Fractal properties of the frontier in Poissonian coloring

Anne-Laure Basdevant, Guillaume Blanc, Nicolas Curien, Arvind Singh

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February 14, 2023

Abstract

We study a model of random partitioning by nearest-neighbor coloring from Poisson rain, introduced independently by Aldous [2] and Preater [6]. Given two initial points in $[0,1]^d$ respectively colored in red and blue, we let independent uniformly random points fall in $[0,1]^d$, and upon arrival, each point takes the color of the nearest point fallen so far. We prove that the colored regions converge in the Hausdorff sense towards two random closed subsets whose intersection — the *frontier* — has Hausdorff dimension strictly between $(d-1)$ and d, thus answering a conjecture raised by Aldous in [2]. However, several topological properties of the frontier remain elusive.

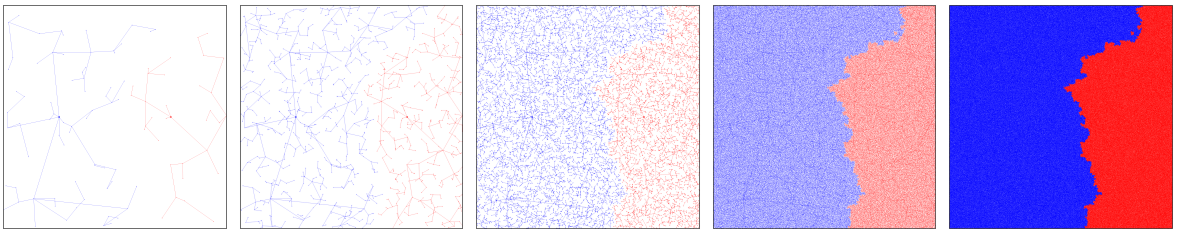

Figure 1: Simulation of the Poisson coloring of space where a new incoming point takes the color of the nearest neighbor in the process so far, from left to right with $10^2, 10^3, 10^4, 10^6$ and 10^7 points.

Introduction and main results

We consider a model of Poissonian coloring which is based on a dynamical construction in the d-dimensional hypercube $[0,1]^d$. Initially, two points $R_0 \neq B_0$ are planted in $[0,1]^d$: think of R_0 as an initial red seed, and of B_0 as an initial blue seed. All the randomness in the construction comes from a sequence $(X_n)_{n\in\mathbb{N}^*}$ of independent random variables, uniformly distributed in $[0,1]^d$. Picturing $X_1, X_2, \ldots$ as points falling consecutively in $[0,1]^d$, we let each point take the color of the closest point already present (nearest neighbor for the usual Euclidean metric d). Formally, define the initial red and blue sets as $\mathcal{R}_0 = \{R_0\}$ and $\mathcal{B}_0 = \{B_0\}$, respectively. Then, by induction, for each $n \in \mathbb{N}$ such that the red and blue sets $\mathcal{R}_n$ and $\mathcal{B}_n$ have been constructed, proceed as follows: almost surely, we have $d(X_{n+1}, \mathcal{R}_n) \neq d(X_{n+1}, \mathcal{B}_n)$, and

- if $d(X_{n+1}, \mathcal{R}_n) < d(X_{n+1}, \mathcal{B}_n)$, then set $\mathcal{R}_{n+1} = \mathcal{R}_n \cup \{X_{n+1}\}$ and $\mathcal{B}_{n+1} = \mathcal{B}_n$;
- otherwise, if $d(X_{n+1}, \mathcal{R}_n) > d(X_{n+1}, \mathcal{B}_n)$, then set $\mathcal{R}_{n+1} = \mathcal{R}_n$ and $\mathcal{B}_{n+1} = \mathcal{B}_n \cup \{X_{n+1}\}$.

Letting $n \to \infty$, the red and blue sets $\mathcal{R}_n$ and $\mathcal{B}_n$ respectively converge, for the Hausdorff distance between closed subsets of $[0,1]^d$, to

$$\mathcal{R}_\infty = \overline{\bigcup_{n \geqslant 0} \mathcal{R}_n} \quad \text{and} \quad \mathcal{B}_\infty = \overline{\bigcup_{n \geqslant 0} \mathcal{B}_n}.$$

The object we are interested in is the *frontier* $\mathcal{F}_\infty = \mathcal{R}_\infty \cap \mathcal{B}_\infty$, which is also easily shown to be the limit for the Hausdorff distance of the discrete frontier $\mathcal{F}_n = \{x \in [0,1]^d : d(x, \mathcal{R}_n) = d(x, \mathcal{B}_n)\}$, as $n \to \infty$ (*c.f.* Proposition 12). See Figure 1 for a simulation of the coloring process.

This very natural model can be found in Aldous [2], which attributes it to Penrose & Wade [5, Section 7.6.8], although it may have been considered by other authors before. Recently, Lichev and Mitsche [4] studied the combinatorial properties of genealogical trees induced by the coloring procedure. Here, we focus instead on the geometric and topological properties of the model. After completion of this work, we learned from Aldous that Preater [6] had considered the same model, and in particular answered [2, Conjecture 3], showing that the frontier $\mathcal{F}_\infty$ has zero Lebesgue measure (see [6, Theorem 2]). Our main result is the following, and settles a conjecture of Aldous [2, Section 5.3.3].

Theorem 1 (The frontier is fractal)**.** Almost surely, the Hausdorff dimension of the frontier $\mathcal{F}_\infty$ satisfies

$$d - 1 < \dim_H \mathcal{F}_\infty < d.$$

The proof is divided in two main steps, which we summarize below.

- Upper bound**.** We first show that for every $x \in [0,1]^d$ and $r > 0$ such that the ball $\overline{B}(x,r)$ does not contain the seeds R_0 and B_0, there is a positive probability that the smaller ball $\overline{B}(x, r/6)$ is monochromatic at the end of the coloring (Lemma 2). Together with a multi-scale argument, this shows that the Hausdorff dimension of the frontier $\mathcal{F}_\infty$ is strictly less than d (see Proposition 4). A result with a similar flavor, also using a first-passage percolation argument, was obtained by Preater (see [6, Theorem 1]), who showed that $\mathcal{F}_\infty$ has zero Lebesgue measure.

- Lower bound**.** The lower bound on the Hausdorff dimension of the frontier is based on ideas and techniques developed by Aizenman & Burchard in [1, Sections 5 and 6], where they introduce general conditions which allow to lower bound the Hausdorff dimension of random curves (see [1, Theorem 1.3]. Their result applies in particular to scaling limits of interfaces from critical statistical physics models such as percolation; random curves which have a positive probability, at each scale, of oscillating. Unfortunately, it is not clear that our frontier $\mathcal{F}_\infty$ even contains curves, see Open question 1 below. We find a workaround by adapting the ideas of Aizenman & Burchard to get a Hausdorff dimension lower bound result for connected random closed subsets of $[0,1]^d$. The exact statement is given in Theorem 7. We hope that this extension will prove to be of independent interest.

A natural variant. There are natural variants of this coloring model, such as the following "segment" model (as opposed to the original "point" model): still thinking of R_0 and B_0 as initial red and blue seeds, and of $X_1, X_2, \ldots$ as points falling consecutively in $[0,1]^d$, let as before $\mathcal{R}_0 = \{R_0\}$ and $\mathcal{B}_0 = \{B_0\}$ be the initial red and blue sets, respectively. Then, by induction, for each $n \in \mathbb{N}$ such that the red and blue sets $\mathcal{R}_n$ and $\mathcal{B}_n$ have been constructed, proceed as follows: almost surely, we have $d(X_{n+1}, \mathcal{R}_n) \neq d(X_{n+1}, \mathcal{B}_n)$, and

- if $d(X_{n+1}, \mathcal{R}_n) < d(X_{n+1}, \mathcal{B}_n)$, then set $\mathcal{R}_{n+1} = \mathcal{R}_n \cup [Y_n, X_{n+1}]$ and $\mathcal{B}_{n+1} = \mathcal{B}_n$, where Y_n denotes the point on $\mathcal{R}_n$ which is closest to X_{n+1};

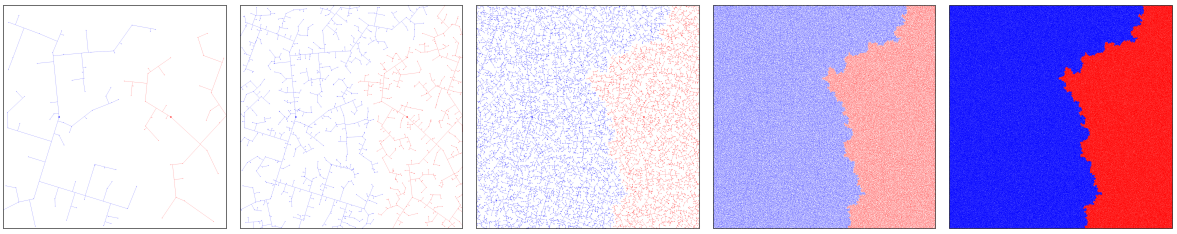

Figure 2: Simulation of the variant Poisson coloring of space where a new incoming point is linked by a monochromatic segment to the nearest point in the process so far, from left to right with $10^2, 10^3, 10^4, 10^6$ and 10^7 points. The arrivals points X_i are the same as those used for Figure 1. Notice that the red "island" on the top right part of the figure present in the original point model has disappeared.

- otherwise, if $d(X_{n+1}, \mathcal{R}_n) > d(X_{n+1}, \mathcal{B}_n)$, then set $\mathcal{R}_{n+1} = \mathcal{R}_n$ and $\mathcal{B}_{n+1} = \mathcal{B}_n \cup [Y_n, X_{n+1}]$, where Y_n denotes the point on $\mathcal{B}_n$ which is closest to X_{n+1}.

Note that, by construction, the red and blue sets $\mathcal{R}_n$ and $\mathcal{B}_n$ are connected finite unions of line segments, so that Y_n is always well defined (such a point is almost surely unique because X_{n+1} is uniform and independent of $X_1, \dots, X_n$). Upon minor technical modifications in the proofs, Theorem 1 holds for this coloring process as well.

Elusive topological properties of the frontier. Although our results show the convergence in a strong sense of the colored regions and establish the fractal nature of the frontier, many questions remain open, such as the existence of a 0/1-law for the Hausdorff dimension of $\mathcal{F}_\infty$. We focus here on the planar case $d = 2$, which concentrates the most interesting topological questions. Notice first that almost surely, the frontier $\mathcal{F}_\infty$ is not connected, the reason being that it is possible for a point to get surrounded by points of the opposite color, thus eventually creating an "island" in the coloring. See [6, Theorem 3], and Figure 3. This island creation is not possible in the segment model, where the limiting frontier is almost surely connected.

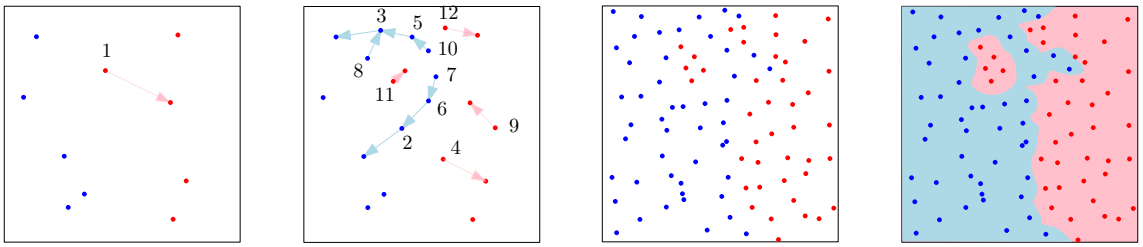

Figure 3: Illustration of the creation of an "island". Such an island can be seen on Figure 1 on the top right corner. This shows that the limiting frontier $\mathcal{F}_\infty$ is not connected almost surely.

Open Question 1 (Curves)**.** Is the frontier $\mathcal{F}_\infty$ a countable union of curves?

It is natural to believe that $\mathcal{F}_\infty$ is a countable union of curves (i.e, images of continuous paths from $[0,1]$ to $\mathbb{R}^2$), or that the limiting frontier in the segment model is a curve. Although Aizenman & Burchard [1] provide sufficient conditions (namely [1, Hypothesis **H1**]) which would allow to show that $\mathcal{F}_\infty$ *contains* a curve[1], checking those estimates seems hard in our setup due to the lack of a

[1]There are connected compact subsets of $\mathbb{R}^2$ which do not contain any non-trivial curves, such as the pseudo-arc.

