \{0, \dots, \ell - 1\}, \quad (12a)$$

$$\sum_{i=0}^{\ell-1} h_i \mu_i \leq 0, \quad (12b)$$

where  $h_0 \geq 1$  and  $h_i = d_{i+1} - d_i$ , for all  $i = \{0, \dots, \ell - 1\}$  with  $d_\ell \in \mathbb{N}$  satisfying  $d_{\ell-1} + 1 \leq d_\ell \leq k$ .

The proof of Theorem 3 is postponed to Section VI-B. To provide some intuition relative to Theorem 3, we anticipate that the constants  $\mu_i$  are bounding the real part of the eigenvalues of the matrix  $A$ . That is,  $\mu_0$  bounds the eigenvalue with largest real part of  $A$ , while  $\mu_1$  bounds the second largest eigenvalue with real part different from the first, and so on. Then, (12b) can be interpreted as a bound in the partial sum of eigenvalues of  $A$ , considering their multiplicities, in turn implying  $k$ -contraction (see the proof in Section VI-B for more details).

**Remark 5** A particular case in which the former Theorem applies is when  $\ell = 1$ . In that case, the former condition reduces to the existence of a real number  $\mu_0 \leq 0$  and a symmetric positive definite matrix  $P_0 \succ 0$  such that  $A^\top P_0 + P_0 A \prec 2\mu_0 P_0$ . This condition is satisfied if and only if  $A$  is Hurwitz, which implies that system (11) is  $k$ -contractive for all  $k \in \{1, \dots, n\}$  by means of Lemma 1.

**Remark 6** The inertia requirements in (12a) cannot be represented as semidefinite constraints. However, these constraints can be dropped without a significant impact on the solution of the inequality. Indeed, by Lemma 10 in Section VI, a given constant  $\mu_i$  imposes a specific inertia on the matrix  $P_i$ , depending on the eigenvalues of  $A$ .

**Remark 7** Other attempts at dropping the requirement of matrix compound computations appeared in the literature [32]. However, to the best of our knowledge, Theorem 3 is the first result proposing necessary and sufficient conditions [19, Section IV.C].

#### B. Computational burden of Theorem 3

We now compare Theorem 2 and Theorem 3 in terms of the computational burden imposed by the solution of the respective matrix inequalities. We focus on the result in Theorem 3 for the case  $\ell = k$  and  $d_i = d_{i-1} + 1$ , since it provides the largest set of matrix inequalities. Let  $M \in \mathbb{R}^{r \times r}$  be an arbitrary square matrix and  $Q \in \mathbb{R}^{r \times r}$  be a symmetric matrix. Since  $Q$  is symmetric, each condition of the form  $QM + M^\top Q \preceq \mu Q$  requires the computation of  $N = r(r - 1)/2 + 1$  variables, namely the entries of the top triangular portion of  $Q$  and the scalar  $\mu$ . Then, Theorem 2 requires  $N_1 = \binom{n}{k} (\binom{n}{k} + 1)/2 + 1$  variables, while Theorem 3 requires  $N_2 = kn(n - 1)/2 + k$  variables.

To better understand how the size of the problem scales with different values of  $k$  and  $n$ , we refer to Fig. 3. Clearly, for large dimensional systems and low  $k$ , the conditions in Theorem 3 ask for a significantly smaller number of variables. Moreover, even in the

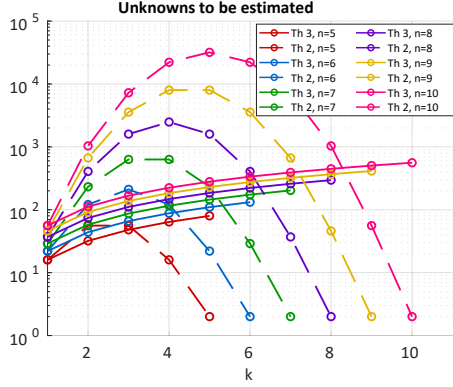


Fig. 3. Number of variables to be estimated by Theorem 2 (dashed) and by Theorem 3 (solid) in function of  $k$ . Colors refer to different  $n$ .

worst case of  $k = n$ , Theorem 3 typically requires between  $10^2$  and  $10^3$  variables. Differently, Theorem 2 can easily reach  $10^4$  variables.

This computation shows that conditions in Theorem 3 do not grow in dimension as fast as the condition in Theorem 2. Moreover, for  $k$  sufficiently smaller than  $n$ , we have  $N_2 \leq N_1$ . We recall that the cases  $k = \{1, 2, 3\}$  are interesting from a control viewpoint, see Section IV-D.

### C. $k$ -order stabilizability

Consider a linear system of the form (10). It is well-known that stabilizability of the pair  $(A, B)$  is a necessary and sufficient condition for the existence of a stabilizing controller. A similar property can be defined when considering  $k$ -contractive designs. We refer to this property as  $k$ -order stabilizability.

**Definition 5 ( $k$ -order stabilizability)** System (10) is  $k$ -order stabilizable if there exists a matrix  $K \in \mathbb{R}^{n \times m}$  such that the closed-loop system  $\dot{x} = (A - BK)x$  is  $k$ -contractive.

Conditions for  $k$ -order stabilizability can be easily derived by transforming the system into a suitable form. Using standard Kalman decomposition, system (10) is algebraically equivalent to a system of the form

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u \quad (13)$$

where  $x_c \in \mathbb{R}^{n_c}$ ,  $x_u \in \mathbb{R}^{n_u}$ ,  $n_c + n_u = n$  and the pair  $(A_c, B_c)$  is controllable. The non-negative integer  $n_u$  is the dimension of the null-space of the controllability matrix of (10). Consequently, we admit the possibility of  $n_u = 0$  and  $A_u$  being non-existing.

**Lemma 3** System (10) is  $k$ -order stabilizable if and only if its canonical Kalman decomposition (13) satisfies one of the two following conditions:

- the integer  $n_u$  satisfies  $0 \leq n_u < k$ ;
- the autonomous system  $\dot{x}_u = A_u x_u$ , is  $k$ -contractive.

The proof of Lemma 3 is postponed to Section VII-A. Intuitively, Lemma 2 asks for the uncontrollable part to be already  $k$ -contractive (or of dimension smaller than  $k$ ). For the case  $k = 1$ , Lemma 3 reduces to  $n_u = 0$ , a necessary and sufficient condition for controllability in linear systems, or  $\dot{x}_u = A_u x_u$ , being stable, which is a sufficient condition for the classical notion of stabilizability. We also remark that similar definitions could be developed for  $k$ -order controllability, observability, and detectability.

### D. $k$ -contractive feedback design

Starting from a  $k$ -order stabilizability property and the decomposition (13), one can easily derive a  $k$ -contractive feedback design, e.g., via pole placement on the pair  $(A_c, B_c)$ . Nonetheless, in view of an extension of these notions to the nonlinear context, we look for coordinate-free conditions, i.e. that do not rely on change of coordinates and decompositions that would not be easy to extend to nonlinear systems.

Motivated by the result in Theorem 3, we now derive constructive conditions for designing  $k$ -contractive controllers. This section presents a design methodology that follows the philosophy of feedback stabilization based on Lyapunov tests for stabilizability [33, Section 14.5]. That is, we first solve a set of matrix inequalities, which are feasible if and only if the system is  $k$ -order stabilizable. Then, the controller is derived from the result of these inequalities. We start by presenting a generalized Lyapunov test for  $k$ -order stabilizability.

**Theorem 4** System (10) is  $k$ -order stabilizable if and only if there exist:

- a positive integer  $\ell \in \mathbb{N}$  satisfying  $1 \leq \ell \leq k$ ,
- $\ell$  real numbers  $\mu_i \in \mathbb{R}$ , with  $i \in \{0, \dots, \ell - 1\}$ ,
- $\ell$  positive integers  $d_i \in \mathbb{N}$ , with  $i \in \{0, \dots, \ell - 1\}$ , satisfying

$$0 = d_0 < d_1 < \dots < d_{\ell-1} \leq k - 1,$$

- and  $\ell$  symmetric matrices  $W_i \in \mathbb{R}^{n \times n}$  of respective inertia  $(d_i, 0, n - d_i)$ , with  $i \in \{0, \dots, \ell - 1\}$ ,

such that

$$W_i A^\top + A W_i - B B^\top \prec 2\mu_i W_i, \quad \forall i \in \{0, \dots, \ell - 1\}, \quad (14a)$$

$$\sum_{i=0}^{\ell-1} h_i \mu_i \leq 0, \quad (14b)$$

where  $h_0 \geq 1$ ,  $h_i = d_{i+1} - d_i$ , for all  $i = \{1, \dots, \ell - 1\}$  with  $d_\ell \in \mathbb{N}$  satisfying  $d_{\ell-1} + 1 \leq d_\ell \leq k$ .

The proof of Theorem 4 is postponed to Section VII-C. Notice that for  $k = 1$ , inequalities (14a)- (14b) reduce to the existence of a constant  $\mu_0 \leq 0$  and a symmetric positive definite matrix  $W_0 \succ 0$  such that

$$W_0 A^\top + A W_0 - B B^\top \prec 2\mu_0 W_0.$$

Hence, we recover the well-known Lyapunov test for stabilizability [33, Section 14.4]. Differently put, the inequalities (14) can be seen as a generalization of the Lyapunov test for stabilizability to the context of  $k$ -contraction.

Now, based on the presented generalized Lyapunov test for  $k$ -order stabilizability, we can directly derive a  $k$ -contractive feedback controller for the linear system (10). The result is summarized in the following proposition.

**Proposition 1** Assume that (10) is  $k$ -order stabilizable. Then, there exist:

- a positive integer  $\ell \in \mathbb{N}$  satisfying  $1 \leq \ell \leq k$ ,
- $\ell$  real numbers  $\mu_i \in \mathbb{R}$ , with  $i \in \{0, \dots, \ell - 1\}$ ,
- $\ell$  positive integers  $d_i \in \mathbb{N}$ , with  $i \in \{0, \dots, \ell - 1\}$ , satisfying

$$0 = d_0 < d_1 < \dots < d_{\ell-1} \leq k - 1,$$

- and  $\ell$  symmetric matrices  $W_i \in \mathbb{R}^{n \times n}$  of respective inertia  $(d_i, 0, n - d_i)$ , with  $i \in \{0, \dots, \ell - 1\}$ ,

such that (14) is satisfied, and the following colinearity relation holds

$$B^\top W_i^{-1} = B^\top W_0^{-1}, \quad \forall i \in \{0, \dots, \ell - 1\}. \quad (15)$$

Furthermore, with this solution, system (10) is  $k$ -contractive with the feedback law

$$u = -Kx, \quad K = \frac{\rho}{2} B^\top W_0^{-1}, \quad \forall \rho \geq 1. \quad (16)$$

The proof is postponed to Section VII-D. We highlight that, since  $k$ -contraction for the closed-loop system is preserved for all  $\rho \geq 1$ , the proposed controller presents a generalization of the infinite-gain margin property [34, Section 3.2.2] to the framework of partial stabilization. Consequently, this result expands similar infinite-gain margin designs [7] from 1-contraction to  $k$ -contraction.

#### IV. $k$ -CONTRACTION FOR NONLINEAR SYSTEMS

As a follow-up to the linear scenario, we now move to the analysis of  $k$ -contraction for nonlinear systems. The main goal is to provide sufficient conditions inspired by the results in Section III.

##### A. Sufficient conditions

Consider a nonlinear system of the form (1). The following theorem provides sufficient conditions for  $k$ -contraction.

**Theorem 5** *Let  $\mathcal{S} \subsetneq \mathbb{R}^n$  be a compact forward invariant set. Suppose there exist two symmetric matrices  $P_0, P_1 \in \mathbb{R}^{n \times n}$  of respective inertia  $(0, 0, n)$  and  $(k-1, 0, n-k+1)$ , and  $\mu_0, \mu_1 \in \mathbb{R}$  such that*

$$\frac{\partial f}{\partial x}(x)^\top P_0 + P_0 \frac{\partial f}{\partial x}(x) \prec 2\mu_0 P_0, \quad (17a)$$

$$\frac{\partial f}{\partial x}(x)^\top P_1 + P_1 \frac{\partial f}{\partial x}(x) \prec 2\mu_1 P_1, \quad (17b)$$

$$\mu_1 + (k-1)\mu_0 < 0, \quad (17c)$$

for all  $x \in \mathcal{S}$ . Then, system (1) is infinitesimally  $k$ -contractive on  $\mathcal{S}$  (therefore,  $k$ -contractive on  $\mathcal{S}$ ).

