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# Regenerative bootstrap for $\beta$ -null recurrent Markov chains

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**Abstract:** Two regeneration-based bootstrap methods, namely, the *Regeneration based-bootstrap* [3, 19] and the *Regenerative Block bootstrap* [9] are shown to be valid for the problem of estimating the integral of a function with respect to the invariant measure in a  $\beta$ -null recurrent Markov chain with an accessible atom. An extension of the Central Limit Theorem for randomly indexed sequences is also presented.

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In [22], Bradley Efron introduced the Bootstrap as a way to overcome some limitations of classical methods that often relied on strong assumptions about the data's underlying distribution or the model's form. Since then, these techniques, first studied in the i.i.d. case, have been developed and extended to time-series (see [40] for an extensive survey of methods) and applied to a wide range of problems in various fields such as signal processing [57, 58], soil science [56] and econometrics [42, 27]. They can handle any level of complexity in data or models from fully parametric to completely nonparametric cases. These methods are easy to implement with modern computing power and can provide more accurate and reliable inferences than traditional methods in many situations.

Although originally designed for i.i.d. sampling, there has been significant interest in adapting the bootstrap to situations where the data is dependent. Several resampling methods have been proposed for time series data: these include the autoregressive-sieve bootstrap [37], block bootstrap [39], circular bootstrap [51], the stationary bootstrap [52], continuous-path block bootstrap [46], tapered block bootstrap [47], frequency-domain bootstrap [45, 35], and local bootstrap [49]. For detailed reviews and comparisons of these methods see [23, 36, 34, 15] and the references therein.

In the Markovian case, numerous approaches have been developed and examined. In [38], the authors proposed a block resampling scheme that consists in resampling from a nonparametric estimate of the one-step transition matrix of a finite state Markov chain. This method was extended to the countable case in [3]. Extensions of this method have been proposed for the case where the state space is Euclidean, as seen in [54], [48, 50] and [26]. The general concept is to estimate the marginal distribution and the transition probability function using a nonparametric function estimation technique and then resample from those estimates. For a detailed explanation of this approach, refer to Section 4 in [34].

A completely new approach to this problem was introduced in [3]. Instead of using estimated transition probabilities, they exploit the regeneration properties of a Markov chain when an accessible atom is visited infinitely often. The main idea underlying this method consists in dividing the chain into a random number of i.i.d. regeneration blocks and then resampling the same number of regeneration blocks. This method, named *Regeneration based bootstrap*, was proved to be valid for finite state atomic chains in [3], and it was extended to general atomic positive recurrent Markov chains in [19].

It was pointed out in [8] that the *Regeneration based bootstrap* is not second-order correct (its rate is  $O_{\mathbb{P}}(n^{-1/2})$  only). To overcome this limitation, a variation of this method, called *Regenerative Block bootstrap* (RBB), was introduced in [9]. This method consists in imitating the renewal structure of the chain by sampling regeneration data blocks, until the length of the reconstructed bootstrap series is larger than the length  $n$  of the original data series (notice the difference with the *Regeneration based bootstrap*, where the number of sampled blocks is equal to the number of regeneration blocks in the original chain). It was shown in [9] that, for atomic positive recurrent Markov chains, the RBB for estimating the integral of a function with respect to the invariant probability, has a uniform

rate of convergence of order  $O_{\mathbb{P}}(n^{-1})$  (the same as in the i.i.d. case).

Despite all these efforts in the positive recurrent case, up to our knowledge, no bootstrap method has been studied in the null-recurrent scenario. Hence, our objective in this paper is to start this study and show that both *Regeneration based-bootstrap* and *Regenerative Block bootstrap* are valid schemes for estimating integrals with respect to the invariant measure when the Markov chain is  $\beta$ -null recurrent and possesses an accessible atom. The task is challenging because the time returns to an atom of such Markov chains is infinite and we know that the bootstrap of such quantity does not work [4, 33]. Despite this fact, we will show that, by using the correct random normalization, one can obtain a CLT for the mean and the validity of both regeneration methods.

The paper is organized as follows: in section 1 we provide a brief introduction to null recurrent Markov chains, making a special emphasis on atomic ones and presenting the main results that we use throughout the paper. In subsection 1.3 we present an extension of the Central Limit Theorem for randomly indexed sequences (Lemma 1.1). Section 2 is dedicated to the *Regenerative Block bootstrap* in  $\beta$ -null recurrent Markov chains, while Section 3 is devoted to the *Regeneration based-bootstrap*. In section 4 we have added a few simulations to show the behavior of both algorithms in practice. The technical proofs are postponed to Section 5.

## 1. A short introduction to null-recurrent Markov chains

In this section, we introduce some notation and review some important concepts from Markov chain theory that will be used throughout the paper. For more details, please refer to [43, 21].

### 1.1. Notation and definitions

Consider an homogeneous Markov chain  $\mathbf{X} = X_0, X_1, \dots$ , on a countably generated state space  $(E, \mathcal{E})$ , with transition kernel  $P$  and initial probability distribution  $\lambda$ . This means that for any  $B \in \mathcal{E}$  and  $n \in \mathbb{N}$ , we have  $\mathcal{L}(X_0) = \lambda$  and

$$\mathbb{P}(X_{n+1} \in B \mid X_0, \dots, X_n) = P(X_n, B) \quad \text{almost surely.}$$

Note that the assumption of a countably generated state space is commonly used in Markov chain theory to avoid pathological examples known as 'anormal' chains [20]. For more information on this topic, see [20], [29], and [55]. An example of an 'anormal' chain can be found in [13]. This assumption does not significantly limit the generality of our results since most of the time  $\mathcal{E} = \mathcal{B}(\mathbb{R}^d)$ , which is countably generated.

In the following, we use  $\mathbb{P}_{\lambda}$  (or  $\mathbb{P}_x$  for  $x$  in  $E$ ) to denote the probability measure on the underlying space such that  $X_0 \sim \lambda$  (or  $X_0 = x$ ). We use  $\mathbb{E}_{\lambda}(\cdot)$  to represent the  $\mathbb{P}_{\lambda}$ -expectation (or  $\mathbb{E}_x(\cdot)$  to represent the  $\mathbb{P}_x$ -expectation), and  $\mathbb{I}\{\mathcal{A}\}$  to represent the indicator function of event  $\mathcal{A}$ .

A homogeneous Markov chain is said to be irreducible if there exists a  $\sigma$ -finite measure  $\phi$  on  $(E, \mathcal{E})$  such that for all  $x \in E$  and all  $A \in \mathcal{E}$  with  $\phi(A) > 0$ , there exists some  $n \geq 1$  such that  $P^n(x, A) > 0$ . In this case, there exists a maximal irreducibility measure  $\psi$  with respect to which all other irreducibility measures are absolutely continuous. If  $\mathbf{X}$  is  $\psi$ -irreducible, there is  $d' \in \mathbb{N}^*$  and disjoint sets  $D_1, \dots, D_{d'}$   $D_{d'+1} = D_1$  weighted by  $\psi$  such that  $\psi(E \setminus \cup_{1 \leq i \leq d'} D_i) = 0$  and  $\forall x \in D_i, P(x, D_{i+1}) = 1$ . The the g.c.d.  $d$  of such integers is called the *period* of the chain.  $\mathbf{X}$  is said to be *aperiodic* if  $d = 1$ .

In the following, we assume that the Markov chains under consideration are homogeneous, aperiodic, and irreducible with maximal irreducibility measure  $\psi$ .

An irreducible chain possesses an accessible atom, if there is a set  $\alpha \in \mathcal{E}$  such that for all  $x, y$  in  $\alpha$ :  $P(x, \bullet) = P(y, \bullet)$  and  $\psi(\alpha) > 0$ . For instance, when a chain can take a countable number of values, any single point visited by the chain is an atom. Denote by  $\sigma_\alpha$  and  $\tau_\alpha$ , respectively, the times of first visit and first return of the chain to  $\alpha$ , i.e.  $\tau_\alpha = \inf \{n \geq 1 : X_n \in \alpha\}$  and  $\sigma_\alpha = \inf \{n \geq 0 : X_n \in \alpha\}$ . The subsequent visit and return times  $\sigma_\alpha, \tau_\alpha(k)$ ,  $k \geq 1$  are defined inductively as follows:

$$\tau_\alpha(1) = \tau_\alpha \quad , \quad \tau_\alpha(k) = \min \{n > \tau_\alpha(k-1) : X_n \in \alpha\}, \quad (1)$$

$$\sigma_\alpha(1) = \sigma_\alpha \quad , \quad \sigma_\alpha(k) = \min \{n > \sigma_\alpha(k-1) : X_n \in \alpha\}. \quad (2)$$

We use  $T_n(\alpha)$  to represent the random variable that counts the number of times the chain visits the set  $\alpha$  up to time  $n$ , i.e.,  $T_n(\alpha) = \sum_{t=0}^n \mathbb{I}\{X_t \in \alpha\}$ . Similarly, we use  $T(\alpha)$  to represent the total number of visits of chain  $\mathbf{X}$  to  $\alpha$ . An atom  $\alpha$  is called *recurrent* if  $\mathbb{E}_x T(\alpha) = +\infty$  for all  $x \in \alpha$ ; otherwise, it is called *transient*. A notable property of atomic chains is that all accessible atoms are either all recurrent or all transient. Therefore, we say that an atomic chain is recurrent if one (and thus all) of its accessible atoms is recurrent.

