



**HAL**  
open science

## Generalized Lyapunov conditions for k-contraction: analysis and feedback design

Andreu Cecilia, Samuele Zoboli, Daniele Astolfi, Ulysse Serres, Vincent  
Andrieu

### ► To cite this version:

Andreu Cecilia, Samuele Zoboli, Daniele Astolfi, Ulysse Serres, Vincent Andrieu. Generalized Lyapunov conditions for k-contraction: analysis and feedback design. 2023. hal-04300588

**HAL Id: hal-04300588**

**<https://hal.science/hal-04300588>**

Preprint submitted on 24 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Generalized Lyapunov conditions for $k$ -contraction: analysis and feedback design

Andreu Cecilia, Samuele Zoboli, Daniele Astolfi, Ulysse Serres and Vincent Andrieu

**Abstract**—Recently, the concept of  $k$ -contraction has been introduced as a promising generalization of contraction for dynamical systems. However, the study of  $k$ -contraction properties has faced significant challenges due to the reliance on complex mathematical objects called matrix compounds. As a result, related control design methodologies have yet to appear in the literature. In this paper, we overcome existing limitations and propose new sufficient conditions for  $k$ -contraction which do not rely on matrix compounds. Our design-oriented conditions stem from a strong geometrical interpretation and establish a connection between  $k$ -contraction and  $p$ -dominance. Notably, these conditions are also necessary in the linear time-invariant framework. Leveraging on these findings, we propose a feedback design methodology for both the linear and the nonlinear scenarios.

**Index Terms**—Contraction analysis, Nonlinear systems, Linear matrix inequalities, Inertia Theorems, Compound matrices, Linear Systems.

## I. INTRODUCTION

Contraction theory is an emerging topic that has been used in numerous applications, such as observer design [1], multi-agent system synchronization [2]–[4] and controller design [5]–[9]. Nonetheless, many systems cannot present classical contractivity properties, e.g. multi-stable systems or orbitally stable systems. This fact motivated the study of suitable generalizations. Some notable examples are horizontal contraction [10, Section VII], transversal exponential stability [11] and  $p$ -dominance [12], [13]. Motivated by the results of Muldowney [14], the recent work [15] presented the notion of  $k$ -contraction as the generalization to  $k$ -dimensional objects of the standard contraction concept for distances. As such,  $k$ -contraction includes classical contraction as the special case  $k = 1$ . For  $k > 1$ , this property can be used to analyze the asymptotic behavior of systems that are not contractive in the classical sense. For example, for 2-contractive time-invariant systems, every bounded solution converges to an equilibrium point (not necessarily unique).

Existing sufficient conditions for  $k$ -contraction are given in terms of a particular matrix compound of the Jacobian of the vector field dynamics [14]–[16]. Although these conditions are adequate for system analysis, their application for feedback design are limited. First, matrix compounds rapidly explode in dimension for low value of  $k$  and systems of large dimension. This fact drastically increases the computational complexity of potential feedback design algorithms.

Second, the use of matrix compounds hinder the derivation of a tractable matrix inequality problem for feedback design. Consequently, a  $k$ -contractive design methodology has yet to be developed.

Considering these limitations, this work presents alternative design-oriented conditions for  $k$ -contraction that do not rely on matrix compounds, but rather on simple matrix inequalities on the given system dynamics. In particular, we build upon the generalized Lyapunov matrix inequalities studied for instance in [13], [17] and [18, Section 5]. Moreover, the connections between  $k$ -contraction, infinitesimally  $k$ -contraction, and  $p$ -dominance [12], [13] are discussed. By exploiting these novel conditions, we devise a feedback design methodology in both the linear and the nonlinear framework. In the linear time-invariant framework, our design is based on a new generalization of the notion of stabilizability. In the nonlinear framework, we restrict the design to linear feedback laws and propose a controller that guarantees 2-contractivity of the closed-loop.

The remainder of this document is organized as follows. In Section II, we provide a refined definition of  $k$ -contraction which strongly focuses on its geometrical interpretation. Then, we recover the notion of infinitesimal  $k$ -contraction, which has been used in [19], and link it to the proposed definition of  $k$ -contraction. Subsequently, we recall matrix compounds-based sufficient conditions for  $k$ -contraction and discuss their limitations. Section III focuses on linear systems. First, we derive necessary and sufficient conditions for  $k$ -contraction that do not require matrix compounds, but rather on generalized Lyapunov matrix inequalities. Then, based on these results, we propose the notion of  $k$ -order stabilizability together with a new  $k$ -contractive feedback design. In doing so, we also collect and recall in a unified theorem a series of results on inertia theorems (see, e.g. [20, Lemma 1, Section 3] and [21, Theorem 2.5]) and we provide new inertia theorems on algebraic inequalities of the form  $WA^T + AW - BB^T \prec 0$  not requiring any controllability assumption (cfr. [22]). Section IV focuses on extending these results to nonlinear systems. Similarly, we first provide sufficient conditions for  $k$ -contraction in nonlinear systems and, then, propose a design methodology for 2-contraction. In section V, we discuss the similarities, differences and links between  $k$ -contraction and  $p$ -dominance. All the proofs are postponed in Sections VI–VIII and in the Appendix to ease the reading of the article. Finally, some conclusions and future perspectives are drawn in Section IX.

*Notation:*  $\mathbb{R}_{\geq 0} := [0, \infty)$  and  $\mathbb{N} := \{0, 1, 2, \dots\}$ .  $|\cdot|$  denotes the standard Euclidean norm. Given  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^m$ , we set  $(x, y) := (x^T, y^T)^T$ . The operation  $\binom{n}{k} := \frac{n!}{k!(n-k)!}$  depicts the binomial coefficient, with  $n!$  denoting the factorial of  $n \in \mathbb{N}$ . The inertia of a matrix  $P$  [21, Definition 2.1] is defined by the triplet of integers  $\text{In}(P) := (\pi_-(P), \pi_0(P), \pi_+(P))$ , where  $\pi_-(P)$ ,  $\pi_+(P)$  and  $\pi_0(P)$  denote the numbers of eigenvalues of  $P$  with negative, positive and zero real part, respectively, counting multiplicities. The cardinality of a set is denoted as  $\text{card}(\cdot)$ .  $A \succ 0$  (resp.  $A \succeq 0$ ) denotes  $A$  being a positive definite (resp. positive semidefinite) matrix.

<sup>1</sup> A. Cecilia is with Universitat Politècnica de Catalunya, Avinguda Diagonal, 647, 08028 Barcelona, Spain. (andreu.cecilia@upc.edu).

<sup>2</sup> S. Zoboli is with Université Paul Sabatier Toulouse 3, LAAS-CNRS, Toulouse, France (name.surname@laas.fr).

<sup>3</sup> D. Astolfi, U. Serres and V. Andrieu are with Université Claude Bernard Lyon 1, CNRS, LAGEPP UMR 5007, 43 boulevard du 11 novembre 1918, F-69100, Villeurbanne, France (name.surname@univ-lyon1.fr).

The research leading to these results is partially funded by the Spanish Ministry of Universities funded by the European Union - NextGenerationEU (2022UPC-MSC-93823) and ANR via projects DELICIO (ANR-18-CE40-0010) and ALLIGATOR (ANR-22-CE48-0009-01).

## II. PRELIMINARIES ON $k$ -CONTRACTION

### A. Definition of $k$ -contraction

In this work, we consider nonlinear systems of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \quad (1)$$

where  $f$  is sufficiently smooth. The flow of  $f$  is denoted by  $\psi^t$ , and  $\psi^t(x_0)$  is the trajectory of (1) at time  $t$ . By definition,  $\psi^0(x_0) = x_0$ . We now formally define the property of  $k$ -contraction studied in this article. Our definition strongly focuses on a geometrical interpretation and it is related to the notion presented in the works [14], [15]. Moreover, when considering objects of dimension 1 ( $k = 1$ ), it matches the definition of contraction presented in [11], [23].

In [11], [23], 1-contraction expresses the fact that the length of any  $C^1$  curve from  $[0, 1]$  to  $\mathbb{R}^n$  of initial conditions decreases with time. To extend such a notion to any positive integer  $k \in [1, n]$ , with  $n$  being the state dimension of (1), we consider a set of sufficiently smooth functions  $\mathcal{I}_k$  defined on  $[0, 1]^k$ , namely

$$\mathcal{I}_k := \left\{ \Phi : [0, 1]^k \rightarrow \mathbb{R}^n \mid \Phi \text{ is a smooth immersion} \right\}. \quad (2)$$

Let  $P \in \mathbb{R}^{n \times n}$  be a positive definite symmetric matrix. For each  $\Phi$  in  $\mathcal{I}_k$ , we define the volume  $V^k(\Phi)$  of  $\Phi$  as

$$V^k(\Phi) := \int_{[0,1]^k} \sqrt{\det \left\{ \frac{\partial \Phi}{\partial r}(r)^\top P \frac{\partial \Phi}{\partial r}(r) \right\}} dr. \quad (3)$$

Note that, since  $f$  in (1) is sufficiently smooth, for each forward invariant set and for each  $t$  in  $\mathbb{R}_{\geq 0}$  the corresponding flow  $\psi^t$  is also sufficiently smooth in this set. Consequently, for each  $\Phi$  in  $\mathcal{I}_k$  such that  $\text{Im}(\Phi)$  is in a forward invariant set,  $\psi^t \circ \Phi$  is in  $\mathcal{I}_k$ .

**Remark 1** When  $\Phi$  is injective and  $P$  is the identity matrix, the volume  $V^k$  defined in (3) coincides with the standard Euclidean  $k$ -volume of the submanifold  $\Phi([0, 1]^k) \subseteq \mathbb{R}^n$ . Note that 1-volumes are lengths, 2-volumes are areas and 3-volumes are standard volumes.

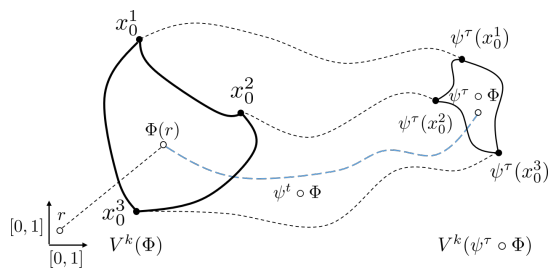
**Remark 2** Note that the volume definition (3) can be generalized to the Riemannian framework by substituting the Euclidean metric  $P$  with a symmetric positive definite 2-tensor  $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ , see [24, Lemma 3.2]. However, in this paper, we will focus on the Euclidean scenario in order to obtain more tractable conditions.

From now on, we let  $k$  be a fixed integer between 1 and  $n$ . We now define  $k$ -contraction properties for nonlinear systems of the form (1), which will be used throughout the article.

**Definition 1 ( $k$ -contraction)** System (1) is said to be  $k$ -contractive on a forward invariant set  $\mathcal{S} \subseteq \mathbb{R}^n$  if there exist real numbers  $a, b > 0$  such that, for every  $\Phi \in \mathcal{I}_k$  satisfying  $\text{Im}(\Phi) \subseteq \mathcal{S}$ , the following holds

$$V^k(\psi^t \circ \Phi) \leq b e^{-at} V^k(\Phi), \quad \forall t \in \mathbb{R}_{\geq 0}. \quad (4)$$

In other words, a system is  $k$ -contractive if, for any parametrized  $k$ -dimensional submanifold of  $\mathbb{R}^n$  from which trajectories are complete, its volume is exponentially shrinking along the system dynamics. An intuitive representation of the required volume convergence condition is presented in Fig. 1. When  $k = 1$ , this means that the length of any sufficiently smooth curve is exponentially decreasing, matching the definition in [11]. Moreover, this definition includes the ones in [14], and [15, Section 3.2]. We remark that Definition 1 is invariant under uniformly bounded diffeomorphic coordinate changes on  $\mathcal{S}$ . This fact is formalized through the following lemma.



**Fig. 1.** Flow of a 2-contractive system. The initial submanifold of initial conditions, described by  $\Phi$ , is some surface with points at  $x_0^1$ ,  $x_0^2$  and  $x_0^3$ . The volume of this submanifold  $V^k(\cdot)$  decreases exponentially along the trajectories of the system.

**Lemma 1** Assume that the system (1) is  $k$ -contractive on a forward invariant set  $\mathcal{S} \subseteq \mathbb{R}^n$  for some positive constants  $a, b > 0$ . Moreover, consider a diffeomorphism  $\varphi : \mathcal{S} \rightarrow \mathcal{S}$ , which satisfies for some positive constant  $\bar{c}, \underline{c} > 0$

$$\underline{c}I \preceq \frac{\partial \varphi}{\partial x}(x)^\top \frac{\partial \varphi}{\partial x}(x) \preceq \bar{c}I, \quad \forall x \in \mathcal{S}. \quad (5)$$

Then, there exists a positive constant  $\bar{b} > 0$  such that for every  $\Phi \in \mathcal{I}_k$  satisfying  $\text{Im}(\Phi) \subseteq \mathcal{S}$ , the following holds

$$V^k(\varphi \circ \psi^t \circ \Phi) \leq \bar{b} e^{-at} V^k(\varphi \circ \Phi),$$

The proof is postponed to Appendix A.

### B. Infinitesimal $k$ -contraction

Inspired by classical works on contraction theory [25], we now provide a result linking the exponential stability properties of the variational system to the  $k$ -contraction property proposed in Definition 1. We remark that the definition of  $k$ -contraction for the variational system was used in [19]. However, to the best of our knowledge, the connection between  $k$ -contraction (as presented in Definition 1) and  $k$ -contraction of the variational system has not been properly characterized. In this section, we recall this definition and we provide further geometrical interpretation of it, along the lines of Definition 1. We start by recalling the dynamics of the variational system, which is defined as the system describing the evolution of an infinitesimal displacement along the trajectories of the system. The linearization of (1) about the trajectory  $\psi^t(x_0)$  is

$$\dot{v} = \frac{\partial f}{\partial x}(\psi^t(x_0)) v, \quad (6)$$

where  $v$  belongs to the tangent space  $T_{\psi^t(x_0)} \mathbb{R}^n = \mathbb{R}^n$ . Then,  $\frac{\partial \psi^t}{\partial x}(x_0)v_0$  is a trajectory of (6) at time  $t$  initialized at  $v_0$  at  $t = 0$ . From linearity, it can be deduced that  $\frac{\partial \psi^t}{\partial x}(x_0)$  is the state transition matrix of (6). Then,  $\frac{\partial \psi^t}{\partial x}(x_0)v_0$  depicts the infinitesimal displacement with respect to the solution  $\psi^t(x_0)$  induced by the initial condition  $x_0 + v_0$ .

We recall that the trajectory  $\psi^t(x_0)$  is locally exponentially stable, that is, the trajectory generated from any initial condition close enough to  $x_0$  will exponentially converge to  $\psi^t(x_0)$ , if and only if the variational system (6) is exponentially stable [26, Theorem 3.13]. In classical contraction theory [25], this property is generalized by considering simultaneously all the trajectories in a set. That is, the system (1) is contracting in a forward invariant set  $\mathcal{S} \subseteq \mathbb{R}^n$  if the variational system (6) is exponentially stable for all  $x_0 \in \mathcal{S}$ . Then, contraction on a forward invariant set  $\mathcal{S}$  implies that every solution in  $\mathcal{S}$  converge to the same trajectory [25], or equivalently, the distance between any pair of trajectories shrinks to zero. In a sense, contraction exemplifies how the linearization along trajectories can be used to

derive incremental properties of the original system. In this section, we generalize this idea by considering a  $k$ -contracting property on the variational system (6).

Precisely, pick any  $x_0 \in \mathbb{R}^n$  and  $k$  initial conditions of the variational system in (6)  $v_0^1, \dots, v_0^k$ . We define the following matrix

$$\Psi(t, x_0) := \begin{bmatrix} \frac{\partial \psi^t}{\partial x}(x_0)v_0^1 & \dots & \frac{\partial \psi^t}{\partial x}(x_0)v_0^k \end{bmatrix} \in \mathbb{R}^{n \times k}.$$

Note that  $\Psi(0, x_0) = \frac{\partial \Phi_{\text{loc}}}{\partial r}(r)$ , where  $\Phi_{\text{loc}}$  is an immersion parameterized by the variable  $r \in [0, 1]^k$  whose image is an infinitesimal  $k$ -order parallelotope with vertices at  $x_0$  and  $v_0^i + x_0$ , namely

$$\Phi_{\text{loc}}(r) = \sum_{i=1}^k r_i(v_0^i + x_0) + \left(1 - \sum_{i=1}^k r_i\right)x_0,$$

with  $r_i \in [0, 1]$  for  $i \in \{1, \dots, k\}$  being the  $i$ -th component of  $r$ . For  $k = 1$ ,  $\Phi_{\text{loc}}(r)$  defines a straight line between  $x_0$  and  $x_0 + v_0^1$ . The volume of the infinitesimal parallelotope can be computed by means of the multiplicative compound which is defined as follows.