A detailed discussion of Theorem 5 is postponed to Section VIII-B, along with the associated proof. Intuitively, inequality (17a) bounds the expansion rate for the variational system (6) by a factor  $\mu_0$ . Differently, the second inequality (17b) bounds the contraction rate for the variational system on a subspace by a factor  $\mu_1$ . Consequently, inequality (17c) constraints the contraction rate to be faster than the expansion rate. This resembles the eigenvalue bounding approach of Section III. However, a simple eigenvalue interpretation is not applicable in the nonlinear framework. Hence, we directly bound the fastest diverging direction and the slowest converging one, asking for the latter to be sufficiently fast through (17c).

Notice that Theorem 5 considers constant matrices  $P_0, P_1$ . In view of recent results on Riemannian contraction analysis, e.g. [1], [9], [31], we claim that the use of constant metrics is restrictive and that state-dependant metrics can help in widening the result to more general cases. A direct consequence of this observation is the fact that, contrarily to the linear case, the conditions of Theorem 5 are in general not equivalent to those of Theorem 2. Nonetheless, we highlight that restricting ourselves to the case of constant matrices can help to derive some new asymptotic behavior for  $k$ -contractive systems, as discussed at the end of Section II-C. Finally, we remark that, under linear dynamics, Theorem 5 recovers Theorem 3 for the particular case  $\ell = 2$  and  $d_1 = k - 1$ .

**Remark 8** *Theorem 5 requires solving an infinite set of matrix inequalities. Nonetheless, there are multiple strategies that can be used to reduce it to a feasible problem. For instance, one could exploit convex relaxation, as explained in [14, Section VI]. Alternatively, for systems with a semi-linear structure (namely,  $f(x) = Ax + g(x)$ ), one can obtain a finite set of LMIs if the nonlinear term satisfies a monotonic or a sector-bounded condition [35].*

##### B. Relaxing conditions for the planar case

Note that, differently from Theorem 2, in Theorem 5 we require the set  $\mathcal{S}$  to be compact. We conjecture this compactness assumption can also be dropped in Theorem 5. This conjecture is motivated by the following result for the planar case  $n = k = 2$ .

**Lemma 4** *Let  $\mathcal{S} \subseteq \mathbb{R}^2$  and assume there exist symmetric matrices  $P_0, P_1 \in \mathbb{R}^{2 \times 2}$  of inertia  $\text{In}(P_0) = (0, 0, 2)$ ,  $\text{In}(P_1) = (1, 0, 1)$  and  $\mu_0, \mu_1 \in \mathbb{R}$  such that, for all  $x \in \mathcal{S}$ , inequalities in (17) are satisfied. Then, system (1) is 2-contractive on  $\mathcal{S}$ .*

The proof of Lemma 4 is postponed to Section VIII-C. This result shows that the inequalities in (17) may be valid on the whole  $\mathbb{R}^n$ . Consequently, in future works, we aim at exploring if Theorem 5 can be expanded to the whole  $\mathbb{R}^n$ . Currently, the technical obstruction that prevents us to conclude the conjecture is the use of Theorem 6 and Lemma 16 in Theorem 5 proof, which require  $\mathcal{S} \subsetneq \mathbb{R}^n$  in order to guarantee a bounded invariant subspace splitting.

##### C. $k$ -contractive feedback design

Following the lines of the linear results presented in Section III, we now elaborate on the conditions for  $k$ -contraction proposed in Theorem 5. We aim at devising  $k$ -contractive controllers for nonlinear systems. Precisely, we consider nonlinear systems of the form

$$\dot{x} = f(x) + Bu \quad (18)$$

where  $u \in \mathbb{R}^m$  and  $f$  is sufficiently smooth. In the next proposition, we provide a result on  $k$ -contractive controller design.

**Proposition 2** *Let  $\mathcal{S} \subsetneq \mathbb{R}^n$  be a compact set, and assume there exist two symmetric matrices  $W_0, W_1 \in \mathbb{R}^{n \times n}$ , with  $W_0 \succ 0$  and inertia  $\text{In}(W_1) = (k-1, 0, n-k+1)$  and a pair of real numbers  $\mu_0, \mu_1 \in \mathbb{R}$ , such that, for all  $x \in \mathcal{S}$ ,*

$$W_0 \frac{\partial f}{\partial x}(x)^\top + \frac{\partial f}{\partial x}(x) W_0 - BB^\top \prec 2\mu_0 W_0 \quad (19a)$$

$$W_1 \left( \frac{\partial f}{\partial x}(x) - \frac{1}{2} BB^\top W_0^{-1} \right)^\top + \left( \frac{\partial f}{\partial x}(x) - \frac{1}{2} BB^\top W_0^{-1} \right) W_1 - BB^\top \prec 2\mu_1 W_1. \quad (19b)$$

Then, there exists a real number  $\omega > 0$  such that if

$$(k-1)\mu_0 + \mu_1 + \omega < 0, \quad (20)$$

the feedback law  $u = -Kx$  with

$$K = \frac{1}{2} B^\top (W_0^{-1} + W_1^{-1}). \quad (21)$$

makes the system (18)  $k$ -contractive on  $\mathcal{S}$ , if  $\mathcal{S}$  is forward invariant for the closed-loop system.

The proof of Proposition 2 is postponed to Section VIII-D. Proposition 2 is an extension of the result for linear systems in Theorem 3 to the nonlinear framework. However, besides the nonlinearities, we highlight some main differences between the two results. First, since we require constant matrices  $W_i$ , the nonlinear result cannot be proven to be necessary in general. Second, even if such constant matrices do exist, there is no guarantee that they satisfy a colinearity condition (15) uniformly on  $x$ . Hence, Proposition 2 proposes an alternative design that trades the colinearity condition (15) for conservativeness in the sum of rates (20), i.e. the addition of  $\omega > 0$ . A similar approach can be used in the linear scenario to avoid the colinearity condition (15) at the price of the necessity result.

#### D. Asymptotic behavior

In this section, we describe some asymptotic behaviors for  $k$ -contractive systems. In particular, for the cases  $k = \{1, 2\}$  we can directly rely on Definition 1 to establish the existence of a unique or multiple equilibria and exclude more complex behaviors (such as limit cycle or chaotic ones). For  $k = 3$ , the sufficient conditions of Theorem 5 allow to conclude the existence of simple attractors, that is, fixed points or limit cycles. Finally, for  $k = n$ , the sufficient condition of Theorem 2 allows to exclude the existence of repelling equilibria.

**1-contractivity:** For  $k = 1$ , we recover the asymptotic behavior of the classical notion of contraction (see, e.g. [29] and references therein). Namely, one can deduce the existence and attractiveness of a unique equilibrium.

**Lemma 5** *Assume that system (1) is 1-contractive (according to Definition 1) in a closed and forward invariant set  $\mathcal{S} \subseteq \mathbb{R}^n$ . Then, system (1) has a unique equilibrium which is exponentially stable with a domain of attraction including  $\mathcal{S}$ .*

Noting that for  $k = 1$  volumes boil down to distances, the previous lemma can be proved by using Banach fixed point theorem as in [6, Lemma 2]. Furthermore, we highlight that  $\mathcal{S}$  is not required to be a compact set, hence it can be equal to  $\mathbb{R}^n$ .

**2-contractivity:** The property of 2-contraction can be used to study phenomena: non-existence of trivial periodic solutions and multi-stability.

**Lemma 6** *Assume that system (1) is 2-contractive (according to Definition 1) in a compact and forward invariant set  $\mathcal{S} \subsetneq \mathbb{R}^n$ . Then, the following holds:*

- 1) [10] system (1) has no non-trivial periodic solutions in  $\mathcal{S}$ ;
- 2) [36] every solution of system (1) initialized in  $\mathcal{S}$  converges to an equilibrium point (which may not be the same for every solution).

The first item of the previous lemma shows that 2-contraction and the conditions in Theorem 5 could be understood as a generalization of the Bendixson's criteria for systems of dimension larger than 2. The second item connects 2-contraction with multi-stability.

**3-contractivity:** For the case  $k = 3$ , the property of 3-contraction can be used to study non-trivial periodic solutions of the system. In this case, however, we don't explicitly rely on the Definition 1, but rather on the (more stringent) sufficient conditions of Theorem 5.

**Lemma 7** *Assume that system (1) satisfies the conditions of Theorem 5 with  $k = 3$  in a compact forward invariant set  $\mathcal{S} \subsetneq \mathbb{R}^n$ . Then, every solution of (1) with initial condition in  $\mathcal{S}$  converges to a simple attractor, namely, a fixed point or a limit cycle (which may not be the same for every solution).*

The proof of Lemma 7 is postponed to Section VIII-E. We remark that this result is not generally true for 3-contractive systems and it is a consequence of the condition (17) with constant matrices  $P_i$ . As an example, consider the Rössler system [37]

$$\begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= -x_1 - x_3, \\ \dot{x}_3 &= 0.5((x_1 - x_1^2) - x_3). \end{aligned} \quad (22)$$

The 3-additive compound of the Jacobian is  $\frac{\partial f}{\partial x}(x)^{[3]} = -0.5$ . Therefore, condition (9) is trivially satisfied and the system (22) is 3-contractive. Nonetheless, the nonlinear term  $x_1^2$  prevents the existence

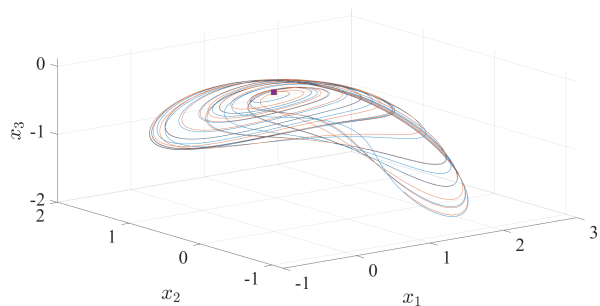


Fig. 4. Evolution of two trajectories of the system (22). The first (blue) has an initial condition  $[0.1, 0.1, 0]$  and the second (red)  $[0.099, 0.1, 0]$ .

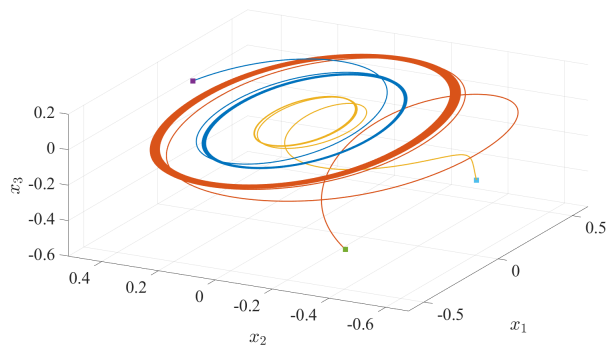


Fig. 5. Evolution of three trajectories of the system (23). The first (blue) has an initial condition  $[0.2, 0.5, 0]$ , the second (red)  $[-0.3, -0.3, -0.5]$  and the third (yellow)  $[0.2, -0.5, -0.3]$ .

of a constant matrix  $P_1$  in (17b). Therefore, even if the system is 3-contractive and evolves in a compact set, (17) cannot be satisfied and there is no guarantee it will converge to a fixed point or limit cycle. Indeed, this system presents chaotic behavior and its trajectories do not converge to a simple attractor, see Figure 4.

Alternatively, consider the following system,

$$\begin{aligned} \dot{x}_1 &= x_2 - 2x_3, & \dot{x}_2 &= -x_1 - x_3, \\ \dot{x}_3 &= 0.5((x_1 - x_1^3) - x_3). \end{aligned} \quad (23)$$

In this case, the 3-additive compound of the Jacobian is  $\frac{\partial f}{\partial x}(x)^{[3]} = -0.5$ , similarly to (22). Then, the system is 3-contractive. Moreover, it evolves in a compact forward invariant set. However, differently from (22), inequalities (17a)-(17c) are satisfied with  $\mu_0 = 0.2$ ,  $\mu_1 = -0.45$  and  $P_0 = \begin{bmatrix} 1.45 & 0.20 & 0.13 \\ 0.20 & 2.09 & 0.26 \\ 0.13 & 0.26 & 0.67 \end{bmatrix}$ ,  $P_1 = \begin{bmatrix} -2.05 & -1.02 & -0.45 \\ -1.02 & 26.75 & 14.47 \\ -0.45 & 14.47 & 6.41 \end{bmatrix}$ . Therefore, by Lemma 7 we can conclude that the system will converge to a simple attractor (which may not be the same for every solution). This behavior is shown in Figure 5, which presents the evolution of three trajectories.

