Numerical bounds on the Crouzeix ratio for a class of matrices

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Abstract

We provide numerical bounds on the Crouzeix ratio for KMS matrices A which have a line segment on the boundary of the numerical range. The Crouzeix ratio is the supremum over all polynomials p of the spectral norm of $p(A)$ divided by the maximum absolute value of p on the numerical range of A. Our bounds confirm the conjecture that this ratio is less than or equal to 2. We also give a precise description of these numerical ranges.

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Keywords : numerical range, spectral set

1 Introduction

We consider n by n matrices A_n with 1's in the strict upper triangle and 0's elsewhere. For $n = 3, 4, 5, 6$, we have numerically determined upper and lower bounds on the value

$$\psi(A_n) := \sup\{\|p(A_n)\| : p \text{ a polynomial with } |p| \leq 1 \text{ in } W(A_n)\},$$

where $W(A_n)$ denotes the numerical range of A_n and $\|\cdot\|$ is the spectral norm in $\mathbf{C}^{n,n}$. (We refer to this quantity as the *Crouzeix ratio* although sometimes this term denotes the reciprocal, $\max_{z\in W(A_n)} |p(z)|/\|p(A_n)\|$, for a given polynomial p [10].) Crouzeix's conjecture is that for all square matrices A, $\psi(A) \leq 2$. A way to determine $\psi(A)$ is to introduce a Riemann mapping g from the interior of $W(A)$ onto the open unit disk $\mathbb{D}$ and to consider the matrix $M := g(A)$. In this case, we can write

$$\psi(A) = \psi_{\mathbb{D}}(M) := \max\{\|f(M)\| : f \text{ holomorphic in } \mathbb{D} \text{ with } |f| \leq 1 \text{ in } \mathbb{D}\}.$$

We know that the maximum is realized by a Blaschke product of order $n-1$ and each choice of such a Blaschke product b provides a lower bound: $\psi(A) = \psi_{\mathbb{D}}(M) \geq \|b(M)\|$. From the von Neumann inequality, we easily deduce

$$\psi_{\mathbb{D}}(M) \leq \psi_{cb,\mathbb{D}}(M) := \min\{\text{cond}(H) : H \in \mathbb{C}^{n,n},\ \|H^{-1}MH\| \leq 1\}.$$

Thus, the exhibition of a matrix H satisfying $\|H^{-1}MH\| \leq 1$ leads to an upper bound $\psi(A) \leq \text{cond}(H) := \|H\| \cdot \|H^{-1}\|$. This approach has been used to prove $\psi(A) \leq 2$ for some classes of matrices [4, 2, 3, 6], but this assumes that one knows the matrix M with sufficient accuracy. This is the case when $W(A)$ is an ellipse since there is an analytic formula for g and also for [2, 3]

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where $g(A) = cA$ for a certain constant c. In general, however, there is no simple expression for the boundary of $W(A)$ and for the Riemann mapping g. There are numerical methods for computing g and thus M with high precision, but to *guarantee* the accuracy would require a complete analysis of all discretization and rounding errors.

In section 2, we consider the matrix

$$A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Our numerical and analytical work suggests that $1.9956978 < \psi(A_3) < 1.9956979$, and we are confident in this range, although it does not provide a proof that $\psi(A_3) \le 2$, since it relies on numerical computation of $g(A)$.

Another approach is to identify a rational function f_1 such that the image $\Omega := f_1(\mathbb{D})$ of the unit disk is a subset of (and close to) $W(A_3)$. Let g_1 be the inverse of f_1. Then with $M_1 = g_1(A_3)$, we can write

$$\psi_\Omega(A_3) := \sup\{\|h(A_3)\| : |h| \le 1 \text{ in } \Omega\} = \psi_{\mathbb{D}}(M_1) := \sup\{\|f(M_1)\| : |f| \le 1 \text{ in } \mathbb{D}\}.$$

Since $\Omega \subset W(A_3)$, we clearly have $\psi(A_3) \le \psi_\Omega(A_3) \le \psi_{cb,\mathbb{D}}(M_1)$. We are now able to compute M_1 analytically and exhibit a matrix H_1 such that $\|H_1^{-1}M_1H_1\| = 1$ and $\operatorname{cond}(H_1) \approx 1.9999514$. However, we must verify numerically that $\Omega \subset W(A_3)$.

In section 3, we consider more generally the matrices A_n for $n > 3$. They belong to the class of KMS matrices [8] and they are the matrices in this class for which the boundary of the numerical range contains a line segment [7]. We derive a simple description of their numerical ranges and determine numerically the following bounds:

$$1.993800 \le \psi(A_4) \le 1.993801, \quad 1.992921 \le \psi(A_5) \le 1.992922, \quad 1.992444 \le \psi(A_6) \le 1.992445.$$

In section 4, we explain the numerical method used to compute the conformal mapping and we provide the Matlab code used for the computation of $M = g(A)$.