**Definition 2 (Multiplicative Compound [27])** Consider a matrix  $Q \in \mathbb{R}^{n \times m}$  and select an integer  $k \in [1, \min\{n, m\}]$ . Moreover, define a minor of order  $k$  of the matrix  $Q$  as the determinant of some  $k \times k$  submatrix of  $Q$ . The  $k$ -th multiplicative compound of  $Q$ , denoted as  $Q^{(k)}$ , is the  $\binom{n}{k} \times \binom{m}{k}$  matrix including all the minors of order  $k$  of  $Q$  in a lexicographic order.

As an example, consider a  $3 \times 3$  matrix  $Q$  with entries  $q_{ij}$  for  $i, j = 1, \dots, 3$ . The  $2^{\text{nd}}$  multiplicative compound  $Q^{(2)}$  is

$$Q^{(2)} = \begin{bmatrix} \det \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} & \det \begin{pmatrix} q_{11} & q_{13} \\ q_{21} & q_{23} \end{pmatrix} & \det \begin{pmatrix} q_{12} & q_{13} \\ q_{22} & q_{23} \end{pmatrix} \\ \det \begin{pmatrix} q_{11} & q_{12} \\ q_{31} & q_{32} \end{pmatrix} & \det \begin{pmatrix} q_{11} & q_{13} \\ q_{31} & q_{33} \end{pmatrix} & \det \begin{pmatrix} q_{12} & q_{13} \\ q_{32} & q_{33} \end{pmatrix} \\ \det \begin{pmatrix} q_{21} & q_{22} \\ q_{31} & q_{32} \end{pmatrix} & \det \begin{pmatrix} q_{21} & q_{23} \\ q_{31} & q_{33} \end{pmatrix} & \det \begin{pmatrix} q_{22} & q_{23} \\ q_{32} & q_{33} \end{pmatrix} \end{bmatrix}.$$

Note that for a  $Q \in \mathbb{R}^{n \times n}$ ,  $Q^{(n)} = \det(Q)$  and  $Q^{(1)} = Q$ .

Thanks to the previous definition, by considering  $P = I$ , we can compute the volume of  $\Phi_{\text{loc}}$  and  $\frac{\partial \psi^t}{\partial x} \circ \Phi_{\text{loc}}$  as follows

$$V^k(\Phi_{\text{loc}}) = |\Psi(0, x_0)^{(k)}|, \quad V^k\left(\frac{\partial \psi^t}{\partial x} \circ \Phi_{\text{loc}}\right) = |\Psi(t, x_0)^{(k)}|.$$

The second equality is a consequence of the linearity of the dynamics of  $\Psi(t, x_0)^{(k)}$  and we postpone further details at the beginning of Appendix C. Given the aforementioned notions, we have the following definition.

**Definition 3 (Infinitesimal  $k$ -contraction)** System (1) is said to be infinitesimally  $k$ -contractive on a forward invariant set  $S \subseteq \mathbb{R}^n$  if there exist real numbers  $a, b > 0$  such that

$$\left| \Psi(t, x_0)^{(k)} \right| \leq b e^{-at} \left| \Psi(0, x_0)^{(k)} \right|, \quad (7)$$

for all  $(t, x_0) \in \mathbb{R}_{\geq 0} \times S$ .

Roughly speaking, the bound in (7) implies that the volume of an infinitesimal parallelotope connected to the trajectory  $\psi^t(x_0)$  and generated by the vectors  $\frac{\partial \psi^t}{\partial x}(x_0)v_0^1, \dots, \frac{\partial \psi^t}{\partial x}(x_0)v_0^k$  exponentially shrinks to zero. An intuitive depiction of this property is presented in Fig. 2.

Notice that, for the case  $k = 1$ , Definition 3 boils down to (6) being exponentially stable for all  $x_0 \in S$ , which is a sufficient condition for the classical notion of contraction [25]. In the next proposition, we link the notion of infinitesimal  $k$ -contraction to  $k$ -contraction as presented in Definition 1.

**Proposition 1** Suppose system (1) is infinitesimally  $k$ -contractive on a forward invariant set  $S$ . Then, it is also  $k$ -contractive on  $S$ .

The proof is postponed to Appendix C.

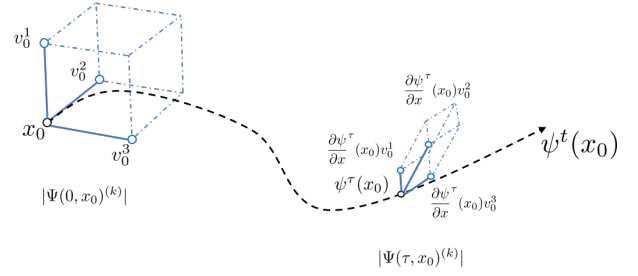


Fig. 2. Flow of an infinitesimally 3-contractive system.

### C. Sufficient conditions based on additive matrix compounds

Sufficient conditions for  $k$ -contraction were originally given in the seminal work by Muldowney [14] and were recently re-proposed in the works [15], [28]. These conditions strongly depend on the use of additive matrix compound, which is defined as follows.

**Definition 4 (Additive Compound [27])** Consider a matrix  $Q \in \mathbb{R}^{n \times n}$  and select an integer  $k \in [1, n]$ . The  $k$ -th additive compound of  $Q$  is the  $\binom{n}{k} \times \binom{n}{k}$  matrix defined as

$$Q^{[k]} := \frac{d}{d\epsilon} \Big|_{\epsilon=0} (I + \epsilon Q)^{(k)}.$$

The additive compound can be explicitly computed in terms of the entries of  $Q$ . For example, for  $Q \in \mathbb{R}^{n \times n}$  we have  $Q^{[n]} = \text{tr}(Q)$  and  $Q^{[1]} = Q$ . More details on this operation can be found in [29].

Bearing this definition in mind, we now reframe the sufficient condition for  $k$ -contraction presented in [14], [15] in the framework of this paper, namely, we view them through the lenses of Definition 1.

**Theorem 1** Let  $S \subseteq \mathbb{R}^n$  be a forward invariant set and suppose there exist a real number  $\eta > 0$  and a symmetric positive definite matrix  $Q \in \mathbb{R}^{\binom{n}{k} \times \binom{n}{k}}$  such that

$$Q \left( \frac{\partial f}{\partial x}(x)^{[k]} \right) + \left( \frac{\partial f}{\partial x}(x)^{[k]} \right)^\top Q \preceq -\eta I, \quad \forall x \in S. \quad (8)$$

Then, system (1) is  $k$ -contractive on  $S$  according to Definition 1.

The proof is postponed to Appendix B. The extension of Theorem 1 to time-varying systems can be found in [30].

**Remark 3** Inequality (8) is equivalent to the condition in [15, Theorem 9] using the logarithmic norm induced by the weighted  $\ell_2$  norm (e.g. [31, Equation 2.56]). However, in our statement, the set  $S$  is allowed to be non-convex. Furthermore, when  $k = 1$ , we recover the well-known Demidovich conditions (see [32]) and the proof in [11] for contraction of lengths in the context of Euclidean metrics.

**Remark 4** Theorem 1 can be generalized to the case of Riemannian volumes, see Remark 2. However, we omit these results to ease the reading of the document. It should be remarked that such generalization also expands on point IV in [33, Proposition 2.5], since we consider volume objects of dimension lower than  $n$ .

### D. Limitations of matrix compound-based conditions

Although Theorem 1 provides a suitable condition for system analysis, we claim that the presence of matrix compounds hinders the process of devising  $k$ -contractive feedback designs. Indeed, consider a linear control system of the form

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (9)$$

where  $u$  is the control input. Assume we want to design a state-feedback controller of the form  $u = -Kx$ , with  $K \in \mathbb{R}^{n \times m}$ , such that the closed-loop system is  $k$ -contractive. Then, Theorem 1 reduces to designing  $K$  such that condition (8) is satisfied for the closed-loop system, namely,

$$Q \left( (A - BK)^{[k]} \right) + \left( (A - BK)^{[k]} \right)^\top Q \preceq -\eta I.$$

However, this is a non-convex matrix inequality, due to the strong coupling between the matrices  $B, K$  imposed by the additive matrix compound. Consequently, even for a simple linear case, a design methodology for the gain  $K$  cannot be straightforwardly derived.

Additionally, notice that matrix compounds rapidly grow in dimension when the order of the system is large and the  $k$  is low, since they involve matrices of dimensions  $\binom{n}{k} \times \binom{n}{k}$ . We remark that, to the best of the authors' knowledge, interesting asymptotic properties of  $k$ -contractive systems have been shown only for small values of  $k$ , specifically  $k \in \{1, 2, 3\}$  (see [15] and Section V-A). Consequently, as highlighted in previous works [34], compound-based conditions often explode in computational complexity.

With this in mind, a consistent portion of the following sections is dedicated to presenting alternative design-oriented conditions for  $k$ -contraction of linear and nonlinear systems which do not depend on matrix compounds. These conditions will be the fundamental building blocks in the derivation of control laws guaranteeing  $k$ -contractivity of the closed-loop.

### III. $k$ -CONTRACTION FOR LINEAR SYSTEMS

We start our analysis by focusing on the linear scenario. This will provide fundamental intuitions on the notion of  $k$ -contraction that will be instrumental in the subsequent analysis of nonlinear dynamics.

#### A. Generalized Lyapunov necessary and sufficient conditions

Consider a linear system of the form

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n. \quad (10)$$

We now provide a set of sufficient and necessary conditions guaranteeing that (10) is  $k$ -contractive according to Definition 1. This result is based on the following two facts:

- a necessary and sufficient condition for system (10) to be  $k$ -contractive is that the sum of the real part of any combination of  $k$ -eigenvalues of  $A$  is negative, see Lemma 5 below in Section VI;
- the generalized Lyapunov matrix inequality (see, e.g [17], [20])

$$PA + A^\top P \prec -2\mu P$$

admits a symmetric solution  $P$  of inertia  $\text{In}(P) = (p, 0, n - p)$  if and only if  $A$  has  $p$  eigenvalues with real part larger than  $\mu$  and  $n - p$  eigenvalues with real part smaller than  $\mu$ , see below Lemma 6 in Section VI.

Consequently, combining the previous two properties, we state now the following main result.

**Theorem 2** *System (10) is  $k$ -contractive if and only if there exist:*

- a positive integer  $\ell \in \mathbb{N}$  satisfying  $1 \leq \ell \leq k$ ,
- $\ell$  real numbers  $\mu_i \in \mathbb{R}$ , with  $i \in \{0, \dots, \ell - 1\}$ ,
- $\ell$  positive integers  $d_i \in \mathbb{N}$ , with  $i \in \{0, \dots, \ell - 1\}$ , satisfying

$$0 = d_0 < d_1 < \dots < d_{\ell-1} \leq k - 1,$$

- and  $\ell$  symmetric matrices  $P_i \in \mathbb{R}^{n \times n}$  of respective inertia  $(d_i, 0, n - d_i)$ , with  $i \in \{0, \dots, \ell - 1\}$ ,

such that

$$A^\top P_i + P_i A \prec 2\mu_i P_i, \quad \forall i \in \{0, \dots, \ell - 1\}, \quad (11a)$$

$$\sum_{i=0}^{\ell-1} h_i \mu_i \leq 0, \quad (11b)$$

where  $h_0 \geq 1$  and  $h_i = d_{i+1} - d_i$ , for all  $i = \{0, \dots, \ell - 1\}$  with  $d_\ell \in \mathbb{N}$  satisfying  $d_{\ell-1} + 1 \leq d_\ell \leq k$ .

The proof of Theorem 2 is postponed to Section VI-B. To provide some intuition relative to Theorem 2, we anticipate that the constants  $\mu_i$  are bounding the eigenvalues of the matrix  $A$ . That is,  $\mu_0$  bounds the largest eigenvalue of  $A$ , while  $\mu_1$  bounds the second largest eigenvalue with real part different from the first, and so on. Then, (11b) can be interpreted as a bound in the partial sum of eigenvalues of  $A$ , considering their multiplicities, in turn implying  $k$ -contraction (see the proof in Section VI-B for more details).

**Remark 5** *A particular case in which the assumption of the former Theorem applies is when  $\ell = 1$ . In that case, the former condition reduces to the existence of a real number  $\mu_0$  and a symmetric positive definite matrix  $P_0 \succ 0$  such that*

$$A^\top P_0 + P_0 A \prec 2\mu_0 P_0,$$

with  $\mu_0 \leq 0$ . This condition is satisfied if and only if  $A$  is Hurwitz, which would imply that system (10) is  $k$ -contractive for all  $k \in \{1, \dots, n\}$ .

**Remark 6** *The inertia constraints in (11a) cannot be represented as semidefinite constraints. However, these constraints can be dropped without a significant impact on the solution of the inequality. Indeed, by Lemma 6 in Section VI, a fixed constant  $\mu_i$  imposes a specific inertia on the matrix  $P_i$ , depending on the eigenvalues of  $A$ . Consequently, by correctly fixing  $\mu_i$  we can obtain a matrix  $P_i$  of the desired inertia, without explicitly imposing it as a constraint. A similar strategy was explored in [35] for discrete-time systems.*

**Remark 7** *Some authors have previously proposed alternative conditions for  $k$ -contractions without matrix compounds, e.g. [34]. Nonetheless, as commented in the conference version of this paper [30], our conditions are necessary and sufficient for linear systems, while the conditions in [34] are only sufficient.*

#### B. Computational burden of Theorem 2

We now compare Theorem 1 and Theorem 2 in terms of the computational burden imposed by the solution of the respective matrix inequalities. We focus on the result in Theorem 2 for the case  $\ell = k$  and  $d_i = d_{i-1} + 1$ , since it provides the largest set of matrix inequalities. Let  $M \in \mathbb{R}^{r \times r}$  be an arbitrary square matrix and  $Q \in \mathbb{R}^{r \times r}$  be a symmetric matrix. Since  $Q$  is symmetric, each condition of the form  $QM + M^\top Q \preceq \mu Q$  requires the computation of  $N = r(r - 1)/2 + 1$  variables, namely the entries of the top triangular portion of  $Q$  and the scalar  $\mu$ . Then, Theorem 1 requires  $N_1 = \binom{n}{k} \left( \binom{n}{k} - 1 \right) / 2 + 1$  variables, while Theorem 2 requires  $N_2 = kn(n - 1)/2 + k$  variables.

To better understand how the size of the problem scales with different values of  $k$  and  $n$ , we refer to Fig. 3. Clearly, for large dimensional systems and low  $k$ , the conditions in Theorem 2 ask for a significantly smaller number of variables. Moreover, even in the worst case of  $k = n$ , Theorem 2 typically requires between  $10^2$  and  $10^3$  variables. Differently, Theorem 1 can easily reach  $10^4$  variables.

This computation shows that conditions in Theorem 2 do not grow in dimension as fast as the condition in Theorem 1. Moreover, for  $k$

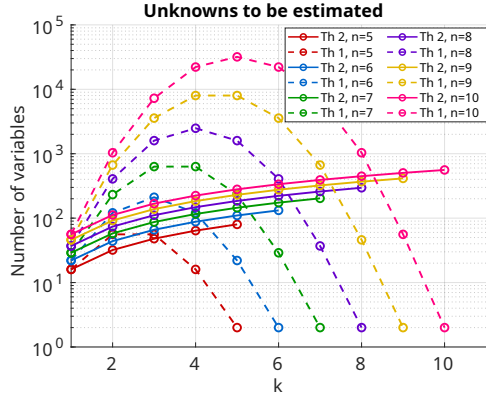


Fig. 3. Number of variables to be estimated by Theorem 1 (dashed) and by Theorem 2 (solid) in function of  $k$ . Colors refer to different  $n$ .

sufficiently smaller than  $n$ , we have  $N_2 \leq N_1$ . We recall that, to the best of our knowledge, only the cases  $k = \{1, 2, 3\}$  are interesting from a control viewpoint, since they are the only ones presenting interesting asymptotic behaviors, see [15] and Section V-A.

### C. $k$ -order stabilizability

Consider a linear system of the form (9). It is well-known that stabilizability of the pair  $(A, B)$  is a necessary and sufficient condition for the existence of a stabilizing controller. A similar property can be defined when considering  $k$ -contractive designs. We refer to this condition as  $k$ -order stabilizability.

**Definition 5 ( $k$ -order stabilizability)** System (9) is  $k$ -order stabilizable if there exists a matrix  $K \in \mathbb{R}^{n \times m}$  such that the closed-loop system  $\dot{x} = (A - BK)x$  is  $k$ -contractive.

