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Complement to Higher Bernstein Polynomials and Multiple Poles of $\frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho\omega \wedge \bar{\omega}'$

Daniel Barlet*

November 22, 2023

ABSTRACT. We give, using higher Bernstein polynomials defined in our paper [2], a stronger version of our previous result in [1] whose converse is proved in [2] and we give some complements to the results in [2] which help to compute these higher order Bernstein polynomials. Then we show some non trivial examples where we determine the root of the second Bernstein polynomial which is not a double root of the full Bernstein polynomial and where the main theorem of [2] applies and localizes where a double pole exists for the meromorphic extension of the (conjugate) analytic functional given by polar parts of $\omega' \mapsto |f|^{2\lambda} \bar{f}^{-h} \rho\omega \wedge \bar{\omega}'$ when $h \in \mathbb{N}$ is large enough.

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Contents

1	Introduction	2
2	Some useful results to complete [2]	3
3	A more precise result than [1]	5
4	Examples	7
5	Appendix: detailed computations	11
6	Bibliography	14

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1 Introduction

The first motivation for this “complement” to the paper [2] is to give some explicit examples of higher Bernstein polynomials in the case where they have roots which are not multiple roots of the (full) Bernstein polynomial of the fresco under consideration in *loc. cit.*, for instance in Theorem 8.5.1.

The first remark is that, in general, it is rather difficult to compute the Bernstein polynomial of the fresco associated to a given pair (f, ω) , even in the case where f has an isolated singularity. Nevertheless, in the case where f is a polynomial in $\mathbb{C}[x_0, \dots, x_n]$ having $(n + 2)$ monomials, we describe in the article [1] a rather elementary method to obtain an estimation for the Bernstein polynomial of the fresco $\mathcal{F}_{f, \omega}$ associated to a monomial $(n + 1)$ -form ω .

Of course, when the full Bernstein polynomial has a root of multiplicity $k \geq 2$ then this root is also a root of the j -th Bernstein polynomial for each $j \in [1, k]$ but when the Bernstein polynomial has only simple roots, the computation of the higher order Bernstein polynomials, even in the special situation of [1], is not obvious. We present here some examples where we show that the second Bernstein polynomial is not trivial in cases where the full Bernstein polynomial has no multiple root.

We concentrate here on the case of the function $f(x, y, z) = xy^3 + yz^3 + zx^3 + \lambda xyz$ where $\lambda \neq 0$ is any complex parameter, and where $\omega = \mu dx \wedge dy \wedge dz$ with μ a monomial in $\mathbb{C}[x, y, z]$. The tools used to estimate the Bernstein polynomial of the associated fresco $\mathcal{F}_{f, \omega}$ are valid for almost all polynomials in $\mathbb{C}[x_0, \dots, x_n]$ having $(n + 2)$ monomials, for any $n \geq 1$ (see the conditions $C1$ and $C2$ in [1]).

The tools used to determine the second Bernstein polynomial in the examples below are also easy to generalize, at least to obtain information on the smallest root of the Bernstein polynomial in the previous setting, knowing that this root is, in general, a root of the k -th Bernstein polynomial of the fresco $\mathcal{F}_{f, \omega}$ where k is the nilpotent order of this fresco.

It seems not difficult, for some colleague fun of computers, to make a program which produces in such a situation, not only estimates for the (full) Bernstein polynomial of $\mathcal{F}_{f, \omega}$ but also, computing also the polynomial in (a,b) annihilating $[\omega]$ in $\mathcal{F}_{f, \omega}$ (described in [1]) to estimate the smallest root of the Bernstein polynomial.

During the computations which lead to the examples presented in Section 4 I realized that some rather easy consequence of the Section 5 in [2] where missing to enlighten the relationship between asymptotic expansions, themes and higher Bernstein polynomials. These results are given in Section 2 and will be add to the second version of my paper [2] (in preparation).

Another point appears also during this period; the fact that the main result of [2] is a converse of a statement which is more precise than the main result in [1]. The reason is that, without the notion of higher Bernstein polynomial of a fresco, this stronger statement cannot be formulate ! So we add this improved version of the main theorem of [1] in Section 3 below.

2 Some useful results to complete [2]

We begin by recalling some facts from [2]. For the definition of a fresco see Definition 5.1.2 or Section 7.1 in [2].

REMINDER.

1. Any $[\alpha]$ -primitive fresco \mathcal{F} has an embedding in some $\Xi_\alpha^{(N)} \otimes V$ thanks to Theorem 5.1.3 in [2].
2. Remind that, if \mathcal{F} is any fresco, for each of its higher Bernstein polynomial, $B_{\mathcal{F}}^k$ for $k \geq 1$, we have

$$B_{\mathcal{F}}^k = \prod_{\alpha} B_{\mathcal{F}^\alpha}^k$$

where $\mathcal{F}^\alpha := \mathcal{F}/\mathcal{F}_{[\neq \alpha]}$ is the $[\alpha]$ -primitive quotient of \mathcal{F} .

3. If a fresco has the root $-\alpha - m$ for its k -th Bernstein polynomial, where $k := d(\mathcal{F}^\alpha)$, there exists a $[\alpha]$ -primitive rank k quotient theme T of \mathcal{F} such that the k -th Bernstein polynomial of T is $(x + \alpha + m)$.
4. Conversely, if T is a $[\alpha]$ -primitive quotient theme of rank k of a $[\alpha]$ -primitive fresco \mathcal{F} such that the k -th Bernstein polynomial of T is $(x + \alpha + m)$, then there exists a root in $[-\alpha - m, -\alpha] \cap \{\alpha - \mathbb{N}\}$ for the k -th Bernstein polynomial of \mathcal{F} . Moreover, when $d(\mathcal{F}) = k$, $(-\alpha - m)$ is a root of the k -th Bernstein polynomial of \mathcal{F} .

We give now a lemma and a remark which will be added in the second version of [2] (in preparation).