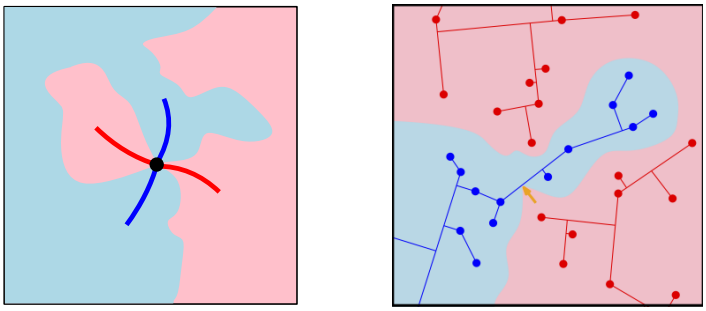

Figure 4: Left. Illustration of a double point in the frontier $\mathcal{F}_\infty$. **Right.** Can the frontier intersect the finite trees in the segment model (orange arrow)?

correlation inequality. Yet, simulations suggest that the connected components of $\mathcal{F}_\infty$ are *simple* curves, meaning that "double points", i.e. points from which four alternating monochromatic non-trivial curves originate, do not exist.

Open Question 2 (Simple curves)**.** If the above question has a positive answer, are those curves almost surely *simple*?

Accordingly, if this is true, then the frontier in the segment model should be made of a single simple curve. In fact, simulations suggest that in that model, the finite red and blue trees $\mathcal{R}_n$ and $\mathcal{B}_n$ are in the interior of the limiting red and blue regions $\mathcal{R}_\infty$ and $\mathcal{B}_\infty$ (it is possible to show that the arrival vertices $X_1, X_2, \dots$ are indeed in the interior of $\mathcal{R}_\infty$ and $\mathcal{B}_\infty$, with minor technical modifications in the proof of Lemma 2 below, but the same results for the whole segments is still out of scope.) A more general question is the following.

Open Question 3 (Safe margin)**.** Suppose that $\mathcal{R}_0$ is made of a segment or a ball instead of a single point. Do we have $\mathbb{P}(\mathcal{R}_0 \cap \mathcal{B}_\infty \neq \emptyset) = 0$?

Our techniques (or those of Preater) only show that the above probability is strictly less than 1, see the discussion before Corollary 1 in [6].

Acknowledgments. We warmly thank David Aldous for discussions about [2] and for providing us with the reference [6]. The first and fourth authors were supported by ANR 19-CE40-0025 ProGraM. The second and third authors were supported by ERC 740943 GeoBrown and ANR RanTanPlan. We are grateful to the participants of the PizzaMa seminar, during which this work was initiated.

1 Monochromatic balls and upper bound on $\dim_H \mathcal{F}_\infty$

In this section we establish our key lemma, Lemma 2, which shows that for every $x \in [0,1]^d$ and $r > 0$ such that $\overline{B}(x,r)$ does not contain the seeds R_0 and B_0, there is a positive probability that the smaller ball $\overline{B}(x,r/6)$ is monochromatic at the end of the coloring. Applying Lemma 2 at all scales yields the upper bound on the dimension of the frontier. In particular, it shows that for every $x \in [0,1]^d$, almost surely there exists an $r > 0$ such that the ball $\overline{B}(x,r)$ is monochromatic at the end of the coloring.

1.1 Key lemma

Before stating the result, let us embed the model in continuous time to gain convenient independence properties.

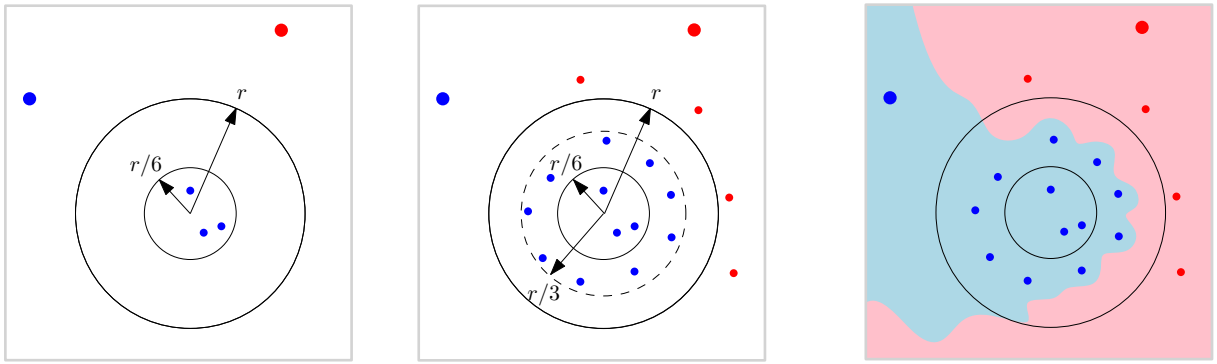

Figure 5: Illustration of the construction of the event $G^x_{r/6,r}$. Suppose that the seeds R_0, B_0 lie outside $\overline{B}(x,r)$.

Left. Notice that before the first point falls in $\overline{A}(x;r/6,r)$, a few points may have fallen in $\overline{B}(x,r/6)$ but, by the choice of $r/6$ versus r, all these points will have the same color (blue here).

Center. Then, we ask that the first few points falling in $\overline{A}(x;r/6,r)$ fall inside $\overline{B}(x,r/3)$, so that they all take the same color (blue here), and are spread out well enough to protect $\overline{B}(x,r/6)$ from being invaded by points of the other color (red here).

Right. A reinforcement property of the process will then entail that with positive probability, the invaders (the red here) cannot penetrate $\overline{B}(x,r/6)$, so that it remains monochromatic.

Poissonisation. Let Λ be a Poisson random measure with intensity $\lambda \otimes \lambda_d$ on $\mathbb{R}_+ \times \mathbb{R}^d$, where λ and λ_d denote the Lebesgue measures on $\mathbb{R}_+$ and $\mathbb{R}^d$, respectively. Let $X_1, X_2, \ldots$ be the points of Λ that fall in $[0,1]^d$, successively at times $\tau_1 < \tau_2 < \ldots$. It is a standard fact that the $(X_n)_{n\in\mathbb{N}^*}$ are independent random variables, uniformly distributed in $[0,1]^d$. Now, the coloring process can be defined in continuous time as follows. The sequence $(\mathcal{R}_n, \mathcal{B}_n)_{n\in\mathbb{N}^*}$ of the discrete setting will here correspond to $(\mathcal{R}_{\tau_n}, \mathcal{B}_{\tau_n})_{n\in\mathbb{N}^*}$, and the sets $\mathcal{R}_t$ and $\mathcal{B}_t$ will be defined at all times $t \in \mathbb{R}_+$ as follows: for each $n \in \mathbb{N}$, we set $\mathcal{R}_t = \mathcal{R}_{\tau_n}$ and $\mathcal{B}_t = \mathcal{B}_{\tau_n}$ for every $t \in [\tau_n, \tau_{n+1}[$, with the convention $\tau_0 = 0$.

For $x \in \mathbb{R}^d$ and $0 < r < R$, we define the annulus $\overline{A}(x;r,R) = \{y \in \mathbb{R}^d : r < |x-y| \leqslant R\}$, and denote by $\mathcal{A}^x_{r,R}$ the σ-algebra generated by the restriction of the Poisson random measure Λ to the set $\mathbb{R}_+ \times \overline{A}(x;r,R)$. The point of Lemma 2 below is to describe an $\mathcal{A}^x_{r/6,r}$-measurable "good event" $G^x_{r/6,r}$, which has probability bounded away from 0 uniformly in x and r, such that if $\overline{B}(x,r)$ does not contain the seeds R_0 and B_0, then on $G^x_{r/6,r}$ the ball $\overline{B}(x,r/6)$ is monochromatic at the end of the coloring. Figure 5 provides an overview of how such a good event is constructed.

Lemma 2. *There is a constant $p \in]0,1[$ for which the following holds. For every $x \in [0,1]^d$ and $r > 0$, there exists an $\mathcal{A}^x_{r/6,r}$-measurable good event $G^x_{r/6,r}$, which has probability $\mathbb{P}\left(G^x_{r/6,r}\right) \geqslant p$, such that if $\overline{B}(x,r)$ does not contain either R_0 or B_0, then on $G^x_{r/6,r}$ the ball $\overline{B}(x,r/6)$ does not meet both $\bigcup_{t\geqslant 0}\mathcal{R}_t$ and $\bigcup_{t\geqslant 0}\mathcal{B}_t$.*

Remark 3. It will be clear from the proof that the event $G^x_{r/6,r}$ also prevents the ball $\overline{B}(x,r/6)$ from bichromaticity whenever $R_0 \in \overline{B}(x,r/3)$ and $B_0 \notin \overline{B}(x,r)$, or $B_0 \in \overline{B}(x,r/3)$ and $R_0 \notin \overline{B}(x,r)$. In particular, Lemma 2 allows to recover the result of Preater (see [6, proof of Theorem 2]) that almost surely, there exists an $r > 0$ such that $\overline{B}(R_0,r)$ does not contain a blue point, and $\overline{B}(B_0,r)$ does not contain a red point.

Proof. Fix $x \in [0,1]^d$ and $r > 0$, and suppose that both R_0 and B_0 lie outside $\overline{B}(x,r)$. We construct an $\mathcal{A}^x_{r/6,r}$-measurable *good event* G on which a "defense" is organized inside the annulus $\overline{A}(x; r/6, r)$, preventing $\overline{B}(x, r/6)$ from meeting both $\bigcup_{t\geqslant 0} \mathcal{R}_t$ and $\bigcup_{t\geqslant 0} \mathcal{B}_t$.

Definition of G. Let $\rho_k = \left(1 + 2^{-k}\right) \cdot r/6$ for all $k \in \mathbb{N}$, and let $(t_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers to be adjusted later, with $T_k := t_0 + \ldots + t_k \to \infty$ as $k \to \infty$. We define $\mathcal{A}^x_{r/6,r}$-measurable events $(G_k)_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$, on $G_0 \cap \ldots \cap G_k$ the ball $\overline{B}(x, \rho_k)$ does not meet both $\bigcup_{0\leqslant t<T_k} \mathcal{R}_t$ and $\bigcup_{0\leqslant t<T_k} \mathcal{B}_t$. The good event G will then be defined as $G = \bigcap_{k\geqslant 0} G_k$. For every $k \in \mathbb{N}$, we denote by A_k the annulus $\overline{A}(x; \rho_{k+1}, \rho_k)$. Let $\delta_k = 1/2 \cdot (\rho_k - \rho_{k+1}) \cdot (k+1)^{-2}$, and let $\mathcal{Z}_k \subset A_k$ be a finite set of points with the following properties:

a. for every $y \in A_k$, there exists $z \in \mathcal{Z}_k$ such that $y \in \overline{B}(z, 3\delta_k/2)$,

b. for any $z \neq z' \in \mathcal{Z}_k$, we have $|z - z'| > \delta_k$,

c. for every $z \in \mathcal{Z}_k$, we have $\overline{B}(z, \delta_k/2) \subset A_k$.

It is clear that such a set $\mathcal{Z}_k$ always exists: we keep adding points satisfying b. and c. until no more point can be added and then a. must also be satisfied by construction. Note also that, because the balls $\left(\overline{B}(z, \delta_k/2)\right)_{z\in\mathcal{Z}_k}$ are disjoint and included in $A_k \subset \overline{B}(x, r/3)$, a volume computation entails that

$$\#\mathcal{Z}_k \leqslant \left(\frac{r/3}{\delta_k/2}\right)^d = \left(8 \cdot 2^{k+1} \cdot (k+1)^2\right)^d. \tag{1}$$

We define G_0 as the event: "for every $z \in \mathcal{Z}_0$, a point of Λ falls in $\overline{B}(z, \delta_0/2)$ over the time interval $[0, t_0[$, meanwhile no point falls in $\overline{A}(x; r/3, r)$". We claim that on G_0, the ball $\overline{B}(x, r/3)$ does not meet both $\bigcup_{0\leqslant t<t_0} \mathcal{R}_t$ and $\bigcup_{0\leqslant t<t_0} \mathcal{B}_t$. In particular, all the points of Λ that have fallen in the spots $\left(\overline{B}(z, \delta_0/2)\right)_{z\in\mathcal{Z}_0}$ over the time interval $[0, t_0[$ have the same *good* color. Indeed, fix a realization of the event G_0. Denote by $y_1, \ldots, y_n$ the points of Λ that fall in $\overline{B}(x, r)$ over the time interval $[0, t_0[$, and by $\tau_1 < \ldots < \tau_n \in [0, t_0[$ their arrival times. Note that by the definition of G_0, the points $y_1, \ldots, y_n$ land in $\overline{B}(x, r/3)$. So y_1 arrives in $\overline{B}(x, r/3)$, with its color. Then when y_2 arrives, it lands at distance at most $2r/3$ of y_1, and at distance more than $2r/3$ of any other point of the process, since these all lie outside $\overline{B}(x, r)$. Therefore, the nearest neighbor of y_2 is y_1, and y_2 inherits it color. The argument iterates, proving the claim.