Conditions for  $k$ -order stabilizability can be easily derived by transforming the system into a suitable form. Using standard Kalman decomposition, system (9) is algebraically equivalent to a system of the form

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u \quad (12)$$

where  $x_c \in \mathbb{R}^{n_c}$ ,  $x_u \in \mathbb{R}^{n_u}$ ,  $n_c + n_u = n$  and the pair  $(A_c, B_c)$  is controllable. The non-negative integer  $n_u$  is the dimension of the null-space of the controllability matrix of (9). Consequently, we admit the possibility of  $n_u = 0$  and  $A_u$  being non-existing.

**Lemma 2** System (9) is  $k$ -order stabilizable if and only if either  $n_u < k$  or the autonomous system  $\dot{x}_u = A_u x_u$ , is  $k$ -contractive otherwise.

The proof of Lemma 2 is postponed to Section VII-A. Intuitively, Lemma 2 asks the uncontrollable part to be already  $k$ -contractive (or of dimension smaller than  $k$ ). For the case  $k = 1$ , Lemma 2 reduces to  $n_u = 0$ , a necessary and sufficient condition for controllability in linear systems, or  $\dot{x}_u = A_u x_u$ , being stable, which is a sufficient condition for the classical notion of stabilizability. We also remark that similar definitions could be developed for  $k$ -order controllability, observability, and detectability, which however are out of this paper's scope.

### D. $k$ -contractive feedback design

Starting from a  $k$ -order stabilizability property and the decomposition (12), one can easily derive a  $k$ -contractive feedback design, e.g., via pole placement on the pair  $(A_c, B_c)$ . Nonetheless, in view

of an extension of these notions to the nonlinear context, we look for coordinate-free conditions, i.e. that do not rely on change of coordinates and decompositions that would not be easy to extend to nonlinear systems.

Motivated by the result in Theorem 2, we now derive constructive conditions for designing  $k$ -contractive controllers. This section presents a design methodology that follows the philosophy of feedback stabilization based on Lyapunov tests for stabilizability [36, Section 14.5]. That is, first, we solve a set of matrix inequalities, which are feasible if and only if the system is  $k$ -order stabilizable. Then, the controller is derived from the result of these inequalities. First, we present a generalized Lyapunov test for  $k$ -order stabilizability.

**Theorem 3** System (9) is  $k$ -order stabilizable if and only if there exist:

- a positive integer  $\ell \in \mathbb{N}$  satisfying  $1 \leq \ell \leq k$ ,
- $\ell$  real numbers  $\mu_i \in \mathbb{R}$ , with  $i \in \{0, \dots, \ell - 1\}$ ,
- $\ell$  positive integers  $d_i \in \mathbb{N}$ , with  $i \in \{0, \dots, \ell - 1\}$ , satisfying
$$0 = d_0 < d_1 < \dots < d_{\ell-1} \leq k - 1,$$
- and  $\ell$  symmetric matrices  $W_i \in \mathbb{R}^{n \times n}$  of respective inertia  $(d_i, 0, n - d_i)$ , with  $i \in \{0, \dots, \ell - 1\}$ ,

such that

$$W_i A^\top + A W_i - B B^\top \prec 2\mu_i W_i, \quad \forall i \in \{0, \dots, \ell - 1\}, \quad (13a)$$

$$\sum_{i=0}^{\ell-1} h_i \mu_i \leq 0, \quad (13b)$$

where  $h_0 \geq 1$ ,  $h_i = d_{i+1} - d_i$ , for all  $i = \{1, \dots, \ell - 1\}$  with  $d_\ell \in \mathbb{N}$  satisfying  $d_{\ell-1} + 1 \leq d_\ell \leq k$ .

The proof of Theorem 3 is postponed to Section VII-C. Notice that for  $k = 1$ , inequalities (13a)- (13b) reduce to the existence of a constant  $\mu_0 \leq 0$  and a symmetric positive definite matrix  $W_0 \succ 0$  such that

$$W_0 A^\top + A W_0 - B B^\top \prec 2\mu_0 W_0.$$

Hence, we recover the well-known Lyapunov test for stabilizability [36, Section 14.4]. Differently put, the inequalities (13) can be seen as a generalization of the Lyapunov test for stabilizability to the context of  $k$ -contraction.

Now, based on the presented generalized Lyapunov test for  $k$ -order stabilizability, we can directly derive a  $k$ -contractive feedback controller for the linear system (9). The result is summarized in the following proposition.

**Proposition 2** Assume that (9) is  $k$ -order stabilizable. Then, there exist:

- a positive integer  $\ell \in \mathbb{N}$  satisfying  $1 \leq \ell \leq k$ ,
- $\ell$  real numbers  $\mu_i \in \mathbb{R}$ , with  $i \in \{0, \dots, \ell - 1\}$ ,
- $\ell$  positive integers  $d_i \in \mathbb{N}$ , with  $i \in \{0, \dots, \ell - 1\}$ , satisfying
$$0 = d_0 < d_1 < \dots < d_{\ell-1} \leq k - 1,$$
- and  $\ell$  symmetric matrices  $W_i \in \mathbb{R}^{n \times n}$  of respective inertia  $(d_i, 0, n - d_i)$ , with  $i \in \{0, \dots, \ell - 1\}$ ,

such that (13) is satisfied, and the following colinearity relation holds

$$B^\top W_i^{-1} = B^\top W_0^{-1}, \quad \forall i \in \{0, \dots, \ell - 1\}. \quad (14)$$

Furthermore, with this solution, the system (9) is  $k$ -contractive with the feedback law

$$u = -Kx, \quad K = \frac{\rho}{2} B^\top W_0^{-1}, \quad \forall \rho \geq 1. \quad (15)$$

The proof is postponed to Section VII-D. We highlight that, since  $k$ -contraction for the closed-loop system is preserved for all  $\rho \geq 1$ , the proposed controller presents a generalization of the infinite-gain margin property [37, Section 3.2.2] to the framework of partial stabilization. Consequently, this result expands similar infinite-gain margin designs [6] from 1-contraction to  $k$ -contraction.

#### IV. $k$ -CONTRACTION FOR NONLINEAR SYSTEMS

As a follow-up to the linear scenario, we now move to the analysis of  $k$ -contraction for nonlinear systems. The main goal is to provide sufficient conditions inspired by the results in Section III.

##### A. Sufficient conditions

Consider a nonlinear system of the form (1). The following theorem provides sufficient conditions for  $k$ -contraction.

**Theorem 4** *Let  $\mathcal{S} \subsetneq \mathbb{R}^n$  be a compact forward invariant set. Suppose there exist two symmetric matrices  $P_0, P_{k-1} \in \mathbb{R}^{n \times n}$  of respective inertia  $(0, 0, n)$  and  $(k-1, 0, n-k+1)$ , and  $\mu_0, \mu_{k-1} \in \mathbb{R}$  such that*

$$\frac{\partial f}{\partial x}(x)^\top P_0 + P_0 \frac{\partial f}{\partial x}(x) \prec 2\mu_0 P_0, \quad (16a)$$

$$\frac{\partial f}{\partial x}(x)^\top P_{k-1} + P_{k-1} \frac{\partial f}{\partial x}(x) \prec 2\mu_{k-1} P_{k-1}, \quad (16b)$$

$$\mu_{k-1} + (k-1)\mu_0 < 0, \quad (16c)$$

for all  $x \in \mathcal{S}$ . Then, system (1) is infinitesimally  $k$ -contractive on  $\mathcal{S}$  (therefore,  $k$ -contractive on  $\mathcal{S}$ ).

A detailed discussion of Theorem 4 is postponed to Section VIII-B, along with the relative proof. Intuitively, inequality (16a) bounds the expansion rate of the variational system (6) by a factor  $\mu_0$ . Differently, the second inequality (16b) bounds the contraction rate of a subspace of the tangent bundle by a factor  $\mu_{k-1}$ . Consequently, inequality (16c) constraints the contraction rate to be faster than the expansion rate. This resembles the eigenvalue bounding approach of Section III. However, a simple eigenvalue interpretation is not applicable in the nonlinear framework. Hence, we directly bound the fastest diverging direction and the slowest converging one, asking for the latter to be sufficiently fast through (16c).

Notice that Theorem 4 considers constant matrices  $P_0, P_1$ . In view of recent results on Riemannian contraction analysis, e.g. [1], [11], [33], we claim that the use of constant metrics is restrictive and that state-dependant metrics can help in widening the result to more general cases. A direct consequence of this observation is the fact that, contrarily to the linear case, the conditions of Theorem 4 are in general not equivalent to those of Theorem 1. Nonetheless, we highlight that restricting ourselves to the case of constant matrices can help to derive some new asymptotic behavior for  $k$ -contractive systems, as discussed at the end of Section II-C.

**Remark 8** *Theorem 4 requires solving an infinite set of matrix inequalities. Nonetheless, there are multiple strategies that can be used to reduce it to a feasible problem. For instance, one could exploit convex relaxation, as explained in [13, Section VI]. Alternatively, for systems with a semilinear structure (namely,  $f(x) = Ax + g(x)$ ), one can obtain a finite set of LMIs if the nonlinear term satisfies a monotonic or a sector-bounded condition, similar to [38].*

##### B. Relaxing conditions for the planar case

Note that, differently from Theorem 1, in Theorem 4 we require the set  $\mathcal{S}$  to be compact. We conjecture this compactness assumption can also be dropped in Theorem 4. This conjecture is motivated by the following result for the planar case  $n = k = 2$ .

**Lemma 3** *Let  $\mathcal{S} \subseteq \mathbb{R}^2$  and assume there exist symmetric matrices  $P_0, P_1 \in \mathbb{R}^{2 \times 2}$  of inertia  $\text{In}(P_0) = (0, 0, 2)$ ,  $\text{In}(P_1) = (1, 0, 1)$  and  $\mu_0, \mu_1 \in \mathbb{R}$  such that, for all  $x \in \mathcal{S}$ , inequalities in (16) are satisfied. Then, system (1) is 2-contractive on  $\mathcal{S}$ .*

The proof of Lemma 3 is postponed to Section VIII-C. This result shows that the inequalities in (16) do not necessarily need  $\mathcal{S} \subsetneq \mathbb{R}^n$  to show  $k$ -contraction. Consequently, in future works, we aim at exploring if Theorem 4 can be expanded to the complete  $\mathbb{R}^n$ . Currently, the technical obstruction that prevents us to conclude the conjecture is the use of Theorem 5 and Lemma 12 in Theorem 4 proof, which require  $\mathcal{S} \subsetneq \mathbb{R}^n$  in order to guarantee a bounded invariant subspace splitting.

##### C. 2-contractive feedback design

Following the lines of the linear results presented in Section III, we now elaborate on the conditions for  $k$ -contraction proposed in Theorem 4. We aim at devising  $k$ -contractive controllers for nonlinear systems. We will focus on the specific case of  $k = 2$ , due to the interesting asymptotic properties shown by 2-contractive systems, see Section V, and the reduced conservativity of Theorem 4 for  $k = 2$ .

Precisely, consider nonlinear systems of the form

$$\dot{x} = f(x) + Bu \quad (17)$$

where  $u \in \mathbb{R}^m$  and  $f$  is sufficiently smooth. In the next proposition, we provide a result on 2-contractive controller design.

**Proposition 3** *Let  $\mathcal{S} \subsetneq \mathbb{R}^n$  be a compact set, and assume there exist a pair of symmetric matrices  $W_0, W_1 \in \mathbb{R}^n$ , with  $W_0 \succ 0$  and inertia  $\text{In}(W_1) = (1, 0, n-1)$  and a pair of real numbers  $\mu_0, \mu_1 \in \mathbb{R}$ , such that, for all  $x \in \mathcal{S}$ ,*

$$W_0 \frac{\partial f}{\partial x}(x)^\top + \frac{\partial f}{\partial x}(x) W_0 - BB^\top \prec 2\mu_0 W_0 \quad (18a)$$

$$W_1 \left( \frac{\partial f}{\partial x}(x) - \frac{1}{2} BB^\top W_0^{-1} \right)^\top + \left( \frac{\partial f}{\partial x}(x) - \frac{1}{2} BB^\top W_0^{-1} \right) W_1 - BB^\top \prec 2\mu_1 W_1. \quad (18b)$$

Then, there exists a real number  $\varepsilon > 0$  such that if

$$\mu_0 + \mu_1 + \varepsilon < 0, \quad (19)$$

the feedback law  $u = -Kx$  with

$$K = \frac{1}{2} B^\top (W_0^{-1} + W_1^{-1}). \quad (20)$$

makes the system (17) 2-contractive on  $\mathcal{S}$ , if  $\mathcal{S}$  is forward invariant for the closed-loop.

The proof of Proposition 3 is postponed to Section VIII-D. Proposition 3 is an extension of the result for linear systems in Theorem 2 to the nonlinear framework, and restricted to the case  $k = 2$ . However, besides the nonlinearities, we highlight some main differences between the two results. First, since we require constant matrices  $W_i$ , the nonlinear result cannot be proven to be necessary in general. Second, even if such constant matrices do exist, there is no guarantee that they satisfy a colinearity condition (14) uniformly on  $x$ . Hence, Proposition 3 proposes an alternative design that trades the colinearity condition (14) for conservativeness in the sum of rates

(19), i.e. the addition of  $\varepsilon > 0$ . Nonetheless, we remark that a similar approach can be used in the linear scenario at the price of the result necessity.

**Remark 9** *Substituting (19) with the condition  $\mu_1 \leq 0$ , we obtain a feedback design methodology for 1-dominance, see Section V-A, which can serve as an alternative to existing feedback design methodologies for  $p$ -dominance, e.g. [39], [40].*

**Remark 10** *The strategy of building a controller by recursively computing matrices  $W$  in (18b) resembles the approach proposed in [41, Lemma 2]. Consequently, Proposition 3 can also be interpreted as an adaptation of the controller design suggested in [41, Lemma 2] to the context of  $k$ -contraction.*

## V. ASYMPTOTIC BEHAVIOURS

This section is dedicated to discussing the asymptotic behaviour derived from a  $k$ -contractive property. Precisely, we discuss the similarities and differences between the behaviour of  $k$ -contractive systems and the recently proposed  $p$ -dominance analysis [12], [13]. Furthermore, this connection allows us to derive novel asymptotic properties for 3-contractive systems.

### A. Relation to $p$ -dominance

Partial stability of dynamical systems via  $p$ -dominance has recently attracted the attention of the control community, e.g., [35], [39]. In what follows, we link our main result to recent developments in  $p$ -dominance analysis [12], [13]. We start by recalling the definition of  $p$ -dominance.

**Definition 6** ( *$p$ -dominance*) *System (1) is said to be strictly  $p$ -dominant on  $\mathcal{S} \subsetneq \mathbb{R}^n$  if<sup>1</sup> there exist a real number  $\mu \geq 0$  and a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  with inertia  $\text{In}(P) = (p, 0, n - p)$  such that*

$$P \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P \prec -2\mu P, \quad \forall x \in \mathcal{S}. \quad (21)$$

In general,  $p$ -dominance and  $k$ -contraction are different properties. As an example, consider a linear system of the form (10) with  $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ . This system satisfies (21) for all  $0 < \mu < 1$  and some symmetric matrix  $P$  with inertia  $\text{In}(P) = (1, 0, 1)$ . However, we have  $A^{[2]} = 1$ . Therefore, this system is 1-dominant, yet it is not 2-contractive. An example of a  $k$ -contractive system that does not satisfy  $p$ -dominance conditions can be found in equation (23). Hence, there are  $p$ -dominant systems that are not  $k$ -contractive and vice versa. Nonetheless, condition (16b) sheds light on a link between these two properties. Interestingly,  $p$ -dominance has been related to various differential properties [13, Section V], such as differential positiveness [42] and monotonicity [43]. However, to the best of the authors' knowledge, the link between  $k$ -contraction and  $p$ -dominance has not been properly clarified in the literature.

Precisely, consider the variational system of (1), presented in (6). Then, the  $p$ -dominance condition (21) splits the tangent space in a vertical subspace of dimension  $p$  and a horizontal subspace of dimension  $n - p$ . Namely, for each initial condition  $x_0 \in \mathcal{S}$  the tangent space can be divided in a horizontal distribution  $\mathcal{H}_x$  and a vertical distribution  $\mathcal{V}_x$ . The property of  $p$ -dominance can be interpreted as a form of horizontal contraction [10, Section VII], in the sense that contraction is only imposed in the horizontal subspace.

<sup>1</sup>The definition can be extended to the full set  $\mathbb{R}^n$  but in this case condition (21) is modified into  $P \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P \preceq -2\mu P - \varepsilon I$  for all  $x \in \mathbb{R}^n$  where the term  $-\varepsilon I$  is added to ensure uniformity.

However, horizontal contraction is not sufficient for  $k$ -contraction [19], and a bound on the expansion rate of the vertical subspace has to be imposed. This bound is obtained via (16a) paired with (16c).