2 Numerical estimates for the matrix A_3

Here we consider the matrix $A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. We will see in the next section that the boundary of its numerical range is the union of a part of an algebraic curve $\left\{ \frac{2\,e^{i\theta}+e^{2i\theta}}{3} \ : -\frac{2\pi}{3} \le \theta \le \frac{2\pi}{3} \right\}$ and of the vertical straight line $[-\frac{1}{2} - i\frac{\sqrt{3}}{6}, -\frac{1}{2} + i\frac{\sqrt{3}}{6}]$. The algebraic curve is a cardioid; its Cartesian equation is $27(x^2{+}y^2)^2 - 18(x^2{+}y^2) - 8x - 1 = 0$.

We denote by g the Riemann mapping from $W(A)$ onto the unit disk $\mathbb{D}$ that satisfies $g(0) = 0$ and $g'(0) > 0$. Then

$$M = g(A) = \begin{pmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{with } a = g'(0),\ b = g'(0) + \tfrac{1}{2}g''(0).$$

From the numerical computations it appears that $a \simeq 1.360374515$, $b \simeq 0.710915425$ with an accuracy that we empirically estimate better than 10^{-8}. Using the Blaschke product $f(z) =$

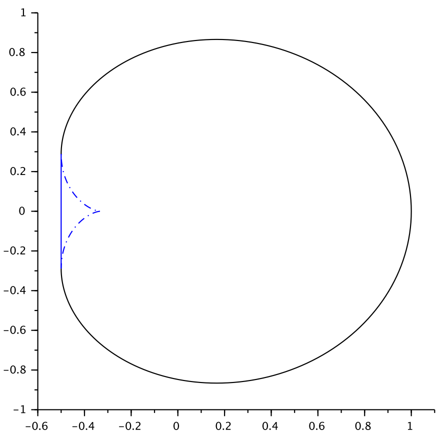

Figure 1: The boundary of the numerical range in black, the remaining part of the algebraic curve in dashed blue

$\dfrac{z+0.5470208}{1+0.5470208z}\,\dfrac{z-0.1465739}{1-0.1465739z}$, we obtain $\|f(M)\| = 1.9956978$, which (numerically) shows that $\psi(A) \geq 1.9956978$.

We now choose the matrix

$$H = \begin{pmatrix} a & b/2a & -b^2/8a^3 \\ 0 & 1 & -b/2a^2 \\ 0 & 0 & 1/a \end{pmatrix}, \quad \text{then} \quad H^{-1}MH = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore $\psi_{cb,\mathbb{D}}(M) \leq \|H\|\|H^{-1}\| = \mathrm{cond}(H) \simeq 1.995697855$.

With the numerical values obtained previously, we believe that we have the two sided estimates $1.9956978 < \psi_M(a,b) < 1.9956979$. Therefore, it appears from the numerical simulation that $W(A)$ is a (complete) 1.9956979-spectral set for A, that the complete bound [11] is the same as the ordinary bound, and that a function which realizes $\psi(A)$ is a Blaschke product of order 2 with 2 real roots. But we have only an empiric estimate of the accuracy that we justify as follows. If we let $a(n)$ be the numerical value of a computed with our program (described further) using n points on the boundary, we have verified that $33 \leq \frac{a(1447)-a(n)}{n^{-4}} \leq 65$ for values of n between 23 and 1205. This suggests that our method is of order n^{-4} and suggests that $|a(1205) - a| \leq 2\cdot 10^{-11}$. Similarly, it appears that our computation of b is of order n^{-4}, $130 \leq \frac{b(n)-b(1447)}{n^{-4}} \leq 350$ and that $|b(1205) - b| \leq 10^{-10}$.

We turn now to the second attempt which is to consider the image $\Omega = f_1(\mathbb{D})$ of the unit disk by the rational function $f_1(z) = (c_1 z + c_2 z^2 + \cdots + c_7 z^7)/(1 + d_1 z + \cdots + d_7 z^7)$, with the values

$$c = (0.734, 0.49736, 0.07268, -0.00521, 0.00013, 0.00061, -0.00251),$$
$$d = (0.32564, -0.03291, 0.01, -0.004, 0.00084, -0.00242, 0.00028).$$

We let g_1 be the inverse function of f_1 and we set $M_1 = g_1(A)$. We will see that if Ω is included in $W(A)$, then

$$\psi(A) \leq \psi_\Omega(A) \leq \psi_{cb,\mathbb{D}}(M_1) \leq \mathrm{cond}(H_1), \quad \text{if} \quad \|H_1^{-1}M_1H_1\| \leq 1.$$