Denote by  $\mathbb{P}_\alpha$  and  $\mathbb{E}_\alpha(\cdot)$  the probability and the expectation conditionally to  $X_0 \in \alpha$ . If  $\mathbf{X}$  possesses an accessible atom and is aperiodic, the probability of returning infinitely often to the atom  $\alpha$  is equal to one, no matter the starting point, i.e.

$$\mathbb{P}_x(T(\alpha) = \infty) = 1 \quad \forall x \in E.$$

A fundamental tool for understanding the long-term behavior of Markov chains is the existence of invariant measures, that is, a measure  $\pi$  such that

$$\pi(A) = \int P(x, A) d\pi(x) \quad \forall A \in \mathcal{E}.$$

Every irreducible and recurrent Markov chain admits a unique (up to a multiplicative constant) invariant measure [43, Theorem 10.4.9]. In the atomic case, the invariant measure is just the occupation measure over the first block  $\mathcal{B}_1 = (X_{\tau_\alpha(1)+1}, \dots, X_{\tau_\alpha(2)})$  [21, Theorem 6.4.2], i.e.

$$\pi_\alpha(A) = \mathbb{E}_\alpha \left( \sum_{j=1}^{\tau_\alpha} \mathbb{I}\{X_j \in A\} \right), \quad \forall A \in \mathcal{E}. \quad (3)$$

An irreducible Markov chain is *positive recurrent* if its invariant measure is finite. When the invariant measure is just  $\sigma$ -finite, then the chain is called *null recurrent*. From (3), it is clear that an atomic Markov chain is positive recurrent if and only if  $\mathbb{E}_\alpha \tau_\alpha < +\infty$ , and in this case, the measure defined by  $\frac{\pi_\alpha}{\mathbb{E}_\alpha \tau_\alpha}$  is an invariant probability for the chain. The existence of this invariant probability makes the theory of positive recurrent Markov chains, very similar to the i.i.d. case [43, Chapter 17].

Conversely, dealing with null recurrent chains is considerably more challenging, and a comprehensive theory of non-parametric estimation for this type of chain does not exist. To address this issue, Karlsen and Tjøstheim introduced in [31] a regularity condition for the tail behavior of the distribution of  $\tau_\alpha$  that renders the problem more tractable. Specifically, a chain is referred to as  $\beta$ -null recurrent (refer to [31, Definition 3.2 and Theorem 3.1]) if there is a constant  $\beta \in (0, 1)$  and a slowly varying function<sup>1</sup>  $L$  such that

$$\mathbb{P}_\alpha(\tau_\alpha > n) \sim \frac{1}{\Gamma(1 - \beta)n^\beta L(n)}. \quad (4)$$

The number  $\beta$ , also known as the *regularity index* (see [16, 17]) satisfies

$$\beta = \sup \{p > 0 : \mathbb{E}_\alpha(\tau_\alpha^p) < +\infty\}.$$

Some of the most well-known examples of  $\beta$ -null recurrent Markov chain are the random walks in  $\mathbb{R}$ , which are  $1/2$ -null recurrent [30], the Bessel random walks [1], [18] and some types of threshold autoregressive (TAR) [24] and vector autoregressive processes (VAR) [44].

## 1.2. Renewal properties and Block decomposition

The *strong Markov property* implies that the sample paths of an atomic Markov chain can be partitioned into independent blocks of random length corresponding to consecutive visits to  $\alpha$ , given by:

$$\begin{aligned} \mathcal{B}_0 &= (X_0, X_1, \dots, X_{\tau_\alpha(1)}) \\ \mathcal{B}_1 &= (X_{\tau_\alpha(1)+1}, \dots, X_{\tau_\alpha(2)}) \\ &\dots \\ \mathcal{B}_n &= (X_{\tau_\alpha(n)+1}, \dots, X_{\tau_\alpha(n+1)}) \\ &\dots \end{aligned}$$

Note that the distribution of  $\mathcal{B}_0$  depends on the initial measure, and thus it does not have the same distribution as  $\mathcal{B}_j$  for  $j \geq 1$ . The sequence  $\{\tau_\alpha(j)\}_{j \geq 1}$  defines successive times at which the chain forgets its past, which are called

<sup>1</sup>A measurable and positive function  $L$  is said to be *slowly varying at  $+\infty$*  if it is defined in  $[a, +\infty)$  for some  $a \geq 0$ , and satisfies  $\lim_{x \rightarrow +\infty} \frac{L(xt)}{L(x)} = 1$  for all  $t \geq a$ . For a detailed discussion on these types of functions, refer to [12].

*regeneration times*. Similarly, the sequence of i.i.d. blocks  $\{\mathcal{B}_j\}_{j \geq 1}$  is called *regeneration blocks*. As customary in the  $\beta$ -null recurrent Markov chain literature, we will use  $T(n)$  to denote the number of complete regeneration blocks up to time  $n$ , i.e.  $T(n) = \max(T_n(\alpha) - 1, 0)$ . We will denote by  $\ell(\mathcal{B}_i)$  the length of the  $i$ -th block, therefore,

$$\ell(\mathcal{B}_j) = \begin{cases} \tau_\alpha & , \quad j = 0 \\ \tau_\alpha(j+1) - \tau_\alpha(j) & , \quad j \geq 1 \end{cases} \quad (5)$$

The random variable  $T(n)$ , and its relationship with  $\sum_{j=0}^k \ell(\mathcal{B}_j)$ , is crucial in the theory we will develop in this paper, therefore, we will state in this section its main properties in the  $\beta$ -null recurrent scenario.

Assume  $\mathbf{X}$  is a  $\beta$ -null recurrent Markov chain with an accessible atom  $\alpha$ . By (3.27) in [31], the function  $L$  in (4) can be normalized in such a way that

$$u(z) = z^\beta L(z) \quad (6)$$

is a continuous function that is strictly increasing in the interval  $[z_0, +\infty)$  for some  $z_0 \in \mathbb{R}_+$ . Define  $v(z)$  as

$$v(z) = u^{(-1)}(z) = \inf \{s : u(s) > z\}, \quad (7)$$

then,  $u(v(z)) = v(u(z)) = z$  for  $z \geq z_0$ .

Consider the space of càdlàg functions defined on the interval  $[0, +\infty)$ , denoted by  $\mathcal{D}_{[0, +\infty)}$ . This space consists of the real functions that are right-continuous with left limits and defined over  $[0, +\infty)$ . More precisely, a function  $g \in \mathcal{D}_{[0, +\infty)}$  if and only if  $g$  is right-continuous, has left limits at all points  $t > 0$ , and  $\lim_{t \downarrow 0} g(t) = g(0)$ . The space  $\mathcal{D}_{[0, +\infty)}$  is equipped with the Skorokhod<sup>2</sup> topology, making it a completely separable metric space. We will use  $\xrightarrow{\mathcal{D}_{[0, +\infty)}}$  to denote weak convergence in this space, and  $\xrightarrow{\text{fd}}$  for convergence of finite-dimensional laws. Two stochastic processes  $Y_n, Z_n$  in  $\mathcal{D}_{[0, +\infty)}$  are said to be *equivalent* if  $Y_n - Z_n$  converges weakly to the zero process. If  $Y_n \xrightarrow{\mathcal{D}_{[0, +\infty)}} Y$  and  $Y_n$  and  $Z_n$  are equivalent, then  $Z_n \xrightarrow{\mathcal{D}_{[0, +\infty)}} Y$  (see Lemma 3.31 in [28]).

Define the following processes

$$T_n(t) = \frac{T(\lfloor nt \rfloor)}{u(n)}, \quad C_n(t) = \frac{1}{v(n)} \sum_{k=0}^{\lfloor nt \rfloor} \ell(\mathcal{B}_k), \quad (8)$$

and  $C_n^{(-1)}(t) = \inf \{x : C_n(x) > t\}$ . The following Theorem, proved in [31], shows that these three processes converge in  $\mathcal{D}_{[0, +\infty)}$  and that  $T_n$  and  $C_n^{(-1)}$  are equivalent.

**Theorem 1.1.** *Assume  $\mathbf{X}$  is a  $\beta$ -null recurrent atomic Markov chain. Then,*

<sup>2</sup>See Chapter 6 of [28] or Chapter 3 in [11] for more details about this space.

i)  $C_n \xrightarrow{\mathcal{D}_{[0,+\infty)}} S_\beta$  where  $S_\beta$  is the one-sided stable Levy process defined by the marginal characteristics

$$\mathbb{E} [\exp (i s S_\beta (t))] = \exp \left( i s^\beta t \right) \quad s \in \mathbb{R}, t \in [0, +\infty].$$

ii)  $C_n^{(-1)}$  and  $T_n$  are equivalent processes and both converge in  $\mathcal{D}_{[0,+\infty)}$  to the Mittag-Leffler process of parameter  $\beta$ .