With these similarities and differences in mind, the next section is devoted to compare the asymptotic behaviour of  $k$ -contractive systems and  $p$ -dominant ones.

### B. 1-dominance and 2-contractive behaviours

We remark that  $k$ -contractive and  $p$ -dominant systems share interesting guarantees on their asymptotic behaviors. In fact, the asymptotic dynamics of both  $p$  dominant and  $k$ -contractive systems (with  $k = p + 1$ ) have been proven to evolve on a  $p$ -dimensional object. This relationship explains why 1-dominance and 2-contraction share similar convergence results in systems evolving in a bounded set. More precisely, consider system (1) and assume  $\mathcal{S}$  is compact and forward invariant. In [44] it is shown that any bounded solution converges to an equilibrium point if the system is 2-contractive. Similarly, in [13, Corollary 1], it is proven that any bounded solution converges to a fixed point if the system is 1-dominant.

Now, considering these similar asymptotic behaviors, one can question the difference between  $p$ -dominance (21) and the proposed conditions (16a)-(16c). For example, since any trajectory of a 1-dominant system evolving in a compact set will eventually converge to an equilibrium point, the area of any surface defined as in (3) will eventually converge to zero. However, a uniform exponential decay of such a 2-order volume is not guaranteed. As an example, consider the following system

$$\begin{aligned} \dot{x}_1 &= \theta x_1 - x_1^3, \\ \dot{x}_2 &= -x_2, \end{aligned} \quad (22)$$

and its associated Jacobian  $\frac{\partial f}{\partial x} = \begin{bmatrix} \theta - 3x_1^2 & 0 \\ 0 & -1 \end{bmatrix}$ , where  $\theta$  is a positive scalar. Consider any compact forward invariant set  $\mathcal{S}$  which includes the origin. Additionally, consider an initial condition with  $x_1(0) \approx 0$ . Then, the system satisfies (21) in  $\mathcal{S}$  with  $0 < \mu < 1$  and some symmetric matrix  $P$  with  $\text{In}(P) = (1, 0, 1)$ . In other words, the system is 1-dominant and converges to a (non-unique) equilibrium point. Indeed, the system converges to  $[x_1, x_2]^\top = [\pm\sqrt{\theta}, 0]$ . Moreover, system (22) satisfies (16a) with  $\mu_0 > \theta$  and (16b) with  $k = 2$  and  $\mu_1 > -1$ . Therefore, if  $\theta \geq 1$ , condition (16c) is not satisfied and there is no guarantees that the system is uniformly  $k$ -contractive in  $\mathcal{S}$ . To be more specific, the second additive compound of the system is  $\frac{\partial f^{[2]}}{\partial x} = \theta - 3x_1^2 - 1$ . If  $\theta > 1$ , the compound is non-negative when  $|x_1| \leq \sqrt{\frac{\theta-1}{3}}$  and the area of any surface of initial conditions will expand. Indeed, the system becomes 2-contractive once  $|x_1|$  becomes large enough, even if the system was uniformly 1-dominant in all the considered set.

### C. 2-dominance and 3-contractive behaviours

Even if the asymptotic trajectories of  $p$ -dominant and  $k$ -contractive systems (with  $k = p + 1$ ) lie on an object of the same dimension, they may show different asymptotic behaviors. However, as (16a)-(16c) are implemented with constant  $P_i$ , we inherit asymptotic properties from  $p$ -dominance that are not typically obtained in more general  $k$ -contractive systems. This fact is formalized in the following lemma.

**Lemma 4** *Assume that system (1) satisfies (16) with  $k = 3$  in a forward invariant set  $\mathcal{S} \subsetneq \mathbb{R}^n$ . Then, any trajectory of (1) with initial condition in  $\mathcal{S}$  converges to a simple attractor, that is, a fixed point or a limit cycle.*



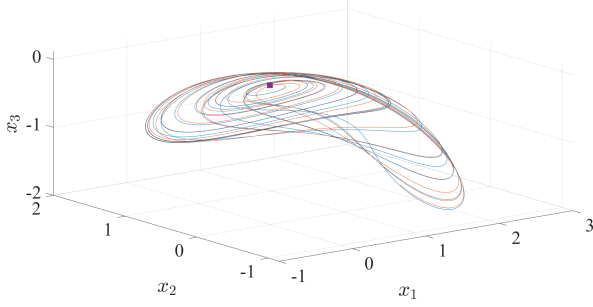


Fig. 4. Evolution of two trajectories of the system (23). The first (blue) has an initial condition  $[0.1, 0.1, 0]$ , the second (red) has an initial condition  $[0.099, 0.1, 0]$ . The trajectories do not converge to any specific limit cycle.

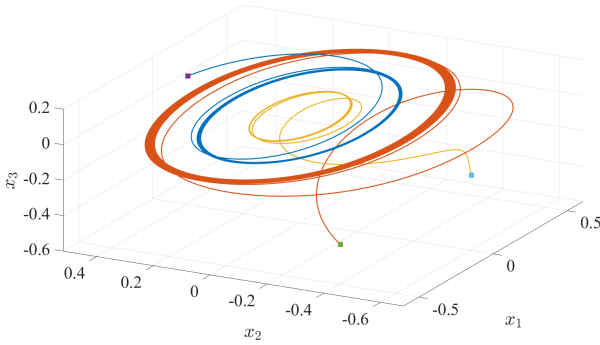


Fig. 5. Evolution of three trajectories of the system (24). The first (blue) has an initial condition  $[0.2, 0.5, 0]$ , the second (red) has an initial condition  $[-0.3, -0.3, -0.5]$ , the third (yellow) has an initial condition  $[0.2, -0.5, -0.3]$ .

The proof of Lemma 4 is postponed to Section VIII-E. We remark that this result is not generally true for 3-contractive systems. As an example, consider the Rössler system [45]

$$\begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= -x_1 - x_3, \\ \dot{x}_3 &= 0.5((x_1 - x_1^2) - x_3). \end{aligned} \quad (23)$$

The 3-additive compound of the Jacobian is  $\frac{\partial f}{\partial x}(x)^{[3]} = -0.5$ . Therefore, condition (8) is trivially satisfied and the system (23) is 3-contractive. Nonetheless, the nonlinear term  $x_1^2$  is not monotonic nor sector bounded. Consequently, this term prevents the existence of a constant matrix  $P_2$  in (16b). Therefore, even if the system is 3-contractive and evolves in a compact set, there is no guarantee that it will converge to a fixed point or limit cycle. Indeed, this system presents chaotic behavior and its trajectories do not converge to a simple attractor, as shown in Figure 4.

Alternatively, consider the following system,

$$\begin{aligned} \dot{x}_1 &= x_2 - 2x_3, & \dot{x}_2 &= -x_1 - x_3, \\ \dot{x}_3 &= 0.5((x_1 - x_1^3) - x_3). \end{aligned} \quad (24)$$

In this case, the 3-additive compound of the Jacobian is  $\frac{\partial f}{\partial x}(x)^{[3]} = -0.5$ , similarly to (23). Then, the system is 3-contractive. Moreover, it evolves in a compact forward invariant set. However, differently from (23), inequalities (16a)-(16c) are satisfied with  $\mu_0 = 0.2, \mu_2 = -0.45$  and  $P_0 = \begin{bmatrix} 1.45 & 0.20 & 0.13 \\ 0.20 & 2.09 & 0.26 \\ 0.13 & 0.26 & 0.67 \end{bmatrix}$ ,  $P_2 = \begin{bmatrix} -2.05 & -1.02 & -0.45 \\ -1.02 & 26.75 & 14.47 \\ -0.45 & 14.47 & 6.41 \end{bmatrix}$ . Therefore, by Lemma 4 we can conclude that the system will

converge to a (non-unique) simple attractor. This behavior is shown in Figure 5, which presents the evolution of three trajectories.

#### D. An illustration

Consider the following nonlinear system:

$$\dot{x}_1 = x_2 - x_3, \quad \dot{x}_2 = -x_1 - x_3 + u, \quad \dot{x}_3 = x_1(x_1^2 - 0.25). \quad (25)$$

In the absence of input ( $u = 0$ ), the system trajectories present an oscillatory behavior. Moreover, for the same conditions, the 3-additive compound is  $\frac{\partial f}{\partial x}(x)^{[3]} = 0$ . Therefore, the system is not 3-contractive (nor 2-contractive or 1-contractive). The objective is to design a linear state-feedback controller  $u = -Kx$ , such that the system converges to an equilibrium point. Note that we do not need the system to reach a specific point, we only require the existence of (at least) one attractive equilibrium. For this example, we restrict ourselves to linear controllers.

A quick computation shows that the closed-loop system presents 3 equilibrium points,

$$x_1^* = \{0, -0.5, 0.5\}, \quad x_2^* = x_3, \quad x_3^* = \frac{-(1+k_1)}{1+k_3+k_2}x_1,$$

where  $k_1, k_2, k_3$  are the components of the feedback gain  $K$ . The existence of multiple equilibrium points prevents the design of a linear controller that achieves 1-contraction, e.g., [6]. Nonetheless, it may still be possible to design a controller that guarantees 2-contraction of the closed-loop. Since the system dynamics evolve in a compact set, 2-contraction guarantees convergence to a (possibly non-unique) equilibrium point, see Section V-B.

With the aim of exploiting the results in Proposition 3, the matrix inequalities (18a)-(18b) are satisfied with  $\mu_0 = 0.3, \mu_1 = -0.6$  and  $W_0 = \begin{bmatrix} 1.86 & -1.21 & -0.92 \\ -1.21 & 2.13 & 0.86 \\ -0.92 & 0.86 & 0.96 \end{bmatrix}$ ,  $W_1 = \begin{bmatrix} 2.84 & 0.06 & 2.65 \\ 0.06 & 0.27 & -5.16 \\ 2.65 & -5.19 & -2.29 \end{bmatrix}$ . With this specific selection of constant and matrices, we have  $\varepsilon = 0.048$ , which implies that (19) is satisfied. Therefore, according to Proposition 3 the state-feedback law  $u = -Kx$  with gain (20), namely

$$K = [0.8978 \quad 2.1567 \quad -1.1765], \quad (26)$$

makes the closed-loop system 2-contractive. Indeed, the closed-loop system satisfies (8) with  $\eta = 0.091$  and  $Q = \begin{bmatrix} 1.26 & -0.06 & 0.46 \\ -0.05 & 2.74 & -1.34 \\ 0.41 & -1.34 & 2.33 \end{bmatrix}$ , which validates the result by means of Theorem 1. More precisely, the closed-loop system presents 3 equilibrium points, one unstable at the origin and 2 (locally) asymptotically stable.

## VI. PROOF OF LINEAR RESULTS (ANALYSIS)

In the rest of this section, given a matrix  $A \in \mathbb{R}^{n \times n}$ , we number its eigenvalues in such a way that

$$\Re\{\lambda_1\} \geq \Re\{\lambda_2\} \geq \dots \geq \Re\{\lambda_n\}. \quad (27)$$

### A. $k$ -contraction properties and inertia theorem

The following result is a direct consequence of the additive compound definition.

**Lemma 5** A system  $\dot{x} = Ax$ , with  $x \in \mathbb{R}^n$  is  $k$ -contractive if and only if the eigenvalues  $\lambda_i$  of the matrix  $A$  satisfy

$$\sum_{i=1}^k \Re\{\lambda_i\} < 0, \quad (28)$$

according to the above mentioned numbering (27).

*Proof:* A necessary and sufficient condition for  $k$ -contraction of a linear system  $\dot{x} = Ax$  is that the matrix  $A^{[k]}$  is Hurwitz [15].

Moreover, recall that a spectral property of the additive compound matrix is that the eigenvalues of the matrix  $A^{[k]}$  are all the possible sums of the form  $\lambda_{i_1} + \lambda_{i_2} + \dots + \lambda_{i_k}$ , with  $1 \leq i_1 < \dots < i_k \leq n$ , see [15]. That is, a necessary and sufficient condition for  $k$ -contraction is that the sum of any combination of  $k$  eigenvalues of  $A$  is negative. In particular, this holds true if and only if the first  $k$  eigenvalues satisfy condition (28) according to the numbering (27). ■

Second, we state a lemma about inertia properties of the Lyapunov equation (11a), collecting together various results from the literature, e.g., [20, Lemma 1, Section 3], [21, Theorem 2.5].

**Lemma 6** *Given a matrix  $A \in \mathbb{R}^{n \times n}$ , a real constant  $\mu$ , and an integer  $p \in \{0, \dots, n\}$ , the following statements are equivalent:*

- 1)  $A$  has  $p$  eigenvalues with real part larger than  $\mu$  and  $n - p$  eigenvalues with real part smaller than  $\mu$ ,
- 2) the matrix  $A - \mu I$  has inertia  $(n - p, 0, p)$ ,
- 3) there exists a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  with inertia  $\text{In}(P) = (p, 0, n - p)$  satisfying

$$A^\top P + PA \prec 2\mu P \quad (29)$$

- 4) given a symmetric positive definite matrix  $Q \succ 0$  there exists a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  with inertia  $\text{In}(P) = (p, 0, n - p)$  satisfying  $(A - \mu I)^\top P + P(A - \mu I) = -Q$ .

*Proof:* The eigenvalues of  $A - \mu I$  are the shifted eigenvalues  $\lambda_1 - \mu, \dots, \lambda_n - \mu$ , numbered as in (27). If  $A - \mu I$  has inertia  $(n - p, 0, p)$ , it implies that  $\Re(\lambda_{p+1}) - \mu < 0$ . This shows that (1)  $\Leftrightarrow$  (2). The implication (3)  $\Leftrightarrow$  (2) is due to [20, Lemma 1, Section 3]. The implication (2)  $\Leftrightarrow$  (4) is due to [21, Theorem 2.5]. Finally, we have that (4)  $\Leftrightarrow$  (3). Indeed, (4) implies

$$A^\top P + PA = -2\mu P - Q \prec -2\mu P$$

because  $Q$  is positive definite. ■

## B. Proof of Theorem 2

The proof of Theorem 2 is obtained by combining the two previous lemmas. To begin with, we introduce a new notation in order to represent the eigenvalues of  $A$  and their associated multiplicities. Precisely, consider the matrix  $A$  in (10) and let  $\Pi : \mathbb{C} \rightarrow \mathbb{R}$  denote the canonical projection onto the real axis. Let  $\sigma(A)$  be the spectrum of  $A$  and suppose  $\Pi(\sigma(A)) = \{\alpha_1, \alpha_2, \dots, \alpha_q\}$  ( $q \leq n$ ) with  $\alpha_1 > \alpha_2 > \dots > \alpha_q$ . Set  $\bar{h}_i = \text{card}(\Pi^{-1}(\alpha_{i+1}) \cap \sigma(A))$ , where eigenvalues have been counted with their algebraic multiplicities (so that  $\bar{h}_0 + \bar{h}_1 + \dots + \bar{h}_{q-1} = n$ ). Finally, let  $\bar{d}_0 = 0$  and  $\bar{d}_i = \sum_{j=0}^{i-1} \bar{h}_j$ , for each  $i \in \{1, \dots, q-1\}$ .

We remark that the variables  $\bar{d}_i, \bar{h}_i$  are related to variables  $d_i, h_i$  of Theorem 2, hence the similar notation. To see this relation, notice that  $\bar{h}_i$  represents the number of eigenvalues projected to  $\alpha_{i+1}$  and  $\bar{d}_i$  represents the amount of eigenvalues with real part strictly larger than  $\alpha_{i+1}$ . Therefore, for any constant  $\mu \in \mathbb{R}$  such that  $\alpha_1 < \mu$ , we have  $\text{In}(-A + \mu I) = (0, 0, n)$ . Similarly, if  $\alpha_{i+1} < \mu < \alpha_i$ , we have that  $\text{In}(-A + \mu I) = (\bar{d}_i, 0, n - \bar{d}_i)$ . Consequently, if we avoid the singularity case  $\pi_0(-A + \mu I) > 0$ , the matrix  $-A + \mu I$  can only present a particular set of inertia  $(\bar{d}_i, 0, n - \bar{d}_i)$  with  $i \in \{0, \dots, q-1\}$ . From this fact and Lemma 6, we obtain that the matrix inequalities in (11a) are only feasible for the particular set of inertia  $\text{In}(P_i) = (\bar{d}_i, 0, n - \bar{d}_i)$  with  $i \in \{0, \dots, q-1\}$ . A direct consequence of this result is that  $\ell \leq q$ .

We now present the main arguments proving sufficiency and necessity of the result in Theorem 2.