Next, in order to define G_k for $k \geqslant 1$, we start with the following deterministic observation. Suppose by induction that, on the event $G_0 \cap \ldots \cap G_{k-1}$, the following holds: at time T_{k-1}, each cell $\left(\overline{B}(z, \delta_{k-1}/2)\right)_{z\in\mathcal{Z}_{k-1}}$ contains a point of the good color, and $\overline{B}(x, \rho_{k-1})$ does not contain any point of the other bad color. Then, every $y \in A_{k-1}$ is at distance at most $2\delta_{k-1}$ from a point of the good color, and the only way of bringing a point of the bad color inside $\overline{B}(x, \rho_k)$ before time T_k is to have points of Λ — say $y_1, \ldots, y_j$ — falling in A_{k-1}, at times say $\tau_1 < \ldots < \tau_j \in [T_{k-1}, T_k[$, with:

- $d\left(y_1; \mathbb{R}^d \backslash \overline{B}(x, \rho_{k-1})\right) < 2\delta_{k-1}$,
- $|y_{i+1} - y_i| < 2\delta_{k-1}$ for each $i \in [\![1, j[\![$,
- $d\left(y_j, \overline{B}(x, \rho_k)\right) < 2\delta_{k-1}$.

Now, let us discretise this information. First, it follows from the inequality $\rho_{k-1} - \rho_k < (j+1) \cdot 2\delta_{k-1}$ that such a path must have length $j \geqslant k^2$. Then, for each $i \in [\![1, k^2]\!]$, let $z_i \in \mathcal{Z}_{k-1}$ be such that $y_i \in \overline{B}(z_i, 3\delta_{k-1}/2)$. The following holds:

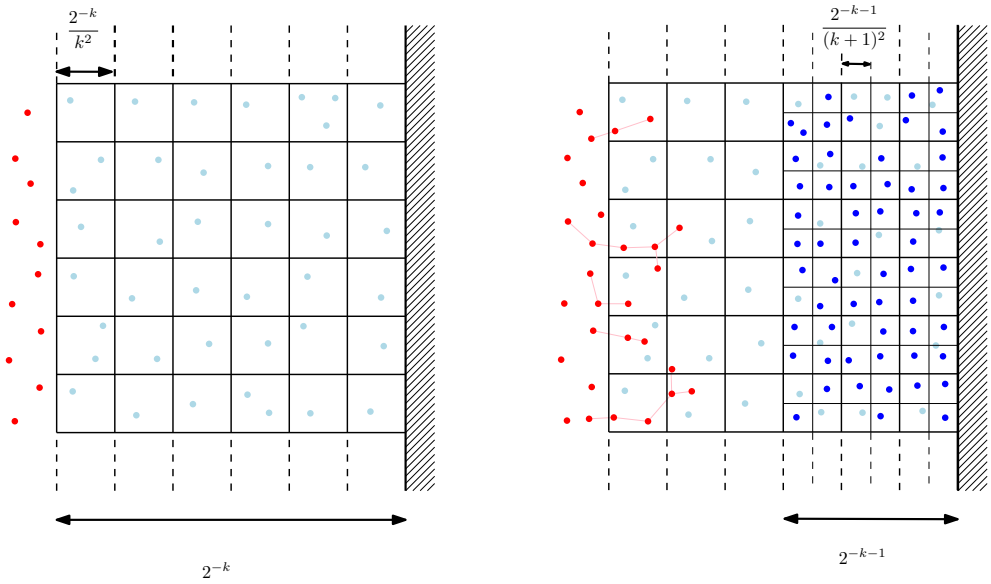

Figure 6: Schematic description of the induction procedure.
Left. We assume that a blue point has fallen in each cell of size $2^{-k}/k^2$ in a annulus of width 2^{-k} before a first red point enters this annulus.
Right. Assuming that the left event holds, the red clusters need to create a red path of a least k^2 points navigating between the blue points in order to reach the smaller annulus of width 2^{-k-1}. By the time this happens, with very high probability, a blue point will have fallen in each cell of size $2^{-k-1}/(k+1)^2$ inside the smaller annulus.

- for every $i \in [\![1, k^2]\!]$, we have $z_i \in \mathcal{Z}_{k-1}$,
- for each $i \in [\![1, k^2[\![$, we have $|z_{i+1} - z_i| \leqslant 5\delta_{k-1}$.

A sequence $z_1, \dots, z_{k^2}$ satisfying the two properties above is said to be *admissible of order* k. Moreover, since for each $i \in [\![1, k^2]\!]$, a point of Λ falls in $\overline{B}(z_i, 3\delta_{k-1}/2)$ at time τ_i, with $\tau_1 < \ldots < \tau_n \in [T_{k-1}, T_k[$, we say that $z_1, \dots, z_{k^2}$ *ring consecutively* over the time interval $[T_{k-1}, T_k[$. We can now formally define the event G_k by "for every $z \in \mathcal{Z}_k$, a point of Λ falls in $\overline{B}(z, \delta_k/2)$ over the time interval $[T_{k-1}, T_k[$, meanwhile no admissible sequence of order k rings consecutively". By induction on k, we see that on $G_0 \cap \ldots \cap G_k$, the ball $\overline{B}(x, \rho_k)$ does not meet both $\bigcup_{0 \leqslant t < T_k} \mathcal{R}_t$ and $\bigcup_{0 \leqslant t < T_k} \mathcal{B}_t$. Finally, we set $G = \bigcap_{k \geqslant 0} G_k$.

The probability $\mathbb{P}(G)$ is bounded away from 0. Because of the disjointness of the time intervals over which they are defined, the events $(G_k)_{k \in \mathbb{N}}$ are independent:

$$\mathbb{P}(G) = \prod_{k \geqslant 0} \mathbb{P}(G_k) = \mathbb{P}(G_0) \cdot \prod_{k \geqslant 1} [1 - \mathbb{P}(F_k)],$$

where F_k is the complement of the event G_k. On F_k,

- either there exists $z \in \mathcal{Z}_k$ such that $\Lambda\left([T_{k-1}, T_k[\times \overline{B}(z, \delta_k/2)\right) = 0$, let us call B_k the corresponding event,
- or there exists an admissible sequence of order k that rings consecutively. We call C_k the corresponding event.

We have $\mathbb{P}(F_k) \leqslant \mathbb{P}(B_k) + \mathbb{P}(C_k)$. For the second term, a union bound and the Markov property for Λ show that

$$\begin{aligned}\mathbb{P}(C_k) &\leqslant \sum_{z_1,\dots,z_{k^2} \text{ admissible of order } k} \mathbb{P}(z_1,\dots,z_{k^2} \text{ ring consecutively}) \\ &= \#\{\text{admissible sequences of order } k\} \cdot \mathbb{P}(\tau_1 + \ldots + \tau_{k^2} < t_k),\end{aligned}$$

where the τ_i's are independent exponential random variables with parameter (*i.e,* inverse mean) $\lambda_k = v_d(3\delta_{k-1}/2)^d$. We set $t_k = \alpha \cdot k^2 \cdot \lambda_k^{-1}$ for all $k \geqslant 1$, where $\alpha \in]0,1[$ is a parameter to be adjusted later. On the one hand, a standard Chernoff bound yields

$$\mathbb{P}(\tau_1 + \ldots + \tau_{k^2} < t_k) \leqslant e^{(1-\alpha+\ln\alpha)k^2}.$$

On the other hand, in order to choose an admissible sequence $z_1,\dots,z_{k^2}$ of order k, there is no more than $\#\mathcal{Z}_{k-1}$ possibilities for the choice of z_1, and then for each $i \in [\![1,k^2[\![$, there is at most $\#\left(\mathcal{Z}_{k-1} \cap \overline{B}(z_i, 5\delta_{k-1})\right) \leqslant 11^d$ possibilities for the choice of z_{i+1}, this upper bound is since the disjoint balls $\left(\overline{B}(z,\delta_{k-1}/2);\ z \in \mathcal{Z}_{k-1} \cap \overline{B}(z_i,5\delta_{k-1})\right)$ are included in $\overline{B}(z_i, 11\delta_{k-1}/2)$. Thus, we find that

$$\mathbb{P}(C_k) \leqslant \#\mathcal{Z}_{k-1} \cdot \left(11^d\right)^{k^2-1} \cdot e^{(1-\alpha+\ln\alpha)k^2}.$$

We now fix α such that $11^d \cdot e^{1-\alpha+\ln\alpha} \leqslant e^{-1}$. Recalling (1), we obtain

$$\mathbb{P}(C_k) \leqslant \left(8 \cdot 2^k \cdot k^2\right)^d \cdot 11^{-d} \cdot e^{-k^2} =: c_k.$$

Next, we upper bound $\mathbb{P}(B_k)$. We have $\mathbb{P}(B_k) \leq \#\mathcal{Z}_k \cdot p_k$, where

$$p_k = \exp\left[-v_d(\delta_k/2)^d \cdot t_k\right] = \exp\left[-\alpha \cdot \left(\frac{\delta_k}{3\delta_{k-1}}\right)^d \cdot k^2\right].$$

Using again (1), we see that

$$\#\mathcal{Z}_k \cdot p_k \leqslant \left(8 \cdot 2^{k+1} \cdot (k+1)^2\right)^d \cdot \exp\left[-\alpha \cdot 24^{-d} \cdot k^2\right] =: b_k,$$

which finally yields

$$\mathbb{P}(F_k) \leqslant b_k + c_k =: a_k.$$

We finally check that $\mathbb{P}(G)$ is bounded away from 0 uniformly in x and r. Since $\sum_{k\geqslant 1} a_k < \infty$, we can find $K \in \mathbb{N}$ (not depending on x or r) such that $\prod_{k\geqslant K+1}(1-a_k) \geqslant 1/2$. With that choice, we have

$$\mathbb{P}(G) = \mathbb{P}(G_0) \cdot \ldots \cdot \mathbb{P}(G_K) \cdot \prod_{k\geqslant K+1} [1 - \mathbb{P}(F_k)] \geqslant \frac{\mathbb{P}(G_0) \cdot \ldots \cdot \mathbb{P}(G_K)}{2}.$$

Note that we have yet to specify the value of t_0, which we now set to $t_0 = r^{-d}$. Given this choice, the probability $\mathbb{P}(G_0)$ is bounded away from 0 uniformly in x and r. Next, for each $k \in [\![1,K]\!]$, we claim that $\mathbb{P}(G_k)$ is also bounded away from 0 uniformly in x and r, because the same is true for the probability of the sub-event: "for each $z \in \mathcal{Z}_k$, a point of Λ falls in $\overline{B}(z,\delta_k/2)$ over the time interval $[T_{k-1}, T_k[$, meanwhile no point falls in A_{k-1}". Thus, the quantity $\mathbb{P}(G_0) \cdot \ldots \cdot \mathbb{P}(G_K) \cdot 1/2$ is bounded away from zero uniformly in x and r, which completes the proof of the lemma. □

1.2 Hausdorff dimension: upper bound

Proposition 4. There exist constants $C, \alpha > 0$ such that for every $x \in [0,1]^d$,

$$\mathbb{P}\left(\mathcal{F}_\infty \text{ meets } \overline{B}(x,\delta)\right) \leqslant C \cdot \delta^\alpha \quad \text{for all } \delta \in]0,1[\text{ small enough so that } R_0, B_0 \notin \overline{B}\left(x, \sqrt{\delta}\right).$$

Proof. Fix $x \in [0,1]^d$, and let $\delta \in]0,1[$ be small enough so that $R_0, B_0 \notin \overline{B}\left(x, \sqrt{\delta}\right)$. Set $r_k = \sqrt{\delta} \cdot 6^{-k}$ for all $k \in \mathbb{N}$, and denote by K the largest integer k such that $r_k > \delta$. By Lemma 2, we have the inclusion

$$\left(\mathcal{F}_\infty \text{ meets } \overline{B}(x,\delta)\right) \subset \left(\text{for every } k \in [\![1,K]\!], \text{ the event } G^x_{r_k,r_{k-1}} \text{ fails to be realized}\right).$$