Lemma 2.0.1 *Let \mathcal{F} be a $[\alpha]$ -primitive fresco and assume that $-\alpha - m$ is a root of its k -th Bernstein polynomial. Then for each $j \in [1, k]$ there exists an integer $m_j \in [0, m]$ such that $-\alpha - m_j$ is a root of the j -th Bernstein polynomial of \mathcal{F} .*

PROOF. By definition, if the nilpotent order for \mathcal{F} is strictly bigger than k then the Bernstein polynomial of $S_k(\mathcal{F})$ has a root which is strictly bigger than $-\alpha - m$. So it is enough to prove the lemma when k is the nilpotent order of \mathcal{F} . Then, by a descendant induction on $j \in [1, k - 1]$ it is enough to prove the case $j = k - 1$.

Taking the quotient by $S_{k-2}(\mathcal{F})$ we reduce the question in the case $k = 2$.

In this case, there exists a quotient theme T with rank 2 whose Bernstein polynomial has the root $-\alpha - m$ as its minimal root. Then the other root $-\alpha - m'$ of the Bernstein polynomial of T satisfies $-\alpha - m' \geq -\alpha - m$. Since T is a quotient of \mathcal{F} , $-\alpha - m'$ is a root of the Bernstein polynomial of \mathcal{F} . If it is a root of the first

Bernstein polynomial of \mathcal{F} we are done. If this is not the case, $-\alpha - m'$ is a root of the second Bernstein polynomial of \mathcal{F} . But in this case $m' < m$ since the roots of the second Bernstein polynomial of \mathcal{F} are simple. Then we can play the same game as before, but with the root $-\alpha - m'$. Since there is only finitely integer in $[0, m]$ we finally reach a root $-\alpha - m''$ of the first Bernstein polynomial of \mathcal{F} such that m'' is in $[0, m]$. \blacksquare

REMARK. Let T be a $[\alpha]$ -primitive theme with rank k . Then its j -th Bernstein polynomial has degree 1 for $j \in [1, k]$ and is equal to $(x + \alpha + m_j)$ where $-\alpha - m_1, \dots, -\alpha - m_k$ are the roots of its Bernstein polynomial in decreasing order. So $-\alpha - m_k$ is the smallest root of its Bernstein polynomial.

Note that, for any $[\alpha]$ -primitive fresco, the smallest root of the Bernstein polynomial is always a root of the k -th Bernstein polynomial where $k = d(\mathcal{F})$ is the nilpotent order of the fresco \mathcal{F} . But for a “general” $[\alpha]$ -primitive fresco, we do not know other relation between the order of the roots of the Bernstein polynomial of \mathcal{F} and the roots of the j -th Bernstein polynomial of \mathcal{F} than the fact, given by the Lemma above.

Proposition 2.0.2 *Let \mathcal{F} be a semi-simple fresco. Then $-\lambda$ is a root of the Bernstein polynomial of \mathcal{F} if and only if there exists a $\tilde{\mathcal{A}}$ -linear surjective map*

$$\pi : \mathcal{F} \rightarrow \mathcal{E}_\lambda \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}(a - \lambda b).$$

PROOF. The existence of π is sufficient because the Bernstein polynomial of a quotient of \mathcal{F} divides the the Bernstein polynomial of \mathcal{F} .

Conversely, if λ is a root of the Bernstein polynomial of \mathcal{F} , since \mathcal{F} is semi-simple, there exists a Jordan-Hölder sequence for \mathcal{F} such its last quotient is \mathcal{E}_λ , thanks to Proposition 7.2.1 in [2]. So the proof is complete. \blacksquare .

Corollary 2.0.3 *Let \mathcal{F} be a $[\alpha]$ -primitive fresco with nilpotent order k . Assume that $\mathcal{F} = \tilde{\mathcal{A}}e \subset \Xi_\alpha^{(k-1)} \otimes V^1$. Let p be the rank of $\mathcal{F}/S_{k-1}(\mathcal{F})$. Then there exists p linearly independent vectors v_1, \dots, v_p in V such that e may be written*

$$e = \sum_{j=1}^p S_j(b) s^{\alpha+m_j-1} (\text{Log } s)^{k-1} \otimes v_j + \psi.$$

where ψ is in $\Xi_\alpha^{(k-2)} \otimes V$, and where the S_j are invertible elements in $\mathbb{C}[[b]]$. Moreover we may choose the vectors v_1, \dots, v_p such that $m_1 < \dots < m_p$.

When this condition is fulfilled the k -th Bernstein polynomial of \mathcal{F} is equal to $\prod_{j=1}^p (x + \alpha + m_j)$.

¹This is not restrictive thanks to the results of Section 5 in [2].

For $\alpha = 1$ it is convenient to replace $\Xi_1^{(k-1)}$ by $\Xi_1^{(k)}/\Xi_1^{(0)}$ to consider only the singular part of the asymptotic expansions. This is the case in the examples computed in Section 4.

PROOF. Since $S_{k-1}(\mathcal{F}) = \mathcal{F} \cap (\Xi_\alpha^{(k-2)} \otimes V)$, it is enough to treat the semi-simple case. In this case, since each $\mathcal{E}_{\alpha+m}$ is embedded in $\Xi_\alpha^{(0)}$ we may assume that $e = \sum_{j=1}^q S_j(b) s^{\alpha+\mu_j-1} \otimes v_j$ where v_1, \dots, v_q is a basis of V (by definition of semi-simplicity), where m_1, \dots, m_q are non negative integers and where S_j are invertible elements in $\mathbb{C}[[b]]$ or vanish identically. Moreover, since the saturation \mathcal{F}^\sharp is a direct sum of $\mathcal{E}_{\alpha+m}$ and has the same rank than \mathcal{F} , we may assume that the vector v_j for which $S_j \neq 0$ generate a subspace W of dimension p in V , where p is the rank of \mathcal{F} . If the integer μ_1, \dots, μ_p are pairwise distinct we may order the v_1, \dots, v_p such that $\mu_1 < \dots < \mu_p$ and put $m_j := \mu_j$. If this is not the case, consider m_1 the infimum of the μ_j and when $\mu_j = m_1$ let $w_1 = v_1 + \sum_{j \geq 2} c_j v_j$ where the sum is on each $j \geq 2$ such that $\mu_j = m_1$ and where $c_j = S_j(0) S_1(0)^{-1}$ with $\mu_j = m_1$. Now we obtain a new expression for e in the basis w_1, v_2, \dots, v_p of W , where m_1 is strictly less than all μ'_j which appear for $j \geq 2$. Continuing in this way we obtain that (w_1, \dots, w_p) is a new basis of W and $m_1 < \dots < m_p$.