**Remark 9** *The sufficient conditions in Theorem 5 resemble conditions for dominance analysis through cone fields, see, e.g., [14], [38]. However, this does not imply the topics are strongly related. For instance, the proofs of the asymptotic properties of 2-contractive systems and systems that admit an invariant cone of rank 1 are very different. In the first case, one first proves that there are no periodic solutions, see Lemma 6, and then the Pugh's closing-lemma implies that any bounded solution converges to an equilibrium. In the second case, (almost all) the trajectories of the system can be projected in*

one-to-one way onto a 1-dimensional linear space, which implies that (almost all) trajectories converge to an equilibrium.

***n*-contractivity:** Finally, for the case  $k = n$ , one can exclude the existence of repelling equilibria, that is, equilibria such that any infinitesimally small perturbation acting in an arbitrary direction results in a diverging trajectory. This is formalized as follows.

**Definition 6 (Infinitesimally repelling equilibrium)** Given a dynamical system  $\dot{x} = f(x)$ , we say that  $x^\circ \in \mathbb{R}^n$  is an infinitesimally repelling equilibrium if  $f(x^\circ) = 0$  and  $-\frac{\partial f}{\partial x}(x^\circ)$  is Hurwitz.

**Lemma 8** Assume that system (1) satisfies the conditions of Theorem 2 with  $k = n$  in a set  $\mathcal{S} \subseteq \mathbb{R}^n$ . Then, system (1) has no infinitesimally repelling equilibria in  $\mathcal{S}$ .

The proof of Lemma 8 is postponed to Section VIII-F. A similar result could be obtained using the conditions of Theorem 5. However, for  $k = n$ , the conditions of Theorem 5 are more restrictive and computationally heavier, see Section III-B. Therefore, we omit the corresponding statement.

## V. ILLUSTRATIONS

### A. Multi-stability of a grid-connected synchronverter

Synchronverters are electrical inverters controlled to behave as a synchronous generator. In this section, we will consider the fourth-order model of the synchronverter presented in [21],

$$\begin{aligned} \dot{x}_1 &= -\frac{R}{L}x_1 + x_2x_3 + \frac{V}{L}\sin x_4 \\ \dot{x}_2 &= -x_1x_3 - \frac{R}{L}x_2 - \frac{m}{L}i_f x_3 + \frac{V}{L}\cos x_4 \\ \dot{x}_3 &= \frac{m}{J}i_f x_2 - \frac{D_p}{J}(x_3 - w_n) + \frac{T_m}{J} \\ \dot{x}_4 &= x_3 - w_n. \end{aligned} \quad (24)$$

Due to space restrictions, we omit the physical meaning of the equations, and we refer to [21] for a detailed presentation.

Synchronverters properly operate when synchronized to the grid. Using the notation in (24), this occurs when  $x_3 = w_n$ . With this in mind, one may wonder if for any initial condition (inside a predefined set) the synchronverter will converge to a state of synchronization. Note that the  $x_4$  dynamics are just an integral of the error  $x_3 - w_n$ . Hence, any equilibrium point of (24) presents the required synchronization behavior.

We highlight that (24) may present multiple equilibrium points [21]. Therefore, the study of the eventual system's synchronization to the grid boils down to the study of its convergence properties to any of its equilibria. Nonetheless, the complexity of such analysis led to several works dedicated to answering this specific question [20].

In this section, we will use  $k$ -contraction as a simpler tool to prove the convergence properties of the synchronverter model (24). In particular, we will show that Theorem 5 can be used to prove 2-contraction of system (24). Then, by boundedness of the system's trajectories, the convergence of the system to a (possibly non-unique) equilibrium point is guaranteed by item 2) of Lemma 6.

We consider the model parameters in [21], that is  $w_n = w_g = 100\pi$ ,  $V = 230\sqrt{3}$ ,  $J = 0.2$ ,  $R = 1.875$ ,  $L = 0.05675$ ,  $D_p = 10$ ,  $m = 3.5$ , with  $T_m = 0$ ,  $i_f = 1$ . Moreover, we consider the forward invariant set  $\mathcal{S}$  defined as  $x_1 \in [-81 \ 5]$ ,  $x_2 \in [-67 \ 10.5]$ ,  $x_3 \in [298 \ 315]$ ,  $x_4 \in [1 \ -0.2]$ . Then, by means of the convex relaxation approach mentioned in Remark 8, the matrix inequalities (17a)-(17c) can be proven to hold with  $\mu_0 = 14$ ,  $\mu_1 =$

$$-15 \text{ and } P_0 = \begin{bmatrix} 0.00007 & 0.000001 & -0.0005 & 0.00145 \\ 0.000001 & 0.00006 & -0.000076 & 0.001036 \\ -0.0005 & -0.000076 & 0.011 & -0.02135 \\ 0.00145 & 0.001036 & -0.02135 & 0.9882 \end{bmatrix}, P_1 = \begin{bmatrix} 0.0007 & 0.00001 & -0.00851 & -0.0692 \\ 0.00001 & 0.00027 & -0.000382 & -0.00009 \\ -0.00851 & -0.000382 & 0.157 & 1.308 \\ -0.0692 & -0.00009 & 1.308 & 0.842 \end{bmatrix}. \text{ Therefore, by Theo-} \\ \text{rem 17, (24) is 2-contractive on } \mathcal{S} \text{ and the desired stability property} \\ \text{follows by item 2) of Lemma 6.}$$

### B. Multi-stability via 2-contractive feedback

Consider the following nonlinear system:

$$\dot{x}_1 = x_2 - x_3, \dot{x}_2 = -x_1 - x_3 + u, \dot{x}_3 = x_1(x_1^2 - 0.25). \quad (25)$$

In the absence of input ( $u = 0$ ), the system trajectories present non-trivial periodic solutions and the 3-additive compound is  $\frac{\partial f}{\partial x}(x)^{[3]} = 0$ . Therefore, the system is not 3-contractive (nor 2-contractive nor 1-contractive). The objective is to design a linear state-feedback controller  $u = -Kx$ , such that the system converges to a non-necessarily unique equilibrium point.

The closed-loop system has 3 equilibrium points,

$$x_1^* = \{0, -0.5, 0.5\}, \quad x_2^* = x_3, \quad x_3^* = \frac{-(1+k_1)}{1+k_3+k_2}x_1,$$

where  $k_1, k_2, k_3$  are the components of the feedback gain  $K$ . The existence of multiple equilibrium points prevents the design of a 1-contractive linear controller, e.g., [7]. Nonetheless, it may still be possible to design a controller that guarantees 2-contraction of the closed-loop. Since the system dynamics evolve in a compact set, 2-contraction guarantees convergence to a (possibly non-unique) equilibrium point by means of item 2) of Lemma 6.

With the aim of exploiting the results in Proposition 2, via the convex relaxation technique mentioned in Remark 8 one can verify that the matrix inequalities (19a)-(19b) are satisfied with  $\mu_0 = 0.3$ ,  $\mu_1 = -0.6$  and  $W_0 = \begin{bmatrix} 1.86 & -1.21 & -0.92 \\ -1.21 & 2.13 & 0.86 \\ -0.92 & 0.86 & 0.96 \end{bmatrix}$ ,  $W_1 = \begin{bmatrix} 2.84 & 0.06 & 2.65 \\ 0.06 & 0.27 & -5.16 \\ 2.65 & -5.19 & -2.29 \end{bmatrix}$ . With this specific selection, we have  $\omega = 0.048$ , which implies (20) is satisfied. Therefore, according to Proposition 2 the state-feedback law  $u = -Kx$  with gain (21), namely  $K = [0.89 \ 2.16 \ -1.18]$ , makes the closed-loop system 2-contractive. Indeed, the closed-loop system satisfies (9) with  $\eta = 0.091$  and  $Q = \begin{bmatrix} 1.26 & -0.06 & 0.46 \\ -0.05 & 2.74 & -1.34 \\ 0.41 & -1.34 & 2.33 \end{bmatrix}$ , which validates the result by means of Theorem 2. More precisely, the closed-loop system presents 3 equilibrium points, one unstable at the origin and 2 (locally) asymptotically stable.

## VI. PROOF OF LINEAR RESULTS (ANALYSIS)

In the rest of this section, given a matrix  $A \in \mathbb{R}^{n \times n}$ , we order its eigenvalues in such a way that

$$\Re(\lambda_1) \geq \Re(\lambda_2) \geq \dots \geq \Re(\lambda_n). \quad (26)$$

### A. $k$ -contraction properties and inertia theorem

The following result is a direct consequence of the additive compound definition.

**Lemma 9** A system  $\dot{x} = Ax$ , with  $x \in \mathbb{R}^n$  is  $k$ -contractive if and only if the eigenvalues  $\lambda_i$  of the matrix  $A$  satisfy

$$\sum_{i=1}^k \Re(\lambda_i) < 0, \quad (27)$$

according to the above mentioned ordering (26).

*Proof:* A necessary and sufficient condition for  $k$ -contraction of a linear system  $\dot{x} = Ax$  is that the matrix  $A^{[k]}$  is Hurwitz [11].



The necessity part can be proven by adapting [39, Lemma 1] to continuous time. Moreover, recall that a spectral property of the additive compound matrix is that the eigenvalues of the matrix  $A^{[k]}$  are all the possible sums of the form  $\lambda_{i_1} + \lambda_{i_2} + \dots + \lambda_{i_k}$ , with  $1 \leq i_1 < \dots < i_k \leq n$ , see [11]. That is, a necessary and sufficient condition for  $k$ -contraction is that the sum of any combination of  $k$  eigenvalues of  $A$  is negative. In particular, this holds true if and only if the first  $k$  eigenvalues satisfy condition (27) with the ordering (26). ■

Additionally, we state a lemma about inertia properties of the Lyapunov equation (12a), collecting together various results from the literature, e.g., [16, Lemma 1, Section 3], [17, Theorem 2.5].

**Lemma 10** *Given a matrix  $A \in \mathbb{R}^{n \times n}$ , a real constant  $\mu$ , and an integer  $p \in \{0, \dots, n\}$ , the following statements are equivalent:*

- 1)  $A$  has  $p$  eigenvalues with real part larger than  $\mu$  and  $n - p$  eigenvalues with real part smaller than  $\mu$ ,
- 2) the matrix  $A - \mu I$  has inertia  $(n - p, 0, p)$ ,
- 3) there exists a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  with inertia  $\text{In}(P) = (p, 0, n - p)$  satisfying  $A^\top P + PA \prec 2\mu P$ .
- 4) given a symmetric positive definite matrix  $Q \succ 0$  there exists a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  with inertia  $\text{In}(P) = (p, 0, n - p)$  satisfying  $(A - \mu I)^\top P + P(A - \mu I) = -Q$ .

*Proof:* The eigenvalues of  $A - \mu I$  are the shifted eigenvalues  $\lambda_1 - \mu, \dots, \lambda_n - \mu$ , ordered as in (26). If  $A - \mu I$  has inertia  $(n - p, 0, p)$ , it implies that  $\Re(\lambda_{p+1}) - \mu < 0$ . This shows that (1) and (2) are equivalent. The equivalence between (3) and (2) is due to [16, Lemma 1, Section 3] and the one between (2) and (4) is due to [17, Theorem 2.5]. Finally, we have that (4) implies (3). Indeed, since  $Q \succ 0$ , (4) implies  $A^\top P + PA = -2\mu P - Q \prec -2\mu P$ . ■

### B. Proof of Theorem 3

The proof of Theorem 3 is obtained by combining the two previous lemmas. To begin with, we introduce a new notation in order to represent the eigenvalues of  $A$  and their associated multiplicities. Precisely, consider the matrix  $A$  in (11) and let  $\Pi : \mathbb{C} \rightarrow \mathbb{R}$  denote the canonical projection onto the real axis. Let  $\sigma(A)$  be the spectrum of  $A$  and suppose  $\Pi(\sigma(A)) = \{\alpha_1, \alpha_2, \dots, \alpha_q\}$  ( $q \leq n$ ) with  $\alpha_1 > \alpha_2 > \dots > \alpha_q$ . Set  $\bar{h}_i = \text{card}(\Pi^{-1}(\alpha_{i+1}) \cap \sigma(A))$ , where eigenvalues have been counted with their algebraic multiplicities (so that  $\bar{h}_0 + \bar{h}_1 + \dots + \bar{h}_{q-1} = n$ ). Finally, let  $\bar{d}_0 = 0$  and  $\bar{d}_i = \sum_{j=0}^{i-1} \bar{h}_j$ , for each  $i \in \{1, \dots, q-1\}$ .