Thus, since those are independent events, we obtain

$$\mathbb{P}\left(\mathcal{F}_\infty \text{ meets } \overline{B}(x,\delta)\right) \leqslant (1-p)^K.$$

Plugging in the equality $K = \left\lceil \log_6\left(\delta^{-1/2}\right)\right\rceil - 1$, we find that

$$(1-p)^K \leqslant (1-p)^{-1} \cdot \delta^\alpha, \quad \text{with } \alpha = -\frac{\ln(1-p)}{2\ln 6} > 0,$$

which yields the required upper bound. □

Proposition 5. There exists $\varepsilon > 0$ such that, almost surely,

$$\dim_H \mathcal{F}_\infty \leqslant d - \varepsilon < d.$$

Proof. Let $\varepsilon = \alpha \wedge (d/2)$, where α is the exponent of Proposition 4. For each $k \in \mathbb{N}$, set $\delta_k = 2^{-k}$, and let $\left(\overline{B}(x,\delta_k)\right)_{x\in\mathcal{X}_k}$ be a covering of $[0,1]^d$ by balls of radius δ_k, with centres $x \in [0,1]^d$ more than δ_k apart so that the $\left(\overline{B}(x,\delta_k/2)\right)_{x\in\mathcal{X}_k}$ are disjoint. In particular, there exists a constant $C' = C'(d) > 0$ such that $\#\mathcal{X}_k \leqslant C' \cdot \delta_k^{-d}$. By definition, the $(d-\varepsilon)$-dimensional Hausdorff measure of $\mathcal{F}_\infty$ is bounded from above by the random variable

$$H = \varliminf_{k\to\infty} \sum_{x\in\mathcal{X}_k} (2\delta_k)^{d-\varepsilon} \cdot \mathbf{1}\left(\mathcal{F}_\infty \text{ meets } \overline{B}(x,\delta_k)\right).$$

We claim that H is almost surely finite, which implies that $\dim_H \mathcal{F}_\infty \leqslant d-\varepsilon$ almost surely. Indeed, using Fatou's lemma, we get

$$\mathbb{E}[H] \leqslant \varliminf_{k\to\infty} \sum_{x\in\mathcal{X}_k} (2\delta_k)^{d-\varepsilon} \cdot \mathbb{P}\left(\mathcal{F}_\infty \text{ meets } \overline{B}(x,\delta_k)\right),$$

and for every k we have, using Proposition 4:

$$\begin{aligned}\sum_{x\in\mathcal{X}_k} (2\delta_k)^{d-\varepsilon} \cdot \mathbb{P}\left(\mathcal{F}_\infty \text{ meets } \overline{B}(x,\delta_k)\right) \\ \leqslant \#\left\{x \in \mathcal{X}_k : R_0 \in \overline{B}\left(x,\sqrt{\delta_k}\right) \text{ or } B_0 \in \overline{B}\left(x,\sqrt{\delta_k}\right)\right\} \cdot (2\delta_k)^{d-\varepsilon} + \#\mathcal{X}_k \cdot (2\delta_k)^{d-\varepsilon} \cdot C \cdot \delta_k^\alpha.\end{aligned}$$

For the first term, we have

$$\begin{aligned}\#\left\{x \in \mathcal{X}_k : R_0 \in \overline{B}\left(x,\sqrt{\delta_k}\right) \text{ or } B_0 \in \overline{B}\left(x,\sqrt{\delta_k}\right)\right\} &\leqslant \#\mathcal{X}_k \cap \overline{B}\left(R_0,\sqrt{\delta_k}\right) + \#\mathcal{X}_k \cap \overline{B}\left(B_0,\sqrt{\delta_k}\right) \\ &\leqslant 2 \cdot \left(\frac{\sqrt{\delta_k}}{\delta_k/2} + 1\right)^d = 2\left(\frac{2}{\sqrt{\delta_k}} + 1\right)^d.\end{aligned}$$

Recalling the assumption that $\varepsilon \leqslant d/2$, we deduce:

$$\varlimsup_{k\to\infty} \#\left\{x \in \mathcal{X}_k : R_0 \in \overline{B}\left(x, \sqrt{\delta_k}\right) \text{ or } B_0 \in \overline{B}\left(x, \sqrt{\delta_k}\right)\right\} \cdot (2\delta_k)^{d-\varepsilon} < \infty.$$

For the second term, we use that $\#\mathcal{X}_k \leqslant C' \cdot \delta_k^{-d}$, and $\varepsilon \leqslant \alpha$ to check:

$$\varlimsup_{k\to\infty} \#\mathcal{X}_k \cdot (2\delta_k)^{d-\varepsilon} \cdot C \cdot \delta_k^{\alpha} < \infty.$$

Combining these two inequalities, we conclude that $\mathbb{E}[H] < \infty$. □

2 Hausdorff dimension lower bounds

In this section, we prove that the Hausdorff dimension of $\mathcal{F}_\infty$ is strictly greater than $(d-1)$. A substantial part of this work consists in the adaptation of the lower bound [1, Theorem 1.3] of Aizenman & Burchard. Indeed, as the knowledgeable reader has undoubtedly noticed, it is not possible to invoke the above mentioned result directly because we do not know whether the frontier $\mathcal{F}_\infty$ contains non-trivial curves. So instead, we modify the proof of Aizenman & Burchard to obtain a general Hausdorff dimension lower bound result for connected random closed subsets which satisfy Property $(\varnothing)$, with the following definition.

Definition 6 (Property $(\varnothing)$)**.** Let $\mathcal{F}$ be a random closed subset of $[0,1]^d$. We say that $\mathcal{F}$ satisfies Property $(\varnothing)$ if there exists a constant $\zeta > 1$ and two constants $Q > 0$ and $q \in]0,1[$ such that the following holds: for every collection $\left(\overline{B}(x_i, r_i);\ i \in [\![1,n]\!]\right)$ of balls with centres $x_1, \dots, x_n \in [0,1]^d$ such that the dilated balls $\left(\overline{B}(x_i, \zeta r_i);\ i \in [\![1,n]\!]\right)$ are disjoint (we say that the balls are ζ-separated), we have

$$\mathbb{P}\left(\text{for each } i \in [\![1,n]\!], \text{ the set } \mathcal{F} \text{ meets } \overline{B}(x_i, r_i)\right) \leqslant Q \cdot q^n.$$

We point out that Property $(\varnothing)$ is very similar to [1, Hypothesis **H2**].

Theorem 7. Let $\mathcal{F}$ be a random closed subset of $[0,1]^d$. Assume that it satisfies Property $(\varnothing)$, and that almost surely $\mathcal{F}$ has a connected component which is not reduced to a point. Then, there exists a constant $s > 1$ such that, almost surely,

$$\dim_H \mathcal{F} \geqslant s > 1.$$

The empty set, or the points of a homogeneous Poisson process, are obvious examples of random closed subsets of $[0,1]^d$ which satisfy Property $(\varnothing)$. Both have Hausdorff dimension 0 but their connected components are singletons. The above result says that, as soon as we request a random closed subset to have a non-trivial connected component (and thus, Hausdorff dimension at least 1), then the fact that it satisfies the property $(\varnothing)$ implies that it is "delocalized" in some sense, and entails that its Hausdorff dimension is, in fact, strictly greater than 1.

Remark 8. As it will be clear from the proof of Theorem 7, and used later on, the following actually holds: there exists a constant $s > 1$ such that almost surely, for any closed subset $\mathcal{G} \subset \mathcal{F}$ with a non-trivial connected component, we have $\dim_H \mathcal{G} \geqslant s$.

In Subsection 2.2, we will first apply Theorem 7 to the frontier $\mathcal{F}_\infty$ in dimension $d = 2$, showing that $\dim_H \mathcal{F}_\infty > 1$ almost surely. The lower bound $\dim_H \mathcal{F}_\infty > d-1$ in higher dimensions will then follow from Theorem 7 together with a slicing lemma, as detailed at the end of Section 2.2. Let us now present the proof of Theorem 7.

2.1 Proof of Theorem 7

As mentioned before, the proof is adapted from [1] and thus uses similar ingredients. Still, we provide here a self-contained proof, recalling and adapting the necessary results from [1] whenever required. At its core, the proof employs the usual "energy method" (see [3, Theorem 6.4.6]) to lower bound the Hausdorff dimension of a set. There are two main parts:

1. We first describe a deterministic splitting procedure for curves which produces, when the curves are oscillating enough, an important number of disjoint sub-curves.

2. Next, we show that if a connected random closed subset $\mathcal{F}$ satisfies Property $(\varnothing)$, then curves located in a shrinking neighborhoods of $\mathcal{F}$ will necessarily oscillate enough so that we can use the splitting procedure above to create many sub-curves. This will enable us to create a sequence of measures with good integrability properties and finally, by compactness, extract a measure ν supported on $\mathcal{F}$ such that $\iint |x-y|^{-s}\mathrm{d}\nu(y)\mathrm{d}\nu(x) < \infty$ for some $s > 1$, which in turn implies that $\dim_H \mathcal{F} \geqslant s$.

2.1.1 A deterministic splitting procedure for curves

Given a small parameter $\alpha \in]0,1[$, we describe the splitting procedure (P_α) mentioned in [1, Lemma 5.2]. It takes as input a continuous path $\gamma : [0,1] \to \mathbb{R}^d$ with $\gamma(1) \neq \gamma(0)$, and outputs a collection $\gamma_1, \dots, \gamma_\kappa$ of subpaths of γ, with the following properties:

- for every $i \in [\![1,\kappa]\!]$, we have $|\gamma_i(0) - \gamma_i(1)| = \alpha \cdot |\gamma(0) - \gamma(1)| =: \delta$,
- for any $i \neq j \in [\![1,\kappa]\!]$, we have $d(\gamma_i[0,1]; \gamma_j[0,1]) \geqslant \alpha\delta$.

The splitting procedure (P_α) goes as follows, see Figure 7 for an illustration. Set $\Delta = |\gamma(0) - \gamma(1)| > 0$ and let $\delta = \alpha \cdot \Delta < |\gamma(0) - \gamma(1)|$. Initially, set $\sigma_1 = 0$, and let

$$\tau_1 = \inf\left\{t \in [0,1] : \gamma(t) \notin \overline{B}(\gamma(0), \delta)\right\}.$$

By induction, for $i \in \mathbb{N}^*$, assuming that $\sigma_1, \tau_1; \dots; \sigma_i, \tau_i$ have been constructed, if

$$d(\gamma(t); \gamma[\sigma_1, \tau_1] \cup \dots \cup \gamma[\sigma_i, \tau_i]) \leqslant (1+\alpha)\delta \quad \text{for all } t \in [\tau_i, 1],$$

then we set $\sigma_{i+1} = 1$ and $\tau_{i+1} = 1$. Otherwise, we set

$$\tau_{i+1} = \inf\{t \in [\tau_i, 1] : d(\gamma(t); \gamma[\sigma_1, \tau_1] \cup \dots \cup \gamma[\sigma_i, \tau_i]) > (1+\alpha)\delta\},$$

and let $\sigma_{i+1} = \sup\left\{t \in [\tau_i, \tau_{i+1}[: \gamma(t) \notin \overline{B}(\gamma(\tau_{i+1}), \delta)\right\}$. Finally, let κ be the largest integer $i \in \mathbb{N}^*$ such that $\tau_i < 1$, and for each $i \in [\![1,\kappa]\!]$ denote by γ_i the path $\theta \in [0,1] \mapsto \gamma((1-\theta)\sigma_i + \theta\tau_i)$. For every $i \in [\![1,\kappa]\!]$, we have $|\gamma_i(0) - \gamma_i(1)| = \delta$, and for any $i \neq j \in [\![1,\kappa]\!]$, we have $d(\gamma_i[0,1]; \gamma_j[0,1]) \geqslant \alpha\delta$.

Definition 9. We say that a continuous path $\gamma : [0,1] \to \mathbb{R}^d$, with $\gamma(1) \neq \gamma(0)$, *deviates by a factor* $\rho > 0$ *from being a straight line* when there exists $t \in [0,1]$ such that

$$\gamma(t) \notin S(\gamma(0), \gamma(1); \rho|\gamma(0) - \gamma(1)|),$$

where $S(x,y;r) = \left\{z \in \mathbb{R}^d : d(z;[x,y]) \leqslant r\right\}$ denotes the sausage of radius r around the line segment $[x,y]$.