Then consider the $\tilde{\mathcal{A}}$ -linear maps given by the linear forms $l_j \in V^*$ defined by $l_j(w_h) = \delta_{j,h}$, $h \in [1, p]$. The $\tilde{\mathcal{A}}$ -linear map $\text{id} \otimes l_j$ for $j \in [1, p]$ sends surjectively \mathcal{F} to $\mathcal{E}_{\alpha+m_j}$ and this implies that $-(\alpha + m_j)$ is a root of the Bernstein polynomial of \mathcal{F} for each $j \in [1, p]$. But since \mathcal{F} has rank p we obtain all the roots of its Bernstein polynomial since the m_j are pair-wise distinct². This completes the proof. ■

REMARK. As a consequence of the previous corollary we have the following characterization of the roots of the k -th Bernstein polynomial of a $[\alpha]$ -primitive fresco with nilpotent order k :

- $-\alpha - m$ is a root of the $[\alpha]$ -primitive fresco with nilpotent order k if and only if there exists a $\tilde{\mathcal{A}}$ -linear surjective map of \mathcal{F} to a rank k theme T_k such its k -th Bernstein polynomial is $(x + \alpha + m)$.

3 A more precise result than [1]

Here is the strengthened version of the main result in [1] announced in the introduction, which uses the higher order Bernstein polynomials of the concerned fresco. The reader may remark that the main result of [2] is a precise converse of this theorem.

Theorem 3.0.1 *Let $\alpha \in]0, 1]$ and assume the hypothesis $H(\alpha, 1)$ ³ for the germ at the origin in \mathbb{C}^{n+1} of holomorphic function $\tilde{f} : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$. Assume that ω*

²Note that the initial μ_j gives also roots of the k -th Bernstein polynomial of \mathcal{F} but they may not give all the roots.

³Remind that this mean that $\exp(2i\pi\alpha)$ is not a root of the local monodromy of f acting on the reduced cohomology of the Milnor's fiber of f at any point outside the origin.

in Ω_0^{n+1} has the following property:

there exists an integer $h \in \mathbb{Z}$ and a form $\omega' \in \Omega_0^{n+1}$ such that $F_h^{\omega, \omega'}(\lambda)$ has a pole of order $p \geq 1$ at some point $\xi \in -\alpha - \mathbb{N}$.

Note $\xi_p = -\alpha - m$ the biggest among such numbers $\xi \in -\alpha - \mathbb{N}$ for any choice of ω' and $h \in \mathbb{Z}$. Then the p -th Bernstein polynomial of the fresco $\mathcal{F}_{f, \omega}$ has a root in $[-\alpha - m, -\alpha] \cap \mathbb{Z}$.

PROOF. Note $P := P_1 P_2$ the annihilator of the class of $[\omega]$ in the $[\alpha]$ -primitive quotient

$$\mathcal{F}^\alpha := \mathcal{F}_{f, \omega} / (\mathcal{F}_{f, \omega})_{\neq \alpha}$$

of the fresco $\mathcal{F}_{f, \omega} := \tilde{\mathcal{A}}[\omega]$ inside the (a,b)-module H_0^{n+1} associated to f , where P_2 is the annihilator of $[\omega]$ in $\mathcal{F}^\alpha / S_{p-1}(\mathcal{F}^\alpha)$. If $F_h^{\omega, \omega'}(\lambda)$ has a pole of order at least equal to p at the point $-\alpha - m$ and if $-\alpha - m$ is not a root of the q -th Bernstein polynomial of \mathcal{F}^α for each $q \geq p$, then $-\alpha - m$ is not a root of the (usual) Bernstein polynomial of the fresco $\tilde{\mathcal{A}} / \tilde{\mathcal{A}}P_1$ which is isomorphic to $S_{p-1}(\mathcal{F}^\alpha)$. In this situation, using Corollary 8.2.5 in [2] we see that $F_{h+p_2}^{P_2 \omega, \omega'}(\lambda)$ has a pole of order at least equal to p at $-\alpha - m$, where k is the rank of the fresco $\mathcal{F}^\alpha / S_{p-1}(\mathcal{F}^\alpha)$. But this is impossible, according to Corollary 8.2.6 in *loc. cit.* since the nilpotent order of $S_{p-1}(\mathcal{F}^\alpha)$ is $p - 1$ and since the image of the class of $P_2 \omega$ in \mathcal{F}^α generates $S_{p-1}(\mathcal{F}^\alpha)$.

So there exists integers $j \geq 0$ and $m_j \in [0, m]$ such that the $(p + j)$ -th Bernstein polynomial of $\mathcal{F}_{f, \omega}^\alpha$ has the root $-\alpha - m_j$. Then the p -th Bernstein polynomial of $\mathcal{F}_{f, \omega}$ has a root in $[-\alpha - m_j, -\alpha] \subset [-\alpha - m, -\alpha]$ (thanks to the remark following Lemma 2.0.1). \blacksquare

The end of Theorem 3.1.2 in [1] is also improved as follows:

Corollary 3.0.2 *In the situation of the previous theorem, let, for each integer s in $[1, p]$, ξ_s be the biggest element in $-\alpha - \mathbb{N}$ for which there exists $h \in \mathbb{Z}$ and $\omega' \in \Omega_0^{n+1}$ such that $F_h^{\omega, \omega'}(\lambda)$ has a pole of order at least equal to s at ξ_s . Then ξ_s is a root of some $(s + j)$ -th Bernstein polynomial of the fresco $\mathcal{F}_{f, \omega}^\alpha$ for some $j \in \mathbb{N}$.*

Moreover, if $\xi_s = \xi_{s+1} = \dots = \xi_{s+p}$, then there exists at least p distinct values of $j \in \mathbb{N}$ such that ξ_s is root of the $(s + j)$ -th Bernstein polynomial of the fresco $\mathcal{F}_{f, \omega}^\alpha$.