We remark that the variables  $\bar{d}_i, \bar{h}_i$  are related to variables  $d_i, h_i$  of Theorem 3, hence the similar notation. To see this relation, notice that  $\bar{h}_i$  represents the number of eigenvalues that project to  $\alpha_{i+1}$  and  $\bar{d}_i$  represents the amount of eigenvalues with real part strictly larger than  $\alpha_{i+1}$ . Therefore, for any constant  $\mu \in \mathbb{R}$  such that  $\alpha_1 < \mu$ , we have  $\text{In}(-A + \mu I) = (0, 0, n)$ . Similarly, if  $\alpha_{i+1} < \mu < \alpha_i$ , we have that  $\text{In}(-A + \mu I) = (\bar{d}_i, 0, n - \bar{d}_i)$ . Consequently, if we avoid the singular case  $\pi_0(-A + \mu I) > 0$ , the matrix  $-A + \mu I$  can only present a particular set of inertia  $(\bar{d}_i, 0, n - \bar{d}_i)$  with  $i \in \{0, \dots, q-1\}$ . From this fact and Lemma 10, we obtain that the matrix inequalities in (12a) are only feasible for the particular set of inertia  $\text{In}(P_i) = (\bar{d}_i, 0, n - \bar{d}_i)$  with  $i \in \{0, \dots, q-1\}$ . A direct consequence of this result is that  $\ell \leq q$ .

We now present the main arguments proving sufficiency and necessity of the result in Theorem 3.

*Sufficiency.* In order to prove the sufficiency, we will show that the set of inequalities (12) implies the condition (27). To this end, notice that a solution of (12a) for some  $\mu_i \in \mathbb{R}$  and  $P_i$  with inertia  $\text{In}(P_i) = (d_i, 0, n - d_i)$  implies that  $\text{In}(-A + \mu I) = (d_i, 0, n - d_i)$  by means of

Lemma 10. That is,  $A$  has only  $d_i$  eigenvalues with real part strictly larger than  $\mu_i$ . This is equivalent to the bound

$$\Re(\lambda_{d_{i+1}}) \leq \Re(\lambda_{d_i+1}) < \mu_i, \quad \forall i \in \{0, \dots, \ell-1\}. \quad (28)$$

Now, due to Lemma 10, we have that  $\mu_{i+1} < \mu_i$  for all  $i \in \{0, \dots, \ell-2\}$ . Therefore, the bound (12b) implies  $\mu_{\ell-1} \leq 0$ , which combined with (28), implies  $\lambda_{d_\ell} < 0$ . Additionally, because the eigenvalues are ordered as per (26), the following bound trivially holds for all  $i \in \{0, \dots, \ell-1\}$

$$\sum_{j=d_i+1}^{d_{i+1}} \Re(\lambda_j) \leq (d_{i+1} - d_i) \Re(\lambda_{d_i+1}) = h_i \Re(\lambda_{d_i+1}),$$

where the definition  $h_i = (d_{i+1} - d_i)$  has been used. Combining this bound with the fact that  $d_\ell \leq k$ , the bound  $\lambda_{d_\ell} < 0$  and the fact that the eigenvalues are ordered as in (26), we obtain the following,

$$\begin{aligned} \sum_{i=1}^k \Re(\lambda_i) &\leq \sum_{i=1}^{d_\ell} \Re(\lambda_i) = \sum_{i=0}^{\ell-1} \sum_{j=d_i+1}^{d_{i+1}} \Re(\lambda_j) \\ &\leq \sum_{i=0}^{\ell-1} h_i \Re(\lambda_{d_i+1}). \end{aligned} \quad (29)$$

Then, combining (28), (29) and (12b) we have

$$\sum_{i=1}^k \Re(\lambda_i) < \sum_{i=0}^{\ell-1} h_i \mu_i \leq 0. \quad (30)$$

Consequently, the system is  $k$ -contractive by Lemma 9.

*Necessity.* Define

$$\begin{aligned} p_k &:= \max \left( \{\bar{d}_0, \bar{d}_1, \dots, \bar{d}_{q-1}\} \cap [0, k-1] \right), \\ c_k &:= \text{card} \left( \{\bar{d}_0, \bar{d}_1, \dots, \bar{d}_{q-1}\} \cap [0, k-1] \right). \end{aligned} \quad (31)$$

Then, the following equality holds

$$(k - p_k) \alpha_{c_k} + \sum_{i=0}^{c_k-2} h_i \alpha_{i+1} = \sum_{i=1}^k \Re(\lambda_i). \quad (32)$$

Hence, combining Lemma 9 and (32), if the system is  $k$ -contractive, the next bound is satisfied

$$(k - p_k) \alpha_{c_k} + \sum_{i=0}^{c_k-2} h_i \alpha_{i+1} < 0. \quad (33)$$

Then, by continuity, there exists a scalar  $\varepsilon > 0$ , such that

$$(k - p_k)(\alpha_{c_k} + \varepsilon) + \sum_{i=0}^{c_k-2} h_i (\alpha_{i+1} + \varepsilon) \leq 0.$$

Next, fix  $\ell = c_k$ ,  $d_\ell = k - p_k + d_{\ell-1}$ ,  $d_i = \bar{d}_i$  for all  $i \in \{0, \dots, \ell-1\}$  and select  $\mu_{i-1} = \varepsilon + \alpha_i$ , for all  $i \in \{1, \dots, \ell\}$ . We have

$$\sum_{i=0}^{\ell-1} h_i \mu_i = (k - p_k)(\alpha_{c_k} + \varepsilon) + \sum_{i=0}^{c_k-2} h_i (\alpha_{i+1} + \varepsilon) \leq 0,$$

thus showing (12b). Now, define matrices  $\hat{A}_i := A - \mu_i I$  with  $i \in \{0, \dots, \ell-1\}$ . It is clear that, since  $\mu_{i-1} > \alpha_i$  by definition, each matrix  $\hat{A}_i$  has  $\bar{d}_i$  eigenvalues with positive real part and  $n - \bar{d}_i$  eigenvalues with negative real part. That is  $\text{In}(\hat{A}_i) = (n - \bar{d}_i, 0, \bar{d}_i)$ . Then, by Lemma 10, there exist symmetric matrices  $P_i$  with  $\text{In}(P_i) = \text{In}(-\hat{A}_i) = (\bar{d}_i, 0, n - \bar{d}_i)$  such that

$$A^\top P_i + P_i A \prec 2\mu_i P_i \quad \forall i = 0, \dots, \ell-1,$$

thus concluding the proof.

## VII. PROOF OF LINEAR RESULTS (DESIGN)

This section follows the next notation. First, we will consider that every pair  $(A, B)$ , is algebraically equivalent to the form in (13). Second, we define  $\Re(\lambda_1^u) \geq \dots \geq \Re(\lambda_{n_u}^u)$ , as the ordered set of eigenvalues of  $A_u$ .

### A. Proof of Lemma 3

*Sufficiency.* Without loss of generality, consider a system in the form (13). After a feedback design  $u = -Kx = -[K_c \ K_u](x_c^\top, x_u^\top)^\top$  is selected, the closed-loop eigenvalues are given by  $\sigma(A) = \sigma(A_c - B_c K_c) \cup \sigma(A_u)$ . Then, we can arbitrarily assign the eigenvalues of the closed-loop matrix  $A_c - B_c K_c$ , since the pair  $(A_c, B_c)$  is controllable. As a consequence, let  $c = k\lambda_1^u$  and select  $K_c$  such that the largest eigenvalue of  $A_c - B_c K_c$  has real part smaller than  $-|\Re(c)|$ . Then, the conditions of Lemma 9 are satisfied either if  $A_u$  is  $k$ -contractive or  $n_u < k$  by construction.

*Necessity.* If  $n_u \geq k$  and  $\dot{x} = A_u x$  is not  $k$ -contractive, then, there is a sum of  $k$  eigenvalues in the spectrum of  $A_u$  that is positive, see Lemma 9. Therefore, since the spectrum of  $A_u$  is invariant to the controller gain, the closed-loop system  $A - BK$  cannot be  $k$ -contractive (invoking again Lemma 9), which proves necessity.

### B. Inertia theorems generalizing stabilizability conditions

We state a set of new technical lemmas related to the feasibility and the inertia of the generalized stabilizability-like inequality (14a). Note that the following lemmas do not require the pair  $(A, B)$  to be controllable, contrarily to [18].

**Lemma 11** Consider a pair of matrices  $(A, B)$  and its canonical decomposition (13). Suppose that for some  $\mu \in \mathbb{R}$ ,  $\text{In}(A_u - \mu I) = (n_u - \varrho, 0, \varrho)$  with  $\varrho \in \{0, \dots, n_u\}$ . Then, there exists a symmetric matrix  $W \in \mathbb{R}^{n \times n}$ , with inertia  $\text{In}(W) = (\varrho, 0, n - \varrho)$ , satisfying

$$WA^\top + AW - BB^\top \prec 2\mu W. \quad (34)$$

*Proof:* Without loss of generality, suppose that the pair  $(A, B)$  is in the form (13). Moreover, define the shifted matrices  $\hat{A}_c := A_c - \mu I$  and  $\hat{A}_u := A_u - \mu I$ . Then, since  $\text{In}(\hat{A}_u) = (n_u - \varrho, 0, \varrho)$  and by Lemma 10, there exist some symmetric matrix  $W_u \in \mathbb{R}^{n_u \times n_u}$  with inertia  $\text{In}(W_u) = (\varrho, 0, n_u - \varrho)$  such that

$$W_u \hat{A}_u^\top + \hat{A}_u W_u = -Q, \quad (35)$$

with  $Q \succ 0$ . Furthermore, since the pair  $(A_c, B_c)$  is controllable, the pair  $(-\gamma I - A_c, B_c)$  is also controllable for any  $\gamma \in \mathbb{R}$ . Hence, for  $\gamma > 0$  large enough and from the Lyapunov test of controllability [33, Theorem 12.4], there exist some positive symmetric matrix  $W_c \succ 0$

$$W_c(-\gamma I - \hat{A}_c)^\top + (-\gamma I - \hat{A}_c)W_c = -B_c B_c^\top$$

which implies

$$W_c \hat{A}_c^\top + \hat{A}_c W_c - B_c B_c^\top = -2\gamma W_c. \quad (36)$$

With this in mind, consider a symmetric matrix  $W$  with inertia  $\text{In}(W) = (\varrho, 0, n - \varrho)$  of the form

$$W = \begin{bmatrix} W_c & 0 \\ 0 & \kappa W_u \end{bmatrix}$$

where  $\kappa > 0$  has to be fixed, with  $W_u$  and  $W_c$  satisfying (35) and (36). Now, by subtracting  $2\mu W$  from the left-hand side of (34), we

get the following equality

$$\begin{bmatrix} W_c & 0 \\ 0 & \kappa W_u \end{bmatrix} \begin{bmatrix} \hat{A}_c^\top & A_{12}^\top \\ 0 & \hat{A}_u^\top \end{bmatrix} + \begin{bmatrix} \hat{A}_c & A_{12} \\ 0 & \hat{A}_u \end{bmatrix} \begin{bmatrix} W_c & 0 \\ 0 & \kappa W_u \end{bmatrix} - \begin{bmatrix} B_c B_c^\top & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2\gamma W_c & \kappa A_{12} W_u \\ \kappa W_u A_{12}^\top & -\kappa Q \end{bmatrix}. \quad (37)$$

Since  $W_c$  and  $Q$  are positive definite, the right hand side of identity (37) can be made negative definite by taking  $\kappa > 0$  sufficiently small, see e.g. [33, Section 14.4]. ■

Additionally, we present the following technical lemma.