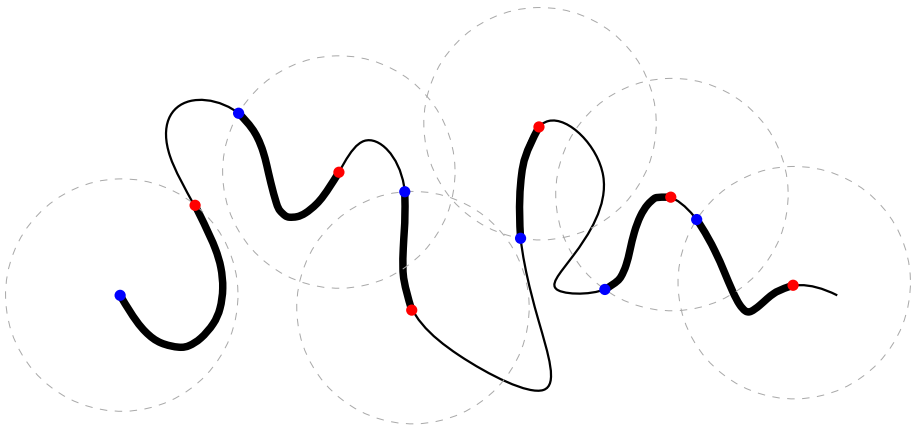

Figure 7: Illustration of the procedure (P_α). The times σ_i are in blue, and the τ_i in red. Several subpaths (in thick line), spanning a distance δ and being $\alpha\delta$ apart, are created from the initial path. The balls have radius δ.

Intuitively — recall that α is small — the number κ of subpaths produced by the procedure (P_α) must be at least of order $\Delta/\delta = 1/\alpha$ (a lower bound which can be attained by a straight line). However, when the input path deviates of a straight line, one would expect the procedure to produce additional paths. This is the meaning of the next proposition.

Proposition 10. Let $\gamma : [0,1] \to \mathbb{R}^d$ be a path with $\gamma(1) \neq \gamma(0)$.

1. The number of subpaths produced by the procedure (P_α) always satisfies
$$\kappa \geqslant \frac{1-\alpha}{(1+\alpha)\alpha}.$$
2. If γ deviates by a factor $\rho > 0$ from being a straight line, the number of subpaths produced by the procedure (P_α) satisfies
$$\kappa \geqslant \frac{\frac{1}{2}\left(1+\sqrt{1+2\rho^2}\right)-(4+\alpha)\alpha}{(1+\alpha)\alpha}.$$

Proof. Recall that $\Delta = |\gamma(0)-\gamma(1)| > 0$ and $\delta = \alpha\Delta$.

1. By the definition of (P_α), there exists $i_1 \in [\![1,\kappa]\!]$ such that $d(\gamma(1);\gamma[\sigma_{i_1},\tau_{i_1}]) \leqslant (1+\alpha)\delta$, and therefore $t_1 \in [\sigma_{i_1},\tau_{i_1}]$ such that $|\gamma(1)-\gamma(t_1)| \leqslant (1+\alpha)\delta$. Then, by induction, for $k \in \mathbb{N}^*$ such that $i_1, t_1; \ldots; i_k, t_k$ have been constructed, proceed as follows. If $i_k = 1$, then set $i_{k+1} = 1$ and let $t_{k+1} = 0$. Otherwise, by the definition of (P_α), there exists $i_{k+1} \in [\![1, i_k[\![$ such that $d\left(\gamma(t_k);\gamma\left[\sigma_{i_{k+1}},\tau_{i_{k+1}}\right]\right) \leqslant (1+\alpha)\delta$, and $t_{k+1} \in \left[\sigma_{i_{k+1}},\tau_{i_{k+1}}\right]$ such that $|\gamma(t_k)-\gamma(t_{k+1})| \leqslant (1+\alpha)\delta$. Finally, let m be the smallest integer $k \in \mathbb{N}^*$ such that $i_k = 1$. We have
$$|\gamma(0)-\gamma(1)| \leqslant |\gamma(0)-\gamma(t_m)| + \sum_{k=1}^{m-1} |\gamma(t_{k+1})-\gamma(t_k)| + |\gamma(t_1)-\gamma(1)| \leqslant \delta + m\cdot(1+\alpha)\delta,$$
hence
$$m \geqslant \frac{\Delta-\delta}{(1+\alpha)\delta} = \frac{1-\alpha}{(1+\alpha)\alpha}.$$
The result follows, since $i_1, \ldots, i_m$ are distinct elements of $[\![1,\kappa]\!]$.

2. Suppose that there exists $t \in [0,1]$ such that $\gamma(t) \notin S(\gamma(0),\gamma(1);\rho\Delta)$. We still denote by $i_1,t_1;\ldots;i_m,t_m$ the sequence of indices and times defined above. We construct another sequence $j_1,u_1;\ldots;j_p,u_p$ in the exact same manner but now obtained by backtracking from time t instead of time 1. By construction, we have $|\gamma(t)-\gamma(u_1)| \leqslant (1+\alpha)\delta$ and $|\gamma(u_k)-\gamma(u_{k+1})| \leqslant (1+\alpha)\delta$ for all $k \leqslant p-1$. Finally, let n be the smallest integer such that $j_n \in \{i_1,\ldots,i_m\}$, and denote by l the index such that $j_n = i_l$.

   The indices $i_1,\ldots,i_m$ and $j_1,\ldots,j_{n-1}$ are all distinct, hence $\kappa \geqslant m+n-1$. Now, on the one hand, with the same argument as above, we have:

$$|\gamma(0)-\gamma(t_l)| + |\gamma(t_l)-\gamma(1)| \leqslant \delta + m\cdot(1+\alpha)\delta.$$

   On the other hand,

$$|\gamma(t_l)-\gamma(t)| \leqslant |\gamma(t_l)-\gamma(u_n)| + \sum_{k=1}^{n-1}|\gamma(u_{k+1})-\gamma(u_k)| + |\gamma(u_1)-\gamma(t)| \leqslant 2\delta + n\cdot(1+\alpha)\delta.$$

   Summing these inequalities, we get

$$|\gamma(0)-\gamma(t_l)| + |\gamma(1)-\gamma(t_l)| + |\gamma(t)-\gamma(t_l)| \leqslant (m+n)(1+\alpha)\delta + 3\delta,$$

   hence

$$m+n-1 \geqslant \frac{\inf_{x\in\mathbb{R}^d}\{|\gamma(0)-x| + |\gamma(1)-x| + |\gamma(t)-x|\} - (4+\alpha)\delta}{(1+\alpha)\delta}.$$

   It remains to lower bound the infimum in the right hand side. First, using the triangle inequality, we get

$$|\gamma(0)-x| + |\gamma(1)-x| + |\gamma(t)-x| \geqslant \frac{|\gamma(0)-\gamma(1)| + |\gamma(0)-\gamma(t)| + |\gamma(t)-\gamma(1)|}{2}.$$

   Then, we make use of the fact that $\gamma(t) \notin S(\gamma(0),\gamma(1);\rho\Delta)$, to get

$$|\gamma(0)-\gamma(t)| + |\gamma(t)-\gamma(1)| \geqslant \sqrt{1+2\rho^2}\cdot\Delta.$$

   Altogether, we obtain

$$\inf_{x\in\mathbb{R}^2}\{|\gamma(0)-x| + |\gamma(1)-x| + |\gamma(t)-x|\} \geqslant \frac{1+\sqrt{1+2\rho^2}}{2}\cdot\Delta,$$

   and the proof is complete.

□

For the rest of the proof of Theorem 7, we set $\rho = \sqrt{18\alpha}$, and denote by $\beta = \beta(\alpha)$ the inverse geometric mean of the two lower bounds in Proposition 10:

$$\frac{1}{\beta} = \sqrt{\frac{1-\alpha}{(1+\alpha)\alpha}\cdot\frac{\frac{1}{2}\left(1+\sqrt{1+2\rho^2}\right)-(4+\alpha)\alpha}{(1+\alpha)\alpha}}. \tag{2}$$

With that choice for ρ, we have $\beta = \alpha - \alpha^2 + o(\alpha^2)$ as $\alpha \to 0^+$, and therefore $\beta < \alpha$ for all sufficiently small α.

2.1.2 Core of the proof

Proof of Theorem 7. Let $\mathcal{F}$ be a random closed subset of $[0,1]^d$. Assume that $\mathcal{F}$ satisfies Property $(\varnothing)$ with constants $\zeta > 1$, and $Q > 0$ and $q \in]0,1[$, and that almost surely $\mathcal{F}$ has a non-trivial connected component. We prove that there exists a constant $s > 1$ such that $\dim_H \mathcal{F} \geqslant s$ almost surely. To this end, by the "energy method" (see, e.g, [3, Theorem 6.4.6]), it suffices to construct a Borel probability measure ν supported on $\mathcal{F}$ such that

$$\int \int \frac{\mathrm{d}\nu(y)}{|x-y|^s} \mathrm{d}\nu(x) < \infty. \tag{3}$$

(We note here, in view of Remark 8, that $\mathcal{F}$ could be replaced with any closed subset $\mathcal{G} \subset \mathcal{F}$ having a non-trivial connected component without affecting the deterministic part of the reasoning. Then, the definition of the events $(E_m)_{m\in\mathbb{N}^*}$ below would not change, and the same constant s would work for all subsets $\mathcal{G}$.)

Fix a realization of $\mathcal{F}$. We claim that it is possible to find a sequence $\left(\gamma_n : [0,1] \to \mathbb{R}^d\right)_{n\in\mathbb{N}}$ of paths, with $\Delta := \inf_{n\geqslant 0} |\gamma_n[0,1]| > 0$, such that:

$$\text{for each } n \in \mathbb{N}, \text{ we have } \gamma_n[0,1] \subset (\mathcal{F})_{1/(n+1)}, \tag{4}$$

where $(\mathcal{F})_\varepsilon = \left\{x \in \mathbb{R}^d : d(x,\mathcal{F}) \leqslant \varepsilon\right\}$ denotes the ε-neighborhood of $\mathcal{F}$. Indeed, denote by $\mathcal{C}$ a non-trivial connected component of $\mathcal{F}$, and let $a \neq b$ be two distinct elements of $\mathcal{C}$. For each $n \in \mathbb{N}$, the points a and b belong to the same connected component of $\mathcal{O}_n = \left\{x \in \mathbb{R}^d : d(x,\mathcal{F}) < 1/(n+1)\right\}$. Since $\mathcal{O}_n$ is open, any connected component of $\mathcal{O}_n$ is path connected, hence there exists a continuous path $\gamma_n : [0,1] \to \mathcal{O}_n$ that connects a to b. In particular, we have $\gamma_n[0,1] \subset (\mathcal{F})_{1/(n+1)}$, and the diameter $|\gamma_n[0,1]|$ of $\gamma_n[0,1]$ is at least $|a-b| > 0$.

We now use the Aizenman & Burchard splitting procedure (P_α) recursively on the path γ_n, and derive a collection $\left(\mu_l^n\right)_{l\in\mathbb{N}}$ of Borel probability measures supported on $\gamma_n[0,1]$. Making use of the fact that $\mathcal{F}$ satisfies Property $(\varnothing)$, we will then show that for almost every realization of $\mathcal{F}$, it is possible to extract a sequence $\left(\nu_n = \mu_{L_n}^n\right)_{n\in\mathbb{N}}$ of which any subsequential weak limit ν is a Borel probability measure supported $\mathcal{F}$ such that (3) holds.

Fix $\alpha \in \left(0, d^{-1/2}\right)$ small enough so that the parameter $\beta = \beta(\alpha)$ defined in (2) satisfies $\beta < \alpha$. Set $\delta_k = \alpha^k$ for all $k \in \mathbb{N}$, and denote by $k_0 = k_0(\omega)$ the smallest integer k such that $\delta_k \leqslant \Delta$. Let us note that, since $\alpha < d^{-1/2}$ and $\Delta \leqslant \sqrt{d}$, we have $\delta_{k_0} > \alpha\Delta$ even when $k_0 = 0$. For each $n \in \mathbb{N}$, we split the path γ_n into a collection $(\gamma_u^n,\ u \in \mathbb{T}^n)$ of subpaths, indexed by a plane tree $\mathbb{T}^n$ with root denoted by o, as follows. First, by the definition of Δ and k_0, we have $|\gamma_n[0,1]| \geqslant \delta_{k_0}$. Thus, there exists $s < t \in [0,1]$ such that $|\gamma_n(s) - \gamma_n(t)| = \delta_{k_0}$, and we let γ_o^n be the path $\theta \in [0,1] \mapsto \gamma_n((1-\theta)s + \theta t)$. Then, by induction, having constructed the paths indexed by $\partial\mathbb{T}_l^n = \{u \in \mathbb{T}^n : |u| = l\}$, we apply for each $u \in \partial\mathbb{T}_l^n$ the procedure (P_α) to the path γ_u^n, and denote by $\gamma_{u1}^n, \ldots, \gamma_{u\kappa_n(u)}^n$ the subpaths generated. The children of u in $\mathbb{T}^n$ are the nodes $u1, \ldots, u\kappa_n(u)$. By construction, the following holds:

- for every $u \in \mathbb{T}^n$, we have $|\gamma_u^n(0) - \gamma_u^n(1)| = \delta_{k_0+|u|}$,
- for any nodes u, v that are not descendants of one another in $\mathbb{T}^n$, we have

$$d(\gamma_u^n[0,1]; \gamma_v^n[0,1]) \geqslant \alpha\delta_{k_0+|u\wedge v|+1},$$

  where $u \wedge v$ denotes the lowest common ancestor of u and v.