PROOF. the proof of the first assertion is analogous to the proof of the theorem above.

The second assertion is an immediate consequence of the fact that the roots of the Bernstein polynomial of a semi-simple fresco are simple, applied to the successive semi-simple quotients

$$S_d(\mathcal{F}^\alpha) / S_{d-1}(\mathcal{F}^\alpha)$$

for $d = s + 1, s + 2, \dots, s + p$. \blacksquare

REMARKS.

1. If for ω given in Ω_0^{n+1} , the maximal order of a pole at some point in $-\alpha - \mathbb{N}$ is equal to d for any choice of $\omega' \in \Omega_0^{n+1}$ and any $h \in \mathbb{Z}$, then ξ_d is a root of the d -th Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega}$, because of the “converse theorem” proved in [2] (see Theorem 8.5.1).
2. To consider a form $\psi \in \mathcal{C}_c^\infty(\mathbb{C}^{n+1})^{0,n+1}$ with small enough support and such that $d\psi = 0$ in a neighborhood of 0 is equivalent to consider $\rho\bar{\omega}'$ where ω' is in Ω_0^{n+1} and ρ is a function in $\mathcal{C}_c^\infty(\mathbb{C}^{n+1})$ with small enough support which is identically 1 near the origin.

Indeed any such ψ may be written as $\psi = \bar{\omega}'$ for some $\omega' \in \Omega^{n+1}$ near the origin thanks to Dolbeault’s Lemma, and then $\psi - \rho\bar{\omega}'$ is identically 0 near the origin, so replacing ψ by $\rho\bar{\omega}'$ do not change the poles which may appear in $-\alpha - \mathbb{N}$ for the functions we are looking at (what ever is the choice of $h \in \mathbb{Z}$ thanks to our hypothesis $H(\alpha, 1)$).

4 Examples

It is, in general, rather difficult to compute the Bernstein of the fresco associated to a given pair (f, ω) , even in the case where f has an isolated singularity. Nevertheless, in the case where f is a polynomial in $\mathbb{C}[x_0, \dots, x_n]$ having $(n + 2)$ monomials, we describe in the article [1], a rather elementary method to obtain an estimation for the Bernstein polynomial of the fresco $\mathcal{F}_{f,\omega}$ associated to a monomial $(n + 1)$ -form ω .

Of course, when the full Bernstein polynomial has a root of multiplicity $k \geq 2$ then this root is also a root of the j -th Bernstein polynomial for each $j \in [1, k]$ but when the Bernstein polynomial has only simple roots, the computation of the higher order Bernstein polynomials, even in the special situation of [1], is not easy. We present in below some examples where we show that the second Bernstein polynomial is not trivial but where the full Bernstein polynomial has no multiple root.

Proposition 4.0.1 *Let $f(x, y, z) := xy^3 + yz^3 + zx^3 + \lambda xyz$ where $\lambda \neq 0$ is any complex number which is a parameter, and consider the holomorphic forms*

$$\omega_1 := dx \wedge dy \wedge dz, \quad \omega_2 := y^3 z^2 \omega_1, \quad \omega_3 = y^7 \omega_1, \quad \text{and} \quad \omega_4 := xy^3 \omega_1.$$

Then, in each of these cases, the fresco \mathcal{F}_{f,ω_i} is a rank 2 theme and the second Bernstein polynomial is equal respectively to $x + 1$, $x + 4$, $x + 5$ and $x + 3$.

Moreover, for $i = 3, 4$ the corresponding (full) Bernstein polynomial of the corresponding frescos has only simple roots.

Note that this proposition allows to apply Theorem 8.5.1 in [2] to conclude that for each $i \in \{1, 2, 3, 4\}$, there exists some integer h and some germ $\omega' \in \Omega_0^3$ such that

the meromorphic extension of

$$F_h^{\omega_i, \omega'}(\lambda) = \frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho \omega_i \wedge \bar{\omega}'$$

has a double pole at the point λ_i equal to the root of the second Bernstein polynomial of the fresco $\mathcal{F}_{f, \omega_i}$.

The proof of this proposition uses several lemmas and the technic of computation described in [1] (see paragraph 4.3.2) .

Lemma 4.0.2 *Let e be a generator of the rank 2 theme $T := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}(a - 2b)(a - b)$ (which is the unique fresco with Bernstein polynomial $(x + 1)^2$). Assume that we have three homogeneous polynomials P, Q and R in \mathcal{A} of respective degrees 3, 4 and k with the following conditions*

1. P, Q and R are monic in a .
2. Then exists a non zero constant c such that $P + cQ$ kills e in T .
3. The Bernstein polynomial of Q^4 is not a multiple of $(x + 1)$ or of $(x + 2)$.
4. The Bernstein polynomial of R is not a multiple of $(x + 3)(x + 2)(x + 1)$

Then Re generates a rank two sub-theme in T .

PROOF. First, remark that our hypothesis implies that $P = (a - \nu b)(a - 2b)(a - b)$ for some $\nu \in \mathbb{C}$ since T is isomorphic to $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}(a - 2b)(a - b)$. We may realize T in the simple pole asymptotic expansion module with rank 2 which is isomorphic to T^\sharp

$$\Theta := \Xi_1^2/\Xi_1^0 \simeq \mathbb{C}[[s]](\text{Log } s)^2 \oplus \mathbb{C}[[s]\text{Log } s$$

where a is the multiplication by s and b is defined by $ab - ba = b^2$ and

$$b(\text{Log } s) = s \text{Log } s \quad \text{and} \quad b((\text{Log } s)^2) = s(\text{Log } s)^2 - 2s \text{Log } s.$$

Then let us prove that image of e in Ξ_1^2/Ξ_1^0 may be written

$$e = u(\text{Log } s)^2 + vs(\text{Log } s)^2 + ws^3(\text{Log } s)^2 + s^4\mathbb{C}[[s]](\text{Log } s)^2 + \mathbb{C}[[s]](\text{Log } s) \quad (\textcircled{a})$$

where φ is in Ξ_1^2/Ξ_1^0 and where $uvw \neq 0$ are complex numbers.