**Lemma 12** Consider a pair of matrices  $(A, B)$  and its canonical decomposition (13). Moreover, assume there exists a (non-singular) symmetric matrix  $W \in \mathbb{R}^{n \times n}$  and a constant  $\mu \in \mathbb{R}$  such that

$$WA^\top + AW - BB^\top \prec 2\mu W. \quad (38)$$

Then,  $\pi_-(W) \geq \pi_-(-A_u + \mu I)$ .

*Proof:* Without loss of generality, we suppose that the pair  $(A, B)$  is in the form (13). Moreover, notice that the inequality (38) can be re-arranged as follows

$$W\bar{A}^\top + \bar{A}W \prec 2\mu W, \quad (39)$$

where  $\bar{A} := A - \frac{1}{2}BB^\top W^{-1}$ . Now, recall the notation in (13) and notice that the eigenvalues in the spectrum of  $A_u$  cannot be modified by the term  $\frac{1}{2}BB^\top P^{-1}$ , thus, we have  $\sigma(A_u) \subsetneq \sigma(\bar{A})$ , or, equivalently,  $\sigma(A_u - \mu I) \subsetneq \sigma(\bar{A} - \mu I)$ . From this fact we obtain that  $\pi_-(-\bar{A} + \mu I) \geq \pi_-(-A_u + \mu I)$ . Then, by Lemma 10 we have that any (non-singular)  $W$  that satisfies (39) necessarily implies  $\pi_-(-\bar{A} + \mu I) = \pi_-(W)$ , which concludes the proof. ■

Finally, we present a technical lemma that relates the colinearity condition in (15) and the stabilizability-like inequality (14a).

**Lemma 13** Consider a pair of matrices  $(A, B)$  and its canonical decomposition (13). Suppose that for some  $\mu_1, \mu_2 \in \mathbb{R}$  with  $\mu_1 \leq \mu_2$ ,  $\text{In}(A_u - \mu_1 I) = (n_u - \varrho_1, 0, \varrho_1)$ ,  $\text{In}(A_u - \mu_2 I) = (n_u - \varrho_2, 0, \varrho_2)$  with  $\varrho_1, \varrho_2 \in \{0, \dots, n_u\}$ . Then, there exist a pair of symmetric matrices  $W_1, W_2 \in \mathbb{R}^{n \times n}$ , with inertia  $\text{In}(W_1) = (\varrho_1, 0, n - \varrho_1)$ ,  $\text{In}(W_2) = (\varrho_2, 0, n - \varrho_2)$ , satisfying

$$W_1 A^\top + AW_1 - BB^\top \prec 2\mu_1 W_1, \quad (40a)$$

$$W_2 A^\top + AW_2 - BB^\top \prec 2\mu_2 W_2, \quad (40b)$$

$$B^\top W_2^{-1} = B^\top W_1^{-1}. \quad (41)$$

*Proof:* Without loss of generality, we suppose the pair  $(A, B)$  to be in the form (13). This is not a restrictive assumption since the colinearity condition (41) is preserved under linear coordinate changes  $z = Tx$ , for any non-singular constant matrix  $T \in \mathbb{R}^{n \times n}$ .

Now, since  $\text{In}(A_u - \mu_1 I) = (n_u - \varrho_1, 0, \varrho_1)$ , we can follow similar arguments as in Lemma 11 proof, to show that (40a) can be satisfied with a symmetric matrix  $W_1$  of the form

$$W_1 = \begin{bmatrix} W_c & 0 \\ 0 & \kappa_1 W_{1,u} \end{bmatrix} \quad (42)$$

where  $\kappa_1 > 0$  is a sufficiently small constant,  $W_{1,u} \in \mathbb{R}^{n_u \times n_u}$  is a symmetric matrix with inertia  $\text{In}(W_{1,u}) = (\varrho_1, 0, n_u - \varrho_1)$  and  $W_c \succ 0$  is a positive definite symmetric matrix computed from

$$W_c(A_c - \mu_1 I)^\top + (A_c - \mu_1 I)W_c - B_c B_c^\top = -2\gamma W_c. \quad (43)$$

for some positive  $\gamma > 0$ .

With this in mind, we can construct a solution to (40b) such that  $W_1$  and  $W_2$  are colinear according to (41). Since  $\text{In}(A_u - \mu_2 I) = (n_u - \varrho_2, 0, \varrho_2)$  and by Lemma 10, there exists some symmetric matrix  $W_{2,u} \in \mathbb{R}^{n_u \times n_u}$  with inertia  $\text{In}(W_{2,u}) = (\varrho_2, 0, n_u - \varrho_2)$  such that, for some  $Q \succ 0$ ,

$$W_{2,u}(A_u - \mu_2 I)^\top + (A_u - \mu_2 I)W_{2,u} = -Q. \quad (44)$$

Moreover, recall the relation (43), then, we derive the following

$$\begin{aligned} & W_c(A_c - \mu_2)^\top + (A_c - \mu_2 I)W_c - B_c B_c^\top \\ &= W_c(A_c - \mu_1 I)^\top + (A_c - \mu_1 I)W_c + 2(\mu_1 - \mu_2)W_c - B_c B_c^\top \\ &= -2(\gamma + \mu_2 - \mu_1)W_c. \end{aligned} \quad (45)$$

With this in mind, consider a symmetric matrix  $W_2$  of the form

$$W_2 = \begin{bmatrix} W_c & 0 \\ 0 & \kappa_2 W_{2,u} \end{bmatrix} \quad (46)$$

where  $\kappa_2 > 0$  has to be fixed, with  $W_{2,u}$  and  $W_c$  satisfying (44) and (45). Now, by subtracting  $2\mu_2 W_2$  from the left-hand side of (40b) and defining  $\hat{A}_c := A_c - \mu_2 I$  and  $\hat{A}_u := A_u - \mu_2 I$ , we get the following equality

$$\begin{aligned} & \begin{bmatrix} W_c & 0 \\ 0 & \kappa_2 W_{2,u} \end{bmatrix} \begin{bmatrix} \hat{A}_c^\top & A_{12}^\top \\ 0 & \hat{A}_u^\top \end{bmatrix} + \begin{bmatrix} \hat{A}_c & A_{12} \\ 0 & \hat{A}_u \end{bmatrix} \begin{bmatrix} W_c & 0 \\ 0 & \kappa_2 W_{2,u} \end{bmatrix} \\ & - \begin{bmatrix} B_c B_c^\top & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2(\gamma + \mu_2 - \mu_1)W_c & \kappa_2 A_{12} W_{2,u} \\ \kappa_2 W_{2,u} A_{12}^\top & -\kappa_2 Q \end{bmatrix}. \end{aligned} \quad (47)$$

Recall that  $\mu_1 \leq \mu_2$  by assumption and  $\gamma > 0, W_c \succ 0$  by design. Consequently,  $-2(\gamma + \mu_2 - \mu_1)W_c$  is negative definite and the right hand side of identity (47) can be made negative definite by taking  $\kappa_2 > 0$  sufficiently small.

Finally, since the system is in the form (13), we have that  $W_1, W_2$  constructed as in the block-diagonal form (42) and (46) satisfy (41).  $\blacksquare$

### C. Proof of Theorem 4

Now, similar to the proof in Section VI-B, we introduce a notation in order to represent the eigenvalues of  $A_u$  and their associated multiplicities. Precisely, consider the matrix  $A_u$  in (13) and  $\sigma(A_u)$  its spectrum and suppose  $\Pi(\sigma(A_u)) = \{\alpha_1, \alpha_2, \dots, \alpha_q\}$  ( $q \leq n_u$ ) where again  $\Pi : \mathbb{C} \rightarrow \mathbb{R}$  denote the canonical projection onto the real axis and with  $\alpha_1 > \alpha_2 > \dots > \alpha_q$ . Set  $\bar{h}_i = \text{card}(\Pi^{-1}(\alpha_{i+1}) \cap \sigma(A_u))$ , where eigenvalues have been counted with their algebraic multiplicities (so that  $\bar{h}_1 + \bar{h}_2 + \dots + \bar{h}_{q-1} = n_u$ ). Finally, let  $\bar{d}_0 = 0, \bar{d}_i = \sum_{j=1}^{i-1} \bar{h}_j, i \in \{1, \dots, q-1\}$ . Similar to Section VI-B,  $\bar{h}_i$  represents the number of eigenvalues of  $A_u$  projected to  $\alpha_{i+1}$  and  $\bar{d}_i$  represents the amount of eigenvalues with real part strictly than  $\alpha_{i+1}$ .

Finally, by recalling Lemma 3 and Lemma 9, a necessary and sufficient condition for  $k$ -order stabilizability is either  $n_u < k$  or

$$\sum_{j=1}^k \Re(\lambda_j^u) < 0. \quad (48)$$

We now present the main arguments proving that the inequalities (14) are necessary and sufficient for  $k$ -order stabilizability.

*Sufficiency.* The goal of this proof is to show that if (14) is satisfied, then either (48) is satisfied or  $n_u < k$ , hence showing the result invoking Lemmas 3 and 9. Without loss of generality, we suppose that the pair  $(A, B)$  is in the form (13).

Firstly, we assume the case  $k \leq n_u$  and we show that (14) implies (48). To this end and by means of Lemma 12, we have that inequality

(14a) implies that  $\pi_-(W_i) \geq \pi_-(-A_u + \mu_i I)$  for all  $i \in \{0, \dots, \ell-1\}$ . Recalling that  $W_i$  has inertia  $\text{In}(W_i) = (d_i, 0, n - d_i)$ , we have that  $A_u$  has at most  $d_i$  eigenvalues with real part strictly larger than  $\mu_i$ . This is equivalent to the bound

$$\Re(\lambda_{d_{i+1}}^u) \leq \Re(\lambda_{d_i+1}^u) < \mu_i, \quad \forall i \in \{0, \dots, \ell-1\}. \quad (49)$$

Then, following similar arguments as in the sufficiency part of Section VI-B, the next bound can be obtained.

$$\sum_{i=1}^k \Re(\lambda_i^u) < \sum_{i=0}^{\ell-1} h_i \mu_i \leq 0. \quad (50)$$

Thus, (48) is satisfied and the pair  $(A, B)$  is  $k$ -order stabilizable invoking Lemmas 3 and 9.

Finally, for the case  $k > n_u$ , we have  $k$ -order stabilizability directly from Lemma 3, thus ending the sufficiency proof.

*Necessity.* As stated before, if the pair  $(A, B)$  is  $k$ -order stabilizable then, either (48) or  $n_u < k$  is verified. The goal of this proof is to show that if one of these conditions are satisfied, then, there exists a solution to the inequalities (14). We begin by assuming the case  $k \leq n_u$  and (48) is verified. Now, let the scalars  $p_k$  and  $c_k$  be defined as in (31).

Then, the following equality holds

$$(k - p_k)\alpha_{c_k} + \sum_{i=0}^{c_k-2} h_i \alpha_{i+1} = \sum_{j=1}^k \Re(\lambda_j^u). \quad (51)$$

Hence, combining (48) and (51), if the system is  $k$ -order stabilizable (with  $k \leq n_u$ ), the next bound is satisfied

$$(k - p_k)\alpha_{c_k} + \sum_{i=0}^{c_k-2} h_i \alpha_{i+1} < 0. \quad (52)$$

Then, by continuity, there exist a scalar  $\varepsilon > 0$ , such that

$$(k - p_k)(\alpha_{c_k} + \varepsilon) + \sum_{i=0}^{c_k-2} h_i (\alpha_{i+1} + \varepsilon) \leq 0$$

Now, fix  $\ell = c_k, d_\ell = k - p_k + d_{\ell-1}, d_i = \bar{d}_i$  for all  $i \in \{0, \dots, \ell-1\}$  and select  $\mu_{i-1} = \varepsilon + \alpha_i$ , for all  $i \in \{1, \dots, \ell\}$ . We have

$$\sum_{i=0}^{\ell-1} h_i \mu_i = (k - p_k)(\alpha_{c_k} + \varepsilon) + \sum_{i=0}^{c_k-2} h_i (\alpha_{i+1} + \varepsilon) \leq 0,$$

thus showing (14b). Now, since  $\mu_i > \alpha_{i+1}$  for all  $i \in \{0, \dots, \ell-1\}$  we have that  $A_u$  has only  $\bar{d}_i$  eigenvalues strictly larger than  $\mu_i$  and the rest are strictly smaller. That is,  $\text{In}(A_u - \mu_i I) = (n_u - \bar{d}_i, 0, \bar{d}_i)$  for all  $i \in \{0, \ell-1\}$ . Then, by Lemma 11, there exist symmetric matrices  $W_i$  with  $\text{In}(W_i) = \{\bar{d}_i, 0, n - \bar{d}_i\}$  such that

$$A^\top W_i + W_i A - B B^\top \prec 2\mu_i W_i \quad \forall i \in \{0, \dots, \ell-1\},$$

thus concluding the proof if (48) is verified and  $k \leq n_u$ .