Now, set $\pi_n(u) = \prod_{v \prec u} \kappa_n(u)^{-1}$ for all $u \in \mathbb{T}^n$, and let $\mu_l^n = \sum_{u \in \partial \mathbb{T}_l^n} \pi_n(u) \cdot (\gamma_u^n)_* \lambda$ for all $l \in \mathbb{N}$, where $(\gamma_u^n)_* \lambda$ denotes the push forward by γ_u^n of the Lebesgue measure on $[0,1]$. By construction, the measure μ_l^n is a probability supported on $\gamma_n[0,1]$, since $\sum_{u \in \partial \mathbb{T}_l^n} \pi_n(u) = 1$ (this is easily checked by induction).

Let $\varepsilon_l := \alpha \delta_{k_0+l+1} = \alpha^{l+2} \cdot \delta_{k_0}$ for all $l \in \mathbb{N}$, and note that $\varepsilon_l \geqslant \alpha^{l+3} \Delta$. For every $L \in \mathbb{N}$, we have

$$\begin{aligned}
\int\int \frac{\mathrm{d}\mu_L^n(y)}{(\varepsilon_L \vee |x-y|)^s} \mathrm{d}\mu_L^n(x) &= \sum_{u,v \in \partial \mathbb{T}_L^n} \int_{\gamma_u^n[0,1]} \int_{\gamma_v^n[0,1]} \frac{\mathrm{d}\mu_L^n(y)}{(\varepsilon_L \vee |x-y|)^s} \mathrm{d}\mu_L^n(x) \\
&\leqslant \sum_{u,v \in \partial \mathbb{T}_L^n} \varepsilon_{|u \wedge v|}^{-s} \cdot \pi_n(u) \cdot \pi_n(v) \\
&= \sum_{l=0}^{L} \varepsilon_l^{-s} \cdot \sum_{\substack{u,v \in \partial \mathbb{T}_L^n \\ |u \wedge v| = l}} \pi_n(u) \cdot \pi_n(v) \\
&\leqslant \sum_{l=0}^{L} \varepsilon_l^{-s} \cdot \sum_{t \in \partial \mathbb{T}_l^n} \sum_{\substack{u,v \in \partial \mathbb{T}_L^n \\ u,v \succeq t}} \pi_n(u) \cdot \pi_n(v) \qquad (5) \\
&= \sum_{l=0}^{L} \varepsilon_l^{-s} \cdot \sum_{t \in \partial \mathbb{T}_l^n} \pi_n(t)^2 \\
&\leqslant \sum_{l=0}^{L} \varepsilon_l^{-s} \cdot \max_{t \in \partial \mathbb{T}_l^n} \pi_n(t) \cdot \sum_{t \in \partial \mathbb{T}_l^n} \pi_n(t) \\
&\leqslant \left(\alpha^3 \Delta\right)^{-s} \cdot \sum_{l=0}^{L} \max_{t \in \partial \mathbb{T}_l^n} \pi_n(t) \cdot \alpha^{-sl}.
\end{aligned}$$

Recall that we chose α small enough so as to have $\beta < \alpha$. Thus, we can fix a constant $s > 1$ such that $\beta < \alpha^s$. We claim now that for almost every realization of $\mathcal{F}$, it is possible to choose $L = L_n(\omega)$, with $L_n \to \infty$ as $n \to \infty$, so that

$$\overline{\lim_{n \to \infty}} \sum_{l=0}^{L_n} \max_{t \in \partial \mathbb{T}_l^n} \pi_n(t) \cdot \alpha^{-sl} < \infty. \qquad (6)$$

It is here that the probabilistic machinery comes into play, through the fact that $\mathcal{F}$ satisfies Property $(\varnothing)$. We will introduce a family $(E_m)_{m \in \mathbb{N}^*}$ of events, with $\sum_{m \geqslant 1} \mathbb{P}(E_m) < \infty$, such that the event E_{k_0+l} holds whenever there exists a node $u \in \partial \mathbb{T}_l^n$ with $\pi_n(u) > \beta^l$. The Borel-Cantelli Lemma will imply that almost surely this cannot happen for l large enough, and in turn prove (6).

To get there, let $l \in \mathbb{N}^*$ and suppose that there exists $u \in \mathbb{T}_l^n$ such that $\pi_n(u) > \beta^l$. Denoting by $u_0, \ldots, u_l$ the geodesic from the root to u in $\mathbb{T}^n$, this can be reformulated as $\prod_{0 \leqslant k < l} \kappa_n(u_k) < (1/\beta)^l$. By the definition of β given in (2) as the inverse of the geometric mean of the two lower bounds for $\kappa_n(\cdot)$ obtained in Proposition 10, we see that there must exist a number $j > l/2$ of indices $l_1 < \ldots < l_j \in [\![0, l[\![$ such that, for each $i \in [\![1, j]\!]$, the path $\gamma_{u_{l_i}}^n$ does not deviate of a factor $\rho = \sqrt{18\alpha}$ from being a straight line. In particular, there exists $\sigma_1 \leqslant \ldots \leqslant \sigma_j < \tau_j \leqslant \ldots \leqslant \tau_1 \in [0,1]$ such that, for every $i \in [\![1, j]\!]$:

$$|\gamma_n(\sigma_i) - \gamma_n(\tau_i)| = \delta_{k_0+l_i} \quad \text{and} \quad \gamma_n[\sigma_i, \tau_i] \subset S(\gamma_n(\sigma_i), \gamma_n(\tau_i); \rho \delta_{k_0+l_i}).$$

Now, writing $m_i = k_0 + l_i$ for all $i \in [\![1, j]\!]$, let us discretise this information.

Discretisation step. For each $m \in \mathbb{N}$, let $\left(\overline{B}(z, \rho\delta_m)\right)_{z \in \mathcal{Z}_m}$ be a covering of $[0,1]^d$ by balls of radius $\rho\delta_m$, with centres $z \in [0,1]^d$ more than $\rho\delta_m$ apart so that the $\left(\overline{B}(z, \rho\delta_m/2)\right)_{z \in \mathcal{Z}_m}$ are disjoint. For each $i \in [\![1, j]\!]$, we can find $x_i, y_i \in \mathcal{Z}_{m_i}$ such that $\gamma_n(\sigma_i) \in \overline{B}(x_i, \rho\delta_{m_i})$ and $\gamma_n(\tau_i) \in \overline{B}(y_i, \rho\delta_{m_i})$, and we have

$\gamma_n[\sigma_i, \tau_i] \subset S(x_i, y_i; 2\rho\delta_{m_i})$. Discretising further, let us place a number $H \in \mathbb{N}^*$ (to be adjusted soon) of points

$$\left(z_h^i = \left(1 - \frac{h}{H}\right) \cdot x_i + \frac{h}{H} \cdot y_i,\ h \in [\![0, H]\!] \right),$$

spread evenly on the line segment $[x_i, y_i]$. By construction, the path γ_n must meet each one of the balls $\left(\overline{B}\left(z_h^i, 2\rho\delta_{m_i}\right);\ h \in [\![0, H]\!]\right)$. Now, since $\gamma_n[0,1] \subset (\mathcal{F})_{1/(n+1)}$, a similar statement holds for $\mathcal{F}$: namely, for all n sufficiently large so that $1/(n+1) \leqslant \rho\delta_{m_j}$, the set $\mathcal{F}$ must meet each one of the balls $\left(\overline{B}\left(z_h^i, 3\rho\delta_{m_i}\right);\ i \in [\![1, j]\!], h \in [\![0, H]\!]\right)$; i.e, the intersection event

$$A^{m_1,\dots,m_j}_{x_1,y_1;\dots;x_j,y_j}\text{: ``for each } i \in [\![1, j]\!] \text{ and every } h \in [\![0, H]\!], \text{ the set } \mathcal{F} \text{ meets } \overline{B}\left(z_h^i, 3\rho\delta_{m_i}\right)\text{''}$$

must be realized. Here, the sequence $x_1, y_1; \dots; x_j, y_j$ has the following properties:

- for every $i \in [\![1, j]\!]$, we have $x_i, y_i \in \mathcal{Z}_{m_i}$, with $(1 - 2\rho)\delta_{m_i} \leqslant |x_i - y_i| \leqslant (1 + 2\rho)\delta_{m_i}$,
- for every $i \in [\![1, j[\![$, we have $x_{i+1}, y_{i+1} \in S(x_i, y_i; 2\rho\delta_{m_i})$.

We shall call any sequence satisfying those two properties *admissible with respect to* $m_1, \dots, m_j$.

Summing up the previous reasoning, we have shown that, if there exists a node $u \in \partial\mathbb{T}_l^n$ such that $\pi_n(u) > \beta^l$, then for all n sufficiently large so that $1/(n+1) \leqslant \rho\delta_{k_0+l-1}$, there must exist a number $j > l/2$ of indices $l_1 < \dots < l_j \in [\![0, l[\![$, and a sequence $x_1, y_1; \dots; x_j, y_j$ which is admissible with respect to $k_0 + l_1, \dots, k_0 + l_j$, such that the event $A^{k_0+l_1,\dots,k_0+l_j}_{x_1,y_1;\dots;x_j,y_j}$ is realized. Let us now define, for all $m \in \mathbb{N}^*$, the event:

$$E_m = \bigcup_{m/3 \leqslant j \leqslant m} \ \bigcup_{m_1 < \dots < m_j \in [\![0, m[\![} \ \bigcup_{x_1, y_1; \dots; x_j, y_j \text{ admissible}} A^{m_1,\dots,m_j}_{x_1,y_1;\dots;x_j,y_j}.$$

If, for some $l \geqslant 2k_0$ (this ensures that $l/2 \geqslant (k_0 + l)/3$), small enough so that $\rho\delta_{k_0+l-1} \geqslant 1/(n+1)$, there exists a node $u \in \partial\mathbb{T}_l^n$ such that $\pi_n(u) > \beta^l$, then the event E_{k_0+l} must be realized. Now, let us show that it is possible to adjust α in order to have $\sum_{m \geqslant 1} \mathbb{P}(E_m) < \infty$.