Remark that the only restrictive condition for writing e as in (\textcircled{a}) is the condition $uvw \neq 0$. The condition $u \neq 0$ is easy because we assume that e is a generator of

⁴By definition B_P is defined by the formula

$$(-b)^p B_P(-b^{-1}a) = P$$

where P is in \mathcal{A} , is homogeneous in (a, b) of degree p and monic in a . This is the Bernstein polynomial of the fresco $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}P$.

T with Bernstein polynomial $(x+1)^2$, so writing e as a $\mathbb{C}[[b]]$ -linear combination of the $\mathbb{C}[[b]]$ -basis $e_1 = (\text{Log } s)^2$ and $e_2 = \text{Log } s$ of T we see that the coefficient of e_1 must be invertible in $\mathbb{C}[[b]]$.

But the condition $(P + cQ)(e) = 0$ implies, since the Bernstein element of T is $(a - 2b)(a - b)$, that we may write⁵ $P = (a - \nu b)(a - 2b)(a - b)$.

The annihilator of $(\text{Log } s)^2$ in Ξ_1^2/Ξ_1^0 is the ideal $\tilde{A}(a - 2b)(a - b)$ so we have $P((\text{Log } s)^2) = 0$ in T . Since $Q((\text{Log } s)^2)$ has a non zero term in $s^4(\text{Log } s)^2$, because -1 is not a root of B_Q , only the term coming from

$$P(s(\text{Log } s)^2) = \frac{4 - \nu}{24} s^4 (\text{Log } s)^2 \quad \text{modulo } \mathbb{C}[[s]] \text{Log } s$$

can compensate for this term, in order to obtain the equality $(P + cQ)(e) = 0$. Then $u \neq 0$ implies $v \neq 0$.

But now, the only term which can kill the non zero term in $s^5(\text{Log } s)^2$ coming from $Q(vs(\text{Log } s)^2)$ (using that B_Q is not a multiple of $(x+2)$) can only come from $P(ws^2(\text{Log } s)^2)$ and this proves that $w \neq 0$. So the assertion (\textcircled{a}) holds true.

Now if R is homogeneous of degree k in (a, b) a necessary condition on R such that Re has no term in $s^{k+i}(\text{Log } s)^2$, for $i = 0, 1, 2$, is that B_R divides $(x+1)(x+2)(x+3)$. So, when it is not the case Lemma 5.2.4 in [2] implies that Re is a rank 2 theme and its second Bernstein polynomial has a (unique) root equal to $-(k+j)$ where $-j$ is the smallest integer among $\{-1, -2, -3\}$ which is not a root of B_R (see Corollary 2.0.3). ■

Note that the Lemma above may be easily generalized to many $[\alpha]$ -primitive frescos provided that the nilpotent order is known and that it has a generator which admits a enough simple element in its annihilator.

Lemma 4.0.3 *In the situation of Proposition 4.0.1, the frescos generated by the forms*

$$\omega_1 := dx \wedge dy \wedge dz, \quad \omega_2 := y^3 z^2 \omega_1, \quad \omega_3 := y^7 \omega_1, \quad \text{and} \quad \omega_4 := xy^3 \omega_1$$

generate rank 2 [1]-primitive themes. Their Bernstein polynomials are respectively equal to

$$(x+1)^2, \quad (x+3)^2 \text{ or } (x+2)(x+3), \quad (x+3)(x+5) \quad \text{and} \quad (x+2)(x+3)$$

and their respective 2-Bernstein polynomials are $(x+1)$, $(x+3)$, $(x+5)$ and $(x+3)$. In the cases $i = 3, 4$ there is no double root for the Bernstein polynomial of $\mathcal{F}_{f, \omega_i}$.

PROOF. The first point is to show that $\mathcal{F}_{f, \omega_1}$ has rank 2. Since f has an isolated singularity at the origin, we have $\text{Ker } df^n = df \wedge \Omega^{n-1}$ and then $H^{n+1}/bH^{n+1} \simeq$

⁵In our choice of f and ω_1 , $\mu = 3$.

$\mathcal{O}_0/J(f)$ and H^{n+1} has no b -torsion and no a -torsion. Since f is not⁶ in $J(f)$ the image of ω_1 and $a\omega_1 = f\omega_1$ in H^{n+1} are linearly independent (over \mathbb{C}) and then the rank of $\tilde{\mathcal{A}}\omega_1$ is at least equal to 2. Now the computation in [1] (see 4.3.2) shows that the Bernstein polynomial of this fresco divides $(x+1)^3$ (see also the detailed computation below). So it is a theme of rank 2 or 3. But using our main result, the rank 3 would imply that there exists a pole of order 3 for some $F_h^{\omega_1, \omega'}(\lambda)$ which is impossible⁷ in \mathbb{C}^3 . So $\mathcal{F}_{f, \omega_1}$ is a rank 2 theme with Bernstein polynomial $(x+1)^2$. The computation in [1] gives that $P_3 + c\lambda^{-4}P_4$ kills ω_1 in H^{n+1} where

$$P_3 := (a-3b)(a-2b)(a-b), P_4 = (a-(13/4)b)(a-(5/2)b)(a-(7/4)b)a, \text{ and } c = 4^4$$

This is easily obtain by using the technic of the computation of *loc.cit.* (see the detailed computation in the Appendix below). Then we may apply Lemma 4.0.2 to see that $\lambda m_1 m_2 \omega_1 = \lambda(a-2b)(a-b)\omega_1$ generates rank 2 themes in H^{n+1} . But the identity $\lambda m_1 m_2 = m_4 y^3 z^2$ shows that ω_2 generates also rank 2 in H^{n+1} since $m_4 \omega_2 = \lambda m_1 m_2 \omega_1 = \lambda(a-2b)(a-b)\omega_1$ applying Lemma 4.0.2 with $R = (a-2b)(a-b)$ whose Bernstein polynomial is $(x+1)^2$. Moreover we see that Re has a non zero term in $s^3(\text{Log } s)^2$.