*Remark 1.1.* The Mittag-Leffler process with parameter  $\beta$  is defined as the inverse of  $S_\beta$ . It is a strictly increasing continuous stochastic process defined as

$$M_\beta(t) = t^\beta M_\beta(1) \quad , \quad \mathbb{E} (M_\beta^m(1)) = \frac{m!}{\Gamma(1+m\beta)} \quad m \geq 0.$$

Theorem 1.1 shows a striking difference between positive and null recurrent Markov chains. While in the former the existence of moments for  $\ell(B_j)$  implies that  $C_n$  and  $T_n$  (taking  $u(n) = n$ ) converge almost surely respectively to  $t\mathbb{E}_\alpha \tau_\alpha$  and  $\frac{t}{\mathbb{E}_\alpha \tau_\alpha}$ , and therefore,  $T(n)$  can be approximated almost surely by the deterministic quantity  $n$ , in the latter, we only have weak convergence, hence  $T(n)$  can only be controlled by the deterministic quantity  $u(n)$  in distribution.

### 1.3. Properties of linear functionals defined on $\beta$ -null recurrent chains

For a measurable function  $f : E \rightarrow \mathbb{R}$ , and an atomic Markov chain  $\mathbf{X}$  with an accessible atom  $\alpha$ , consider the problem of estimating  $\pi_\alpha(f) = \int f d\pi_\alpha$ , where  $\pi_\alpha$  is as in (3) and  $\pi_\alpha(f) < +\infty$ . Denote by  $S_n(f)$  the partial sums of  $f$  over the chain, that is

$$S_n(f) = \sum_{k=0}^n f(X_k). \tag{9}$$

The Ratio Limit Theorem for atomic chains [21, Theorem 6.6.2] shows that if  $g$  is a measurable function, then, for every invariant measure  $\pi$  we have

$$\frac{S_n(f)}{S_n(g)} \xrightarrow{\text{a.s.}} \frac{\pi(f)}{\pi(g)}, \tag{10}$$

as long as  $\pi(g) \neq 0$ .

*Remark 1.2.* From (10) is clear that  $\frac{S_n(f)}{T(n)}$  is a strongly consistent estimator of  $\pi_\alpha(f)$ , and, in the positive recurrent case,  $\frac{S_n(f)}{n} \xrightarrow{\text{a.s.}} \frac{\pi_\alpha(f)}{\mathbb{E}_\alpha \tau_\alpha}$ . In the null recurrent case, however,  $\frac{S_n(f)}{n} \xrightarrow{\text{a.s.}} 0$  (see Corollary 6.6.3 in [21]) and there is no deterministic sequence  $a(n)$  such that  $\frac{S_n(f)}{a(n)}$  converges almost surely to a non-zero limit [16].

Given that our interest in this paper is to apply the bootstrap method to the estimation of  $\pi_\alpha(f)$  we need to find a series of i.i.d. random variables whose



mean strongly converges to  $\pi_\alpha(f)$ . To do this, define the following random variables

$$f(\mathcal{B}_j) = \begin{cases} \sum_{i=0}^{\tau_\alpha} f(X_i) & , \quad j = 0 \\ \sum_{i=\tau_\alpha(j)+1}^{\tau_\alpha(j+1)} f(X_i) & , \quad j \geq 1 \end{cases}.$$

The strong Markov property implies that under  $\mathbb{P}_\alpha$ , the sequence  $\{f(\mathcal{B}_j)\}_{j \geq 0}$  is i.i.d. Moreover, for every initial measure  $\lambda \in \mathcal{M}(\mathcal{E})_+$  such that  $\mathbb{P}_\lambda(\tau_\alpha < \infty) = 1$ , the random variables  $f(\mathcal{B}_j), j \geq 0$  are independent and for  $j \geq 1$  they are i.i.d. Therefore,  $S_n(f)$  can now be written as a sum of independent random variables as follows:

$$S_n(f) = f(\mathcal{B}_0) + \sum_{j=1}^{T(n)} f(\mathcal{B}_j) + \sum_{i=\tau_\alpha(T(n)+1)+1}^n f(X_i), \quad (11)$$

with the convention that the sum of an empty set is 0. As customary in the  $\beta$ -null recurrent literature, we will denote the last term in (11) by  $f(\mathcal{B}_{(n)})$ .

Equation (3) indicates that

$$\mathbb{E}_\alpha f(\mathcal{B}_j) = \pi_\alpha(f), \quad j = 1, \dots \quad (12)$$

hence, if we assume that  $\pi_\alpha(|f|) < +\infty$ , the Law of Large Numbers for randomly indexed sequences [25, Theorem 8.2, pp 302] shows that

$$\frac{1}{T(n)} \sum_{j=1}^{T(n)} f(\mathcal{B}_j) \xrightarrow{\text{a.s.}} \pi_\alpha(f). \quad (13)$$

*Remark 1.3.* The almost sure convergence of both  $\frac{S_n(f)}{T(n)}$  and  $\frac{1}{T(n)} \sum_{j=1}^{T(n)} f(\mathcal{B}_j)$  to  $\pi_\alpha(f)$  and the decomposition (11) shows that  $\frac{f(\mathcal{B}_0)}{T(n)}$  and  $\frac{f(\mathcal{B}_{(n)})}{T(n)}$  both converge almost surely to 0. This allows us to only consider in our estimations the i.i.d. blocks  $f(\mathcal{B}_j), j \geq 1$ .

If we suppose further that  $f(\mathcal{B}_1)$  has finite second moment and we denote by  $\sigma^2$  the variance of  $f(\mathcal{B}_1)$ , then

$$\frac{1}{T(n)} \sum_{j=1}^{T(n)} \left( f(\mathcal{B}_j) - \frac{1}{T(n)} \sum_{i=1}^{T(n)} f(\mathcal{B}_i) \right)^2 \xrightarrow{\text{a.s.}} \sigma^2. \quad (14)$$

Much of the work carried out in this investigation deals with sequences indexed by the sequence of random variables  $T(n)$ . As explained at the end of Section 1.2, this sequence, although it converges almost surely to  $+\infty$ , can not be deterministically approximated in probability, it only admits an approximation in distribution. This creates huge problems, even for simple tasks, as to obtaining a CLT, because, CLTs for randomly indexed sequences (see [2] for

the original formulation and Th. 17.2 in [11] for its more general form) require being able to control deterministically, at least in probability, the sequence of the number of terms. The result we present below, extends this CLT, replacing the requirement of the control in probability by the existence of the limit of a stochastic process defined in terms of the sequence of the number of terms.

**Lemma 1.1** (CLT for randomly indexed sequences). *Let  $X_1, X_2 \dots$  be i.i.d. random variables such that  $E(X_1) = \mu$  and  $\text{Var } X_1 = \sigma^2 > 0$ . Let  $N(n)$  be a sequence of integer-valued random variables. If there exists an unbounded increasing sequence of real numbers  $u_n$  such that the process  $N_n(t) = \frac{N(\lfloor nt \rfloor)}{u_n}$  satisfy the following conditions:*

- *Exists a process  $S_n$  in  $\mathcal{D}_{[0,+\infty)}$  that is non-negative and non-decreasing for each  $n$ .*
- *$S_n \xrightarrow{\mathcal{D}_{[0,+\infty)}} S$  where  $S$  is a strictly increasing non-negative process with independent increments, no fixed jumps, and  $S(0) \equiv 0$ .*
- *$N_n$  is equivalent to  $S_n^{(-1)}$ .*

Then,  $N_n$  converges to  $S^{(-1)}$ ,

$$\sqrt{N(n)} \left( \frac{\sum_{j=1}^{N(n)} (X_j - \mu)}{N(n)\sigma} \right), \quad (15)$$

converges weakly to a standard Normal distribution and this distribution is independent of  $S^{(-1)}(1)$ .

**Corollary 1.1.** [Theorem 17.2 in [11]] *Suppose  $X_1, \dots, X_n$  are i.i.d. with  $\mathbb{E}X_1 = \mu$  and  $\text{Var } X_1 = \sigma^2$ . If  $N(n)$  is a sequence of integer-valued random variables such that*

$$\frac{N(n)}{u_n} \xrightarrow{p} \theta, \quad (16)$$

where  $\theta$  is a positive random variable and the  $u_n$  is sequence of positive numbers going to infinity, then

$$\sqrt{N(n)} \left( \frac{\sum_{j=1}^{N(n)} (X_j - \mu)}{N(n)\sigma} \right)$$

converges in distribution to a standard normal random variable.

Using Lemma 1.1 and Theorem 1.1 we can provide a different proof of the following Central Limit Theorem for  $\beta$ -null recurrent atomic Markov chains, which was originally proved in [5].

**Theorem 1.2.** *Let  $\mathbf{X}$  be a  $\beta$ -null recurrent Markov chain, with an accessible atom  $\alpha$ . For every  $\pi_\alpha$ -measurable function  $f$  such that  $\mathbb{E} \left[ (f(\mathcal{B}_1))^2 \right] < +\infty$ ,*

$$\sqrt{T(n)} \left( \frac{\sum_{j=1}^{T(n)} f(\mathcal{B}_j)}{T(n)} - \int f d\pi_\alpha \right) \tag{17}$$

*converges in distribution to a Normal random variable with mean 0 and variance  $\sigma^2$ . Moreover,  $\frac{T(n)}{n^\beta L(n)}$  converges to a Mittag-Leffler distribution with parameter  $\beta$  that is independent of the limiting distribution of (17).*

The following corollary is a direct consequence of Theorem 1.2, equation (14) and Slutsky's theorem.