We now proceed with the necessity proof for the case  $k > n_u$ . For this proof, we remark that since a  $k$ -contractive system is also  $\bar{k}$ -contractive for all  $\bar{k} \in \{k, \dots, n\}$  [11], we have that if a pair  $(A, B)$  is  $k$ -order stabilizable, then, it is also  $\bar{k}$ -order stabilizable for all  $\bar{k} \in \{k, \dots, n\}$ . Additionally, by means of Lemma 3, a pair  $(A, B)$  is always  $k$ -order stabilizable if  $k = n_u + 1$ . Therefore, for all  $k > n_u$ ,  $k$ -order stabilizability necessarily implies  $\bar{k}$ -order stabilizability with  $\bar{k} = n_u + 1$ . With this fact in mind, this proof is based on showing that, if  $k = n_u + 1$ , then, there always exists a pair of matrices  $W_0, W_1$  and constants  $\mu_0, \mu_1$  such that (14) is satisfied. We highlight that this result does not require (48) to be satisfied.

Precisely, assume  $k = n_u + 1$ . Notice that we can always guarantee  $\text{In}(A_u - \mu_0 I) = (n_u, 0, 0)$  for any  $\mu_0 \in \mathbb{R}$  large enough. Therefore,

by considering this sufficiently large  $\mu_0$  and by means of Lemma 11, we know that there exists a symmetric matrix  $W_0$  with inertia  $\text{In}(W_0) = (0, 0, n)$  solution of (14a). Furthermore, we can always find a sufficiently negative constant  $\mu_1 < 0$ , such that

$$\mu_1 + n_u \mu_0 \leq 0, \quad (53)$$

and  $\text{In}(A_u - \mu_1 I) = (0, 0, n_u)$ . Therefore, by considering this  $\mu_1$  and by means of Lemma 11, there exists a symmetric matrix  $W_1$  with inertia  $\text{In}(W_1) = (n_u, 0, n - n_u)$  solution of (14a). Finally, fix  $\ell = 2$  and select the aforementioned pair of matrices  $W_0, W_1$  and pair of constants  $\mu_0, \mu_1$  (these matrices and constants satisfy (14a)). Moreover, fix  $d_\ell = n_u + 1$ . With this selection, we have  $d_0 = 0, d_1 = n_u$  and  $h_0 = n_u, h_1 = 1$ . Thus, (53) implies (14b).

#### D. Proof of Proposition 1

The first part of the proof focuses on proving the existence of solutions for the inequalities (14) considering the assumptions stated in the theorem and in particular the colinearity condition in (15). An immediate result of Lemma 13 is that there always exist a set of constants  $\mu_i$  and  $W_i$  such that (14b) and the colinearity condition in (15) is simultaneously satisfied for all  $i \in \{1, \dots, \ell - 1\}$ . Moreover, notice that Lemma 13 preserves the relation between the inertia of  $A_u - \mu_i I$  and  $W_i$  as in Lemma 11. Consequently, the arguments in the necessity part of Section VII-C could be repeated to obtain the existence of a solution from a  $k$ -order stabilizability assumption.

The second part of the proof consist in showing how the state-feedback law (16) makes the closed-loop system

$$\dot{x} = (A - BK)x = (A - \frac{\rho}{2} BB^\top W_0^{-1})x, \quad (54)$$

$k$ -contractive for all  $\rho \geq 1$ . Note that, since  $W_i$  is non-singular and symmetric for all  $i \in \{0, \dots, \ell - 1\}$  and by means of the colinearity condition (15), the left-hand side of (14a) can be rearranged as follows for all  $i \in \{0, \dots, \ell - 1\}$

$$\begin{aligned} & W_i A^\top + A W_i - B B^\top \\ &= W_i (A - \frac{1}{2} B B^\top W_i^{-1})^\top + (A - \frac{1}{2} B B^\top W_i^{-1}) W_i \\ &= W_i (A - \frac{1}{2} B B^\top W_0^{-1})^\top + (A - \frac{1}{2} B B^\top W_0^{-1}) W_i. \end{aligned}$$

Combining this result with the right-hand side of (14a), we obtain that for all  $i \in \{0, \dots, \ell - 1\}$

$$W_i (A - \frac{1}{2} B B^\top W_0^{-1})^\top + (A - \frac{1}{2} B B^\top W_0^{-1}) W_i \prec 2\mu_i W_i. \quad (55)$$

Now, by adding  $(1 - \rho) B B^\top$  in both sides of (55) and considering the fact that  $(1 - \rho) B B^\top \preceq 0$  for all  $\rho \geq 1$ , by (15) we get,

$$\begin{aligned} & W_i (A - \frac{\rho}{2} B B^\top W_0^{-1})^\top + (A - \frac{\rho}{2} B B^\top W_0^{-1}) W_i \\ & \prec 2\mu_i W_i + (1 - \rho) B B^\top \preceq 2\mu_i W_i. \end{aligned} \quad (56)$$

By post-multiplying and pre-multiplying both side of (56) by  $W_i^{-1}$  and defining  $P_i := W_i^{-1}$  we get for all  $i \in \{0, \dots, \ell - 1\}$

$$(A - \frac{\rho}{2} B B^\top W_0^{-1})^\top P_i + P_i (A - \frac{\rho}{2} B B^\top W_0^{-1}) \prec 2\mu_i P_i. \quad (57)$$

Finally, combining (57) and (14b) with Theorem 3 proves that the closed-loop system (54) is  $k$ -contractive.

## VIII. PROOFS OF NONLINEAR RESULTS

### A. Preliminary results

We provide in this section some preliminary results that will be used in the proof of Theorem 5. We start by recalling the definition of  $p$ -dominance [14].

**Definition 7 ( $p$ -dominance)** System (1) is said to be strictly  $p$ -dominant on  $\mathcal{S} \subseteq \mathbb{R}^n$  if<sup>1</sup> there exist a real number  $\mu \geq 0$  and a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  with inertia  $\text{In}(P) = (p, 0, n - p)$  such that

$$P \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P \prec -2\mu P, \quad \forall x \in \mathcal{S}. \quad (58)$$

Then, we recall (with a mild reformulation) the following result on  $p$ -dominance [14, Theorem 1].

**Theorem 6** Suppose that system (1) is strictly  $p$ -dominant on a compact forward invariant set  $\mathcal{A} \subseteq \mathbb{R}^n$  with rate  $\mu > 0$  and symmetric matrix  $P$  with inertia  $\text{In}(P) = (p, 0, n - p)$ . Then, for each  $x \in \mathcal{A}$ , there exists an invariant splitting  $T_x \mathbb{R}^n = \mathcal{V}_x \oplus \mathcal{H}_x$ , i.e. there exists a continuous mapping  $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  invertible for any  $x \in \mathcal{A}$  and satisfying

$$\mathbf{T}(x) := [\mathbf{T}_h(x) \quad \mathbf{T}_v(x)], \quad (59a)$$

where  $\mathbf{T}_h : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n-p}$  and  $\mathbf{T}_v : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$  satisfy

$$\text{Im } \mathbf{T}_h(x) = \mathcal{H}_x, \quad \text{Im } \mathbf{T}_v(x) = \mathcal{V}_x. \quad (59b)$$

Moreover, there exist a scalar  $c_h > 0$  such that

$$\left| \frac{\partial \psi^t}{\partial x}(x) [\mathbf{T}_h(x) \quad 0] v \right| \leq c_h e^{-\mu t} \left| [\mathbf{T}_h(x) \quad 0] v \right| \quad (59c)$$

holds for all  $(t, x, v) \in \mathbb{R}_{\geq 0} \times \mathcal{A} \times T_x \mathbb{R}^n$ .

With this in mind, it is clear that if  $\mu_1$  is strictly negative, the matrix inequality (17b) imposes a form of horizontal contraction on the system [8, Section VII]. Nonetheless, horizontal contraction is not a sufficient condition for  $k$ -contraction [15]. This motivates (17a). We clarify the effects of (17a) via the following Lemma.

**Lemma 14** Consider system (1) and assume there exist a forward invariant compact set  $\mathcal{A} \subseteq \mathbb{R}^n$ , a symmetric positive definite matrix  $P_0 \in \mathbb{R}^{n \times n}$  and a scalar  $\mu_0$  satisfying (17a) for all  $x \in \mathcal{A}$ . Then there exists a constant  $c_v > 0$  such that

$$\left| \frac{\partial \psi^t}{\partial x}(x) [0 \quad \mathbf{T}_v(x)] v \right| \leq c_v e^{\mu_0 t} \left| [0 \quad \mathbf{T}_v(x)] v \right| \quad (60)$$

for all  $(t, x, v) \in \mathbb{R}_{\geq 0} \times \mathcal{A} \times T_x \mathbb{R}^n$ , with  $\mathbf{T}_v$  as in (59b).

*Proof:* Consider the function,  $W := v^\top P_0 v$ . It satisfies

$$\underline{\lambda}(P_0) |v|^2 \leq W(v) \leq \bar{\lambda}(P_0) |v|^2, \quad (61)$$

where  $\underline{\lambda}(\cdot)$  and  $\bar{\lambda}(\cdot)$  represent the minimum and maximum eigenvalue of their argument, respectively. By (6), its time-derivative satisfies

$$\begin{aligned} \dot{W} &= v^\top \left( P_0 \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P_0 \right) v \\ &< 2\mu_0 v^\top P_0 v = 2\mu_0 W. \end{aligned}$$

Then, by Grönwall–Bellman inequality, we obtain

$$W(t) \leq e^{2\mu_0 t} W(0), \quad \forall t \in \mathbb{R}_{\geq 0}.$$

Invoking (61), we obtain for all  $(t, x, v) \in \mathbb{R}_{\geq 0} \times \mathcal{A} \times T_x \mathbb{R}^n$

$$\left| \frac{\partial \psi^t}{\partial x}(x) v \right| \leq \sqrt{\frac{\bar{\lambda}(P_0)}{\underline{\lambda}(P_0)}} e^{\mu_0 t} |v|.$$

As  $[0 \quad \mathbf{T}_v(x)] v \in T_x \mathbb{R}^n$ , the result trivially follows.  $\blacksquare$

Given the above results, condition (17c) can be seen as imposing a bound on the maximum expansion rate of the vertical subspace with

<sup>1</sup>The definition can be extended to the full set  $\mathbb{R}^n$  but in this case condition (58) is modified into  $P \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P \preceq -2\mu P - \varepsilon I$  for all  $x \in \mathbb{R}^n$  where the term  $-\varepsilon I$  is added to ensure uniformity.

respect to the contraction rate of the horizontal one. In particular, (17c) holds if the first is smaller than the latter. We now relate this property to infinitesimal  $k$ -contraction. As a first step, we present a technical lemma related to matrix compounds.