Summability of the $\mathbb{P}(E_m)$. Let $m \in \mathbb{N}^*$. By a union bound, we have

$$\mathbb{P}(E_m) \leqslant \sum_{m/3 \leqslant j \leqslant m} \ \sum_{m_1 < \dots < m_j \in [\![0, m[\![} \ \sum_{x_1, y_1; \dots; x_j, y_j \text{ admissible}} \mathbb{P}\left(A^{m_1,\dots,m_j}_{x_1,y_1;\dots;x_j,y_j}\right). \tag{7}$$

Now, fix an integer j such that $m/3 \leqslant j \leqslant m$, fix indices $m_1 < \dots < m_j \in [\![0, m[\![$, and let $x_1, y_1; \dots, x_j, y_j$ be an admissible sequence with respect to $m_1, \dots, m_j$. We control the probability of the intersection event $A^{m_1,\dots,m_j}_{x_1,y_1;\dots;x_j,y_j}$ with the fact that $\mathcal{F}$ satisfies Property $(\varnothing)$. To that end, we extract a collection of ζ-separated balls from the $\left(\overline{B}\left(z_h^i, 3\rho\delta_{m_i}\right);\ i \in [\![1, j]\!], h \in [\![0, H]\!]\right)$. At this point, we choose $H = \lfloor (1 - 2\rho)/((6\zeta + 1)\rho) \rfloor$. With that choice, we have $|x_i - y_i|/H \geqslant (6\zeta + 1)\rho\delta_{m_i}$ for each $i \in [\![1, j]\!]$, hence for any $h_1 \neq h_2 \in [\![0, H]\!]$, the following holds:

$$\text{for every } z_1 \in \overline{B}\left(z_{h_1}^i, 3\zeta\rho\delta_{m_i}\right) \text{ and } z_2 \in \overline{B}\left(z_{h_2}^i, 3\zeta\rho\delta_{m_i}\right), \text{ we have } |z_1 - z_2| \geqslant \rho\delta_{m_i}. \tag{8}$$

In particular, the balls $\left(\overline{B}\left(z_h^j, 3\rho\delta_{m_j}\right);\ h \in [\![0, H]\!]\right)$ are ζ-separated: let us add them all to our collection. To continue, note that the dilated balls $\left(B\left(z_h^j, \zeta \cdot 3\rho\delta_{m_j}\right);\ h \in [\![0, H]\!]\right)$ are all included in the sausage $S\left(x_j, y_j; 3\zeta\rho\delta_{m_j}\right)$, which has diameter at most $(1 + (6\zeta + 2)\rho)\delta_{m_j}$ (since the diameter of $S(x, y; r)$ is $|x - y| + 2r$, and $|x_j - y_j| \leqslant (1 + 2\rho)\delta_{m_j}$). Therefore, assuming now that α is small enough

so as to have $(1+(6\zeta+2)\rho) < \rho/\alpha$: by (8), the sausage $S\left(x_j, y_j; 3\zeta\rho\delta_{m_j}\right)$ meets the dilated ball $\overline{B}\left(z_h^{j-1}, \zeta\cdot 3\rho\delta_{m_{j-1}}\right)$ for at most one $h_0 \in [\![0,H]\!]$. We add all the balls

$$\left(\overline{B}\left(z_h^{j-1}, 3\rho\delta_{m_{j-1}}\right);\ h\in[\![0,H]\!]\setminus\{h_0\}\right)$$

to our collection. We iterate this argument, noticing that, as the sausages $(S(x_i,y_i;3\zeta\rho\delta_{m_i});\ i\in[\![1,j]\!])$ are nested (without loss of generality, we may assume that α is small enough so as to have $2+3\zeta\alpha\leqslant 3\zeta$) we only have to worry about intersections with the previous sausage at each step. At the end of the construction, we obtain a collection of ζ-separated balls that $\mathcal{F}$ must meet on the event $A^{m_1,\dots,m_j}_{x_1,y_1;\dots;x_j,y_j}$, which has cardinality at least $(H+1)+(j-1)\cdot H \geqslant Hj$. Since $\mathcal{F}$ satisfies Property $(\varnothing)$, we deduce that

$$\mathbb{P}\left(A^{m_1,\dots,m_j}_{x_1,y_1;\dots;x_j,y_j}\right)\leqslant Q\cdot q^{Hj}.$$

Going back to the union bound (7), we get

$$\begin{aligned}\mathbb{P}(E_m) &\leqslant \sum_{m/3\leqslant j\leqslant m}\ \sum_{m_1<\dots<m_j\in[\![0,m[\![}\ \sum_{x_1,y_1;\dots;x_j,y_j\text{ admissible}} Q\cdot q^{Hj}\\ &= \sum_{m/3\leqslant j\leqslant m}\ \sum_{m_1<\dots<m_j\in[\![0,m[\![} \#\{\text{admissible sequences with respect to } m_1,\dots,m_j\}\cdot Q\cdot q^{Hj}.\end{aligned}$$

Now, given an integer j such that $m/3\leqslant j\leqslant m$, and indices $m_1<\dots<m_j\in[\![0,m[\![$, let us control the number of admissible sequences with respect to $m_1,\dots,m_j$. First, there exists a constant $C=C(d,\alpha)>0$ such that $\#\mathcal{Z}_{m_1}\leqslant C\cdot\delta_{m_1}^{-d}$, this because the balls $\left(\overline{B}\left(z,\rho\delta_{m_1}/2\right);\ z\in\mathcal{Z}_{m_1}\right)$ are disjoint and included in the $\rho\delta_{m_1}/2$-neighborood of $[0,1]^d$. Next, we claim that there exists a constant $c=c(d)$ such that for each $i\in[\![1,j[\![$,

$$\#\mathcal{Z}_{m_{i+1}}\cap S(x,y;2\rho\delta_{m_i})\leqslant\frac{c}{\rho}\cdot\left(\frac{\delta_{m_i}}{\delta_{m_{i+1}}}\right)^d\quad\text{for all } x,y\in\mathcal{Z}_{m_i}\text{ such that } |x-y|\leqslant(1+2\rho)\delta_{m_i}.$$

This is because the balls $\left(\overline{B}\left(z,\rho\delta_{m_{i+1}}/2\right);\ z\in\mathcal{Z}_{m_{i+1}}\cap S(x,y;2\rho\delta_{m_i})\right)$ are disjoint and included in the sausage $S\left(x,y;2\rho\delta_{m_i}+\rho\delta_{m_{i+1}}/2\right)$. Thus, we obtain that the number of admissible sequences with respect to $m_1,\dots,m_j$ is bounded from above by

$$\left(C\cdot\delta_{m_1}^{-d}\right)^2\cdot\prod_{i=1}^{j-1}\left(\frac{c}{\rho}\cdot\left(\frac{\delta_{m_i}}{\delta_{m_{i+1}}}\right)^d\right)^2 = C^2\cdot\frac{(c/\rho)^{2(j-1)}}{\delta_{m_j}^{2d}} = \frac{C^2}{(c/\rho)^2}\cdot\frac{(c/\rho)^{2j}}{\alpha^{2dm_j}} =: C'\cdot\frac{(c/\rho)^{2j}}{\alpha^{2dm_j}},$$

where the constant C' depends only on d and α. Plugging this inequality into the union bound, we find

$$\begin{aligned}\mathbb{P}(E_m) &\leqslant \sum_{m/3\leqslant j\leqslant m}\ \sum_{m_1<\dots<m_j\in[\![0,m[\![} C'\cdot\frac{(c/\rho)^{2j}}{\alpha^{2dm_j}}\cdot Q\cdot q^{Hj}\\ &\leqslant \sum_{m/3\leqslant j\leqslant m}\ \sum_{m_1<\dots<m_j\in[\![0,m[\![} C'\cdot\frac{(c/\rho)^{2m}\vee 1}{\alpha^{2dm}}\cdot Q\cdot q^{Hm/3}\\ &= C'\cdot Q\cdot\sum_{m/3\leqslant j\leqslant m}\binom{m}{j}\cdot\left(\frac{(c/\rho)^2\vee 1}{\alpha^{2d}}\cdot q^{H/3}\right)^m\\ &\leqslant C'\cdot Q\cdot 2^m\cdot\left(\frac{(c/\rho)^2\vee 1}{\alpha^{2d}}\cdot q^{H/3}\right)^m = C'\cdot Q\cdot\left(2\cdot\frac{(c/\rho)^2\vee 1}{\alpha^{2d}}\cdot q^{H/3}\right)^m.\end{aligned}$$

Recalling that $\rho=\sqrt{18\alpha}$ and $H=\lfloor(1-2\rho)/((6\zeta+1)\rho)\rfloor$, a straightforward analysis shows that the $2\cdot\left((c/\rho)^2\vee 1\right)\cdot\alpha^{-2d}\cdot q^{H/3}$ term can be made strictly smaller than 1 by choosing α small enough. For such α, we get $\sum_{m\geqslant 1}\mathbb{P}(E_m)<\infty$.

Concluding the proof. By the Borel–Cantelli lemma, almost surely, the event E_m fails to be realized for all sufficiently large m. Therefore, to almost every realization of $\mathcal{F}$ corresponds some $l_0 \geqslant 2k_0$ such that E_m fails to be realized for all $m \geqslant k_0 + l_0$. Now, define $L_n = L_n(\omega)$ as the largest integer $l \in \mathbb{N}^*$ such that $\rho\delta_{k_0+l-1} \geqslant 1/(n+1)$ (note that L_n is well defined for all sufficiently large n, and that $L_n \to \infty$ as $n \to \infty$), and let $\nu_n = \mu^n_{L_n}$. Recalling (5), we have

$$\int\int \frac{\mathrm{d}\nu_n(y)}{(\varepsilon_{L_n} \vee |x-y|)^s}\mathrm{d}\nu_n(x) \leqslant \left(\alpha^3\Delta\right)^{-s} \cdot \sum_{l=0}^{L_n} \max_{u\in\partial\mathbb{T}^n_l} \pi_n(u)\cdot\alpha^{-sl}.$$

By all the above work, if for some $l \in [\![2k_0, L_n]\!]$, there exists a node $u \in \partial\mathbb{T}^n_l$ such that $\pi_n(u) > \beta^l$, then the event E_{k_0+l} must be realized. Now we can write, recalling that s was chosen such that $\beta < \alpha^s$:

$$\sum_{l=0}^{L_n} \max_{u\in\partial\mathbb{T}^n_l} \pi_n(u)\cdot\alpha^{-sl} \leqslant \sum_{0\leqslant l<l_0} \alpha^{-sl} + \sum_{l=l_0}^{L_n} \beta^l\cdot\alpha^{-sl} \leqslant \sum_{0\leqslant l<l_0} \alpha^{-sl} + \sum_{l\geqslant l_0} \left(\frac{\beta}{\alpha^s}\right)^l < \infty.$$

This proves that

$$\overline{\lim_{n\to\infty}} \int\int \frac{\mathrm{d}\nu_n(y)}{(\varepsilon_{L_n} \vee |x-y|)^s}\mathrm{d}\nu_n(x) < \infty.$$

Since we are working on the compact space $[0,1]^d$, the sequence of probability measures $(\nu_n)_{n\in\mathbb{N}}$ is automatically tight: let ν be any subsequential weak limit of $(\nu_n)_{n\in\mathbb{N}}$. For each $\varepsilon > 0$, by the Portmanteau theorem, we have $\nu((\mathcal{F})_\varepsilon) \geqslant \overline{\lim}_{n\to\infty} \nu_n((\mathcal{F})_\varepsilon) = 1$ (the last equality holds because the support of ν_n is included in $(\mathcal{F})_\varepsilon$ for all sufficiently large n thanks to (4)). Since $\mathcal{F}$ is closed, we deduce that the probability measure ν is supported on $\mathcal{F}$. Furthermore, since ν is the weak limit of some subsequence $(\nu_{n_k})_{k\in\mathbb{N}}$, we have

$$\begin{aligned}
\int\int \frac{\mathrm{d}\nu(y)}{(\varepsilon \vee |x-y|)^s}\mathrm{d}\nu(x) &= \lim_{k\to\infty} \int\int \frac{\mathrm{d}\nu_{n_k}(y)}{(\varepsilon \vee |x-y|)^s}\mathrm{d}\nu_{n_k}(x) \\
&\leqslant \overline{\lim_{n\to\infty}} \int\int \frac{\mathrm{d}\nu_n(y)}{(\varepsilon \vee |x-y|)^s}\mathrm{d}\nu_n(x) \\
&\leqslant \overline{\lim_{n\to\infty}} \int\int \frac{\mathrm{d}\nu_n(y)}{(\varepsilon_{L_n} \vee |x-y|)^s}\mathrm{d}\nu_n(x) < \infty.
\end{aligned}$$

This upper bound does not depend on ε, and thus letting $\varepsilon \to 0^+$, we conclude by the monotone convergence theorem that the integrability condition (3) holds, completing the proof of Theorem 7. □

2.2 Lower bound for the Hausdorff dimension of $\mathcal{F}_\infty$

The upper bound for the Hausdorff dimension of $\mathcal{F}_\infty$ stated in Theorem 1 was established in Proposition 5, we now come to the lower bound. First, we check that $\mathcal{F}_\infty$ satisfies Property $(\varnothing)$ and almost surely has a non-trivial connected component. With Theorem 7, this directly yields the result in dimension $d = 2$, which bootstraps to any dimension with a slicing lemma.

Proposition 11. The frontier $\mathcal{F}_\infty$ satisfies Property $(\varnothing)$: there exists constants $Q > 0$ and $q \in]0,1[$ such that, for every collection $\left(\overline{B}(x_i, r_i);\ i \in [\![1,n]\!]\right)$ of 7-separated balls, we have

$$\mathbb{P}\left(\text{for each } i \in [\![1,n]\!], \text{ the frontier } \mathcal{F}_\infty \text{ meets } \overline{B}(x_i,r_i)\right) \leqslant Q\cdot q^n.$$

Proof. This result is a consequence of Lemma 2. Let $\left(\overline{B}(x_i, r_i);\ i \in [\![1,n]\!]\right)$ be a collection of 7-separated balls, and fix a realization of the intersection event: "for each $i \in [\![1,n]\!]$, the frontier $\mathcal{F}_\infty$ meets $\overline{B}(x_i,r_i)$".