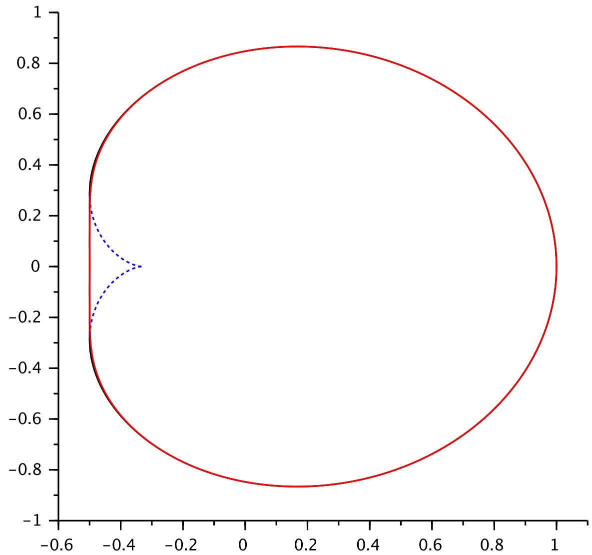

Figure 2: The boundary of Ω in red, of $W(A)$ in black.

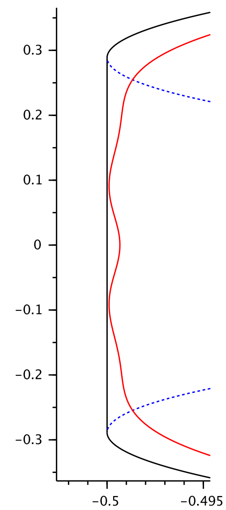

Figure 3: Zoom close to the straight line.

We are now able to compute $a_1 = g_1'(0)$, $b_1 = g_1'(0) + \frac{1}{2}g_1''(0)$, and thus M_1 with an accuracy better than 10^{-14}. We choose

$$H_1 = \begin{pmatrix} a_1 & b_1/2a_1 & -b_1^2/8a_1^3 \\ 0 & 1 & -b_1/2a_1^2 \\ 0 & 0 & 1/a_1 \end{pmatrix}, \quad \text{then} \quad H_1^{-1} M_1 H_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives an estimate $\psi(A) \le \psi_{cb,\mathbb{D}}(M_1) \le 1.9996222$ with an accuracy better than 10^{-12} which ensures that $W(A)$ is a 2-spectral set for A.

It remains to show that $W(A)$ contains Ω. For that, we first remark that the set $\{z : p(z) < 0\}$ with $p(z) := 27|z|^4 - 18|z|^2 - 8\,\mathrm{Re}\, z - 1 < 0$ is the interior of the cardioid and that the rectangle $\{z : -\frac{1}{2} \le \mathrm{Re}\, z \le 0 \text{ and } |\,\mathrm{Im}\, z| \le \sqrt{3}/6\}$ is contained in $W(A)$. Taking into account the symmetry with respect to the real axis, in order to show that Ω is contained in $W(A)$, it suffices to show that the set $\{z = f_1(e^{i\theta}) : 0 \le \theta \le \frac{3\pi}{4}\}$ is interior to the cardioid and that the set $\{z = f_1(e^{i\theta}) : \frac{3\pi}{4} \le \theta \le \pi\}$ is interior to the rectangle.

a) With $\theta_j = \frac{j\pi}{1000}$, $0 \le j \le 750$, we have computed $\max\limits_{0\le j\le 750} p(f_1(e^{i\theta_j})) = -0.0008777\ldots$ and $\max\limits_{0\le j\le 749} \Big|\dfrac{p(f_1(e^{i\theta_{j+1}})) - p(f_1(e^{i\theta_j}))}{\theta_{j+1}-\theta_j}\Big| = 0.0174\ldots$ This gives us, for $0 \le \theta \le 3\pi/4$, an estimate of $\max\big|\frac{d}{d\theta}p(f_1(e^{i\theta}))\big| \le 0.018$ and thus $\max p(f_1(e^{i\theta})) \le -0.0008777 + 0.018\pi/2000 < -0.000849$. This shows that the set $\{z = f_1(e^{i\theta}) : 0 \le \theta \le \frac{3\pi}{4}\}$ is interior to the cardioid, thus interior to $W(A)$.

b) We turn now to the part $\frac{3\pi}{4} \le \theta \le \pi$. We have computed $\min\limits_{750\le j\le 1000} \mathrm{Re}(f_1(e^{i\theta_j})) = -0.4998968$ and $\max\limits_{750\le j\le 1000} \Big|\dfrac{\mathrm{Re}(f_1(e^{i\theta_{j+1}})) - \mathrm{Re}(f_1(e^{i\theta_j}))}{\theta_{j+1}-\theta_j}\Big| = 0.01485\ldots$ This gives us, for $3\pi/4 \le$

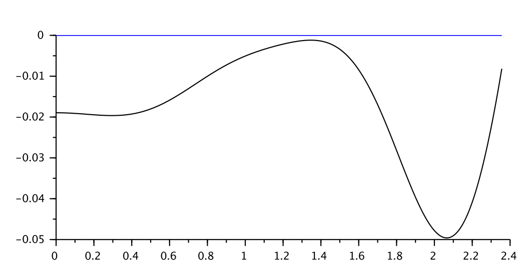

Figure 4: Curve $p(f_1(e^{i\theta}))$, $0 \leq \theta \leq \frac{3\pi}{4}$.