**Corollary 1.2.** *Under the same hypothesis of Proposition 1.2,*

$$\frac{\sqrt{T(n)}}{s_n} \left( \frac{\sum_{j=1}^{T(n)} f(\mathcal{B}_j)}{T(n)} - \int f d\pi_\alpha \right) \tag{18}$$

*converges weakly to a Normal distribution with mean 0 and variance 1. Here  $s_n^2 = \frac{1}{T(n)} \sum_{j=1}^{T(n)} \left( f(\mathcal{B}_j) - \frac{1}{T(n)} \sum_{j=1}^{T(n)} f(\mathcal{B}_j) \right)^2$ .*

## 2. The regenerative block-bootstrap algorithm

Let  $\mathbf{X}^{(n)} = (X_0, \dots, X_n)$  be observations drawn from a  $\beta$ -null recurrent Markov chain  $\mathbf{X}$  with an *a priori* known accessible atom  $\alpha$ . As in the previous section, let  $f$  be a  $\pi_\alpha$ -integrable function such that  $f(\mathcal{B}_1)$  has a finite second moment. Denote by  $\sigma^2$  the variance of  $f(\mathcal{B}_1)$ .

The bootstrap method we study in this section was introduced in [9] for positive recurrent Markov chains. In the atomic case, it was shown to have a uniform rate of convergence of  $O_p(n^{-1})$  under mild conditions.

In this section, we show that the method is also applicable in the  $\beta$ -null recurrent case, although, we have not been able to obtain a rate.

Proposition 3.1 in [7] shows that for positive recurrent chains, in the nonstationary case (when the initial law  $\lambda$  is not the invariant probability measure), the first data block  $\mathcal{B}_0$  induces a bias of order  $O(n^{-1})$ , which cannot be estimated from a single realization  $\mathbf{X}^{(n)}$  of the chain starting from  $\lambda$ . The last block  $\mathcal{B}_{(n)}$  (which is incomplete) induces a first-order term in the bias too. This led the authors in [9] to only consider statistics based on the regenerative data blocks  $\mathcal{B}_1, \dots, \mathcal{B}_{T(n)}$ .

In the  $\beta$ -null recurrent case, the lack of finite first moment for the block sizes suggests that considering the non-regenerative blocks will incur in an even worst bias, hence, as in [9], we will only consider statistics based on the regenerative data blocks  $\mathcal{B}_1, \dots, \mathcal{B}_{T(n)}$ .

While our asymptotic results are specifically stated for integrals with respect to the invariant measure, the algorithm can be applied to a broader range of statistics  $G_n$  that have an appropriate standardization  $S_n$ . This includes non-degenerate  $U$ -statistics and differentiable functionals.

The RBB procedure is performed in four steps as follows:

1. Count the number of visits  $T_n(\alpha)$  to the atom  $\alpha$  up to time  $n$ . And divide the observed sample path  $\mathbf{X}^{(n)} = (X_0, \dots, X_n)$  into  $T_n(\alpha) + 1$  blocks,  $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{T_n(\alpha)-1}, \mathcal{B}_{T_n(\alpha)}^{(n)}$  valued in the torus  $\mathbb{T} = \cup_{n=1}^{\infty} E^n$ , corresponding to the pieces of the sample path between consecutive visits to the atom  $\alpha$ . Drop the first and last (non-regenerative) blocks. Denote by  $T(n)$  the number of remaining blocks.
2. Draw sequentially bootstrap data blocks  $\mathcal{B}_{1, T(n)}^*, \dots, \mathcal{B}_{k, T(n)}^*$  independently from the empirical distribution  $F_n = T(n)^{-1} \sum_{j=1}^{T(n)} \delta_{\mathcal{B}_j}$  of the blocks  $\{\mathcal{B}_j\}_{1 \leq j \leq T(n)}$  conditioned on  $\mathbf{X}^{(n)}$ , until the length  $\ell^*(k) = \sum_{j=1}^k \ell(\mathcal{B}_{j, T(n)}^*)$  of the bootstrap data series is larger than  $n$ . Let  $T_n^*(\alpha) = \inf\{k \geq 1, \ell^*(k) > n\}$  and  $T^*(n, T(n)) = T_n^*(\alpha) - 1$ .
3. From the data blocks generated in step 2, reconstruct a pseudo-trajectory of size  $l^*(T^*(n, T(n)))$  by binding the blocks together

$$X^{*(n)} = (\mathcal{B}_{1, T(n)}^*, \dots, \mathcal{B}_{T^*(n, T(n)), T(n)}^*).$$

Compute the *RBB statistic*  $G_n^* = G_n(X^{*(n)})$ .

4. If  $S_n = S(\mathcal{B}_1, \dots, \mathcal{B}_{T(n)})$  is an appropriate standardization of the original statistic  $G_n$ , compute  $S_n^* = S(\mathcal{B}_{1, T(n)}^*, \dots, \mathcal{B}_{T^*(n, T(n)), T(n)}^*)$ .

The *RBB distribution* is then given by

$$H_{RBB}(x) = \mathbb{P}^*\left(S_n^{*-1}(G_n^* - G_n) \leq x\right)$$

where  $\mathbb{P}^*(\bullet) = \mathbb{P}(\bullet \mid \mathbf{X}^{(n)})$  denotes the conditional probability given  $\mathbf{X}^{(n)}$ .

Our main asymptotic result, in the case of integrals concerning the invariant measure, is the following.

**Theorem 2.1.** *Let  $\mathbf{X}$  be a  $\beta$ -null recurrent Markov chain with an accessible atom  $\alpha$ , and let  $f$  be a  $\pi_\alpha$ -integrable function such that  $\mathbb{E}[(f(\mathcal{B}_1))^2] < +\infty$ . Then we have,*

$$\sqrt{T^*(n, T(n))} \left( \frac{\sum_{j=1}^{T^*(n, T(n))} \left( f(\mathcal{B}_{j, T(n)}^*) - \frac{1}{T(n)} \sum_{i=1}^{T(n)} f(\mathcal{B}_i) \right)}{T^*(n, T(n)) \sigma_{T(n)}} \right) \xrightarrow{d^*} N(0, 1),$$

in probability along the data, where  $d^*$  denotes the convergence in distribution conditionally to the data and

$$\sigma_{T(n)}^2 = \frac{1}{T(n)} \sum_{j=1}^{T(n)} \left( f(\mathcal{B}_j) - \frac{1}{T(n)} \sum_{i=1}^{T(n)} f(\mathcal{B}_i) \right)^2.$$

This theorem yields that the bootstrap distribution of the standardized sum has asymptotically the same distribution as the statistics  $\frac{\sum_{j=1}^{T(n)} f(\mathcal{B}_j)}{T(n)}$  estimating  $\int f d\pi_\alpha$ . The regenerative block bootstrap is thus first-order correct. In particular, this justifies the use of the quantiles of the bootstrap distribution (with or without standardizing) to obtain confidence intervals for  $\int f d\pi_\alpha$ .

### 3. The regeneration-based bootstrap algorithm

In this section, we adapt the *Regeneration-base bootstrap* to the  $\beta$ -null recurrent Markov chain scenario.

Similarly to Section 2, consider observations  $\mathbf{X}^{(n)} = (X_0, \dots, X_n)$  drawn from a  $\beta$ -null recurrent Markov chain  $\mathbf{X}$  that has an accessible atom  $\alpha$  known beforehand. Suppose that  $f$  is a function such  $\pi_\alpha(f)$  is finite and the second moment of  $f(\mathcal{B}_1)$  is also finite. Let  $\sigma^2$  represent the variance of  $f(\mathcal{B}_1)$ .

The algorithm we present in this section was introduced in [3, 19] for positive recurrent Markov chains with an accessible known atom. Similarly to the RBB, it consists on dividing the chain into  $\mathcal{B}_1, \dots, \mathcal{B}_{T(n)}$  regenerative blocks and then resampling blocks to form the empirical distribution of  $\mathcal{B}_1, \dots, \mathcal{B}_{T(n)}$ . The main difference between the Regeneration-based bootstrap and the RBB is that in the former, the number of bootstrapped blocks is  $T(n)$ , hence, non-random conditionally to  $\mathbf{X}^{(n)}$ , while in the latter is random.

The full algorithm is as follows:

1. Count the number of visits  $T_n(\alpha)$  to the atom  $\alpha$  up to time  $n$ . And divide the observed sample path  $\mathbf{X}^{(n)} = (X_0, \dots, X_n)$  into  $T_n(\alpha) + 1$  blocks,  $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{T_n(\alpha)-1}, \mathcal{B}_{T_n(\alpha)}^{(n)}$  valued in the torus  $\mathbb{T} = \cup_{n=1}^\infty E^n$ , corresponding to the pieces of the sample path between consecutive visits to the atom  $\alpha$ . Drop the first and last (non-regenerative) blocks. Denote by  $T(n)$  the number of remaining blocks.
2. Draw  $T(n)$  bootstrap data blocks  $\mathcal{B}_{1,T(n)}^*, \dots, \mathcal{B}_{T(n),T(n)}^*$  independently from the empirical distribution  $F_n = T(n)^{-1} \sum_{j=1}^{T(n)} \delta_{\mathcal{B}_j}$  of the blocks  $\{\mathcal{B}_j\}_{1 \leq j \leq T(n)}$  conditioned on  $\mathbf{X}^{(n)}$ .
3. From the bootstrap data blocks generated at step 2, reconstruct a trajectory by binding the blocks together, getting the reconstructed sample path

$$X^{*(n)} = (\mathcal{B}_{1,T(n)}^*, \dots, \mathcal{B}_{T(n),T(n)}^*).$$

Compute the statistic  $G_n^* = G_n(X^{*(n)})$ .