*Sufficiency.* In order to prove the sufficiency, we will show that the set of inequalities (11) implies the condition (28). To this end, notice that a solution of (11a) for some  $\mu_i \in \mathbb{R}$  and  $P_i$  with inertia  $\text{In}(P_i) = (d_i, 0, n - d_i)$  implies that  $\text{In}(-A + \mu I) = (d_i, 0, n - d_i)$  by means of Lemma 6. That is,  $A$  has only  $d_i$  eigenvalues with real part strictly larger than  $\mu_i$ . This is equivalent to the bound

$$\Re(\lambda_{d_{i+1}}) \leq \Re(\lambda_{d_i+1}) < \mu_i, \quad \forall i \in \{0, \dots, \ell-1\}. \quad (30)$$

Now, due to Lemma 6, we have that  $\mu_{i+1} < \mu_i$  for all  $i \in \{0, \dots, \ell-2\}$ . Therefore, the bound (11b) implies  $\mu_{\ell-1} \leq 0$ , which combined with (30), implies  $\lambda_{d_\ell} < 0$ . Additionally, because the eigenvalues are numbered as per (27), the following bound trivially holds for all  $i \in \{0, \dots, \ell-1\}$

$$\sum_{j=d_{i+1}}^{d_{i+1}} \Re(\lambda_j) \leq (d_{i+1} - d_i) \Re(\lambda_{d_i+1}) = h_i \Re(\lambda_{d_i+1}),$$

where the definition  $h_i = (d_{i+1} - d_i)$  has been used. Combining this bound with the fact that  $d_\ell \leq k$ , the bound  $\lambda_{d_\ell} < 0$  and the fact that the eigenvalues are numbered as in (27), we obtain the following,

$$\begin{aligned} \sum_{i=1}^k \Re(\lambda_i) &\leq \sum_{i=1}^{d_\ell} \Re(\lambda_i) = \sum_{i=0}^{\ell-1} \sum_{j=d_{i+1}}^{d_{i+1}} \Re(\lambda_j) \\ &\leq \sum_{i=0}^{\ell-1} h_i \Re(\lambda_{d_{i+1}}). \end{aligned} \quad (31)$$

Then, combining (30), (31) and (11b) we have

$$\sum_{i=1}^k \Re(\lambda_i) < \sum_{i=0}^{\ell-1} h_i \mu_i \leq 0. \quad (32)$$

Consequently, by Lemma 5, we obtain that the system is  $k$ -contractive.

*Necessity.* Define

$$\begin{aligned} p_k &:= \max\left(\{\bar{d}_0, \bar{d}_1, \dots, \bar{d}_{q-1}\} \cap [0, k-1]\right), \\ c_k &:= \text{card}\left(\{\bar{d}_0, \bar{d}_1, \dots, \bar{d}_{q-1}\} \cap [0, k-1]\right). \end{aligned} \quad (33)$$

Then, the following equality holds

$$(k - p_k) \alpha_{c_k} + \sum_{i=0}^{c_k-2} h_i \alpha_{i+1} = \sum_{i=1}^k \Re(\lambda_i). \quad (34)$$

Hence, combining Lemma 5 and (34), if the system is  $k$ -contractive, the next bound is satisfied

$$(k - p_k) \alpha_{c_k} + \sum_{i=0}^{c_k-2} h_i \alpha_{i+1} < 0. \quad (35)$$

Then, by continuity, there exist a scalar  $\varepsilon > 0$ , such that

$$(k - p_k)(\alpha_{c_k} + \varepsilon) + \sum_{i=0}^{c_k-2} h_i (\alpha_{i+1} + \varepsilon) \leq 0.$$

Next, fix  $\ell = c_k$ ,  $d_\ell = k - p_k + d_{\ell-1}$ ,  $d_i = \bar{d}_i$  for all  $i \in \{0, \dots, \ell-1\}$  and select  $\mu_{i-1} = \varepsilon + \alpha_i$ , for all  $i \in \{1, \dots, \ell\}$ . We have

$$\sum_{i=0}^{\ell-1} h_i \mu_i = (k - p_k)(\alpha_{c_k} + \varepsilon) + \sum_{i=0}^{c_k-2} h_i (\alpha_{i+1} + \varepsilon) \leq 0,$$

thus showing (11b). Now, define matrices  $\hat{A}_i := A - \mu_i I$  with  $i \in \{0, \dots, \ell-1\}$ . It is clear that, since  $\mu_{i-1} > \alpha_i$  by definition, each matrix  $\hat{A}_i$  has  $\bar{d}_i$  eigenvalues with positive real part and  $n - \bar{d}_i$  eigenvalues with negative real part. That is  $\text{In}(\hat{A}_i) = (n - \bar{d}_i, 0, \bar{d}_i)$ .

Then, by Lemma 6, there exist symmetric matrices  $P_i$  with  $\text{In}(P_i) = \text{In}(-\hat{A}_i) = \{\bar{d}_i, 0, n - \bar{d}_i\}$  such that

$$A^\top P_i + P_i A \prec 2\mu_i P_i \quad \forall i = 0, \dots, \ell - 1,$$

thus concluding the proof.

## VII. PROOF OF LINEAR RESULTS (DESIGN)

This section follows the next notation. First, we will consider that every pair  $(A, B)$ , is algebraically equivalent to the form in (12). Second, we define  $\Re(\lambda_1^u) \geq \dots \geq \Re(\lambda_{n_u}^u)$ , as the numbered set of eigenvalues of  $A_u$ .

### A. Proof of Lemma 2

*Sufficiency.* Without loss of generality, consider a system in the form (12). After a feedback design  $u = -Kx = -[K_c \ K_u](x_c^\top, x_u^\top)^\top$  is selected, the closed-loop eigenvalues are given by  $\sigma(A) = \sigma(A_c - B_c K_c) \cup \sigma(A_u)$ . Then, we can arbitrarily assign the eigenvalues of the closed-loop matrix  $A_c - B_c K_c$ , since the pair  $(A_c, B_c)$  is controllable. As a consequence, let  $c = k\lambda_1^u$  and select  $K_c$  such that the largest eigenvalue of  $A_c - B_c K_c$  has real part smaller than  $-|\Re(c)|$ . Then, the conditions of Lemma 5 are satisfied either if  $A_u$  is  $k$ -contractive or  $n_u < k$  by construction.

*Necessity.* If  $n_u \geq k$  and  $\dot{x} = A_u x$  is not  $k$ -contractive, then, there is a sum of  $k$  eigenvalues in the spectrum of  $A_u$  that is positive, see Lemma 5. Therefore, since the spectrum of  $A_u$  is invariant to the controller gain, the closed-loop system  $A - BK$  cannot be  $k$ -contractive (invoking again Lemma 5), which proves necessity.

### B. Inertia theorems generalizing stabilizability conditions

We state a set of new technical lemmas related to the feasibility and the inertia of the generalized stabilizability-like inequality (13a). Note that the following lemmas do not require the pair  $(A, B)$  to be controllable, contrarily to [22].

**Lemma 7** Consider a pair of matrices  $(A, B)$  and its canonical decomposition (12). Suppose that for some  $\mu \in \mathbb{R}$ ,  $\text{In}(A_u - \mu I) = (n_u - \varrho, 0, \varrho)$  with  $\varrho \in \{0, \dots, n_u\}$ . Then, there exists a symmetric matrix  $W \in \mathbb{R}^{n \times n}$ , with inertia  $\text{In}(W) = (\varrho, 0, n - \varrho)$ , satisfying

$$WA^\top + AW - BB^\top \prec 2\mu W. \quad (36)$$

*Proof:* Without loss of generality, suppose that the pair  $(A, B)$  is in the form (12). Moreover, define the shifted matrices  $\hat{A}_c := A_c - \mu I$  and  $\hat{A}_u := A_u - \mu I$ . Then, since  $\text{In}(\hat{A}_u) = (n_u - \varrho, 0, \varrho)$  and by Lemma 6, there exist some symmetric matrix  $W_u \in \mathbb{R}^{n_u \times n_u}$  with inertia  $\text{In}(W_u) = (\varrho, 0, n_u - \varrho)$  such that

$$W_u \hat{A}_u^\top + \hat{A}_u W_u = -Q, \quad (37)$$

with  $Q \succ 0$ . Furthermore, since the pair  $(A_c, B_c)$  is controllable, the pair  $(-\gamma I - A_c, B_c)$  is also controllable for any  $\gamma \in \mathbb{R}$ . Hence, for  $\gamma > 0$  large enough and from the Lypaunov test of controllability [36, Theorem 12.4], there exist some positive symmetric matrix  $W_c \succ 0$

$$W_c(-\gamma I - \hat{A}_c)^\top + (-\gamma I - \hat{A}_c)W_c = -B_c B_c^\top$$

which implies

$$W_c \hat{A}_c^\top + \hat{A}_c W_c - B_c B_c^\top = -2\gamma W_c. \quad (38)$$

With this in mind, consider a symmetric matrix  $W$  with inertia  $\text{In}(W) = (\varrho, 0, n - \varrho)$  of the form

$$W = \begin{bmatrix} W_c & 0 \\ 0 & \kappa W_u \end{bmatrix}$$

where  $\kappa > 0$  has to be fixed, with  $W_u$  and  $W_c$  satisfying (37) and (38). Now, by subtracting  $2\mu W$  from the left-hand side of (36), we get the following equality

$$\begin{bmatrix} W_c & 0 \\ 0 & \kappa W_u \end{bmatrix} \begin{bmatrix} \hat{A}_c^\top & A_{12}^\top \\ 0 & \hat{A}_u^\top \end{bmatrix} + \begin{bmatrix} \hat{A}_c & A_{12} \\ 0 & \hat{A}_u \end{bmatrix} \begin{bmatrix} W_c & 0 \\ 0 & \kappa W_u \end{bmatrix} - \begin{bmatrix} B_c B_c^\top & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2\gamma W_c & \kappa A_{12} W_u \\ \kappa W_u A_{12}^\top & -\kappa Q \end{bmatrix}. \quad (39)$$

Since  $W_c$  and  $Q$  are positive definite, the right hand side of identity (39) can be made negative definite by taking  $\kappa > 0$  sufficiently small, see, e.g. [36, Section 14.4]. ■

Additionally, we present the following technical lemma.

**Lemma 8** Consider a pair of matrices  $(A, B)$  and its canonical decomposition (12). Moreover, assume there exists a (non-singular) symmetric matrix  $W \in \mathbb{R}^{n \times n}$  and a constant  $\mu \in \mathbb{R}$  such that

$$WA^\top + AW - BB^\top \prec 2\mu W. \quad (40)$$

Then,  $\pi_-(W) \geq \pi_-(-A_u + \mu I)$ .

*Proof:* Without loss of generality, we suppose that the pair  $(A, B)$  is in the form (12). Moreover, notice that the inequality (40) can be re-arranged as follows

$$W\bar{A}^\top + \bar{A}W \prec 2\mu W, \quad (41)$$

where  $\bar{A} := A - \frac{1}{2}BB^\top W^{-1}$ . Now, recall the notation in (12) and notice that the eigenvalues in the spectrum of  $A_u$  cannot be modified by the term  $\frac{1}{2}BB^\top P^{-1}$ , thus, we have  $\sigma(A_u) \subsetneq \sigma(\bar{A})$ , or, equivalently,  $\sigma(A_u - \mu I) \subsetneq \sigma(\bar{A} - \mu I)$ . From this fact we obtain that  $\pi_-(-\bar{A} + \mu I) \geq \pi_-(-A_u + \mu I)$ . Then, by Lemma 6 we have that any (non-singular)  $W$  that satisfies (41) necessarily implies  $\pi_-(-\bar{A} + \mu I) = \pi_-(W)$ , which concludes the proof. ■

Finally, we present a technical lemma that relates the colinearity condition in (14) and the generalized stabilizability-like inequality (13a).

**Lemma 9** Consider a pair of matrices  $(A, B)$  and its canonical decomposition (12). Suppose that for some  $\mu_1, \mu_2 \in \mathbb{R}$  with  $\mu_1 \leq \mu_2$ ,  $\text{In}(A_u - \mu_1 I) = (n_u - \varrho_1, 0, \varrho_1)$ ,  $\text{In}(A_u - \mu_2 I) = (n_u - \varrho_2, 0, \varrho_2)$  with  $\varrho_1, \varrho_2 \in \{0, \dots, n_u\}$ . Then, there exist a pair of symmetric matrices  $W_1, W_2 \in \mathbb{R}^{n \times n}$ , with inertia  $\text{In}(W_1) = (\varrho_1, 0, n - \varrho_1)$ ,  $\text{In}(W_2) = (\varrho_2, 0, n - \varrho_2)$ , satisfying

$$W_1 A^\top + AW_1 - BB^\top \prec 2\mu_1 W_1, \quad (42a)$$

$$W_2 A^\top + AW_2 - BB^\top \prec 2\mu_2 W_2, \quad (42b)$$

$$B^\top W_2^{-1} = B^\top W_1^{-1}. \quad (43)$$

*Proof:* Without loss of generality, we suppose the pair  $(A, B)$  to be in the form (12). This is not a restrictive assumption since the colinearity condition (43) is preserved under linear coordinate changes  $z = Tx$ , for any non-singular constant matrix  $T \in \mathbb{R}^{n \times n}$ .

Now, since  $\text{In}(A_u - \mu_1 I) = (n_u - \varrho_1, 0, \varrho_1)$ , we can follow similar arguments as in Lemma 7 proof, to show that (42a) can be satisfied with a symmetric matrix  $W_1$  of the form

$$W_1 = \begin{bmatrix} W_c & 0 \\ 0 & \kappa_1 W_{1,u} \end{bmatrix} \quad (44)$$

where  $\kappa_1 > 0$  is a sufficiently small constant,  $W_{1,u} \in \mathbb{R}^{n_u \times n_u}$  is a symmetric matrix with inertia  $\text{In}(W_{1,u}) = (\varrho_1, 0, n_u - \varrho_1)$  and  $W_c \succ 0$  is a positive definite symmetric matrix computed from

$$W_c(A_c - \mu_1 I)^\top + (A_c - \mu_1 I)W_c - B_c B_c^\top = -2\gamma W_c. \quad (45)$$

for some positive  $\gamma > 0$ .

With this in mind, we can construct a solution to (42b) such that  $W_1$  and  $W_2$  are colinear according to (43). Since  $\text{In}(A_u - \mu_2 I) = (n_u - \varrho_2, 0, \varrho_2)$  and by Lemma 6, there exists some symmetric matrix  $W_{2,u} \in \mathbb{R}^{n_u \times n_u}$  with inertia  $\text{In}(W_{2,u}) = (\varrho_2, 0, n_u - \varrho_2)$  such that

$$W_{2,u}(A_u - \mu_2 I)^\top + (A_u - \mu_2 I)W_{2,u} = -Q, \quad (46)$$

for some  $Q \succ 0$ . Moreover, recall the relation (45), then, we can derive the following set of equalities

$$\begin{aligned} & W_c(A_c - \mu_2)^\top + (A_c - \mu_2 I)W_c - B_c B_c^\top \\ &= W_c(A_c - \mu_1 I)^\top + (A_c - \mu_1 I)W_c + 2(\mu_1 - \mu_2)W_c - B_c B_c^\top \\ &= -2(\gamma + \mu_2 - \mu_1)W_c. \end{aligned} \quad (47)$$

With this in mind, consider a symmetric matrix  $W_2$  of the form

$$W_2 = \begin{bmatrix} W_c & 0 \\ 0 & \kappa W_{2,u} \end{bmatrix} \quad (48)$$

where  $\kappa_2 > 0$  has to be fixed, with  $W_{2,u}$  and  $W_c$  satisfying (46) and (47). Now, by subtracting  $2\mu_2 W_2$  from the left-hand side of (42b) and defining  $\hat{A}_c := A_c - \mu_2 I$  and  $\hat{A}_u := A_u - \mu_2 I$ , we get the following equality

$$\begin{aligned} & \begin{bmatrix} W_c & 0 \\ 0 & \kappa W_{2,u} \end{bmatrix} \begin{bmatrix} \hat{A}_c^\top & A_{12}^\top \\ 0 & \hat{A}_u^\top \end{bmatrix} + \begin{bmatrix} \hat{A}_c & A_{12} \\ 0 & \hat{A}_u \end{bmatrix} \begin{bmatrix} W_c & 0 \\ 0 & \kappa W_{2,u} \end{bmatrix} \\ & - \begin{bmatrix} B_c B_c^\top & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2(\gamma + \mu_2 - \mu_1)W_c & \kappa A_{12} W_{2,u} \\ \kappa W_{2,u} A_{12}^\top & -\kappa_2 Q \end{bmatrix}. \end{aligned} \quad (49)$$

Recall that  $\mu_1 \leq \mu_2$  by assumption and  $\gamma > 0, W_c \succ 0$  by design. Consequently,  $-2(\gamma + \mu_2 - \mu_1)W_c$  is negative definite and the right hand side of identity (49) can be made negative definite by taking  $\kappa_2 > 0$  sufficiently small.