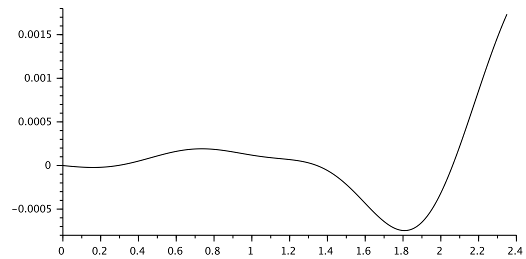

Figure 5: Curve $\frac{d}{d\theta}p(f_1(e^{i\theta}))$, $0 \leq \theta \leq \frac{3\pi}{4}$.

$\theta \leq \pi$, an estimate of $\max\left|\frac{d}{d\theta}p(f_1(e^{i\theta}))\right| \leq 0.015$ and thus $\min \mathrm{Re}(f_1(e^{i\theta})) \geq -0.4998968 - 0.015\pi/2000 > -0.499921$. Note also that this part of the curve clearly satisfies $\mathrm{Re}(f_1(e^{i\theta})) \leq 0$ and $\mathrm{Im}\, f_1(e^{i\theta})) \leq \sqrt{3}/6$. This shows that the set $\{z = f_1(e^{i\theta}) : \frac{3\pi}{4} \leq \theta \leq \pi\}$ is interior to $W(A)$.

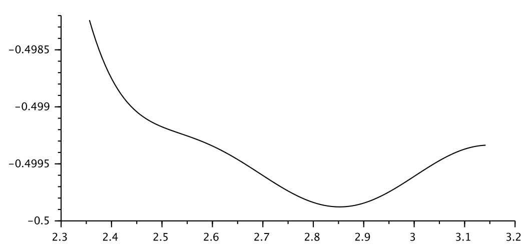

Figure 6: Curve $\mathrm{Re}(f_1(e^{i\theta})), \frac{3\pi}{4} \leq \theta \leq \pi$.

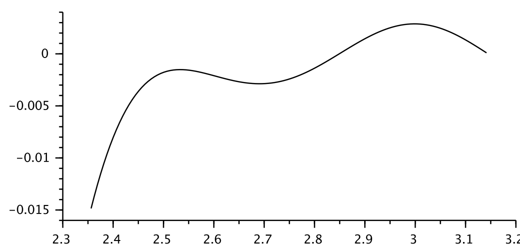

Figure 7: Curve $\frac{d}{d\theta}\,\mathrm{Re}(f_1(e^{i\theta}))$, $\frac{3\pi}{4} \leq \theta \leq \pi$.

3 Estimates for the class of matrices A_k

Recall [9] that the boundary of the numerical range $W(A)$ is the convex hull of the algebraic curve with tangential equation $T(u, v, w) := \det(uB+vC+wI) = 0$ where we have written $A = B+i\,C$, with B and C self-adjoint. For the matrix A_k, the corresponding tangential equation $T_k(u, v, w) = 0$, can be obtained from the recursion

$$T_1(u, v, w) = w, \quad T_{k+1}(u, v, w) = w\,T_k(u, v, w) + \frac{(w - \frac{u+iv}{2})^k - (w - \frac{u-iv}{2})^k}{i\,v}\,\frac{u^2+v^2}{4}.$$

For instance,

$$T_3(u, v, w) = w^3 - \frac{3}{4}w(u^2+v^2) + \frac{1}{4}(u^2+v^2)u,$$
$$T_4(u, v, w) = w^4 - \frac{3}{2}w^2(u^2+v^2) + w(u^2+v^2)u - \frac{1}{16}(u^2+v^2)(3u^2-v^2).$$

We can see, by recursion, that

$$T_k(\cos\varphi, \sin\varphi, w) = \frac{(-1)^k}{\sin\varphi}\,\mathrm{Im}\left(e^{-i\varphi}(\tfrac{1}{2}e^{i\varphi}-w)^k\right).$$

We now remark that, if $\varphi = \frac{k\theta}{2}$ and $w = -\frac{1}{2}\frac{\sin((k-1)\theta/2)}{\sin(\theta/2)}$, then $e^{-i\varphi/k}(\frac{1}{2}e^{i\varphi} - w) = \frac{1}{2}\frac{\sin(k\theta/2)}{\sin\theta/2} \in \mathbb{R}$, whence $T_k(\cos\varphi, \sin\varphi, w) = 0$. Now, we consider the algebraic curve $\{f_k(e^{i\theta}) : |\theta| \le \pi\}$ with $f_k(z) = (\sum_{j=1}^{k-1} j z^{k-j})/k = \frac{z^{k+1}-z-k(z^2-z)}{k(z-1)^2}$. We remark that