**Lemma 15** Consider a time-varying matrix  $M(t) \in \mathbb{R}^{n \times n}$

$$M(t) = [H(t) \quad V(t)],$$

with  $H(t) \in \mathbb{R}^{n \times n-p}$ ,  $V(t) \in \mathbb{R}^{n \times p}$  and  $p \in [0, n]$ . Assume there exist real numbers  $c_h, c_v, \alpha, \beta > 0$  such that

$$|H(t)| \leq c_h e^{-\alpha t}, \quad |V(t)| \leq c_v e^{\beta t}, \quad \forall t \in \mathbb{R}_+. \quad (62)$$

If  $\alpha > (k-1)\beta$  for some integer  $k \in [p+1, n]$ , there exist some real numbers  $c, \varepsilon > 0$  such that

$$|M(t)^{(k)}| \leq c e^{-\varepsilon t}, \quad \forall t \in \mathbb{R}_+. \quad (63)$$

*Proof:* Consider the elements of the compound matrix  $M(t)^{(k)}$ . Each one is a  $k^{th}$ -order minor of the original matrix  $M(t)$ , i.e., it is the determinant of a  $k \times k$  submatrix of  $M(t)$ , see Definition 2. Since  $k \geq p+1$ , each  $k \times k$  submatrix contains at least one column composed of elements of  $H(t)$ . That is, in the minimum case

$$M_k(t) = [h(t) \quad v_1(t) \quad \dots \quad v_{k-1}(t)], \quad (64)$$

where  $M_k(t) \in \mathbb{R}^{k \times k}$  is a submatrix of  $M(t)$ ,  $h(t) \in \mathbb{R}^k$  is a vector with components of  $H(t)$  and  $v_i(t) \in \mathbb{R}^k$  for  $i = 1, \dots, k-1$  is a vector with components of  $V(t)$ . In what follows, we show the elements of  $M(t)^{(k)}$  are bounded. Hence, we focus on submatrices of the form (64), since their determinant represents the worst-case scenario in a stability sense. By definition of the wedge product,

$$\det(M_k(t)) = h(t) \wedge v_1(t) \wedge \dots \wedge v_{k-1}(t).$$

The wedge product can be represented using a basis  $e^i$ , where  $e^i$  depicts the  $i$ th canonical vector of  $\mathbb{R}^n$ . More specifically, by bilinearity of the wedge product, we have

$$\det(M_k(t)) = \sum_{i=1}^n h^i(t) (e^i \wedge v_1(t) \wedge \dots \wedge v_{k-1}(t)),$$

where  $h^i(t)$  is the  $i$ th element of  $h(t)$ . By performing similar operations on the remaining vectors we deduce

$$\det(M_k(t)) = \sum_{i_1=1}^k \dots \sum_{i_k=1}^k h^{i_1}(t) v_2^{i_2}(t) \dots v_{k-1}^{i_{k-1}}(t) E_k, \quad (65)$$

where  $E_k := (e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_k})$ . By (62), we have

$$|h^i(t)| \leq c_h e^{-\alpha t}, \quad |v^i(t)| \leq c_v e^{\beta t}.$$

Moreover, the factor  $E_k$  will be either zero or an element of the canonical basis in  $\mathbb{R}^n$  multiplied by plus or minus one. Thus, using the triangle inequality, one obtains

$$|\det(M_k(t))| \leq \kappa c_h c_v^{k-1} e^{-(\alpha + (k-1)\beta)t}$$

where  $\kappa > 0$  is a positive constant related to the number of non-zero instances of  $E_k$ . Now, since  $\alpha - (k-1)\beta > 0$  by assumption, by continuity there always exists  $\varepsilon > 0$  such that  $\alpha - (k-1)\beta - \varepsilon > 0$ . Then,

$$|M(t)^{(k)}| = |e^{-\varepsilon t} e^{\varepsilon t} M(t)^{(k)}| \leq e^{-\varepsilon t} |e^{\varepsilon t} M(t)^{(k)}|.$$

By considering the worst-case (64), we have

$$e^{\varepsilon t} |\det(M_k(t))| \leq \bar{c} e^{(-\alpha + (k-1)\beta + \varepsilon)t},$$

for some  $\bar{c} > 0$ . Hence, since  $\alpha - (k-1)\beta - \varepsilon > 0$ , each element of  $e^{\varepsilon t} M(t)^{(k)}$  is exponentially decreasing and the norm  $|e^{\varepsilon t} M(t)^{(k)}|$  is uniformly bounded for all  $t \in \mathbb{R}_{\geq 0}$ , thus concluding the proof. ■

Leveraging on the previous lemmas, we provide a bound on the compound of the variational system (6) state transition matrix.

**Lemma 16** Consider system (1) and assume there exist two symmetric matrices  $P_0, P_1 \in \mathbb{R}^{n \times n}$  of respective inertia  $(0, 0, n)$  and  $(k-1, 0, n-k+1)$ , and  $\mu_0, \mu_1 \in \mathbb{R}$  such that (17) is satisfied. Then, there exist  $\varepsilon, c > 0$  such that

$$\left| \frac{\partial \psi^t}{\partial x}(x)^{(k)} \right| \leq c e^{-\varepsilon t}, \quad \forall (t, x) \in \mathbb{R}_{\geq 0} \times \mathcal{A}. \quad (66)$$

*Proof:* Consider (59a) in Theorem 6. Invertibility of  $\mathbf{T}(x)$  yields

$$\frac{\partial \psi^t}{\partial x}(x) = \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}(x) \mathbf{T}(x)^{-1} = \boldsymbol{\psi}^t(x) \mathbf{T}(x)^{-1},$$

with  $\boldsymbol{\psi}^t(x) := \begin{bmatrix} \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) & \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_v(x) \end{bmatrix}$ . Given any  $v \in T_x \mathbb{R}^n$ , consider the decomposition  $v = (v^h, v^v)$ , where  $v^h \in \mathbb{R}^{n-p}$  and  $v^v \in \mathbb{R}^p$ . Then, for an arbitrary  $v^h$ , inequality (59c) of Theorem 6 implies

$$\left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) v^h \right| \leq c_h e^{\mu_1} |\mathbf{T}_h(x) v^h|.$$

Recall the definition of matrix norm,

$$\left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) \right| := \max_{|u|=1} \left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) u \right|.$$

By selecting vector  $u^*$  such that  $|u^*| = 1$ , the previous exponential relation and the triangular inequality yield

$$\begin{aligned} \left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) \right| &= \left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) u^* \right| \\ &\leq c_h e^{\mu_1} |\mathbf{T}_h(x) u^*| \leq c_h e^{\mu_1} |\mathbf{T}_h(x)|. \end{aligned}$$

Since  $\mathcal{A}$  is compact and  $\mathbf{T}$  is continuous,  $|\mathbf{T}_h(x)|$  is bounded for all  $x \in \mathcal{A}$ . Then, by (59c), and by (60) we obtain

$$\begin{aligned} \left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) \right| &\leq c_h e^{\mu_1} |\mathbf{T}_h(x)| \leq \bar{c}_h e^{-\mu_1} \\ \left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_v(x) \right| &\leq c_v e^{\mu_0} |\mathbf{T}_v(x)| \leq \bar{c}_v e^{\mu_0} \end{aligned}$$

for all  $x \in \mathcal{A}$ . Finally, by boundedness of  $\mathbf{T}(x)$  and Lemma 15,

$$\left| \frac{\partial \psi^t}{\partial x}(x)^{(k)} \right| \leq |\boldsymbol{\psi}^t(x)^{(k)}| |\mathbf{T}(x)^{-1(k)}| \leq c e^{-\varepsilon t}, \quad \forall x \in \mathcal{A}.$$

concluding the proof. ■

## B. Proof Theorem 5

Consider the  $k$ -th multiplicative compound of matrix  $\Psi(t, x_0)$  defined as in Section II-B. From the Cauchy-Binet formula [40, Chapter 1] we get:

$$\begin{aligned} \Psi(t, x_0)^{(k)} &= \left[ \frac{\partial \psi^t}{\partial x}(x_0) v_0^1 \quad \dots \quad \frac{\partial \psi^t}{\partial x}(x_0) v_0^k \right]^{(k)} \\ &= \frac{\partial \psi^t}{\partial x}(x_0)^{(k)} \Psi(0, x_0)^{(k)}. \end{aligned}$$

From (17) and Lemma 16 we obtain for all  $(x_0, t)$  in  $\mathcal{S} \times \mathbb{R}_{\geq 0}$

$$|(\Psi(t, x_0))^{(k)}| \leq c e^{-\varepsilon t} |(\Psi(0, x_0))^{(k)}|.$$

Hence, the system is infinitesimally  $k$ -contractive on  $\mathcal{S}$  and the  $k$ -contractive property follows from Theorem 1.

### C. Proof of Lemma 4

Let us decompose the Jacobian of the vector field  $f$  as follows

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} F_s(x) & G_{12}(x) \\ G_{21}(x) & F_u(x) \end{bmatrix}.$$

Then, according to Theorem 2, a sufficient condition for 2-contraction in a set  $\mathcal{A}$  is:

$$\frac{\partial f}{\partial x}(x)^{[2]} = F_s(x) + F_u(x) < 0, \quad \forall x \in \mathcal{S}. \quad (67)$$

By subtracting  $2\mu_0 P_0$  in both sides of (17a), the following inequality is obtained

$$\begin{pmatrix} F_s(x) - \mu_0 & G_{12}(x) \\ G_{21}(x) & F_u(x) - \mu_0 \end{pmatrix} P_0 + P_0 \begin{pmatrix} F_s(x) - \mu_0 & G_{21}(x) \\ G_{12}(x) & F_u(x) - \mu_0 \end{pmatrix} < 0.$$

Since  $P_0$  is positive definite, the previous inequality necessarily implies that  $\frac{\partial f}{\partial x}(x) - \mu_0 I$  is Hurwitz for all  $x \in \mathcal{A}$ , and, consequently, its determinant is positive. That is,

$$0 < F_s(x)F_u(x) - \mu_0(F_s(x) + F_u(x)) + \mu_0^2 - G_{12}(x)G_{21}(x). \quad (68)$$

Similarly, by subtracting  $2\mu_1 P_1$  in both sides of (17b), the following inequality is obtained

$$\begin{pmatrix} F_s(x) - \mu_1 & G_{12}(x) \\ G_{21}(x) & F_u(x) - \mu_1 \end{pmatrix} P_1 + P_1 \begin{pmatrix} F_s(x) - \mu_1 & G_{21}(x) \\ G_{12}(x) & F_u(x) - \mu_1 \end{pmatrix} < 0.$$

Since  $\text{In}(P_1) = (1, 0, 1)$ , by Lemma 10, this inequality necessarily implies that  $\text{In}\left(\frac{\partial f}{\partial x}(x) - \mu_1 I\right) = (1, 0, 1)$  for all  $x \in \mathcal{A}$ , and, consequently, its determinant is negative. That is,

$$F_s(x)F_u(x) - \mu_1(F_s(x) + F_u(x)) + \mu_1^2 - G_{12}(x)G_{21}(x) < 0,$$

which can be rearranged as

$$F_s(x)F_u(x) - G_{12}(x)G_{21}(x) < \mu_1(F_s(x) + F_u(x)) - \mu_1^2. \quad (69)$$

Now, combining (68) and (69) we get

$$0 < (\mu_1 - \mu_0)(F_s(x) + F_u(x)) + \mu_0^2 - \mu_1^2. \quad (70)$$

Then, since  $\mu_0 > \mu_1$  we have that  $\mu_1 - \mu_0 < 0$ , which, by combining (17c) and the fact that  $\mu_0, \mu_1$  are real, implies  $\mu_0^2 - \mu_1^2 < 0$ . Additionally, the conditions (17a)-(17c) imply that  $F_s(x) + F_u(x) \neq 0$  for all  $x \in \mathbb{R}^n$  by Lemma 10. Therefore, by combining (70) and (17c) we get (67), which ends the proof.