Finally, since the system is in the form (12), it can be trivially seen that  $W_1, W_2$  constructed as in the block-diagonal form (44) and (48) satisfy (43), which ends the proof.  $\blacksquare$

### C. Proof of Theorem 3

Now, similar to the proof in Section VI-B, we introduce a new notation in order to represent the eigenvalues of  $A_u$  and their associated multiplicities. Precisely, consider the matrix  $A_u$  in (12) and  $\sigma(A_u)$  its spectrum and suppose  $\Pi(\sigma(A_u)) = \{\alpha_1, \alpha_2, \dots, \alpha_q\}$  ( $q \leq n_u$ ) where again  $\Pi: \mathbb{C} \rightarrow \mathbb{R}$  denote the canonical projection onto the real axis and with  $\alpha_1 > \alpha_2 > \dots > \alpha_q$ . Set  $\bar{h}_i = \text{card}(\Pi^{-1}(\alpha_{i+1}) \cap \sigma(A_u))$ , where eigenvalues have been counted with their algebraic multiplicities (so that  $\bar{h}_1 + \bar{h}_2 + \dots + \bar{h}_{q-1} = n_u$ ). Finally, let  $\bar{d}_0 = 0, \bar{d}_i = \sum_{j=1}^{i-1} \bar{h}_j, i \in \{1, \dots, q-1\}$ . Similar to Section VI-B,  $\bar{h}_i$  represents the number of eigenvalues of  $A_u$  projected to  $\alpha_{i+1}$  and  $\bar{d}_i$  represents the amount of eigenvalues with real part strictly than  $\alpha_{i+1}$ .

Finally, by recalling Lemma 2 and Lemma 5, we see that a necessary and sufficient condition for  $k$ -order stabilizability is either  $n_u < k$  or

$$\sum_{j=1}^k \Re(\lambda_j^u) < 0. \quad (50)$$

We now present the main arguments proving that the inequalities (13) are necessary and sufficient for  $k$ -order stabilizability.

*Sufficiency.* The goal of this proof is to show that if (13) is satisfied, then either (50) is satisfied or  $n_u < k$ , hence showing the result invoking Lemmas 2 and 5. Without loss of generality, we suppose that the pair  $(A, B)$  is in the form (12).

Firstly, we assume the case  $k \leq n_u$  and we show that (13) implies (50). To this end and by means of Lemma 8, we have that inequality (13a) implies that  $\pi_-(W_i) \geq \pi_-(-A_u + \mu_i I)$  for all  $i \in \{0, \dots, \ell-1\}$ . Recalling that  $W_i$  has inertia  $\text{In}(W_i) = (d_i, 0, n - d_i)$ , we have that  $A_u$  has at most  $d_i$  eigenvalues with real part strictly larger than  $\mu_i$ . This is equivalent to the bound

$$\Re(\lambda_{d_{i+1}}^u) \leq \Re(\lambda_{d_i+1}^u) < \mu_i, \quad \forall i \in \{0, \dots, \ell-1\}. \quad (51)$$

Then, following similar arguments as in the sufficiency part of Section VI-B, the next bound can be obtained.

$$\sum_{i=1}^k \Re(\lambda_i^u) < \sum_{i=0}^{\ell-1} h_i \mu_i \leq 0. \quad (52)$$

Thus, (50) is satisfied and the pair  $(A, B)$  is  $k$ -order stabilizable invoking Lemmas 2 and 5.

Finally, for the case  $k > n_u$ , we have  $k$ -order stabilizability directly from Lemma 2, thus ending the sufficiency proof.

*Necessity.* As stated before, if the pair  $(A, B)$  is  $k$ -order stabilizable then, either (50) or  $n_u < k$  is verified. The goal of this proof is to show that if one of these conditions are satisfied, then, there exists a solution to the inequalities (13). We begin by assuming the case  $k \leq n_u$  and (50) is verified. Now, let the scalars  $p_k$  and  $c_k$  be defined as in (33).

Then, the following equality holds

$$(k - p_k)\alpha_{c_k} + \sum_{i=0}^{c_k-2} h_i \alpha_{i+1} = \sum_{j=1}^k \Re(\lambda_j^u). \quad (53)$$

Hence, combining (50) and (53), if the system is  $k$ -order stabilizable (with  $k \leq n_u$ ), the next bound is satisfied

$$(k - p_k)\alpha_{c_k} + \sum_{i=0}^{c_k-2} h_i \alpha_{i+1} < 0. \quad (54)$$

Then, by continuity, there exist a scalar  $\varepsilon > 0$ , such that

$$(k - p_k)(\alpha_{c_k} + \varepsilon) + \sum_{i=0}^{c_k-2} h_i (\alpha_{i+1} + \varepsilon) \leq 0$$

Now, fix  $\ell = c_k, d_\ell = k - p_k + d_{\ell-1}, d_i = \bar{d}_i$  for all  $i \in \{0, \dots, \ell-1\}$  and select  $\mu_{i-1} = \varepsilon + \alpha_i$ , for all  $i \in \{1, \dots, \ell\}$ . We have

$$\sum_{i=0}^{\ell-1} h_i \mu_i = (k - p_k)(\alpha_{c_k} + \varepsilon) + \sum_{i=0}^{c_k-2} h_i (\alpha_{i+1} + \varepsilon) \leq 0,$$

thus showing (13b). Now, since  $\mu_i > \alpha_{i+1}$  for all  $i \in \{0, \dots, \ell-1\}$  we have that  $A_u$  has only  $\bar{d}_i$  eigenvalues strictly larger than  $\mu_i$  and the rest are strictly smaller. That is,  $\text{In}(A_u - \mu_i I) = (n_u - \bar{d}_i, 0, \bar{d}_i)$  for all  $i \in \{0, \ell-1\}$ . Then, by Lemma 7, there exist symmetric matrices  $W_i$  with  $\text{In}(W_i) = \{\bar{d}_i, 0, n - \bar{d}_i\}$  such that

$$A^\top W_i + W_i A - B B^\top \prec 2\mu_i W_i \quad \forall i \in \{0, \dots, \ell-1\},$$

thus concluding the proof if (50) is verified and  $k \leq n_u$ .

We now proceed with the necessity proof for the case  $k > n_u$ . For this proof, we remark that since a  $k$ -contractive system is also  $\bar{k}$ -contractive for all  $\bar{k} \in \{k, \dots, n\}$  [15], we have that if a pair  $(A, B)$  is  $k$ -order stabilizable, then, it is also  $\bar{k}$ -order stabilizable for all  $\bar{k} \in \{k, \dots, n\}$ . Additionally, by means of Lemma 2, a pair  $(A, B)$  is always  $k$ -order stabilizable if  $k = n_u + 1$ . Therefore, for all  $k > n_u$ ,  $k$ -order stabilizability necessarily implies  $\bar{k}$ -order stabilizability with  $\bar{k} = n_u + 1$ . With this fact in mind, this proof is based on showing that, if  $k = n_u + 1$ , then, there always exists a pair of matrices  $W_0, W_1$  and constants  $\mu_0, \mu_1$  such that (13) is satisfied. We highlight that this result does not require (50) to be satisfied.

Precisely, assume  $k = n_u + 1$ . Notice that we can always guarantee  $\text{In}(A_u - \mu_0 I) = (n_u, 0, 0)$  for any  $\mu_0 \in \mathbb{R}$  large enough. Therefore, by considering this sufficiently large  $\mu_0$  and by means of Lemma 7, we know that there exists a symmetric matrix  $W_0$  with inertia  $\text{In}(W_0) = (0, 0, n)$  solution of (13a). Furthermore, we can always find a sufficiently negative constant  $\mu_1 < 0$ , such that

$$\mu_1 + n_u \mu_0 \leq 0, \quad (55)$$

and  $\text{In}(A_u - \mu_1 I) = (0, 0, n_u)$ . Therefore, by considering this  $\mu_1$  and by means of Lemma 7, there exists a symmetric matrix  $W_1$  with inertia  $\text{In}(W_1) = (n_u, 0, n - n_u)$  solution of (13a). Finally, fix  $\ell = 2$  and select the aforementioned pair of matrices  $W_0, W_1$  and pair of constants  $\mu_0, \mu_1$  (these matrices and constants satisfy (13a)). Moreover, fix  $d_\ell = n_u + 1$ . With this selection, we have  $d_0 = 0, d_1 = n_u$  and  $h_0 = n_u, h_1 = 1$ . Thus, (55) implies (13b), which ends the proof.

#### D. Proof of Proposition 2

The first part of the proof focuses on proving the existence of solutions for the inequalities (13) considering the assumptions stated in the theorem and in particular the colinearity condition in (14). An immediate result of Lemma 9 is that there always exist a set of constants  $\mu_i$  and  $W_i$  such that (13b) and the colinearity condition in (14) is simultaneously satisfied for all  $i \in \{1, \dots, \ell - 1\}$ . Moreover, notice that Lemma 9 preserves the relation between the inertia of  $A_u - \mu_i I$  and  $W_i$  as in Lemma 7. Consequently, the arguments in the necessity part of Section VII-C could be repeated to obtain the existence of a solution from a  $k$ -order stabilizability assumption.

The second part of the proof consist in showing how the state-feedback law (15) makes the closed-loop system

$$\dot{x} = (A - BK)x = (A - \frac{\rho}{2}BB^\top W_0^{-1})x, \quad (56)$$

$k$ -contractive for all  $\rho \geq 1$ . Note that, since  $W_i$  is non-singular and symmetric for all  $i \in \{0, \dots, \ell - 1\}$  and by means of the colinearity condition (14), the left-hand side of (13a) can be rearranged as follows for all  $i \in \{0, \dots, \ell - 1\}$

$$\begin{aligned} & W_i A^\top + A W_i - B B^\top \\ &= W_i (A - \frac{1}{2} B B^\top W_i^{-1})^\top + (A - \frac{1}{2} B B^\top W_i^{-1}) W_i \\ &= W_i (A - \frac{1}{2} B B^\top W_0^{-1})^\top + (A - \frac{1}{2} B B^\top W_0^{-1}) W_i. \end{aligned}$$

Combining this result with the right-hand side of (13a), we obtain that for all  $i \in \{0, \dots, \ell - 1\}$

$$W_i (A - \frac{1}{2} B B^\top W_0^{-1})^\top + (A - \frac{1}{2} B B^\top W_0^{-1}) W_i \prec 2\mu_i W_i. \quad (57)$$

Now, by adding  $(1 - \rho)BB^\top$  in both sides of (57) and considering the fact that  $(1 - \rho)BB^\top \preceq 0$  for all  $\rho \geq 1$ , by (14) we get,

$$\begin{aligned} & W_i (A - \frac{\rho}{2} B B^\top W_0^{-1})^\top + (A - \frac{\rho}{2} B B^\top W_0^{-1}) W_i \\ & \prec 2\mu_i W_i + (1 - \rho) B B^\top \preceq 2\mu_i W_i. \end{aligned} \quad (58)$$

By post-multiplying and pre-multiplying both side of (58) by  $W_i^{-1}$  and defining  $P_i := W_i^{-1}$  we get for all  $i \in \{0, \dots, \ell - 1\}$

$$(A - \frac{\rho}{2} B B^\top W_0^{-1})^\top P_i + P_i (A - \frac{\rho}{2} B B^\top W_0^{-1}) \prec 2\mu_i P_i. \quad (59)$$

Finally, combining (59) and (13b) with Theorem 2 proves that the closed-loop system (56) is  $k$ -contractive.

## VIII. PROOFS OF NONLINEAR RESULTS

### A. Preliminary results

We provide in this section some preliminary results that will be used in the proof of Theorem 4. First, we recall (with a mild reformulation) the following result on  $p$ -dominance [13, Theorem 1].

**Theorem 5** *Suppose that system (1) is strictly  $p$ -dominant on a compact forward invariant set  $\mathcal{A} \subsetneq \mathbb{R}^n$  with rate  $\mu > 0$  and symmetric matrix  $P$  with inertia  $\text{In}(P) = (p, 0, n - p)$ . Then, for each  $x \in \mathcal{A}$ , there exists an invariant splitting  $T_x \mathbb{R}^n = \mathcal{V}_x \oplus \mathcal{H}_x$ , i.e. there exists a continuous mapping  $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  invertible for any  $x \in \mathcal{A}$  and satisfying*

$$\mathbf{T}(x) := [\mathbf{T}_h(x) \quad \mathbf{T}_v(x)], \quad (60a)$$

where  $\mathbf{T}_h : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n-p}$  and  $\mathbf{T}_v : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$  satisfy

$$\text{Im } \mathbf{T}_h(x) = \mathcal{H}_x, \quad \text{Im } \mathbf{T}_v(x) = \mathcal{V}_x. \quad (60b)$$

Moreover, there exist a scalar  $c_h > 0$  such that

$$\left| \frac{\partial \psi^t}{\partial x}(x) [\mathbf{T}_h(x) \quad 0] v \right| \leq c_h e^{-\mu t} \left| [\mathbf{T}_h(x) \quad 0] v \right| \quad (60c)$$

holds for all  $(t, x, v) \in \mathbb{R}_{\geq 0} \times \mathcal{A} \times T_x \mathbb{R}^n$ .

With this in mind, it is clear that if  $\mu_{k-1}$  is strictly negative, the matrix inequality (16b) imposes a form of horizontal contraction on the system [10, Section VII]. Nonetheless, horizontal contraction is not a sufficient condition for  $k$ -contraction [19]. This motivates (16a). We clarify the effects of (16a) via the following Lemma.

**Lemma 10** *Consider system (1) and assume there exist a forward invariant compact set  $\mathcal{A} \subsetneq \mathbb{R}^n$ , a positive definite matrix  $P_0 \in \mathbb{R}^{n \times n}$  and a scalar  $\mu_0$  satisfying (16a) for all  $x \in \mathcal{A}$ . Then there exists a constant  $c_v > 0$  such that*

$$\left| \frac{\partial \psi^t}{\partial x}(x) [0 \quad \mathbf{T}_v(x)] v \right| \leq c_v e^{\mu_0 t} \left| [0 \quad \mathbf{T}_v(x)] v \right| \quad (61)$$

for all  $(t, x, v) \in \mathbb{R}_{\geq 0} \times \mathcal{A} \times T_x \mathbb{R}^n$ , with  $\mathbf{T}_v$  as in (60b).

*Proof:* Consider the function,  $W := v^\top P_0 v$ . It satisfies

$$\underline{\lambda}(P_0) |v|^2 \leq W(v) \leq \bar{\lambda}(P_0) |v|^2, \quad (62)$$

where  $\underline{\lambda}(\cdot)$  and  $\bar{\lambda}(\cdot)$  represent the minimum and maximum eigenvalue of their argument, respectively. By (6), its time-derivative satisfies

$$\begin{aligned} \dot{W} &= v^\top \left( P_0 \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P_0 \right) v \\ &< 2\mu_0 v^\top P_0 v = 2\mu_0 W. \end{aligned}$$

Then, by Grönwall–Bellman inequality, we obtain

$$W(t) \leq e^{2\mu_0 t} W(0), \quad \forall t \in \mathbb{R}_{\geq 0}.$$

Invoking (62), we obtain for all  $(t, x, v) \in \mathbb{R}_{\geq 0} \times \mathcal{A} \times T_x \mathbb{R}^n$

$$\left| \frac{\partial \psi^t}{\partial x}(x) v \right| \leq \sqrt{\frac{\bar{\lambda}(P_0)}{\underline{\lambda}(P_0)}} e^{\mu_0 t} |v|.$$

As  $[0 \quad \mathbf{T}_v(x)] v \in T_x \mathbb{R}^n$ , the result trivially follows.  $\blacksquare$

Given the above results, condition (16c) can be seen as imposing a bound on the maximum expansion rate of the vertical subspace with respect to the contraction rate of the horizontal one. In particular, (16c) holds if the first is smaller than the latter. We now relate this property to infinitesimal  $k$ -contraction. As a first step, we present a technical lemma related to matrix compounds.