Since $m_4 \omega_2$ generates a rank 2 theme, then ω_2 generates a rank 2 theme also (the rank 3 is again excluded because it would imply that $f^2 \notin J(f)$ which is impossible as explained above).

The technic of computation in [1] applied to ω_2 gives now that the Bernstein polynomial of the rank 2 theme $\tilde{\mathcal{A}}\omega_2$ has to divide⁸ the polynomial $(x+2)(x+3)^2$.

But the fact that $m_4 \omega_2$ has a non zero term in $s^3(\text{Log } s)^2$ (and no term in $(\text{Log } s)^2$ or in $s(\text{Log } s)^2$) implies, since we have

$$m_4 \omega_2 = 4(a-2b)\omega_2$$

ω_2 has a non zero term in $s^2(\text{Log } s)^2$ and then -3 is a root of the second Bernstein polynomial of the fresco $\mathcal{F}_{f, \omega_2}$. So the Bernstein polynomial is either $(x+2)(x+3)$ or $(x+3)^2$.

We know⁹ that the Bernstein polynomial of $\mathcal{F}_{f, \omega_3}$ divides $(x+5)(x+3)(x+2)$. But we know also that $m_1^2 m_4 \omega_1 = \lambda m_3 \omega_3$ has a non zero term in $s^5(\text{Log } s)^2$ (as a consequence of Lemma 4.0.2) and $-m_3 \omega_3 = (a-2b)\omega_3$ implies that ω_3 has a non zero term in $s^4(\text{Log } s)^2$. Then the second Bernstein polynomial of $\mathcal{F}_{f, \omega_3}$ is $x+5$.

Note that, in this case, the Bernstein polynomial of the fresco $\mathcal{F}_{f, \omega_3}$ has two simple roots.

The last case is similar, since we know that $m_1 \omega_1$ has a non zero term in $s^2(\text{Log } s)^2$. So our assertion is consequence of the estimation of the Bernstein polynomial. ■

For the convenience of the reader, we give in the Appndix below some detailed computations for these four examples.

⁶This point is not so easy to check directly. But the rank is not 1 since this would implies that this fresco has a simple pole and the argument used in Lemma 4.0.2 gives then a contradiction.

⁷This would give an order 4 pole for the meromorphic continuation of $|f|^{2\lambda}$!

⁸This computation gives that $Q_3 + d\lambda^{-4}Q_4$ kills ω_2 in H^{n+1} with $Q_3 := (a-4b)(a-4b)(a-3b)$.

⁹see the computation below.

5 Appendix: detailed computations

THE DETAILED COMPUTATION FOR ω_1 . The linear system to compute m_1, \dots, m_4 (we note ω_1 by 1) is given by $MX = {}^t(a, b, b, b)(1)$ which gives

$$\begin{aligned} 0 &= (2m_1 + m_2 - 3m_3) \quad \text{and} \\ 0 &= (3m_1 - 2m_2 - m_3) \quad \text{and so} \\ m_1 &= m_2 = m_3 \quad \text{and} \\ m_4 &= b(1) - 3m_1 - m_2 = b(1) - 4m_1 \\ a(1) &= 3m_1 + b(1) - 4m_1 \quad \text{so} \quad m_1 = -(a - b)(1) = m_2 = m_3 \quad \text{and} \quad m_4 = -(4a - 3b)(1) \end{aligned}$$

The linear system to compute $m_1^2, \dots, m_1 m_4$ is given by $MX = {}^t(a, 2b, 4b, b)(m_1)$ which gives

$$\begin{aligned} 2b(m_1) &= (2m_1 + m_2 - 3m_3)m_1 \quad \text{and} \\ 3b(m_1) &= (3m_1 - 2m_2 - m_3)m_1 \quad \text{and so} \\ m_1^2 &= m_1 m_2 + b(m_1) \quad m_1 m_3 = m_1 m_2 \quad \text{and} \quad m_1 m_4 = b(m_1) - 4m_1 m_2 \\ a(m_1) &= m_1 m_2 + b(m_1) + m_1 m_2 + m_1 m_2 + b(m_1) - 4m_1 m_2 \quad \text{so} \\ -m_1 m_2 &= (a - 2b)(m_1) \quad \text{and} \quad -m_1^2 = (a - 3b)(m_1) = -(a - 3b)(a - b)(1) \end{aligned}$$

The linear system to compute $m_1^2 m_2, m_1 m_2^2, m_1 m_2 m_3, m_1 m_2 m_4$ is given by $MX = {}^t(a, 2b, 5b, 4b)(m_1 m_2)$ which gives

$$\begin{aligned} 3b(m_1 m_2) &= (2m_1^2 m_2 + m_1 m_2^2 - 3m_1 m_2 m_3) \quad \text{and} \\ b(m_1 m_2) &= (3m_1^2 m_2 - 2m_1 m_2^2 - m_1 m_2 m_3) \quad \text{and so} \\ m_1^2 m_2 &= m_1 m_2^2, m_1 m_2 m_3 = m_1^2 m_2 - b(m_1 m_2), m_1 m_2 m_4 = 5b(m_1 m_2) - 4m_1^2 m_2 \\ -m_1^2 m_2 &= (a - 4b)(m_1 m_2) \quad \text{and} \quad -m_1 m_2 m_3 = (a - 3b)(m_1 m_2). \end{aligned}$$

So the Bernstein polynomial of the fresco $\mathcal{F}_{f, \omega_1}$ has to divide the Bernstein polynomial of $P := (a - 3b)(a - 2b)(a - b)$ which is $B_P = (x + 1)^3$.