$$k\, e^{i\theta}\, f_k'(e^{i\theta}) = \sum_{j=1}^{k-1} j(k-j) e^{i(k-j)\theta} = \frac{e^{ik\theta/2}}{2} \sum_{j=1}^{k-1} j(k-j) \cos\tfrac{(k-2j)\theta}{2};$$

hence, the vector $e^{ik\theta/2}$ is the unit normal at the point $f_k(e^{i\theta})$ and the equation of the tangent at this point is $\mathrm{Re}\,\big((e^{-ik\theta/2}(x+iy-f(e^{i\theta})\big) = 0$, i.e. $ux+vy+w = 0$, with $u = \cos\frac{k\theta}{2}$, $v = \sin\frac{k\theta}{2}$, $w = -\,\mathrm{Re}\,\big(e^{-ik\theta/2} f(e^{i\theta})\big) = -\frac{1}{2}\sum_{j=1}^{k-1}\cos(\frac{k-2j}{2}\theta) = -\frac{1}{2}\frac{\sin((k-1)\theta/2)}{\sin(\theta/2)}$. Thus, this shows that the algebraic curve which satisfies the tangential equation $T_k(u, v, w) = 0$ is the set:[1]

$\{z : z = \frac{1}{k}\sum_{j=1}^{k-1} j\, e^{i(k-j)\theta}, \quad -\pi \le \theta \le \pi\}$.

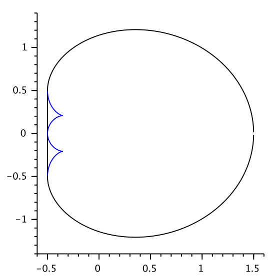

Figure 8: $W(A_4)$.

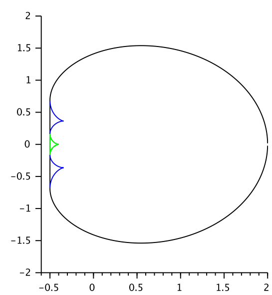

Figure 9: $W(A_5)$.

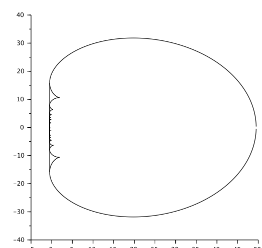

Figure 10: $W(A_{100})$.

The points with horizontal tangent are given by $\theta_j = \frac{(2j-1)\pi}{k}$, $j = 1, \ldots, n$ and are cuspid for $j = 2, \ldots, k-1$; the $k-1$ points of the algebraic curve on the flat part are the points $-\frac{1}{2} + \frac{i}{2\tan(j\pi/k)}$, $j = 1, \ldots, k-1$.

Note that the matrix $A_k + A_k^* + I$, whose entries are all 1's, has the simple eigenvalue k (eigenvector $(1, 1, \cdots, 1)^T$), and the eigenvalue 0 of multiplicity $k-1$ (with orthonormal eigenvectors $\frac{1}{\sqrt{k}}(e^{i\theta_j}, e^{2i\theta_j}, \ldots, e^{ik\theta_j})^T$, $\theta_j = \frac{2j\pi}{k}$, $j = 1 \ldots, k-1$). This implies that the point $\frac{k-1}{2} = f_k(1)$ is the extremal right point of $W(A_k)$, and that the boundary has a flat part on the line $\mathrm{Re}\, z = -\frac{1}{2}$. The boundary of the numerical range is the union of a part of the algebraic curve and of a straight part:

$$\partial W(A_k) = \{z : z = \tfrac{1}{k}\sum_{j=1}^{k-1} j\, e^{i(k-j)\theta}, |\theta| \le \tfrac{2\pi}{k}\} \cup \{z : z = -\tfrac{1}{2}(1+iy),\ -\cot(\tfrac{\pi}{k}) \le y \le \cot(\tfrac{\pi}{k})\}.$$

We turn now to some estimates for the constant corresponding to the matrices A_k, $k \le 6$. For A_4, we have computed the values $g'(0) = 1.1888506$, $g''(0) = -1.6292742$, $g'''(0) = 4.7085601$, which gives $g(A_4) = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}$, with $a = 1.1888506$, $b = 0.3742134$, $c = 0.3443362$.

[1]Thanks to Bernd Beckermann for a useful remark.

We have obtained a lower bound $\psi(A_4) \geq 1.9938003$ with the Blaschke product corresponding to the 3 coefficients $-0.4560323 \pm 0.3891911i$, 0.2474013. We have also an upper bound: with the values $x = b/a^{1.5}$, $y = az - bx/a + c/a^{1.5}$, $z = -0.0735033$, $t = -0.0231366$ and the matrix

$$H = \begin{pmatrix} a^{1.5} & xa & y & t \\ 0 & a^{0.5} & 0 & z \\ 0 & 0 & a^{-0.5} & -xa \\ 0 & 0 & 0 & a^{-1.5} \end{pmatrix}, \text{ it holds } H^{-1}g(A_4)H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, $\mathrm{cond}(H) \simeq 1.9938002$ and $\|H^{-1}g(A_4)H\| = 1$.