**Lemma 11** Consider a time-varying matrix  $M(t) \in \mathbb{R}^{n \times n}$

$$M(t) = [H(t) \quad V(t)],$$

with  $H(t) \in \mathbb{R}^{n \times n-p}$ ,  $V(t) \in \mathbb{R}^{n \times p}$  and  $p \in [0, n]$ . Assume there exist real numbers  $c_h, c_v, \alpha, \beta > 0$  such that

$$|H(t)| \leq c_h e^{-\alpha t}, \quad |V(t)| \leq c_v e^{\beta t}, \quad \forall t \in \mathbb{R}_+. \quad (63)$$

If  $\alpha > (k-1)\beta$  for some integer  $k \in [p+1, n]$ , there exist some real numbers  $c, \varepsilon > 0$  such that

$$|M(t)^{(k)}| \leq c e^{-\varepsilon t}, \quad \forall t \in \mathbb{R}_+. \quad (64)$$

*Proof:* Consider the elements of the compound matrix  $M(t)^{(k)}$ . Each one is a  $k^{th}$ -order minor of the original matrix  $M(t)$ , i.e., it is the determinant of a  $k \times k$  submatrix of  $M(t)$ , see Definition 2. Since  $k \geq p+1$ , each  $k \times k$  submatrix contains at least one column composed of elements of  $H(t)$ . That is, in the minimum case

$$M_k(t) = [h(t) \quad v_1(t) \quad \dots \quad v_{k-1}(t)], \quad (65)$$

where  $M_k(t) \in \mathbb{R}^{k \times k}$  is a submatrix of  $M(t)$ ,  $h(t) \in \mathbb{R}^k$  is a vector with components of  $H(t)$  and  $v_i(t) \in \mathbb{R}^k$  for  $i = 1, \dots, k-1$  is a vector with components of  $V(t)$ . In what follows, we show the elements of  $M(t)^{(k)}$  are bounded. Hence, we focus on submatrices of the form (65), since their determinant represents the worst-case scenario in a stability sense. Recall that, by definition of the wedge product,

$$\det(M_k(t)) = h(t) \wedge v_1(t) \wedge \dots \wedge v_{k-1}(t).$$

The wedge product can be represented using a basis  $e^i$ , where  $e^i$  depicts the  $i$ th canonical vector of  $\mathbb{R}^n$ . More specifically, by bilinearity of the wedge product, we have

$$\det(M_k(t)) = \sum_{i=1}^n h^i(t) (e^i \wedge v_1(t) \wedge \dots \wedge v_{k-1}(t)),$$

where  $h^i(t)$  is the  $i$ th element of  $h(t)$ . By performing similar operations on the remaining vectors we deduce

$$\det(M_k(t)) = \sum_{i_1=1}^k \dots \sum_{i_k=1}^k h^{i_1}(t) v_2^{i_2}(t) \dots v_{k-1}^{i_{k-1}}(t) E_k, \quad (66)$$

where  $E_k := (e^{i_1} \wedge e^{i_2} \wedge \dots \wedge e^{i_k})$ . By (63), we have

$$|h^i(t)| \leq c_h e^{-\alpha t}, \quad |v^i(t)| \leq c_v e^{\beta t}.$$

Moreover, the factor  $E_k$  will be either zero or an element of the canonical basis in  $\mathbb{R}^n$  multiplied by plus or minus one. Thus, using the triangle inequality, one obtains

$$|\det(M_k(t))| \leq \kappa c_h c_v e^{(-\alpha + (k-1)\beta)t}$$

where  $\kappa > 0$  is a positive constant related to the number of non-zero instances of  $E_k$ . Now, since  $\alpha - (k-1)\beta > 0$  by assumption, by continuity there always exists  $\varepsilon > 0$  such that  $\alpha - (k-1)\beta - \varepsilon > 0$ . Then,

$$|M(t)^{(k)}| = |e^{-\varepsilon t} e^{\varepsilon t} M(t)^{(k)}| \leq e^{-\varepsilon t} |e^{\varepsilon t} M(t)^{(k)}|.$$

By considering the worst-case (65), we have

$$e^{\varepsilon t} |\det(M_k(t))| \leq \bar{c} e^{(-\alpha + (k-1)\beta + \varepsilon)t},$$

for some  $\bar{c} > 0$ . Hence, since  $\alpha - (k-1)\beta - \varepsilon > 0$ , each element of  $e^{\varepsilon t} M(t)^{(k)}$  is exponentially decreasing and the norm  $|e^{\varepsilon t} M(t)^{(k)}|$  is uniformly bounded for all  $t \in \mathbb{R}_{\geq 0}$ , thus concluding the proof. ■

Leveraging on the previous lemmas, we now provide a bound on the  $k$  multiplicative compound of the state transition matrix of the variational system (6).

**Lemma 12** Consider system (1) and assume there exist a forward invariant compact set  $\mathcal{A} \subseteq \mathbb{R}^n$ , constants  $\mu_0, \mu_{k-1}$  and matrices  $P_0, P_{k-1} \in \mathbb{R}^{n \times n}$  such that (16) is satisfied. Then, there exist  $\varepsilon, c > 0$  such that

$$\left| \frac{\partial \psi^t}{\partial x}(x)^{(k)} \right| \leq c e^{-\varepsilon t}, \quad \forall (t, x) \in \mathbb{R}_{\geq 0} \times \mathcal{A}. \quad (67)$$

*Proof:* Consider (60a) in Theorem 5. Invertibility of  $\mathbf{T}(x)$  yields

$$\frac{\partial \psi^t}{\partial x}(x) = \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}(x) \mathbf{T}(x)^{-1} = \boldsymbol{\psi}^t(x) \mathbf{T}(x)^{-1},$$

with  $\boldsymbol{\psi}^t(x) := \begin{bmatrix} \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) & \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_v(x) \end{bmatrix}$ . Given any  $v \in \mathbb{T}_x \mathbb{R}^n$ , consider the decomposition  $v = (v^h, v^v)$ , where  $v^h \in \mathbb{R}^{n-p}$  and  $v^v \in \mathbb{R}^p$ . Then, for an arbitrary  $v^h$ , inequality (60c) of Theorem 5 implies

$$\left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) v^h \right| \leq c_h e^{\mu_{k-1} t} |\mathbf{T}_h(x) v^h|.$$

Recall the definition of matrix norm,

$$\left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) \right| := \max_{|u|=1} \left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) u \right|.$$

By selecting vector  $u^*$  such that  $|u^*| = 1$ , the previous exponential relation and the triangular inequality yield

$$\begin{aligned} \left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) \right| &= \left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) u^* \right| \\ &\leq c_h e^{\mu_{k-1} t} |\mathbf{T}_h(x) u^*| \leq c_h e^{\mu_{k-1} t} |\mathbf{T}_h(x)|. \end{aligned}$$

Since  $\mathcal{A}$  is compact and  $\mathbf{T}$  is continuous,  $|\mathbf{T}_h(x)|$  is bounded for all  $x \in \mathcal{A}$ . Then, by (60c), and by (61) we obtain

$$\begin{aligned} \left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_h(x) \right| &\leq c_h e^{\mu_{k-1} t} |\mathbf{T}_h(x)| \leq \bar{c}_h e^{-\mu_{k-1} t} \\ \left| \frac{\partial \psi^t}{\partial x}(x) \mathbf{T}_v(x) \right| &< c_v e^{\mu_0 t} |\mathbf{T}_v(x)| \leq \bar{c}_v e^{\mu_0 t} \end{aligned}$$

for all  $x \in \mathcal{A}$ . Finally, by boundedness of  $\mathbf{T}(x)$  and Lemma 11, we obtain

$$\left| \frac{\partial \psi^t}{\partial x}(x)^{(k)} \right| \leq |\boldsymbol{\psi}^t(x)^{(k)}| |\mathbf{T}(x)^{-1(k)}| \leq c e^{-\varepsilon t}$$

for all  $x \in \mathcal{A}$ , concluding the proof. ■

## B. Proof Theorem 4

Consider the  $k$ -th multiplicative compound of matrix  $\Psi(t, x_0)$  defined as in Section II-B. From the Cauchy-Binet formula [46, Chapter 1] we get:

$$\begin{aligned} \Psi(t, x_0)^{(k)} &= \begin{bmatrix} \frac{\partial \psi^t}{\partial x}(x_0) v_0^1 & \dots & \frac{\partial \psi^t}{\partial x}(x_0) v_0^k \end{bmatrix}^{(k)} \\ &= \frac{\partial \psi^t}{\partial x}(x_0)^{(k)} \Psi(0, x_0)^{(k)}. \end{aligned}$$

From (16) and Lemma 12 we obtain for all  $x \in \mathcal{S}$

$$|(\Psi(t, x_0))^{(k)}| \leq c e^{-\varepsilon t} |(\Psi(0, x_0))^{(k)}|.$$

Hence, the system is infinitesimally  $k$ -contractive on  $\mathcal{S}$  and the  $k$ -contractive property follows from Proposition 1.

## C. Proof of Lemma 3

Let us decompose the Jacobian of the vector field  $f$  as follows

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} F_s(x) & G_{12}(x) \\ G_{21}(x) & F_u(x) \end{bmatrix}.$$

Then, according to Theorem 1, a sufficient condition for 2-contraction in a set  $\mathcal{A}$  is:

$$\frac{\partial f}{\partial x}(x)^{[2]} = F_s(x) + F_u(x) < 0, \quad \forall x \in \mathcal{A}. \quad (68)$$

By subtracting  $2\mu_0 P_0$  in both sides of (16a), the following inequality is obtained

$$\begin{aligned} & \begin{bmatrix} F_s(x) - \mu_0 & G_{12}(x) \\ G_{21}(x) & F_u(x) - \mu_0 \end{bmatrix} P_0 \\ & + P_0 \begin{bmatrix} F_s(x) - \mu_0 & G_{21}(x) \\ G_{12}(x) & F_u(x) - \mu_0 \end{bmatrix} \prec 0. \end{aligned}$$

Since  $P_0$  is positive definite, the previous inequality necessarily implies that  $\frac{\partial f}{\partial x}(x) - \mu_0 I$  is Hurwitz for all  $x \in \mathcal{A}$ , and, consequently, its determinant is positive. That is,

$$0 < F_s(x)F_u(x) - \mu_0(F_s(x) + F_u(x)) + \mu_0^2 - G_{12}(x)G_{21}(x). \quad (69)$$

Similarly, by subtracting  $2\mu_1 P_1$  in both sides of (16b), the following inequality is obtained

$$\begin{aligned} & \begin{bmatrix} F_s(x) - \mu_1 & G_{12}(x) \\ G_{21}(x) & F_u(x) - \mu_1 \end{bmatrix} P_1 \\ & + P_1 \begin{bmatrix} F_s(x) - \mu_1 & G_{21}(x) \\ G_{12}(x) & F_u(x) - \mu_1 \end{bmatrix} \prec 0. \end{aligned}$$

Since  $\text{In}(P_1) = (1, 0, 1)$ , by Lemma 6, this inequality necessarily implies that  $\text{In}\left(\frac{\partial f}{\partial x}(x) - \mu_1 I\right) = (1, 0, 1)$  for all  $x \in \mathcal{A}$ , and, consequently, its determinant is negative. That is,

$$F_s(x)F_u(x) - \mu_1(F_s(x) + F_u(x)) + \mu_1^2 - G_{12}(x)G_{21}(x) < 0,$$

which can be rearranged as

$$F_s(x)F_u(x) - G_{12}(x)G_{21}(x) < \mu_1(F_s(x) + F_u(x)) - \mu_1^2. \quad (70)$$

Now, combining (69) and (70) we get

$$0 < (\mu_1 - \mu_0)(F_s(x) + F_u(x)) + \mu_0^2 - \mu_1^2. \quad (71)$$

Then, since  $\mu_0 > \mu_1$  we have that  $\mu_1 - \mu_0 < 0$ . Moreover, (16c) and the fact that  $\mu_0, \mu_1$  are real implies  $\mu_0^2 - \mu_1^2 < 0$ . Therefore, by combining (71) and (16c) we get the sufficient condition for 2-contraction in (68), which ends the proof.

### D. Proof of Proposition 3

Consider inequality (16a), pre-multiply and post-multiply both sides of the inequality by  $P_0^{-1}$  and fix  $W_0 = P_0^{-1}$ . Similarly, pre-multiply and post-multiply both sides of the inequality (16b) by  $P_1^{-1}$  and fix  $\bar{W}_1 = P_1^{-1}$ . Then, according to Theorem 4, a sufficient condition for the closed-loop system to be 2-contractive is the existence of symmetric matrices  $W_0, W_1 \in \mathbb{R}^{n \times n}$  of inertia  $\text{In}(W_0) = (0, 0, n)$ ,  $\text{In}(W_1) = (1, 0, n-1)$  and  $\bar{\mu}_0, \bar{\mu}_1 \in \mathbb{R}$  such that,

$$W_0 \left( \frac{\partial f}{\partial x}(x) - BK \right)^\top + \left( \frac{\partial f}{\partial x}(x) - BK \right) W_0 \prec 2\bar{\mu}_0 W_0, \quad (72a)$$

$$W_1 \left( \frac{\partial f}{\partial x}(x) - BK \right)^\top + \left( \frac{\partial f}{\partial x}(x) - BK \right) W_1 \prec 2\bar{\mu}_1 W_1, \quad (72b)$$

$$\bar{\mu}_1 + \bar{\mu}_0 < 0, \quad (72c)$$

for all  $x \in \mathcal{S}$ , where  $\mathcal{S}$  is assumed to be compact and forward invariant. In this proof, we show that if the inequalities in (18)-(19) are satisfied and the gain matrix  $K$  is designed as in (20), then, the inequalities in (72a)-(72c) are also satisfied. Thus, the closed-loop system is 2-contractive according to Theorem 4.

To this end, note that the left-hand side of (72a) with  $K$  fixed as in (20) can be rewritten as:

$$\begin{aligned} & W_0 \frac{\partial f}{\partial x}(x)^\top + \frac{\partial f}{\partial x}(x) W_0 - BB^\top - \frac{1}{2} BB^\top W_1^{-1} W_0 \\ & - \frac{1}{2} W_0 W_1^{-1} BB^\top \prec \mu_0 W_0 - \frac{1}{2} BB^\top W_1^{-1} W_0 - \frac{1}{2} W_0 W_1^{-1} BB^\top, \end{aligned} \quad (73)$$

where the right hand side is obtained employing (18a). Now, let  $\varepsilon > 0$  such that

$$\left[ I - \frac{1}{2} BB^\top W_1^{-1} \right] W_0 \left[ I - \frac{1}{2} BB^\top W_1^{-1} \right]^\top \preceq (1 + \varepsilon) W_0.$$

Furthermore, note that we have the identity

$$\begin{aligned} & -\frac{1}{2} BB^\top W_1^{-1} W_0 - \frac{1}{2} W_0 W_1^{-1} BB^\top = \\ & \left[ I - \frac{1}{2} BB^\top W_1^{-1} \right] W_0 \left[ I - \frac{1}{2} BB^\top W_1^{-1} \right]^\top - W_0 \\ & - \frac{1}{4} BB^\top W_1^{-1} W_0 W_1^{-1} BB^\top. \end{aligned}$$

As a consequence, by adding and subtracting  $\varepsilon W_0$  from the right hand side of (73) and using the two previous equations, we obtain

$$\begin{aligned} & W_0 \frac{\partial f}{\partial x}(x)^\top + \frac{\partial f}{\partial x}(x) W_0 - BB^\top \\ & - \frac{1}{2} BB^\top W_1^{-1} W_0 - \frac{1}{2} W_0 W_1^{-1} BB^\top \prec (\mu_0 + \varepsilon) W_0 \end{aligned}$$

Thus, selecting  $\bar{\mu}_0 = \mu_0 + \varepsilon$ , shows (72a). Then, remark that inequality (18b) can be re-organized to obtain the inequality (72b) with  $\bar{\mu}_1 = \mu_1$ . Finally, we have that (19) implies (72c), completing the proof.

### E. Proof of Lemma 4

Due to Lemma 6, a necessary condition for the feasibility of (16a)-(16c) for  $k = 3$  is  $\mu_2 < \mu_0$ . Consequently, (16c) implies  $\mu_2 < 0$ . Therefore, the inequalities (16a)-(16c) for  $k = 3$  in a forward invariant set  $\mathcal{S}$  imply 2-dominance in  $\mathcal{S}$ . Finally, the result follows from [13, Corollary 1].

## IX. CONCLUSIONS

We presented new conditions for  $k$ -contraction based on the use of generalized Lyapunov matrix inequalities. The proposed conditions do not rely on matrix compounds. In the linear case, they reduce the  $k$ -contraction analysis to solving a set of matrix inequalities. In the nonlinear context, they extend the well-known Demidovich conditions based on the Jacobian of the vector field along the flow. Moreover, these conditions provide a direct link between the  $p$ -dominance theory and  $k$ -contraction one, which allows to further characterize the asymptotic behavior of 3-contraction. Finally, we showed that the proposed conditions can be used to develop new tools for  $k$ -contractive feedback design, so that to extend existing conditions for standard 1-contraction, see, e.g. [5], [6], [8] and references therein.

Future works will focus on extending the proposed conditions to the context of time-varying systems and Riemannian metrics, similar to the context of 1-contraction, see, e.g. [11], [25], and to discrete time systems. Another topic of interest is the design of  $k$ -contractive observers and their use in practical applications.