Notice that the initial points R_0 and B_0 belong to at most two distinct balls, with indices say i_R and i_B. For every other $i \neq i_R, i_B$, both R_0 and B_0 lie outside $\overline{B}(x_i, 7r_i)$, hence the event $G_i := G^{x_i}_{7r_i/6,7r_i}$ of Lemma 2 fails to be realized. Indeed, if G_i were realized, then the ball $\overline{B}(x_i, 7r_i/6)$ would be monochromatic at the end of the coloring, hence $\mathcal{F}_\infty$ would not meet $\overline{B}(x_i, r_i)$. Therefore, we have

$$\begin{aligned}\big(\text{for each } i \in [\![1,n]\!], \text{ the frontier } \mathcal{F}_\infty \text{ meets } \overline{B}(x_i,r_i)\big)& \\ \subset \big(\text{for each } i \in [\![1,n]\!] \setminus \{i_R, i_B\}, \text{ the event } G_i \text{ fails to be realized}\big)&.\end{aligned}$$

The events $(G_i)_{i\in[\![1,n]\!]}$ are independent and have probability at least $p > 0$, so we conclude that

$$\mathbb{P}\left(\text{for each } i \in [\![1,n]\!], \text{ the frontier } \mathcal{F}_\infty \text{ meets } \overline{B}(x_i,r_i)\right) \leqslant (1-p)^{n-2}.$$

□

Before proving that $\mathcal{F}_\infty$ contains a non trivial connected component, we first consider the following proposition.

Proposition 12. Almost surely, as $n \to \infty$, the discrete frontier $\mathcal{F}_n = \{x \in [0,1]^d : d(x, \mathcal{R}_n) = d(x, \mathcal{B}_n)\}$ converges towards $\mathcal{F}_\infty$ for the Hausdorff distance.

Proof. By the definition of $\mathcal{R}_\infty$ and $\mathcal{B}_\infty$, as $n \to \infty$ we have $\mathcal{R}_n \to \mathcal{R}_\infty$ and $\mathcal{B}_n \to \mathcal{B}_\infty$ for the Hausdorff distance. Now, fix $\varepsilon > 0$, and let us prove that the inclusions $\mathcal{F}_n \subset (\mathcal{F}_\infty)_\varepsilon$ and $\mathcal{F}_\infty \subset (\mathcal{F}_n)_\varepsilon$ hold for all sufficiently large n.

- For all sufficiently large n, the following holds: for each $x \in [0,1]^d$, the ball $\overline{B}(x,\varepsilon)$ contains an element of $\{R_0, B_0, X_1, \dots, X_n\} = \mathcal{R}_n \cup \mathcal{B}_n$. Then, for each $x \in \mathcal{F}_n$, as $d(x, \mathcal{R}_n) = d(x, \mathcal{B}_n)$, the ball $\overline{B}(x,\varepsilon)$ must contain an element of $\mathcal{R}_n$ and an element of $\mathcal{B}_n$. Since $\mathcal{R}_\infty \cap \overline{B}(x,\varepsilon)$ and $\mathcal{B}_\infty \cap \overline{B}(x,\varepsilon)$ are two non-empty closed subsets whose union forms the connected set $\overline{B}(x,\varepsilon)$, they cannot be disjoint; hence $d(x, \mathcal{F}_\infty) \leqslant \varepsilon$, i.e. $\mathcal{F}_n \subset (\mathcal{F}_\infty)_\varepsilon$.
- Conversely, by the convergence of $\mathcal{R}_n$ and $\mathcal{B}_n$ towards $\mathcal{R}_\infty$ and $\mathcal{B}_\infty$, we have $\sup_{x\in\mathcal{R}_\infty} d(x, \mathcal{R}_n) \leqslant \varepsilon$ and $\sup_{x\in\mathcal{B}_\infty} d(x, \mathcal{B}_n) \leqslant \varepsilon$ for all sufficiently large n. Thus, for every $x \in \mathcal{F}_\infty = \mathcal{R}_\infty \cap \mathcal{B}_\infty$, the ball $\overline{B}(x,\varepsilon)$ contains an element of $\mathcal{R}_n$ and an element of $\mathcal{B}_n$. Since $\{y \in \overline{B}(x,\varepsilon) : d(y, \mathcal{R}_n) \leqslant d(y, \mathcal{B}_n)\}$ and $\{y \in \overline{B}(x,\varepsilon) : d(y, \mathcal{R}_n) \geqslant d(y, \mathcal{B}_n)\}$ are two non-empty closed subsets whose union forms the connected set $\overline{B}(x,\varepsilon)$, they cannot be disjoint; hence $d(x, \mathcal{F}_n) \leqslant \varepsilon$, i.e. $\mathcal{F}_\infty \subset (\mathcal{F}_n)_\varepsilon$.

□

Corollary 13. *If dimension $d = 2$, almost surely the frontier $\mathcal{F}_\infty$ contains a non-trivial connected component.*

Proof. Note that in dimension 2, for each $n \in \mathbb{N}$, the discrete frontier $\mathcal{F}_n$ is a finite union of curves, where each curve is composed of line segments belonging to the boundary of the Voronoi cells of $R_0, B_0, X_1, \dots, X_n$ (see Figure 8 below). By virtue of Lemma 2, almost surely there exists some $r > 0$ such that the balls $\overline{B}(R_0, r)$ and $\overline{B}(B_0, r)$ are monochromatic at the end of the coloring (see Remark 3). In particular, this implies that the union of the red (resp. blue) cells in the Voronoi diagram of the points $R_0, B_0, X_1, \dots, X_n$ contains the ball $\overline{B}(R_0, r/2)$ (resp. $\overline{B}(B_0, r/2)$). Therefore, the discrete frontier $\mathcal{F}_n$ contains a curve, i.e. the image of a continuous path $\gamma_n : [0,1] \to [0,1]^d$, of diameter at least r (Figure 8 does not lie). Finally, recall that for the Hausdorff distance:

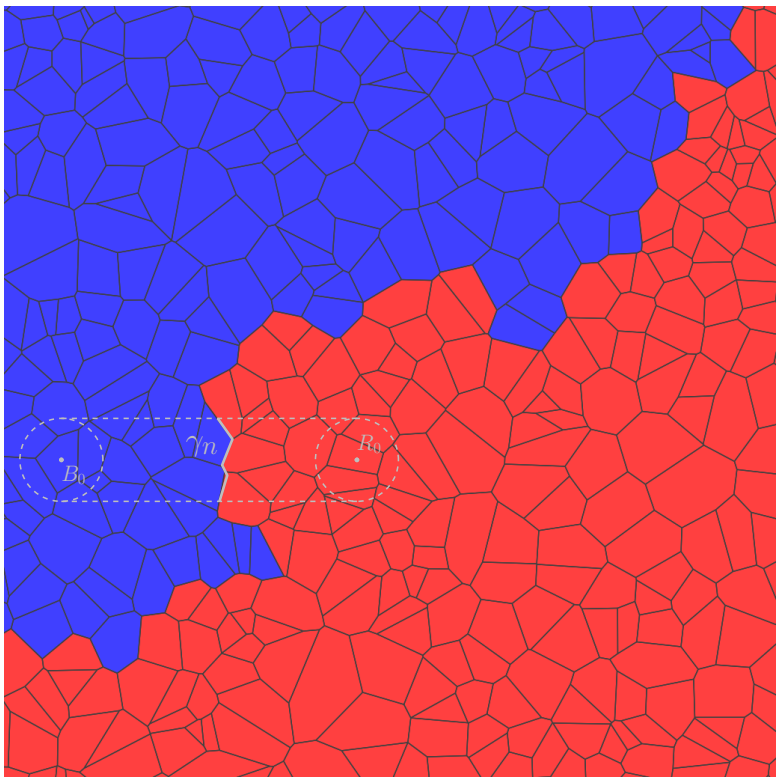

Figure 8: Colored Voronoi diagram of the points $R_0, B_0, X_1, \dots, X_n$. Balls with radius $r/2$ are represented around R_0 and B_0. The path γ_n is also represented, and it has diameter at least r.

(i) the set of closed subsets of $[0,1]^d$ is compact,

(ii) any limit of a sequence of connected closed subsets is connected.

Thereofore, we can extract from $(\gamma_n[0,1])_{n\in\mathbb{N}}$ a subsequence which converges to some limit $\mathcal{C}$, that is necessarily connected and has diameter at least r. Finally, using that $\mathcal{F}_n$ converges to $\mathcal{F}_\infty$, the fact that $\gamma_n[0,1] \subset \mathcal{F}_n$ implies that $\mathcal{C}$ is included in $\mathcal{F}_\infty$. □

Proposition 11 and Corollary 13 show that in dimension $d = 2$, the frontier $\mathcal{F}_\infty$ fulfils the hypotheses of Theorem 7. Thereore, there exists a constant $s > 1$ such that, almost surely,

$$\dim_H \mathcal{F}_\infty \geqslant s > 1.$$

This gives the lower bound stated in Theorem 1 when $d = 2$. Finally, the lower bound in higher dimensions is deduced from the two-dimensional case using a slicing argument, as explained in the next proposition.

Proposition 14. For any dimension $d \geqslant 2$, there exists $\varepsilon > 0$ such that, almost surely,

$$\dim_H \mathcal{F}_\infty \geqslant d - 1 + \varepsilon > d - 1.$$

Proof. Fix a plane $\mathcal{P} \subset \mathbb{R}^d$ that contains the red and blue seeds R_0 and B_0, and denote by $\mathcal{P}^\perp$ its orthogonal. By the slicing theorem [3, Theorem 1.6.2], for $\varepsilon > 0$, we have

$$H^{d-1+\varepsilon}(\mathcal{F}_\infty) \geqslant c_d \int_{\mathcal{P}^\perp} H^{1+\varepsilon}(\mathcal{F}_\infty \cap (z + \mathcal{P}))\mathrm{d}z,$$

where $H^t(A)$ denotes the s-dimensional Hausdorff measure of a subset $A \subset \mathbb{R}^d$. By Proposition 11, the random closed subset $\mathcal{F}_\infty$ satisfies Property $(\varnothing)$. By Theorem 7, there exists a constant $s > 1$ such that almost surely, for any closed subset $\mathcal{G} \subset \mathcal{F}$ with a non-trivial connected component, we have $\dim_H \mathcal{G} \geqslant s$ (see Remark 8). Moreover, by Lemma 2, almost surely there exists some $r > 0$ such that the balls $\overline{B}(R_0, r)$ and $\overline{B}(B_0, r)$ are monochromatic at the end of the coloring (see Remark 3). As in the proof of Corollary 13, this implies that for all $n \in \mathbb{N}$, the union of the red (resp. blue) cells in the Voronoi diagram of the points $R_0, B_0, X_1, \dots, X_n$ contains the ball $\overline{B}(R_0, r/2)$ (resp. $\overline{B}(B_0, r/2)$). Hence, for every $|z| \leqslant r/4$, the intersection between this union of red (resp. blue) cells and the affine plane $(z + \mathcal{P})$ contains a two-dimensional ball of radius $r/4$. Now, using the exact same arguments as in the proof of Proposition 13, we deduce that for every $|z| \leqslant r/4$, the closed subset $\mathcal{F}_\infty \cap (z + \mathcal{P}) \subset \mathcal{F}_\infty$ has a non-trivial connected component. Therefore, almost surely we have $\dim_H \mathcal{F}_\infty \cap (z + \mathcal{P}) \geqslant s$ for every $|z| \leqslant r/4$. In particular, choosing $\varepsilon > 0$ such that $1 + \varepsilon < s$, by the definition of the Hausdorff dimension we have $H^{1+\varepsilon}(\mathcal{F}_\infty \cap (z + \mathcal{P})) = \infty$ for all $|z| \leqslant r/4$, and it follows that

$$H^{d-1+\varepsilon}(\mathcal{F}_\infty) \geqslant \int_{\mathcal{P}^\perp \cap \overline{B}(0, r/4)} H^{1+\varepsilon}(\mathcal{F}_\infty \cap (z + \mathcal{P}))\, \mathrm{d}z = \infty.$$

This achieves to show that $\dim_H \mathcal{F}_\infty \geqslant d - 1 + \varepsilon$. □

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