4. If  $S_n = S(\mathcal{B}_1, \dots, \mathcal{B}_{T(n)})$  is an appropriate standardization of the original statistic  $G_n$ , compute  $S_n^* = S(\mathcal{B}_{1,T(n)}^*, \dots, \mathcal{B}_{T(n),T(n)}^*)$ .

As in the RBB case, the asymptotic result stated below shows the validity of this bootstrap scheme when used in estimations of integrals with respect to the invariant measure.

**Theorem 3.1.** *Let  $\mathbf{X}$  be a  $\beta$ -null recurrent Markov chain with an accessible atom  $\alpha$ , and let  $f$  be a  $\pi_\alpha$ -integrable function such that  $\mathbb{E} \left[ (f(\mathcal{B}_1))^2 \right] < +\infty$ , then*

$$\sqrt{T(n)} \left( \frac{\sum_{j=1}^{T(n)} \left( f(\mathcal{B}_{j,T(n)}^*) - \frac{1}{T(n)} \sum_{i=1}^{T(n)} f(\mathcal{B}_i) \right)}{T(n) \sigma_{T(n)}} \right) \xrightarrow{d^*} N(0, 1),$$

almost surely along the data, where  $d^*$  denotes the convergence in distribution conditionally to the data and

$$\sigma_{T(n)}^2 = \frac{1}{T(n)} \sum_{j=1}^{T(n)} \left( f(\mathcal{B}_j) - \frac{1}{T(n)} \sum_{i=1}^{T(n)} f(\mathcal{B}_i) \right)^2.$$

*Remark 3.1.* In its original formulation for the positive recurrent case, the estimator used was  $\frac{S_n(f)}{n}$ , however, by Remark 1.2, this can not be done in the null recurrent case, hence, we need to use  $\frac{1}{T(n)} \sum_{j=1}^{T(n)} f(\mathcal{B}_j)$ .

#### 4. Simulations

To illustrate the convergence of the regenerative bootstraps method described in the previous two sections we will do the following simulation experiment.

Take  $\mathbf{X}$  as the simple symmetric random walk in  $\mathbb{Z}$ , that is

$$X_t = \begin{cases} 0 & , \quad t = 0 \\ \sum_{k=1}^t Y_k & , \quad t \geq 1 \end{cases} \tag{19}$$

with  $P(Y_i = 1) = P(Y_i = -1) = \frac{1}{2}$ . In this random walk, the state 0 is an atom and the invariant measure is  $\pi_0(i) \equiv 1$  (see pp.1143 in [5]). Consider the function  $f(k) = \frac{1}{k^2}$  if  $k \neq 0$  and  $f(0) = 0$ , then

$$\int f(x) d\pi(x) = 2 \sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi_0^2}{3}.$$

In order to show the validity of the proposed methods, we have simulated the first  $10^8$  points of a simple symmetric random walk (see figure 1). Using this

data, we have applied both the RBB and the Regeneration Based-bootstrap 1000 times each and computed the values of

$$Z_{RBB}^* = \sqrt{T^*(n, T(n))} \left( \frac{\sum_{j=1}^{T^*(n, T(n))} \left( f(\mathcal{B}_{j, T(n)}^*) - \frac{1}{T(n)} \sum_{i=1}^{T(n)} f(\mathcal{B}_i) \right)}{T^*(n, T(n)) \sigma_{T(n)}} \right),$$

$$Z_{RegBB}^* = \sqrt{T(n)} \left( \frac{\sum_{j=1}^{T(n)} \left( f(\mathcal{B}_{j, T(n)}^*) - \frac{1}{T(n)} \sum_{i=1}^{T(n)} f(\mathcal{B}_i) \right)}{T(n) \sigma_{T(n)}} \right).$$

Figure 2 shows the validity of both methods, despite the fact that we observed a huge block (52% of the whole trajectory is inside this block) and 25% of the realization is in the final incomplete block. The 95% confidence interval for  $\int f d\pi_0$  using the RBB is (3.1439, 3.3096) and using the regenerative based-bootstrap is (3.1434, 3.3067). Notice that the true value of  $\int f d\pi_0$  is 3.2899, while the estimation obtained using (13) is 3.2226, and the confidence interval obtained via Proposition 1.2 is (3.1432, 3.302).

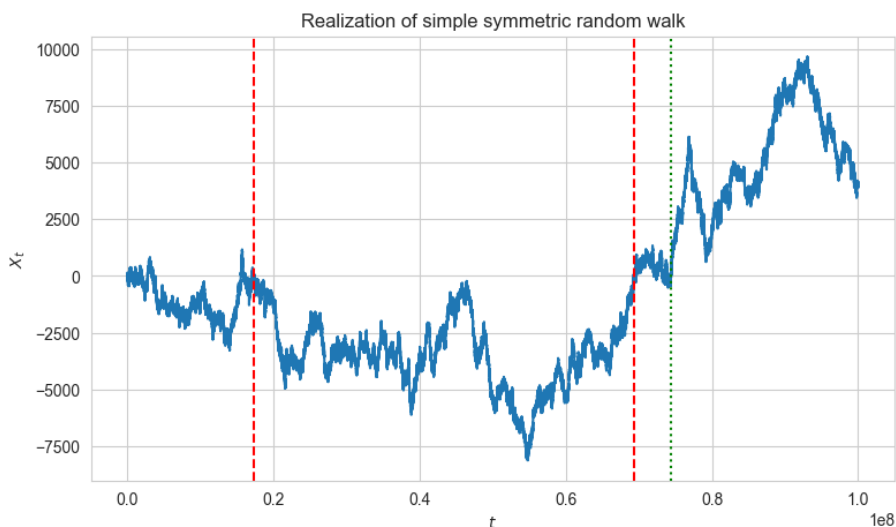


Fig 1: First  $10^8$  points of a realization of a simple symmetric random walk starting at 0. There are 9406 complete blocks in this realization. The red dashed lines delimit the largest block, while the green dotted line marks the end of the last complete block.

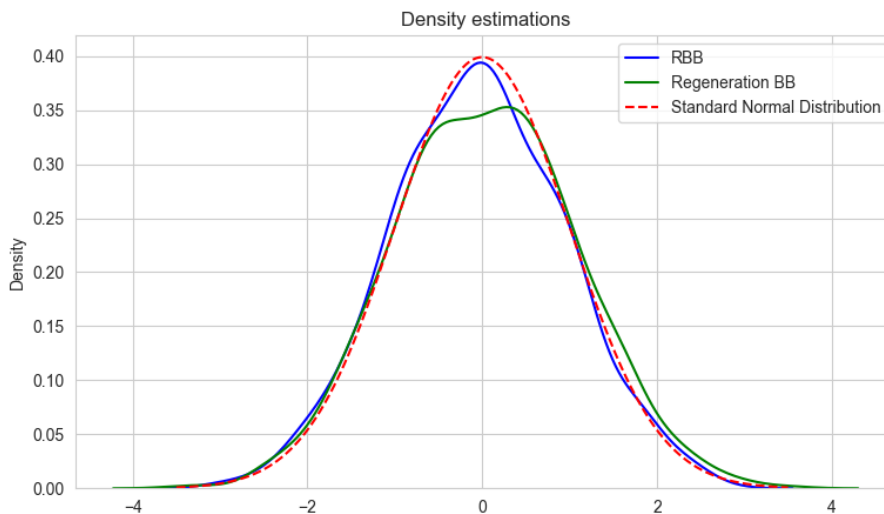


Fig 2: Density estimation of the bootstrap distributions  $Z_{RBB}^*$  and  $Z_{RegBB}^*$  after  $10^3$  simulations.

## 5. Proofs

### 5.1. Proof of Lemma 1.1

For the proof of Lemma 1.1 we need the following result, which appears as part A.3 of Theorem A.1 in [31].

**Lemma 5.1.** *Let  $A_n$  and  $B_n$  be a pair of stochastic processes which are càdlàg, where  $A_n$  is non-negative and non-decreasing. Let  $B$  denote a Brownian motion defined for  $t \in \mathbb{R}$  and let  $A$  denote a strictly increasing non-negative process with independent increments,  $A(0) \equiv 0$  and with no fixed jumps. Assume that  $B_n \xrightarrow{\mathcal{D}_{[0,+\infty)}} B$  and  $A_n \xrightarrow{\mathcal{D}_{[0,+\infty)}} A$ . Then,  $A_n^{(-1)} \xrightarrow{\mathcal{D}_{[0,+\infty)}} A^{(-1)}$  and*

$$\left( A_n^{(-1)}(t), \frac{B_n \circ A_n^{(-1)}(t)}{\sqrt{A_n^{(-1)}(t)}} \right) \xrightarrow{d} \left( A^{(-1)}(t), Z \right) \quad \forall t \in (0, 1],$$

where  $Z$  is standard normal variable independent of  $A^{(-1)}(t)$ .

To prove Lemma 1.1, let  $W_k = \sigma^{-1}(X_k - \mu)$ , then  $\{W_k\}_{k=1}^{\infty}$  is an i.i.d. sequence with  $\mathbb{E}(W_k) = 0$  and  $\text{Var } W_k = 1$  for all  $k$ .