### D. Proof of Proposition 2

Consider inequality (17a), pre-multiply and post-multiply both sides of the inequality by  $P_0^{-1}$  and fix  $W_0 = P_0^{-1}$ . Similarly, pre-multiply and post-multiply both sides of the inequality (17b) by  $P_1^{-1}$  and fix  $W_1 = P_1^{-1}$ . Then, according to Theorem 5, the closed-loop system will be  $k$ -contractive if there exist two symmetric matrices  $W_0, W_1 \in \mathbb{R}^{n \times n}$  of inertia  $\text{In}(W_0) = (0, 0, n)$ ,  $\text{In}(W_1) = (1, 0, n - 1)$  and  $\bar{\mu}_0, \bar{\mu}_1 \in \mathbb{R}$  such that,

$$W_0 \left( \frac{\partial f}{\partial x}(x) - BK \right)^\top + \left( \frac{\partial f}{\partial x}(x) - BK \right) W_0 < 2\bar{\mu}_0 W_0, \quad (71a)$$

$$W_1 \left( \frac{\partial f}{\partial x}(x) - BK \right)^\top + \left( \frac{\partial f}{\partial x}(x) - BK \right) W_1 < 2\bar{\mu}_1 W_1, \quad (71b)$$

$$\bar{\mu}_1 + (k - 1)\bar{\mu}_0 < 0, \quad (71c)$$

for all  $x \in \mathcal{S}$ , where  $\mathcal{S}$  is assumed to be compact and forward invariant. In this proof, we show that if the inequalities in (19)-(20) are satisfied and the gain matrix  $K$  is designed as in (21), then, the inequalities in (71a)-(71c) are also satisfied. Thus, the closed-loop system is  $k$ -contractive according to Theorem 5. To this end,

note that the left-hand side of (71a) with  $K$  fixed as in (21) can be rewritten as:

$$\begin{aligned} W_0 \frac{\partial f}{\partial x}(x)^\top + \frac{\partial f}{\partial x}(x) W_0 - BB^\top \\ - \frac{1}{2} BB^\top W_1^{-1} W_0 - \frac{1}{2} W_0 W_1^{-1} BB^\top \\ < \mu_0 W_0 - \frac{1}{2} BB^\top W_1^{-1} W_0 - \frac{1}{2} W_0 W_1^{-1} BB^\top, \end{aligned} \quad (72)$$

where the right hand side is obtained employing (19a). Now, let  $\bar{\omega} > 0$  such that

$$\left[ I - \frac{1}{2} BB^\top W_1^{-1} \right] W_0 \left[ I - \frac{1}{2} BB^\top W_1^{-1} \right]^\top \preceq (1 + \bar{\omega}) W_0.$$

Furthermore, note that we have the identity

$$\begin{aligned} -\frac{1}{2} BB^\top W_1^{-1} W_0 - \frac{1}{2} W_0 W_1^{-1} BB^\top = \\ \left[ I - \frac{1}{2} BB^\top W_1^{-1} \right] W_0 \left[ I - \frac{1}{2} BB^\top W_1^{-1} \right]^\top - W_0 \\ - \frac{1}{4} BB^\top W_1^{-1} W_0 W_1^{-1} BB^\top. \end{aligned}$$

As a consequence, by adding and subtracting  $\bar{\omega} W_0$  from the right hand side of (72) and using the two previous equations, we obtain

$$\begin{aligned} W_0 \frac{\partial f}{\partial x}(x)^\top + \frac{\partial f}{\partial x}(x) W_0 - BB^\top \\ - \frac{1}{2} BB^\top W_1^{-1} W_0 - \frac{1}{2} W_0 W_1^{-1} BB^\top < (\mu_0 + \bar{\omega}) W_0 \end{aligned}$$

Thus, selecting  $\bar{\mu}_0 = \mu_0 + \bar{\omega}$ , shows (71a). Then, inequality (19b) can be re-organized to obtain the inequality (71b) with  $\bar{\mu}_1 = \mu_1$ . Finally, by fixing  $\omega = (k - 1)\bar{\omega}$ , we have that (20) implies (71c).

### E. Proof of Lemma 7

Due to Lemma 10, a necessary condition for the feasibility of (17a)-(17c) for  $k = 3$  is  $\mu_1 < \mu_0$ . Consequently, (17c) implies  $\mu_1 < 0$ . Therefore, the inequalities (17a)-(17c) for  $k = 3$  in a forward invariant set  $\mathcal{S}$  imply 2-dominance in  $\mathcal{S}$  as defined in Definition 7. Finally, the result follows from [14, Corollary 1] and that  $\mathcal{S} \subsetneq \mathbb{R}^n$ .

### F. Proof of Lemma 8

Let assumptions of Theorem 2 hold for  $k = n$  in a set  $\mathcal{S}$ . We have that  $\frac{\partial f}{\partial x}(x)^{[n]} = \text{tr}\left(\frac{\partial f}{\partial x}(x)\right) < 0$  for all  $x \in \mathcal{S}$ . Therefore, at least one eigenvalue of  $\frac{\partial f}{\partial x}(x)$  is negative for any  $x \in \mathbb{R}^n$  and in particular for all  $x^\circ$  such that  $f(x^\circ) = 0$ , thus showing the result.

## IX. CONCLUSIONS

We presented new conditions for  $k$ -contraction based on the use of generalized Lyapunov matrix inequalities. The proposed conditions can be checked without using matrix compounds. In the linear case, they reduce the  $k$ -contraction analysis to solving a set of matrix inequalities. In the nonlinear context, they extend the well-known Demidovich conditions based on the Jacobian of the vector field along the flow. Finally, we showed that the proposed conditions can be used to develop new tools for  $k$ -contractive feedback design, so that to extend existing conditions for standard 1-contraction.

Future works will focus on further studying the equivalence between  $k$ -contraction and infinitesimally  $k$ -contraction, and extending the proposed conditions to the context of time-varying systems and Riemannian metrics, similar to the context of 1-contraction, see, e.g. [9], [24]. Another topic is the design of  $k$ -contractive observers.

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## APPENDIX

### A. Proof of Lemma 2

The uniformity condition in (5) and the fact that  $P$  is positive definite and symmetric imply the existence of constants  $\bar{\sigma}, \underline{\sigma} > 0$  such that, for all  $(r, x)$  in  $[0, 1]^k \times \mathcal{S}$

$$\begin{aligned} V^k(\varphi \circ \Phi) &= \int_{[0,1]^k} \sqrt{\det \left\{ \frac{\partial \Phi}{\partial r}(r)^\top \frac{\partial \varphi}{\partial x}(x)^\top P \frac{\partial \varphi}{\partial x}(x) \frac{\partial \Phi}{\partial r}(r) \right\}} dr \\ &\leq \sqrt{\bar{\sigma}} \int_{[0,1]^k} \sqrt{\det \left\{ \frac{\partial \Phi}{\partial r}(r)^\top P \frac{\partial \Phi}{\partial r}(r) \right\}} dr = \sqrt{\bar{\sigma}} V^k(\Phi) \end{aligned}$$

and  $V^k(\Phi) \leq V^k(\varphi \circ \Phi) / \sqrt{\bar{\sigma}}$ . Consequently, if the system is  $k$ -contractive in  $\mathcal{S}$  in the original coordinates, we have

$$V^k(\varphi \circ \psi^t \circ \Phi) \leq \sqrt{\bar{\sigma}} V^k(\psi^t \circ \Phi) \leq \frac{\sqrt{\bar{\sigma}}}{\sqrt{\underline{\sigma}}} b e^{-at} V^k(\varphi \circ \Phi),$$

for all  $(r, x)$  in  $[0, 1]^k \times \mathcal{S}$ , thus showing  $k$ -contraction.

### B. Proof of Theorem 1

Consider  $\Phi \in \mathcal{I}_k$ , where  $\mathcal{I}_k$  is defined in (2), satisfying  $\text{Im}(\Phi) \subseteq \mathcal{S}$ . To simplify notation, let us denote for all  $(r, t)$  in  $[0, 1]^k \times \mathbb{R}_{\geq 0}$

$$\Gamma(r, t) := \psi^t \circ \Phi(r), \quad \Gamma_r(r, t) := \frac{\partial \Gamma}{\partial r}(r, t). \quad (73)$$

In words, the factor  $\Gamma(r, t)$  depicts the solution of (1) at time  $t$  taking as a initial condition a point in  $\Phi$  parametrized by  $r$ . Then,  $\Gamma_r(r, t)$  represents the effect of infinitesimal variations in  $r$  on the solution at time  $t$ . For all  $(r, t)$  in  $[0, 1]^k \times \mathbb{R}_{\geq 0}$ , we have

$$\frac{d}{dt} \Gamma(r, t) = f(\Gamma(r, t)).$$

Moreover, since  $\mathcal{S}$  is forward invariant and  $\text{Im}(\Phi) \subseteq \mathcal{S}$ , we have  $\Gamma(r, t) \in \mathcal{S}$  for all  $(r, t)$  in  $[0, 1]^k \times \mathbb{R}_{\geq 0}$ . Additionally, by the chain rule, it follows that the point  $\Gamma_r(r, t)$  evolves according to

$$\frac{d}{dt} \Gamma_r(r, t) = \frac{\partial^2 \Gamma}{\partial r \partial t}(r, t) = \frac{\partial f}{\partial x}(\Gamma(r, t)) \Gamma_r(r, t)$$

for all  $(r, t)$  in  $[0, 1]^k \times \mathbb{R}_{\geq 0}$ . Since these dynamics are linear, following similar steps as in [11, Section 2.5], we obtain

$$\frac{d}{dt} \Gamma_r(r, t)^{(k)} = \frac{\partial f}{\partial x}(\Gamma(r, t))^{[k]} \Gamma_r(r, t)^{(k)}. \quad (74)$$

Similarly, following [11, Section 2.5], it can be shown that the compound matrix of  $\Psi(t, x_0)$  evolves according to the linear dynamics

$$\frac{d}{dt} (\Psi(t, x_0))^{(k)} = \frac{\partial f}{\partial x}(\psi^t(x_0))^{[k]} \Psi(t, x_0)^{(k)}. \quad (75)$$

By (8), dynamics (75) are globally exponentially stable. Thus, considering (74), exponential stability and uniformity of (75) imply

$$\left| \Gamma_r(r, t)^{(k)} \right| \leq b e^{-at} \left| \Gamma_r(r, 0)^{(k)} \right|,$$

for some positive constants  $a, b > 0$ . Now, from the Cauchy-Binet formula [40, Chapter 1], the following equality holds

$$\begin{aligned} \det \left( \Gamma_r(r, t)^\top P \Gamma_r(r, t) \right) \\ = \left( \Gamma_r(r, t)^{(k)} \right)^\top P^{(k)} \Gamma_r(r, t)^{(k)} := v(r, t). \end{aligned} \quad (76)$$

Then, the volume  $V^k(\cdot)$  of  $\psi^t \circ \Phi$  computed according to (3) is

$$V^k(\psi^t \circ \Phi) = \int_{[0,1]^k} \sqrt{v(r, t)} dr.$$

Finally, by selecting  $P$  in (76) as the identity matrix, we obtain

$$\begin{aligned} V^k(\psi^t \circ \Phi) &= \int_{[0,1]^k} \left| \Gamma_r(r, t)^{(k)} \right| dr \leq \int_{[0,1]^k} b e^{-at} \left| \Gamma_r(r, 0)^{(k)} \right| dr \\ &\leq b e^{-at} \int_{[0,1]^k} \left| \Gamma_r(r, 0)^{(k)} \right| dr \leq b e^{-at} V^k(\Phi). \end{aligned}$$

### C. Proof of Theorem 2

Consider the state transition matrix  $\frac{\partial \psi^t}{\partial x}(x_0)$  of the variational system (6). Following [11, Section 2.5], it can be shown that the  $k$ -th order multiplicative compound matrix of  $\frac{\partial \psi^t}{\partial x}(x_0)$  evolves according to the following linear dynamics for all  $(x_0, t)$  in  $\mathcal{S} \times \mathbb{R}_{\geq 0}$

$$\frac{d}{dt} \left( \frac{\partial \psi^t}{\partial x}(x_0) \right)^{(k)} = \frac{\partial f}{\partial x}(\psi^t(x_0))^{[k]} \frac{\partial \psi^t}{\partial x}(x_0)^{(k)}, \quad \frac{\partial \psi^0}{\partial x}(x_0) = I.$$

Therefore, from [11, Proposition 5], it can be shown that (9) implies the existence of some positive constants  $c, \varepsilon > 0$  such that

$$\left| \frac{\partial \psi^t}{\partial x}(x_0) \right| \leq c e^{-\varepsilon t}, \quad \forall (x_0, t) \in \mathcal{S} \times \mathbb{R}_{\geq 0}$$

The proof concludes by repeating the arguments as in Section VIII-B.

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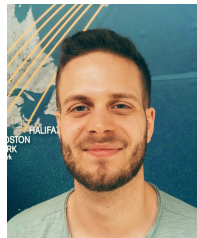
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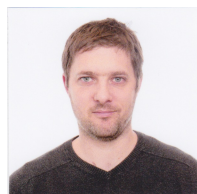
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