We can consider that the estimate $1.993800 \leq \psi(A_4) \leq 1.993801$ is correct.

For A_5, we have computed $g(A_5) = \begin{pmatrix} 0 & a & b & c & d \\ 0 & 0 & a & b & c \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, with $a = 1.1170233$, $b = 0.2325756$, $c = 0.2187502$, $d = 0.1895824$.

We have obtained a lower bound $\psi(A_5) \geq 1.9929216$ with the Blaschke product corresponding to the 4 coefficients $-0.2583004 \pm 0.60451151i$, -0.6247827, 0.3295365. We have also an upper bound: with the values $u = -0.0194597$, $w = -0.0384976$, $g = -0.1091772$, $h = -0.2503045$, $f = ah + ba^{-2}$, $v = ag + bh + ca^{-2}$, $z = af + ba^{-1}$, $t = aw + bg + ch + da^{-2}$, $y = av + bf + ca^{-1}$, $x = az + b$ and the matrix

$$H = \begin{pmatrix} a^2 & x & y & t & u \\ 0 & a & z & v & w \\ 0 & 0 & 1 & f & g \\ 0 & 0 & 0 & a^{-1} & h \\ 0 & 0 & 0 & 0 & a^{-2} \end{pmatrix}, \text{ it holds } H^{-1}g(A_5)H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, $\mathrm{cond}(H) \simeq 1.9929216$ and $\|H^{-1}g(A_5)H\| = 1$.

We can consider that the estimate $1.992921 \leq \psi(A_5) \leq 1.992922$ is correct.

For A_6, we have computed $g(A_6) = \begin{pmatrix} 0 & a & b & c & d & e \\ 0 & 0 & a & b & c & d \\ 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, with $a = 1.0798634$, $b = 0.1590093$, $c = 0.1519169$, $d = 0.1359021$ et $e = 0.1161184$.

We have obtained a lower bound $\psi(A_6) \geq 1.9924447$ with the Blaschke product corresponding to the 5 coefficients $-0.5859775 \pm 0.3199164i$, $-0.0604565 \pm 0.70030221i$, 0.3972632, 0.3295365. We have also a upper bound: with the values
$y = (-0.0163999, -0.0248879, -0.0578414, -0.1294105, -0.243031)$,
$x_9 = a\,y_5 + b\,a^{-2.5}$, $x_8 = a\,y_4 + b\,y_5 + c\,a^{-2.5}$, $x_7 = a\,x_9 + b\,a^{-1.5}$, $x_6 = a\,y_3 + b\,y_4 + c\,y_5 + d\,a^{-2.5}$,
$x_5 = a\,x_8 + b\,x_9 + c\,a^{-1.5}$, $x_4 = a\,x_7 + b\,a^{-0.5}$, $x_3 = a\,y_2 + b\,y_3 + c\,y_4 + d\,y_5 + e\,a^{-2.5}$,
$x_2 = a\,x_6 + b\,x_8 + c\,x_9 + d\,a^{-1.5}$, $x_1 = a\,x_5 + b\,x_7 + c\,a^{-0.5}$; $x_0 = a\,x_4 + b\,a^{0.5}$,
and the matrix

$$H=\begin{pmatrix} a^{2.5} & x_0 & x_1 & x_2 & x_3 & y_1 \\ 0 & a^{1.5} & x_4 & x_5 & x_6 & y_2 \\ 0 & 0 & a^{.5} & x_7 & x_8 & y_3 \\ 0 & 0 & 0 & a^{-.5} & x_9 & y_4 \\ 0 & 0 & 0 & 0 & a^{-1.5} & y_5 \\ 0 & 0 & 0 & 0 & 0 & a^{-2.5} \end{pmatrix}, \text{ it holds } H^{-1}g(A_6)H=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, $\mathrm{cond}(H) \simeq 1.9924445$ and $\|H^{-1}g(A_6)H\| = 1$.

We can consider that the estimate $1.992444 \leq \psi(A_6) \leq 1.992445$ is correct.

Remark. *In each of these cases, the matrix $M = g(A_k)$ satisfies the relation $\psi_{\mathbb{D}}(M) = \psi_{cb,\mathbb{D}}(M)$. This property holds for all $d \times d$ matrices M if $d \leq 2$, but may fail [5] if $d \geq 3$. Also here, for the matrix H which realizes $\psi_{cb,\mathbb{D}}(M)$, $H^{-1}MH$ was a Jordan block, which is not generally the case.*

Table 1 summarizes our results.

n	lower bound	upper bound	difference
3	1.9956978	1.9956979	10^{-7}
4	1.993800	1.993801	10^{-6}
5	1.992921	1.992922	10^{-6}
6	1.992444	1.992445	10^{-6}

Table 1: Upper and Lower Bounds on $\psi(A_n) = \psi_{cb,\mathbb{D}}(M_n)$.