Let's define the following continuous time process for  $t \geq 0$

$$Q_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} W_k. \quad (20)$$



By Theorem 23 and Example 24 in [53],  $Q_n \xrightarrow{\mathcal{D}_{[0,+\infty)}} B$  and given that  $u_n$  is an unbounded increasing sequence, we also have  $Q_{u_n}$  converges weakly to  $B$  in  $\mathcal{D}_{[0,+\infty)}$ .

The conditions imposed to the process  $N_n$  allow us to apply Lemma 5.1 with  $A_n = S_n$  and  $B_n = Q_{u_n}$ . Taking into account that  $N_n$  is equivalent to  $S_n^{(-1)}$  we obtain that for all  $t > 0$

$$\frac{Q_{u_n}(N_n(t))}{\sqrt{N_n(t)}} \xrightarrow{d} N(0, 1). \tag{21}$$

Using that  $N(\lfloor nt \rfloor) = u_n N_n(t)$ , we get

$$Q_{u_n}(N_n(t)) = \frac{\sigma^{-1}}{\sqrt{u_n}} \sum_{j=1}^{N(\lfloor nt \rfloor)} (X_j - \mu), \tag{22}$$

and Lemma 1.1 follows after plugging (22) into (21) and taking  $t = 1$ .

### 5.2. Proof of Corollary 1.1

We assume, at first, that  $\theta$  is bounded, that is, there exists a constant  $K$  such that  $0 < \theta < K$  with probability 1. Without loss of generality, assume the  $u_n$  are integers. Define the process

$$N_n(t) = \begin{cases} \frac{tN(n)}{u_n} & , \text{ if } \frac{N(n)}{u_n} < 1 \\ t\theta & , \text{ otherwise} \end{cases}.$$

As stated in pp. 147 of [11], this process converges to the process  $t\theta$  and trivially satisfies the conditions of Lemma 1.1 (using  $S_n(t) = \frac{t}{\theta}$ ,  $S_n^{-1}(t) = t\theta$ ).

The case when  $K$  is unbounded can be treated by following the same argument as in pp. 148 of [11].

### 5.3. Proof of Theorem 1.2

Recall from Section 1.3 that, by the Strong Markov Property, the sequence  $\{f(\mathcal{B}_j)\}_{j=1}^{+\infty}$  is i.i.d. with mean  $\int f d\pi_\alpha$  and variance  $\sigma^2$ . Consider the processes  $T_n(t)$  and  $C_n$  defined in (8)

$$T_n(t) = \frac{T(\lfloor nt \rfloor)}{u(n)} \quad , \quad C_n(t) = \frac{1}{v(n)} \sum_{k=0}^{\lfloor nt \rfloor} \ell(\mathcal{B}_k).$$

By Theorem 1.1, we can apply Lemma 1.1 with  $X_i = f(\mathcal{B}_i)$ ,  $\mu = \int f d\pi_\alpha$ ,  $N(n) = T(n)$  and  $u_n = n^\beta L(n)$ , which completes the proof.

#### 5.4. Proof of Theorem 2.1

Assume we have observed the chain until time  $n$ , i.e.,  $\mathbf{X}^{(n)} = X_0, X_1, \dots, X_n$ , and we have extracted the  $T(n)$  regeneration blocks:  $\mathcal{B}_1, \dots, \mathcal{B}_{T(n)}$ .

Now we start to sequentially bootstrap data blocks  $\mathcal{B}_{1,T(n)}^*, \dots, \mathcal{B}_{k,T(n)}^*$  independently from the empirical distribution  $F_{T(n)} = T(n)^{-1} \sum_{j=1}^{T(n)} \delta_{\mathcal{B}_j}$  of the blocks  $\{\mathcal{B}_j\}_{1 \leq j \leq T(n)}$ , conditioned on  $\mathbf{X}^{(n)}$ , until the length  $\ell^*(k) = \sum_{j=1}^k \ell(\mathcal{B}_{j,T(n)}^*)$  of the bootstrap data series is larger than  $n$ .

For each  $m$ , define

$$T^*(m, T(n)) = \max \left\{ k : \sum_{j=1}^k \ell \left( f(\mathcal{B}_{j,T(n)}^*) \right) \leq m \right\}, \quad (23)$$

$$U^*(m, T(n)) = \sqrt{m} \left( \frac{\sum_{j=1}^m \left( f(\mathcal{B}_{j,T(n)}^*) - \frac{1}{T(n)} \sum_{i=1}^{T(n)} f(\mathcal{B}_i) \right)}{m\sigma_{T(n)}} \right). \quad (24)$$

Theorem 2.1 will be proved if we show that

$$\mathbb{P}^* \left( U^* \left( T^*(n, T(n)), T(n) \right) \leq x \right) \xrightarrow{P} \mathbb{P}(N \leq x) \quad \forall x \in \mathbb{R}, \quad (25)$$

where  $P$  is a standard normal random variable and  $\mathbb{P}^*(\bullet) = \mathbb{P}(\bullet \mid \mathbf{X}^{(n)})$  denotes the conditional probability given  $\mathbf{X}^{(n)}$ .

Given that we will bootstrap  $T^*(n, T(n))$  terms, which is a random quantity conditionally to the data, we will use Lemma 5.1 to prove (25). In order to do this we need, conditionally to the data:

1. Find a process  $S_{n,T(n)}^*(t)$  that is non-negative, non-decreasing that converges in  $\mathcal{D}_{[0,+\infty)}$  to a process  $S^*$  that is non-negative, strictly increasing, has independent increments, no fixed jumps and  $S^*(0) \equiv 0$ .
2. Show that  $T_{n,T(n)}^*(t) = \frac{T^*(\lfloor nt \rfloor)}{T(n)} = \frac{T^*(\lfloor nt \rfloor, T(n))}{T(n)}$  is equivalent in  $\mathcal{D}_{[0,+\infty)}$  to  $S_{n,T(n)}^{*(-1)}$ .
3. Find a process  $Q_{n,T(n)}^*(t)$  that converges in  $\mathcal{D}_{[0,+\infty)}$  to a Brownian motion when  $n$  goes to  $+\infty$ . This process should satisfy, for some  $t > 0$

$$U^* \left( T^*(n, T(n)), T(n) \right) = \frac{Q_{n,T(n)}^* \circ T_{n,T(n)}^*(t)}{\sqrt{T_{n,T(n)}^*(t)}}. \quad (26)$$

A natural choice for  $Q_{n,T(n)}^*$ , which satisfies (26) for  $t = 1$ , is

$$Q_{n,T(n)}^*(t) = \sqrt{T(n)} \left( \frac{\sum_{j=1}^{\lfloor T(n)t \rfloor} \left( f(\mathcal{B}_{j,T(n)}^*) - \frac{1}{T(n)} \sum_{j=1}^{T(n)} f(\mathcal{B}_i) \right)}{T(n) \sigma_{T(n)}} \right). \quad (27)$$

Take  $S_{n,T(n)}^*(t)$  as

$$S_n^*(t) = \frac{1}{v^*(T(n))} \sum_{i=1}^{\lfloor T(n)t \rfloor} \ell(\mathcal{B}_{i,T(n)}^*), \quad (28)$$

where  $v^*(T(n)) = \sum_{i=0}^{T(n)} \ell(\mathcal{B}_i)$ .

Following the notation of [33], let  $Y_i = l(\mathcal{B}_i)$  and let  $Y_{1,n} \geq Y_{2,n} \geq \dots \geq Y_{n,n}$  be the order statistics of the sizes of the first  $n$  blocks, and take  $Z_{k,n} = \frac{Y_{k,n}}{v(n)}$  where  $v(n)$  is as in (7). By Theorem 1 in [33],

$$Z^{(n)} = (Z_{1,n}, Z_{2,n}, \dots, Z_{n,n}, 0, \dots, 0) \xrightarrow{d} (Z_1, Z_2, \dots) = Z, \quad (29)$$

where  $Z_k = (E_1 + \dots + E_k)^{-\frac{1}{\beta}}$  and  $E_i$  is a sequence of i.i.d. of exponential random variables with mean 1. By Skorokhod-Dudley-Wichura Theorem (see pp. 1171 in [33] and pp. 476 in [6]) we can choose a probability space such that, without changing the distribution of the left hand side of (29),

$$Z^{(n)} \xrightarrow{a.s.} Z. \quad (30)$$

The following Lemma shows that in that space, conditionally to the data, the process  $S_{n,T(n)}^*$  converges in  $\mathcal{D}_{[0,+\infty)}$ .

**Lemma 5.2.** *Suppose that (30) holds, then  $\frac{T(n)}{v(n)}$  converges almost surely to a positive random variable and*

$$S_{n,T(n)}^* \xrightarrow{\mathcal{D}_{[0,+\infty)}} S^* \quad \text{and} \quad S_{n,T(n)}^{*(-1)} \xrightarrow{\mathcal{D}_{[0,+\infty)}} S^{*(-1)} \quad (31)$$

almost surely along the data.

Here,  $S^*(t) = KR^*(t) + t$ ,  $R^*(t) = \sum_{j=1}^{+\infty} Z_j (\lambda_j^*(t) - t)$ ,  $\lambda_j^*(t)$  are independent Poisson processes with parameter 1 and  $K$  is a positive constant. Moreover, the process  $S^*$  is non-negative, strictly increasing, continuous, with independent increments and  $S^*(0) \equiv 0$ .