## APPENDIX

### A. Proof of Lemma 1

The uniformity condition in (5) and the fact that  $P$  is positive definite and symmetric imply the existence of constants  $\bar{\sigma}, \underline{\sigma} > 0$  such that,

$$\begin{aligned} V^k(\varphi \circ \Phi) &= \int_{[0,1]^k} \sqrt{\det \left\{ \frac{\partial \Phi}{\partial r}(r)^\top \frac{\partial \varphi}{\partial x}(x)^\top P \frac{\partial \varphi}{\partial x}(x) \frac{\partial \Phi}{\partial r}(r) \right\}} dr \\ &\leq \sqrt{\bar{\sigma}} \int_{[0,1]^k} \sqrt{\det \left\{ \frac{\partial \Phi}{\partial r}(r)^\top P \frac{\partial \Phi}{\partial r}(r) \right\}} dr = \sqrt{\bar{\sigma}} V^k(\Phi) \end{aligned}$$

and  $V^k(\Phi) \leq V^k(\varphi \circ \Phi)/\sqrt{\bar{\sigma}}$ . Thus, exponential convergence of  $V^k(\Phi)$  implies exponential convergence of  $V^k(\varphi \circ \Phi)$ . Consequently, if the system is  $k$ -contractive in the original coordinates, we have

$$V^k(\varphi \circ \psi^t \circ \Phi) \leq \sqrt{\bar{\sigma}} V^k(\psi^t \circ \Phi) \leq \frac{\sqrt{\bar{\sigma}}}{\sqrt{\underline{\sigma}}} b e^{-at} V^k(\varphi \circ \Phi),$$

thus showing  $k$ -contractivity with  $\bar{b} = \frac{\sqrt{\bar{\sigma}}}{\sqrt{\underline{\sigma}}} b$  in the new coordinates.

### B. Proof of Theorem 1

Consider  $\Phi \in \mathcal{I}_k$ , where  $\mathcal{I}_k$  is defined in (2), satisfying  $\text{Im}(\Phi) \subseteq \mathcal{S}$ . To simplify notation, let us denote for all  $(r, t)$  in  $[0, 1]^k \times \mathbb{R}_{\geq 0}$

$$\Gamma(r, t) := \psi^t \circ \Phi(r), \quad \Gamma_r(r, t) := \frac{\partial \Gamma}{\partial r}(r, t).$$

In plain words, the factor  $\Gamma(r, t)$  depicts the solution of (1) at time  $t$  taking as a initial condition a point in  $\Phi$  parametrized by  $r$ . Then,  $\Gamma_r(r, t)$  represents how a variation in the initial condition  $\Phi(r)$  via a change in  $r$ , modifies the solution at time  $t$ . Since  $\Gamma(r, t)$  represents a solution of the system, for all  $(r, t)$  in  $[0, 1]^k \times \mathbb{R}_{\geq 0}$ , we have

$$\frac{d}{dt} \Gamma(r, t) = f(\Gamma(r, t)).$$

Moreover, since  $\mathcal{S}$  is forward invariant and  $\text{Im}(\Phi) \subseteq \mathcal{S}$ , we have  $\Gamma(r, t) \in \mathcal{S}$  for all  $(r, t)$  in  $[0, 1]^k \times \mathbb{R}_{\geq 0}$ . Additionally, by the chain rule, it follows that the point  $\Gamma_r(r, t)$  evolves according to

$$\frac{d}{dt} \Gamma_r(r, t) = \frac{\partial^2 \Gamma}{\partial r \partial t}(r, t) = \frac{\partial f}{\partial x}(\Gamma(r, t)) \Gamma_r(r, t)$$

for all  $(r, t)$  in  $[0, 1]^k \times \mathbb{R}_{\geq 0}$ . Since these dynamics are linear, following similar steps to the ones presented in [15, Section 2.5], we obtain

$$\frac{d}{dt} \Gamma_r(r, t)^{(k)} = \frac{\partial f}{\partial x}(\Gamma(r, t))^{[k]} \Gamma_r(r, t)^{(k)}. \quad (74)$$

Next, fix a symmetric positive definite matrix  $Q$  such that  $Q = P^{(k)}$ . Then, since  $\Gamma_r(r, t) \in \mathbb{R}^{n \times k}$ , from the Cauchy-Binet formula [46, Chapter 1] the following equality holds

$$\begin{aligned} & \det \left( \Gamma_r(r, t)^\top P \Gamma_r(r, t) \right) \\ &= \left( \Gamma_r(r, t)^{(k)} \right)^\top Q \Gamma_r(r, t)^{(k)} := v(r, t). \end{aligned} \quad (75)$$

Then, the volume  $V^k(\psi^t \circ \Phi)$  of  $\psi^t \circ \Phi$  computed according to (3) takes the form

$$V^k(\psi^t \circ \Phi) = \int_{[0, 1]^k} \sqrt{v(r, t)} dr.$$

In turn, the volume evolves according to

$$\begin{aligned} \frac{d}{dt} V^k(\psi^t \circ \Phi) &= \int_{[0, 1]^k} \frac{d}{dt} \sqrt{v(r, t)} dr \\ &= \int_{[0, 1]^k} \frac{1}{2\sqrt{v(r, t)}} \left[ \left( \Gamma_r(r, t)^{(k)} \right)^\top Q \frac{d}{dt} \Gamma_r(r, t)^{(k)} \right. \\ &\quad \left. + \left( \left( \Gamma_r(r, t)^{(k)} \right)^\top Q \frac{d}{dt} \Gamma_r(r, t)^{(k)} \right)^\top \right] dr. \end{aligned}$$

Hence, for all  $(r, t)$  in  $[0, 1]^k \times \mathbb{R}_{\geq 0}$ , we obtain

$$\begin{aligned} \frac{d}{dt} V^k(\psi^t \circ \Phi) &= \int_{[0, 1]^k} \frac{1}{2\sqrt{v(r, t)}} \left( \Gamma_r(r, t)^{(k)} \right)^\top \\ &\times \left( Q \frac{\partial f}{\partial x}(\Gamma(r, t))^{[k]} + \left( \frac{\partial f}{\partial x}(\Gamma(r, t))^{[k]} \right)^\top Q \right) \Gamma_r(r, t)^{(k)} dr. \end{aligned}$$

Note that in view of (8), there exists  $\mu > 0$  satisfying

$$Q \left( \frac{\partial f}{\partial x}(x)^{[k]} \right) + \left( \frac{\partial f}{\partial x}(x)^{[k]} \right)^\top Q \preceq -\mu Q, \quad \forall x \in \mathcal{S}$$

Then, invoking inequality (8) and recalling that  $\Gamma(r, t) \in \mathcal{S}$  for all  $(r, t)$  in  $[0, 1]^k \times \mathbb{R}_{\geq 0}$ , the previous relation implies

$$\begin{aligned} \frac{d}{dt} V^k(\psi^t \circ \Phi) &\leq \int_{[0, 1]^k} -\frac{\mu v(r, t)}{2\sqrt{v(r, t)}} dr \\ &\leq -\frac{\mu}{2} \int_{[0, 1]^k} \sqrt{v(r, t)} dr \leq -\frac{\mu}{2} V^k(\psi^t \circ \Phi) \end{aligned}$$

for all  $(r, t)$  in  $[0, 1]^k \times \mathbb{R}_{\geq 0}$ . The result follows by Grönwall's lemma.

### C. Proof of Proposition 1

Following [15, Section 2.5], it can be shown that the compound matrix of  $\Psi(t, x_0)$  evolves according to the linear dynamics

$$\frac{d}{dt} (\Psi(t, x_0))^{(k)} = \frac{\partial f}{\partial x}(\psi^t(x_0))^{[k]} (\Psi(t, x_0))^{(k)}. \quad (76)$$

By (7), such dynamics are globally exponentially stable. Consider now an arbitrary  $\Phi \in \mathcal{I}_k$  satisfying  $\text{Im}(\Phi) \subseteq \mathcal{S}$ . By following the first steps of the proof of Theorem 1, dynamics (74) and uniformity of (76) imply

$$\left| \Gamma_r(r, t)^{(k)} \right| \leq b e^{-at} \left| \Gamma_r(r, 0)^{(k)} \right|,$$

Then, by selecting  $P$  in (3) as the identity matrix, by (75) we obtain

$$\begin{aligned} V^k(\psi^t \circ \Phi) &= \int_{[0, 1]^k} \left| \Gamma_r(r, t)^{(k)} \right| dr \leq \int_{[0, 1]^k} b e^{-at} \left| \Gamma_r(r, 0)^{(k)} \right| dr \\ &\leq b e^{-at} \int_{[0, 1]^k} \left| \Gamma_r(r, 0)^{(k)} \right| dr \leq b e^{-at} V^k(\Phi), \end{aligned}$$

and this concludes the proof.

## REFERENCES

- [1] R. G. Sanfelice and L. Praly, "Convergence of Nonlinear Observers on  $\mathbb{R}^n$  With a Riemannian Metric (Part I)," *IEEE Transactions on Automatic Control*, vol. 57, no. 7, pp. 1709–1722, 2012.
- [2] G. Russo and M. di Bernardo, "Solving the rendezvous problem for multi-agent systems using contraction theory," in *48th IEEE Conference on Decision and Control*, 2009, pp. 5821–5826.
- [3] Z. Aminzare and E. D. Sontag, "Synchronization of diffusively-connected nonlinear systems: Results based on contractions with respect to general norms," *IEEE Transactions on Network Science and Engineering*, vol. 1, no. 2, pp. 91–106, 2014.
- [4] M. Giaccagli, S. Zoboli, D. Astolfi, V. Andrieu, and G. Casadei, "Synchronization in Networks of Nonlinear Systems: Contraction Analysis via Riemannian Metrics and Deep-Learning for Feedback Estimation," *under review on IEEE Transactions on Automatic Control*. Preprint: hal-03801100, 2023.
- [5] I. R. Manchester and J.-J. E. Slotine, "Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design," *IEEE Transactions on Automatic Control*, vol. 62, no. 6, pp. 3046–3053, 2017.
- [6] M. Giaccagli, V. Andrieu, S. Tarbouriech, and D. Astolfi, "Infinite gain margin, contraction and optimality: An LMI-based design," *European Journal of Control*, vol. 68, p. 100685, 2022.
- [7] M. Giaccagli, D. Astolfi, V. Andrieu, and L. Marconi, "Sufficient conditions for global integral action via incremental forwarding for input-affine nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 12, pp. 6537–6551, 2022.
- [8] M. Giaccagli, V. Andrieu, S. Tarbouriech, and D. Astolfi, "LMI conditions for contraction, integral action and output feedback stabilization for a class of nonlinear systems," *Automatica*, 2023.
- [9] S. Zoboli, S. Janny, and M. Giaccagli, "Deep learning-based output tracking via regulation and contraction theory," in *22nd IFAC World Congress*, 2023.



- [10] F. Forni and R. Sepulchre, "A differential Lyapunov framework for contraction analysis," *IEEE Transactions on Automatic Control*, vol. 59, no. 3, pp. 614–628, 2014.
- [11] V. Andrieu, B. Jayawardhana, and L. Praly, "Characterizations of global transversal exponential stability," *IEEE Transactions on Automatic Control*, vol. 66, no. 8, pp. 3682–3694, 2020.
- [12] F. Forni and R. Sepulchre, "A dissipativity theorem for  $p$ -dominant systems," in *IEEE 56th Conference on Decision and Control*, 2017, pp. 3467–3472.
- [13] —, "Differential dissipativity theory for dominance analysis," *IEEE Transactions on Automatic Control*, vol. 64, no. 6, pp. 2340–2351, 2019.
- [14] J. Muldowney, "Compound matrices and ordinary differential equations," *Rocky Mountain Journal of Mathematics*, vol. 20, no. 4, pp. 857–872, 1990.
- [15] C. Wu, I. Kanevskiy, and M. Margaliot, " $k$ -contraction: Theory and applications," *Automatica*, vol. 136, p. 110048, 2022.
- [16] R. Ofir, M. Margaliot, Y. Levron, and J.-J. Slotine, "A sufficient condition for  $k$ -contraction of the series connection of two systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 9, pp. 4994–5001, 2022.
- [17] R. A. Smith, "Massera's convergence theorem for periodic nonlinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 120, no. 2, pp. 679–708, 1986.
- [18] L. A. Sanchez, "Cones of rank 2 and the Poincaré–Bendixson property for a new class of monotone systems," *Journal of Differential Equations*, vol. 246, no. 5, pp. 1978–1990, 2009.
- [19] C. Wu and D. V. Dimarogonas, "From partial and horizontal contraction to  $k$ -contraction," *IEEE Transactions on Automatic Control*, pp. 1–8, 2023.
- [20] R. A. Smith, "The Poincaré–Bendixson theorem for certain differential equations of higher order," *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, vol. 83, no. 1-2, pp. 63–79, 1979.
- [21] T. Stykel, "Stability and inertia theorems for generalized Lyapunov equations," *Linear Algebra and its Applications*, vol. 355, no. 1-3, pp. 297–314, 2002.
- [22] H. K. Wimmer, "On the algebraic Riccati equation," *Bulletin of the Australian Mathematical Society*, vol. 14, no. 3, p. 457–461, 1976.
- [23] V. Andrieu, B. Jayawardhana, and L. Praly, "Transverse exponential stability and applications," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3396–3411, 2016.
- [24] M. Lee, *Introduction to Riemannian Manifolds. Second edition*. Springer, 2018.
- [25] W. Lohmiller and J.-J. E. Slotine, "On contraction analysis for non-linear systems," *Automatica*, vol. 34, no. 6, pp. 683–696, 1998.
- [26] H. K. Khalil, *Mathematical Preliminaries*. Prentice Hall, 2002, pp. 57–96.
- [27] E. Bar-Shalom, O. Dalin, and M. Margaliot, "Compound matrices in systems and control theory: a tutorial," *Mathematics of Control, Signals, and Systems*, pp. 1–55, 2023.
- [28] D. Angeli, M. A. Al-Radhawi, and E. D. Sontag, "A robust Lyapunov criterion for nonoscillatory behaviors in biological interaction networks," *IEEE Transactions on Automatic Control*, vol. 67, no. 7, pp. 3305–3320, 2022.
- [29] M. Fiedler, *Special matrices and their applications in numerical mathematics*. Courier Corporation, 2008.
- [30] S. Zobili, A. Cecilia, U. Serres, D. Astolfi, and V. Andrieu, "LMI conditions for  $k$ -contraction analysis: a step towards design," in *62nd IEEE Conference on Decision and Control*, 2023.
- [31] F. Bullo, "Contraction theory for dynamical systems," *Kindle Direct Publishing*, vol. 1, 2022.
- [32] A. Davydov, S. Jafarpour, and F. Bullo, "Non-Euclidean contraction theory for robust nonlinear stability," *IEEE Transactions on Automatic Control*, vol. 67, no. 12, pp. 6667–6681, 2022.
- [33] J. W. Simpson-Porco and F. Bullo, "Contraction theory on Riemannian manifolds," *Systems & Control Letters*, vol. 65, pp. 74–80, 2014.
- [34] O. Dalin, R. Ofir, E. B. Shalom, A. Ovseevich, F. Bullo, and M. Margaliot, "Verifying  $k$ -contraction without computing  $k$ -compounds," *IEEE Transactions on Automatic Control*, pp. 1–15, 2023.
- [35] G. O. Berger and R. M. Jungers, " $p$ -dominant switched linear systems," *Automatica*, vol. 132, p. 109801, 2021.
- [36] J. P. Hespanha, *Linear Systems Theory*. Princeton Press, 2018.
- [37] *Constructive nonlinear control*. Springer Science & Business Media, 2012.
- [38] V. Andrieu and S. Tarbouriech, "LMI conditions for contraction and synchronization," *IFAC-PapersOnLine*, vol. 52, no. 16, pp. 616–621, 2019.
- [39] Y. Sato, Y. Kawano, and N. Wada, "Parametrization of linear controllers for  $p$ -dominance," *IEEE Control Systems Letters*, vol. 7, pp. 1879–1884, 2023.
- [40] W. Che and F. Forni, "Dominant mixed feedback design for stable oscillations," *IEEE Transactions on Automatic Control*, pp. 1–8, 2023.
- [41] R. Sepulchre, M. Arcak, and A. R. Teel, "Trading the stability of finite zeros for global stabilization of nonlinear cascade systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 3, pp. 521–525, 2002.
- [42] F. Forni and R. Sepulchre, "Differentially positive systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 2, pp. 346–359, 2016.
- [43] D. Angeli and E. Sontag, "Monotone control systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 10, pp. 1684–1698, 2003.
- [44] M. Y. Li and J. S. Muldowney, "On R.A. Smith's Autonomous Convergence Theorem," *Rocky Mountain Journal of Mathematics*, vol. 25, no. 1, pp. 365 – 378, 1995.
- [45] O. E. Rössler, "Continuous chaos—four prototype equations," *Annals of the New York Academy of Sciences*, vol. 316, no. 1, pp. 376–392, 1979.
- [46] S. M. Fallat and C. R. Johnson, *Totally nonnegative matrices*. Princeton university press, 2022.