The linear system to compute $m_1 m_4^p, \dots, m_4^{p+1}$ is given by $MX = {}^t(a, (p + 1)b, (p + 1)b, (p + 1)b)(m_4^p)$ which gives:

$$\begin{aligned} 0 &= (2m_1 + m_2 - 3m_3)m_4^p \quad \text{and} \\ 0 &= (3m_1 - 2m_2 - m_3)m_4^p \quad \text{and so} \\ m_1 m_4^p &= m_2 m_4^p = m_3 m_4^p \quad \text{and} \\ m_4^{p+1} &= (p + 1)b(m_4^p) - (3m_1 + m_2)m_4^p = (p + 1)b(m_4^p) - 4m_1 m_4^p \quad \text{so} \\ m_4^{p+1} &= (4a - 3(p + 1)b)(m_4^p) = 4\left(a - \frac{3}{4}(p + 1)b\right)(m_4^p). \end{aligned}$$

So we obtain that

$$Q = 4^4(a - 3b)(a - \frac{9}{4}b)(a - \frac{3}{2}b)(a - \frac{3}{4}b) = 4^4(a - \frac{13}{4}b)(a - \frac{5}{2}b)(a - \frac{7}{4}b)a.$$

So $B_Q = x(x + 1/4)(x + 1/2)(x + 3/4)$.

THE DETAIL COMPUTATION FOR $\omega_2 := y^3z^2\omega_1$. Define $\mu := y^3z^2$. The linear system to compute $m_1\mu, \dots, m_4\mu$ is given by $MX = {}^t(a, b, 4b, 3b)(\mu)$ which gives

$$\begin{aligned} 3b(\mu) &= (2m_1 + m_2 - 3m_3)\mu \quad \text{and} \\ b(\mu) &= (3m_1 - 2m_2 - m_3)\mu \quad \text{and so} \quad m_1\mu = m_2\mu = b(\mu) + m_3\mu \quad \text{and} \\ m_4\mu &= 4b(\mu) - 4m_1\mu \end{aligned}$$

Then we obtain

$$-m_1\mu = -m_2\mu = (a - 3b)(\mu), \quad -m_3\mu = (a - 2b)(\mu)$$

and also $m_4\mu = 4(a - 2b)(\mu)$ which is also used above.

Then we compute $m_1^2\mu, \dots, m_1m_4\mu$ given by the system $MX = {}^t(a, 2b, 7b, 3b)(m_1\mu)$ which gives

$$\begin{aligned} 5b(m_1\mu) &= (2m_1 + m_2 - 3m_3)m_1\mu \quad \text{and} \\ 4b(m_1\mu) &= (3m_1 - 2m_2 - m_3)m_1\mu \quad \text{and so} \\ (m_1 - m_2)m_1\mu &= b(m_1\mu), \quad m_1m_3\mu = m_1m_2\mu - b(m_1\mu) \\ m_1m_4\mu &= -4m_1m_2\mu + 4b(m_1\mu) \end{aligned}$$

Then we obtain

$$-m_1m_2\mu = (a - 4b)(m_1\mu),$$

Finally $m_1^2m_2\mu, m_1m_2^2\mu, m_1m_2m_3\mu, m_1m_2m_4\mu$ are given by the system $MX = {}^t(a, 2b, 8b, 6b)(m_1m_2\mu)$ which gives

$$\begin{aligned} 6b(m_1m_2\mu) &= (2m_1 + m_2 - 3m_3)m_1m_2\mu \quad \text{and} \\ 2b(m_1m_2\mu) &= (3m_1 - 2m_2 - m_3)m_1m_2\mu \quad \text{and so} \\ m_1^2m_2\mu &= m_1m_2^2\mu, \quad m_1m_2m_3\mu = m_1^2m_2\mu - 2b(m_1m_2\mu), \\ m_1m_2m_4\mu &= -4m_1m_2m_3\mu. \end{aligned}$$

This gives $-m_1m_2m_3\mu = (a - 4b)(m_1m_2\mu)$ and so the Bernstein element of the fresco \mathcal{F}_{f,ω_2} is given by

$$m_1m_2m_3\mu = -(a - 4b)(a - 4b)(a - 3b)(\mu) \tag{B}$$

which implies that the Bernstein polynomial divides $(x + 2)(x + 3)^2$. And we know that it is a rank 2 theme with second Bernstein polynomial $x + 3$. In this case it is not clear if the Bernstein polynomial is $(x + 3)^2$ or $(x + 2)(x + 3)$. \square

THE DETAILED COMPUTATION FOR $m_1^2 m_4 \omega_1 = \lambda m_3 \omega_3$. Since we have already obtained $m_1^2 = (a - 3b)(a - b)(1)$ in the first case (computation for ω_1), it is enough to consider the linear system which computes $m_1^3, \dots, m_4 m_1^2$.

It is given by $MX = {}^t(a, 3b, 7b, b)(m_1^2)$ and then

$$4b(m_1^2) = (2m_1 + m_2 - 3m_3)m_1 \quad \text{and}$$

$$6b(m_1^2) = (3m_1 - 2m_2 - m_3)m_1 \quad \text{and so}$$

$$m_1^3 = m_1^2 m_2 + 2b(m_1^2) \quad m_1^2 m_3 = m_1^2 m_2 \quad \text{and}$$

$$m_1^2 m_4 = b(m_1^2) - 4m_1^2 m_2$$

$$a(m_1^2) = m_1^2 m_2 + 2b(m_1^2) + m_1^2 m_2 + m_1^2 m_2 + b(m_1^2) - 4m_1^2 m_2 \quad \text{so} \quad -m_1^2 m_2 = (a - 3b)(m_1^2)$$

$$m_1^2 m_4 = b(m_1^2) + 4(a - 3b)(m_1^2) = (4a - 11b)(m_1^2)$$

So we obtain

$$m_1^2 m_4 = (4a - 11b)(a - 3b)(a - b)(1).$$

An the Bernstein polynomial of $R := (4a - 11b)(a - 3b)(a - b)$ is $(x + 3/4)(x + 2)(x + 1)$ so we may apply Lemma 4.0.2 and (\textcircled{a}) gives that $R\omega_1 = \lambda m_3 \omega_3$ has a non zero term in $s^5(\text{Log } s)^2$.