*Proof.* When (30) holds, by Theorem 1 and Remark 1.3 in [41],

$$\frac{1}{v(n)} \sum_{j=1}^n \ell(\mathcal{B}_j) \xrightarrow{a.s.} \sum_{j=1}^{+\infty} Z_j.$$

The length of the first block,  $\ell(\mathcal{B}_0)$ , is finite with probability 1 and does not depend on  $n$ , hence  $\frac{\ell(\mathcal{B}_0)}{v(n)}$  converges almost surely to 0. This implies that

$$\frac{1}{v(n)} \sum_{j=0}^n \ell(\mathcal{B}_j) \xrightarrow{a.s.} \sum_{j=1}^{+\infty} Z_j. \quad (32)$$

In (7), we defined  $v(z)$  as the inverse of  $u(z) = z^\beta L(z)$ , then, by Proposition 1.5.15 in [12],  $v(z) \sim z^{1/\beta} L_1(z)$  where  $L_1$  is a slowly varying function, hence,

$$\frac{1}{v(n)} \sum_{j=0}^{\lfloor nt \rfloor} \ell(\mathcal{B}_j) \xrightarrow{a.s.} t^{1/\beta} \sum_{j=1}^{+\infty} Z_j \quad \forall t > 0. \quad (33)$$

For each  $t > 0$ , let  $S_n(t) = \frac{1}{v(n)} \sum_{j=0}^{\lfloor nt \rfloor} \ell(\mathcal{B}_j)$ ,  $S_n^{(-1)}(t) = \inf \{x > 0 : S_n(x) > t\}$  and  $S(t) = t^{1/\beta} \sum_{j=1}^{+\infty} Z_j$ , and define the three processes as 0 on  $t = 0$ . By (33) and the Continuous Mapping Theorem,  $S_n^{(-1)} \xrightarrow{a.s.} S^{-1}$ .

Similar to what is described on page 1141 in [5], suppose that  $y$  is such that  $y < S_n^{(-1)}(1)$ . Then, since  $S_n(y) < 1$ , it follows that  $\sum_{j=0}^{\lfloor ny \rfloor} \ell(\mathcal{B}_j) < v(n)$ . Consequently, we have  $T(\lfloor v(n) \rfloor) \geq \lfloor ny \rfloor > ny - 1$ . This in turn implies that  $\frac{T(\lfloor v(n) \rfloor)}{n} \geq y - \frac{1}{n} \geq S_n^{(-1)}(1) - \frac{1}{n}$  for all  $n$ . In a similar way, but taking  $y > S_n^{(-1)}$ , we show that  $\frac{T(\lfloor v(n) \rfloor)}{n} \leq S_n^{(-1)}(1) + \frac{1}{n}$  for all  $n$ . Then,

$$S_{u(n)^{(-1)}(1)} - \frac{1}{u(n)} \leq \frac{T(\lfloor v(u(n)) \rfloor)}{u(n)} \leq S_{u(n)^{(-1)}(1)} + \frac{1}{u(n)}. \quad (34)$$

The first part of the lemma now follows from (34), the convergence of  $S_{u(n)}^{(-1)}(1)$  to  $S^{-1}(1)$  and the fact that  $u(v(n)) = n$  for  $n$  big enough.

To show (31), consider the following process, which was studied in [6],

$$Z_{m,n}^*(t) = \frac{1}{v(n)} \sum_{j=1}^{\lfloor mt \rfloor} \left( \ell(\mathcal{B}_{j,n}^*) - \frac{\sum_{i=1}^n \ell(\mathcal{B}_i)}{n} \right).$$

By Corollary 1.2 in [6] (and its proof<sup>3</sup>), we see that when (30) holds, for any  $m_n$  such that  $\frac{m_n}{n} \rightarrow c$ , conditionally to the data, the process  $Z_{m,n}^*$  converges weakly in  $\mathcal{D}([0, 1])$  to  $R^*(ct)$ . Let  $C > 1$ , on  $[0, C]$  define the process

$$W_n^*(t) = \frac{1}{v(n)} \sum_{j=1}^{\lfloor nt \rfloor} \left( \ell(\mathcal{B}_{j,n}^*) - \frac{\sum_{i=1}^n \ell(\mathcal{B}_i)}{n} \right).$$

<sup>3</sup>In [6], they standardize by  $T_n = \max_{1 \leq k \leq n} l(\mathcal{B}_k)$  but from the proof is clear that the result remains valid if we standardize by  $v(n)$  ( $b_n$  in their notation).

Notice that  $W_n^*(t) = Z_{nC,n}^*\left(\frac{t}{C}\right)$ , hence,  $W_n^* \xrightarrow{\mathcal{D}_{[0,C]}} R^*$  as  $n \rightarrow +\infty$ . Because this convergence holds for arbitrary  $C > 0$ , by Lemma 1.3.ii in [32] we have that  $W_n^* \xrightarrow{\mathcal{D}_{[0,+\infty)}} R^*$ , and therefore,  $W_{T(n)}^* \xrightarrow{\mathcal{D}_{[0,+\infty)}} R^*$ .

The process  $S_{n,T(n)}^*$  can be written as

$$S_{n,T(n)}^*(t) = \frac{v(T(n))}{v^*(T(n))} W_{T(n),T(n)}^*(t) + \frac{\lfloor T(n)t \rfloor}{T(n)} \quad (35)$$

Conditionally to the data,  $\frac{v(T(n))}{v^*(T(n))} = \left(\frac{1}{v(T(n))} \sum_{j=0}^{T(n)} \ell(\mathcal{B}_j)\right)^{-1}$  converges to a positive constant  $K$  by equation (32). Equation (31) now follows from the convergence of  $W_{T(n),T(n)}^*$  and (35).

The continuity of  $S^*$  was shown in pp. 466 of [6], and the rest of the properties are evident from the form of  $R^*$ .  $\square$

The next Lemma handles the equivalence of  $T_{n,T(n)}^*$  and  $S_{n,T(n)}^{*(-1)}$  in  $\mathcal{D}_{[0,+\infty)}$ .

**Lemma 5.3.** *Under the same hypothesis of Lemma 5.2, the processes  $T_{n,T(n)}^*$  and  $S_{n,T(n)}^{*(-1)}$  are equivalent in  $\mathcal{D}_{[0,+\infty)}$ .*

*Proof.* The proof of this result follows the proof of Theorem 3.2 on [31] with slight modifications.

We need to show that, for any  $\varepsilon > 0$  given,

$$\mathbb{P}\left(\sup_{0 < t \leq K} \left|T_{n,T(n)}^*(t) - S_{n,T(n)}^{*(-1)}(t)\right| > \varepsilon\right) \rightarrow 0 \quad \forall K > 0. \quad (36)$$

To prove this, we will show that

$$\mathbb{P}\left(\sup_{0 < t \leq K} \left|T_{v^*(T(n)),T(n)}^*(t) - S_{n,T(n)}^{*(-1)}(t)\right| > \varepsilon\right) \rightarrow 0 \quad \forall K > 0, \quad (37)$$

$$\mathbb{P}\left(\sup_{0 < t < K} \left|T_{v^*(T(n)),T(n)}^*(t) - T_{n,T(n)}^*(t)\right| > \varepsilon\right) \rightarrow 0 \quad \forall K > 0. \quad (38)$$

from where (36) will follow by triangular inequality.

Let  $\eta > 0$

$$\begin{aligned} \left\{S_{n,T(n)}^{*(-1)}(t) < \eta\right\} &\subseteq \left\{S_{n,T(n)}^*(\eta) > t\right\} \\ &= \left\{\frac{1}{v^*(T(n))} \sum_{i=1}^{\lfloor T(n)\eta \rfloor} \ell(\mathcal{B}_{i,T(n)}^*) > t\right\} \\ &= \left\{\sum_{i=1}^{\lfloor T(n)\eta \rfloor} \ell(\mathcal{B}_{i,T(n)}^*) > tv^*(T(n))\right\} \\ &= \left\{\frac{T^*(\lfloor v^*(T(n))t \rfloor, T(n))}{T(n)} < \frac{\lfloor T(n)\eta \rfloor}{T(n)}\right\}. \end{aligned} \quad (39)$$

Because  $T(v^*(n)) = n$ , we can write,

$$T_{v^*(T(n)),T(n)}^*(t) = \frac{T^*([v^*(T(n))t], T(n))}{u^*(v^*(T(n)))} = \frac{T^*([v^*(T(n))t], T(n))}{T(n)},$$

therefore, equation (39) becomes

$$\left\{ S_{n,T(n)}^{*(-1)}(t) < \eta \right\} \subseteq \left\{ T_{v^*(T(n)),T(n)}^*(t) < \frac{\lfloor T(n)\eta \rfloor}{T(n)} \right\}. \quad (40)$$

Similarly, we obtain that

$$\left\{ S_{n,T(n)}^{*(-1)}(t) > \eta \right\} \subseteq \left\{ T_{v^*(T(n)),T(n)}^*(t) \geq \frac{\lfloor T(n)\eta \rfloor}{T(n)} \right\}. \quad (41)$$

Let  $\varepsilon_1 \in (0, 1)$  be fixed and take  $\eta_1 < \eta_2$ , then, by (41) and (40),

$$\begin{aligned} \left\{ \eta_1 \leq S_{n,T(n)}^{*(-1)}(t) < \eta_2 \right\} &\subseteq \left\{ \eta_1(1 - \varepsilon_1) < S_{n,T(n)}^{*(-1)}(t) < \eta_2 \right\} \\ &\subseteq \left\{ \frac{\lfloor T(n)\eta_1(1 - \varepsilon_1) \rfloor}{T(n)} \leq T_{v^*(T(n)),T(n)}^*(t) < \frac{\lfloor T(n)\eta_2 \rfloor}{T(n)} \right\}. \end{aligned}$$