4 About the computation of the conformal mapping g

We may write $g(z) = z\,\exp(u+iv)$, with $u(z)$ and $v(z)$ harmonic real-valued functions. Note that $u(z) = -\log|z|$ on $\partial W(A)$, which determines u in $W(A)$ in a unique way.

Let us consider a representation $\partial W(A) = \{\sigma(\theta)\,;\theta \in [0, 2\pi]\}$ of the boundary and choose $\lambda > 0$. If q is a 2π-periodic real-valued function such that,

$$\int_0^{2\pi} q(\theta)\log\Big|\frac{\sigma(\theta)-\sigma(\varphi)}{\lambda}\Big|\,d\theta = -\log|\sigma(\varphi)|, \qquad \text{for all } \varphi \in [0, 2\pi[,$$

then it holds $u(z) = \int_0^{2\pi} q(\theta)\log\big|\frac{\sigma(\theta)-z}{\lambda}\big|\,d\theta$ since this integral is clearly harmonic and is equal to $-\log|z|$ on $\partial W(A)$. It is known that such a q exists if and only if λ is different from the logarithmic capacity of $W(A)$. Generally we will use this equation with $\lambda = 1$ and then rewrite it as

$$\int_0^{2\pi} q(\theta)\log\Big|\frac{\sigma(\theta)-\sigma(\varphi)}{e^{i\theta}-e^{i\varphi}}\Big|\,d\theta + \int_0^{2\pi} q(\theta)\log|e^{i\theta}-e^{i\varphi}|\,d\theta = -\log|\sigma(\varphi)|, \quad \forall\,\varphi \in [0, 2\pi[.$$

We discretized this equation using a representation $\sigma(\theta)$ of $\partial W(A)$ and approximating $q(\cdot)$ by a trigonometric polynomial $q_n(\cdot)$ of degree n, and employing a collocation method at the points θ_j, $j = 0, 1, \ldots, 2n$ (it is known that an odd number of collocation points is necessary for such a method). So, we get an approximation $q_j = q_n(\theta_j)$ by solving the system

$$\frac{2\pi}{2n+1}\sum_{j=0}^{2n} q_j \log\Big|\frac{\sigma(\theta_j)-\sigma(\theta_i)}{e^{i\theta_j}-e^{i\theta_i}}\Big| + \int_0^{2\pi} q_n(\theta)\log|e^{i\theta}-e^{i\theta_i}|\,d\theta = -\log|\sigma(\theta_i)|,$$

$$\text{for } i = 0, 1, \ldots, 2n.$$

We have approximated the first integral by the trapezoidal formula; of course, if $j = i$, we have to replace $\log\left|\frac{\sigma(\theta_j)-\sigma(\theta_i)}{e^{i\theta_j}-e^{i\theta_i}}\right|$ by $\log|\sigma'(\theta_i)|$. Recall that, for the remaining integral, there holds

$$\int_0^{2\pi} q_n(\theta)\log|e^{i\theta}-e^{i\theta_i}|\,d\theta = -\frac{2\pi}{2n+1}\sum_{j=0}^{2n} c(j-i)\,q_j,$$

$$\text{with } c(k) = c(-k) = \sum_{j=1}^{n}\frac{\cos j\theta_k}{j}.$$

Then, we obtain the approximation of u from

$$u(z) \simeq \frac{2\pi}{2n+1}\sum_{j=0}^{2n} q_j \log|\sigma(\theta_j)-z|$$

and the approximation of the derivatives of g at 0 (note that here $v(x) = 0$ if $x \in \mathbb{R}$)

$$g'(0) = \exp(u(0)),\; g''(0) = 2\,g'(0)u'(0),\; g^{(3)}(0) = 3\,g'(0)(u'(0)^2{+}u''(0)),$$
$$g^{(4)}(0) = 4\;g'(0)(u'(0)^3{+}3u'(0)u''(0){+}u^{(3)}(0)),$$
$$g^{(5)}(0) = 5\;g'(0)(u'(0)^4{+}6u'(0)^2u''(0){+}4u'(0)u^{(3)}(0){+}3u''(0)^2{+}u^{(4)}(0)),$$

via the formulae

$$g'(0) \simeq \exp\Big(\frac{2\pi}{2n+1}\sum_{j=1}^{2n+1} q_j\log|\sigma(\theta_j)|\Big),\quad u^{(k)}(0) \simeq -(k{-}1)!\frac{2\pi}{2n+1}\,\mathrm{Re}\Big(\sum_{j=1}^{2n+1}\frac{q_j}{\sigma(\theta_j)^k})\Big).$$

Remark. *In order to get q, we have to solve a linear system of the form $Mq = b$. But it could appear that the matrix M is not invertible, or badly conditioned, if the logarithmic capacity of $W(A)$ is close to 1. In this case, we can replace this system by $(M{-}E)q = b{-}e$ where E (resp. e) is a matrix (resp. a vector) with all entries equal to 1.*