THE DETAILED COMPUTATION FOR $\omega_3 = y^7 \omega_1$. The linear system to compute $m_1 \nu, \dots, m_4 \nu$, (where $\nu := y^7$) is given by $MX = {}^t(a, b, 8b, b)(y^7)$ which gives:

$$7b(\nu) = (2m_1 + m_2 - 3m_3)\nu \quad \text{and}$$

$$7b(\nu) = (3m_1 - 2m_2 - m_3)\nu \quad \text{and so}$$

$$m_1 \nu = m_2 \nu + 2b(\nu), \quad m_3 \nu = m_2 - b(\nu) \quad \text{and}$$

$$m_4 \nu = -4m_2 \nu + 2b(\nu), \quad \text{so} \quad -m_1 \nu = (a - 5b)(\nu), \quad -m_2 \nu = (a - 3b)(\nu),$$

$$-m_3 \nu = (a - 2b)(\nu) \quad \text{and} \quad m_4 = (4a - 10b)$$

Remark that this computation is already enough to see that the Bernstein polynomial of $\mathcal{F}_{f, \omega_3}$ divides $(x + 5)(x + 3)(x + 2)^{10}$. But we know that $m_1^2 m_4 \omega_1 = \lambda m_3 \omega_3$ has a non zero term in $s^5(\text{Log } s)^2$ and $-m_3 \omega_3 = (a - 2b)\omega_3$ implies that ω_3 has a non zero term in $s^4(\text{Log } s)^2$ and then the second Bernstein polynomial of $\mathcal{F}_{f, \omega_3}$ is $x + 5$. The Bernstein polynomial is $(x + 3)(x + 5)$ because $(a - 2b)\omega_3$ cannot have a non zero term in $s(\text{Log } s)^2$ (note that $m_1^2 m_4 \omega_1$ has a non zero term in $s^3 \text{Log } s$) since $u \neq 0$ and so ω_3 has a non zero term in $s^2 \text{Log } s$.

THE DETAILED COMPUTATION FOR $\omega_4 := m_1 \omega_1$. The linear system to compute $m_1^2, \dots, m_1 m_4^2$ is given by $MX = {}^t(a, 2b, 4b, b)(m_1)$ is already solved above and it gives

$$m_1^2 = (a - 3b)(m_1).$$

¹⁰because we can compute $m_1 m_2 m_3 \nu$ with any order for 1, 2, 3 and then obtain that the P which estimates the Bernstein element of the fresco $\mathcal{F}_{f, \omega_3}$ is right divisible by $(a - 5b)$, $(a - 3)$ and $(a - 2b)$.

The linear system to compute $m_1^2 m_2, \dots, m_1^2 m_4$ is given by $MX = {}^t(a, 3b, 7b, b)(m_1^2)$ which gives:

$$\begin{aligned} 4b(m_1^2) &= (2m_1 + m_2 - 3m_3)m_1 m_4 \quad \text{and} \\ 6b(m_1^2) &= (3m_1 - 2m_2 - m_3)m_1 m_4 \quad \text{and so} \\ m_1^3 &= m_1^2 m_2 + 2b(m_1^2), \quad \text{and } m_1^2 m_3 = m_1^2 m_2 \quad \text{and} \\ m_1^2 m_4 &= 7b(m_1 m_4) - 3m_1^3 - m_1^2 m_2 = b(m_1^2) - 4m_1^2 m_2 \quad \text{so} \quad -m_1^2 m_2 = (a - 3b)(m_1^2). \end{aligned}$$

The linear system to compute $m_1^3 m_2, m_1^2 m_2^2, m_1^2 m_2 m_3, m_1^2 m_2 m_4$ is given by $MX = {}^t(a, 3b, 8b, 4b)(m_1^2 m_2)$ which gives:

$$\begin{aligned} 5b(m_1^2 m_2) &= (2m_1 + m_2 - 3m_3)m_1^2 m_2 \quad \text{and} \\ 4b(m_1) &= (3m_1 - 2m_2 - m_3)m_1^2 m_2 \quad \text{and so} \\ m_1^3 m_2 &= m_1^2 m_2^2 + b(m_1^2 m_2), \quad m_1^2 m_2 m_3 = m_1^2 m_2^2 - b(m_1^2 m_2) \quad \text{and} \\ m_1^2 m_2 m_4 &= 8b(m_1) - 3m_1^2 m_2 - m_1^2 m_2^2 = 5b(m_1^2 m_2) - 4m_1^2 m_2^2 \quad \text{so} \\ m_1^2 m_2 m_3 &= (a - 4b)(m_1^2 m_2). \end{aligned}$$

Finally we find $(a - 4b)(a - 3b)(a - 3b)(m_1 \omega_1) = -m_1 m_2 m_3 (m_1 \omega_1)$. So the Bernstein polynomial of the fresco $\mathcal{F}_{f, \omega_4}$ has to divide $(x + 2)^2(x + 3)$. Since $m_1 \omega_1 = -(a - b)\omega_1$ has clearly a non zero term in $s^2(\text{Log } s)^2$ this fresco is rank 2 theme and its second Bernstein polynomial is $x + 3$. Then its Bernstein polynomial is $(x + 2)(x + 3)$.

6 Bibliography

References

- [1] Barlet, D. *Algebraic differential equations of period-integrals*
Journal of Singularities Volume 25 (2022), pp. 54-77.
- [2] Barlet, D. *Higher Bernstein Polynomials and Multiple Poles of*
 $\frac{1}{\Gamma(\lambda)} \int_X |f|^{2\lambda} \bar{f}^{-h} \rho \omega \wedge \bar{\omega}'$ arXiv:2307.04395 math AG, math. CV

For more references see [2].