This means, that, if  $S_{n,T(n)}^{*(-1)}(t) \in [\eta_1, \eta_2)$ , then

$$\frac{\lfloor T(n)\eta_1(1 - \varepsilon_1) \rfloor}{T(n)} - \eta_2 < T_{v^*(T(n)),T(n)}^*(t) - S_{n,T(n)}^{*(-1)}(t) < \frac{\lfloor T(n)\eta_2 \rfloor}{T(n)} - \eta_1,$$

which implies that, if  $S_{n,T(n)}^{*(-1)}(t) \in [\eta_1, \eta_2)$ , then

$$\left| T_{v^*(T(n)),T(n)}^*(t) - S_{n,T(n)}^{*(-1)}(t) \right| \leq \eta_2 - \eta_1 + \varepsilon_1\eta_1 + \frac{1}{T(n)}. \quad (42)$$

Let  $\varepsilon > 0$  be fixed. For any  $s$  we have

$$\begin{aligned} \mathbb{P} \left( \sup_{t \leq K} \left| \xi_{n,T(n)}^*(t) \right| > \varepsilon \right) &\leq P \left( \sup_{t \leq K} \left| \xi_{n,T(n)}^*(t) \right| > \varepsilon, \sup_{t \leq K} S_{n,T(n)}^{*(-1)}(t) < s \right) \\ &\quad + P \left( \sup_{t \leq K} S_{n,T(n)}^{*(-1)}(t) \geq s \right), \end{aligned}$$

where  $\xi_{n,T(n)}^*(t) = T_{v^*(T(n)),T(n)}^*(t) - S_{n,T(n)}^{*(-1)}(t)$ .

By (31),

$$\lim_{s \uparrow \infty} \lim_{n \rightarrow \infty} P \left( \sup_{t \leq K} S_{n,T(n)}^{*(-1)}(t) \geq s \right) = 0.$$

Therefore, for any  $\delta > 0$  we can choose  $s_0$  such that  $\mathbb{P} \left( \sup_{t \leq K} S_{n,T(n)}^{*(-1)}(t) \geq s_0 \right)$  is smaller than  $\delta$  for  $n$  big enough. By (42),  $\sup_{t \leq K} S_{n,T(n)}^{*(-1)}(t) < s_0$  implies that

$$\left| \xi_{n,T(n)}^*(t) \right| \leq \eta_2 - \eta_1 + \varepsilon_1\eta_1 + \frac{1}{T(n)} \quad \forall t \in [0, K] \quad , \quad \forall \varepsilon_1 \in (0, 1).$$

Choose  $\eta_0, \dots, \eta_L, N_1, \varepsilon_1$  with  $\eta_0 = 0 < \eta_1 < \dots < \eta_{L-1} < \eta_L = s_0$  such that  $\eta_i - \eta_{i+1} < \frac{\varepsilon}{3}$  for all  $i$ . Let  $\varepsilon_1 < \frac{\varepsilon}{s_0}$  and choose  $N_1$  such that  $\frac{1}{T(N_1)} < \frac{\varepsilon}{3}$ .

Notice that for all  $t \in [0, K]$  there is only one  $i_{n,t}$  such that  $S_{n,T(n)}^{*(-1)}(t)$  belongs to  $[\eta_{i_{n,t}}, \eta_{i_{n,t}+1})$ , then, by (42)

$$\left| \xi_{n,T(n)}^*(t) \right| \leq \eta_{i_{n,t}} - \eta_{i_{n,t}+1} + \varepsilon_1 \eta_1 + \frac{1}{T(n)} \leq \varepsilon \quad \forall t \in [0, K], \quad \forall n > N_1,$$

whenever  $S_{n,T(n)}^{*(-1)}(t) < s_0$ . This implies that

$$\mathbb{P} \left( \sup_{t \leq K} \left| \xi_{n,T(n)}^*(t) \right| > \varepsilon, \sup_{t \leq K} S_{n,T(n)}^{*(-1)}(t) < s_0 \right) = 0 \quad \forall n \geq N_1.$$

Hence,

$$\mathbb{P} \left( \sup_{t \leq K} \left| \xi_{n,T(n)}^*(t) \right| > \varepsilon \right) < \delta \quad \forall n > N_1. \quad (43)$$

which implies (37).

Now we turn to the proof of (38).

According to the definition of  $v^*$ ,  $v^*(T(n)) = \sum_{i=0}^{T(n)} l(\mathcal{B}_i) \leq n$ , therefore,

$$\begin{aligned} T_{v^*(T(n)), T(n)}^*(t) &= \frac{T^*(\lfloor v^*(T(n)) t \rfloor, T(n))}{T(n)} \leq \frac{T^*(\lfloor nt \rfloor, T(n))}{T(n)} \\ &\leq T_{n, T(n)}^*(t) \quad \forall n, t. \end{aligned}$$

Notice that  $v^*(T(n) + 1) = \sum_{i=0}^{T(n)+1} l(\mathcal{B}_i) > n$ , therefore,

$$T_{n, T(n)}^*(t) \leq T_{v^*(T(n)+1), T(n)}^*(t) \frac{T(n) + 1}{T(n)} \quad \forall n, t.$$

Hence,

$$T_{v^*(T(n)), T(n)}^*(t) \leq T_n^*(t) \leq T_{v^*(T(n)+1), T(n)}^*(t) \frac{T(n) + 1}{T(n)} \quad \forall n, t.$$

Equation (38) now follows from the convergence of both  $T_{v^*(T(n)), T(n)}^*$  and  $T_{v^*(T(n)+1), T(n)}^*$  to  $S^{*(-1)}$  and the fact that  $\frac{T(n)+1}{T(n)} \xrightarrow{a.s.} 1$ .  $\square$

By (27), Lemmas 5.1, 5.2 and 5.3 we have that, in a space where (30) holds, the convergence in (25) holds almost surely. Therefore, in the original space we have the weakly-weakly (see pp.2550 in [14]) convergence

$$\mathbb{P}^* \left( U^* \left( T^*(n, T(n)), T(n) \right) \leq x \right) \xrightarrow{d} \mathbb{P}(N \leq x) \quad \forall x \in \mathbb{R}. \quad (44)$$

However, given that the right hand side of (44) is a constant for each  $x$ , the convergence in (44) can be improved to convergence in probability, which completes the proof.

**5.5. Proof of Theorem 3.1**

This proof follows the line of the proof of Theorem 2.1 in [10]. As in that paper, let  $\Gamma_2$  be the set of distribution functions  $G$  satisfying  $\int x^2 dG(x) < \infty$  and define the following notion of convergence in  $\Gamma_2$

$$G_n \Rightarrow G \quad \text{iff} \quad G_n \rightarrow G \text{ weakly and } \int x^2 dG_n(x) \rightarrow \int x^2 dG(x). \quad (45)$$

Denote by  $d_2$  a Mallows metric that metricizes the  $\Rightarrow$  convergence in  $\Gamma_2$  (see details in Section 8 of [10])

If  $Y_1, \dots, Y_n$  are i.i.d. random variables with common distribution  $G$ , denote by  $G^{(m)}$  the distribution of

$$m^{-\frac{1}{2}} \sum_{j=1}^m (Y_j - \mathbb{E}Y_j).$$

By pp. 1198 in [10], if  $G, H \in \Gamma_2$  then  $G^{(m)}$  and  $H^{(m)}$  are also in  $\Gamma_2$  and

$$d_2 \left( G^{(m)}, H^{(m)} \right) \leq d_2 (G, H). \quad (46)$$

Let  $F$  be the distribution of  $f(\mathcal{B}_1)$  and denote by  $F_n$  the empirical distribution function of  $f(\mathcal{B}_1), \dots, f(\mathcal{B}_n)$ . By (2.1) in [10] and the fact that  $T(n) \rightarrow +\infty$  a.s.,  $F_{T(n)} \Rightarrow F$  along almost almost all sample paths, hence, conditionally to the data

$$d_2 \left( F_{T(n)}, F \right) \rightarrow 0. \quad (47)$$

Denote by  $N_\sigma$  a standard distribution with mean 0 and variance  $\sigma^2$ . By Proposition 1.2,

$$d_2 \left( F^{(T(n))}, N_\sigma \right) \rightarrow 0. \quad (48)$$

Conditionally to the data, the distribution of

$$\sqrt{T(n)} \left( \frac{\sum_{j=1}^{T(n)} \left( f \left( \mathcal{B}_{j,T(n)}^* \right) - \frac{1}{T(n)} \sum_{i=1}^{T(n)} f \left( \mathcal{B}_i \right) \right)}{T(n)} \right)$$

is  $F_{T(n)}^{(T(n))}$ , then, conditionally to the data,

$$d_2 \left( F_{T(n)}^{(T(n))}, N_\sigma \right) \leq d_2 \left( F_{T(n)}^{(T(n))}, F^{(T(n))} \right) + d_2 \left( F^{(T(n))}, N_\sigma \right)$$

which goes to 0 by (47) and (48). The theorem now follows by (45), (14) and Slutsky's theorem.



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