This method is very efficient for analytic boundary (exponential convergence with respect to n, see for instance [1]), but here we have singularities at the transition points between the straight line and the algebraic part, which reduces the order of convergence to $O(n^{-4})$ which is still good. With $\varphi_k(z) = \frac{1}{k}\sum_{j=1}^{k-1} j\, z^{i(k-j)}$, we have

$$\partial W(A_k) = \{z : z = \varphi_k(\theta), -\tfrac{2\pi}{k} \le \theta \le \tfrac{2\pi}{k}\} \cup \{z : z = -\tfrac{1}{2}(1{+}iy),\; -\cot(\tfrac{\pi}{k}) \le y \le \cot(\tfrac{\pi}{k})\}.$$

We have used $2n{+}1$ points on the algebraic part of $\partial W(A_k)$: $z_j = \varphi_k(\frac{2\pi j}{kn})$ for $j = -n, \ldots, n$ and $2n_2$ equidistant points on the straight part $z_{n+j} = z_n{-}jh\,i$, $j = 1, \ldots, 2n_2$ where $h = 2\cot(\frac{\pi}{k})/(2n_2{+}1)$ and n_2 is chosen such that h is as close as possible to $|z_n{-}z_{n-1}|$.

5 Program in Matlab for the computation of $g(A)$

```
function [gofA,gderivs,nn]= Akstudy(k,n);

% For 3 <= k <= 6, forms the kxk matrix A with ones in the
% strict upper triangle and zeros elsewhere, and computes
% its image gofA under the Riemann mapping from W(A) to the
% unit disk with g(0) = 0, g'(0) > 0.  gderivs(j), j=1,...,5,
```

```
% contains the value of the jth derivative of g at 0.
% W(A) consists of a cardioid and a vertical line segment,
% and 2n+1 points are used to represent the cardioid portion.
% Output argument nn is then the total number of discretization
% points used to represent the boundary of W(A). Note also
% that n may be modified (to make things come out even).

A = triu(ones(k),1); % Form the matrix.

% Discretize W(A). Use n+1 points on the upper part of the
% algebraic curve.
th = 2*[0:n]'*pi/n/k;
z = zeros(size(th));
for j=1:k-1, z = z + (k-j)*exp(1i*j*th)/k; end;

% Choose the same step size on the line segment.
h = abs(z(n+1)-z(n));
nn = fix(imag(z(n+1))/h - 0.5);
h = imag(z(n+1))/(nn+0.5);
for j=1:nn, z(n+1+j) = z(n+1) - 1i*j*h; end;

% Complete by symmetry.
nn = n+1+nn;
zz = conj(z);
z = [z(1:nn); zz(nn:-1:2)];

% Plot W(A).
plot([z; z(1)],'-k','LineWidth',2), axis equal, shg

% Compute the conformal mapping.
nn = length(z); n = (nn-1)/2;
zz = z;
e = [1:n]'; ee = 2*pi/nn * [1:n]';
c0 = sum(ones(size(e)) ./ e);
c = zeros(nn-1,1); d = zeros(nn-1,1);
for j=1:nn-1, c(j) = sum(cos(j*ee)./e); end;
for j=1:nn-1, d(j) = sum(2*sin(j*ee) .* e)/nn; end;
dd = [d; 0];
zzprim = zeros(nn,1);
for j=1:nn, dd = [dd(nn); dd(1:nn-1)]; zzprim(j) = sum(zz.*dd); end;
zzprim = abs(zzprim);

% Compute the matrix M such that Mq = -log |sigma|.
ee = exp(1i*2*pi/nn * [1:nn]');
M = zeros(nn,nn);
for j=1:nn,
  M(j,j) = log(zzprim(j)) - c0;
  for k=j+1:nn,
```

```
    M(k,j) = log(abs((zz(k)-zz(j))/(ee(k)-ee(j)))) - c(k-j);
    M(j,k) = M(k,j);
  end;
end;

% If t < 0, M is badly conditioned, so we translate M.
t = 10∧4-cond(M);
if t < 0, M = M - ones(M); t, pause, end;

% Compute q.
b0 = log(abs(zz));
q = -M\b0;

% Take account of the translation.
if t < 0, b0 = b0 - ones(b0); end;

% Compute derivatives of g at 0.
gp = exp(sum(q.*b0));
b = -real(sum(q./zz));
c = -real(sum(q./zz.∧2));
d = -2*real(sum(q./zz.∧3));
ed = -6*real(sum(q./zz.∧4));
gs = 2*gp*b; gt = 3*gp*(b∧2 + c);
gq = 4*gp*(3*b*c + b∧3 + d);
gc = 5*gp*(b∧4 + 6*b∧2*c + 4*b*d + 3*c∧2 + ed);
gderivs = [gp, gs, gt, gq, gc];
gofA = gp*A + 0.5*gs*A∧2 + (1/6)*gt*A∧3 + (1/24)*gq*A∧4 + (1/120)*gc*A∧5;
```

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