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# On the robustness of networks of heterogeneous semi-passive systems interconnected over directed graphs

Anes Lazri   Elena Panteley   Antonio Loria \*

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## Abstract

In this short note we provide a proof of boundedness of solutions for a network system composed of heterogeneous nonlinear autonomous systems interconnected over a directed graph. The sole assumptions imposed are that the systems are semi-passive [1] and the graph contains a spanning tree.

**Lemma 1** Consider a network containing  $N$  interconnected dynamical systems<sup>1</sup>

$$\dot{x}_i = f_i(x_i) + u_i, \quad i \leq N, \quad x_i \in \mathbb{R}, \quad (1)$$

each of which defines a semi-passive map  $u_i \mapsto x_i$ . Let

$$u_i = - \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j), \quad a_{i,j} \geq 0, \quad (2)$$

where for each  $i \leq N$ ,  $\mathcal{N}_i$  denotes the set of nodes  $v_j$  sending information to the node  $v_i$ . Let this network's topology be defined by a directed graph  $G$  containing a spanning tree. Then, the trajectories  $t \mapsto x(t)$ , where  $x := [x_1 \cdots x_N]$ , solutions to (1)-(2) for all  $i \leq N$ , are globally bounded.  $\square$

*Proof:* The system (1)-(2) in compact form, *i.e.*, defining  $x := [x_1 \cdots x_N]^\top$ , becomes

$$\dot{x} = F(x) - Lx, \quad (3)$$

where  $F(x) := [f_1(x_1) \cdots f_N(x_N)]^\top$  and

$$[L]_{i,j} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{\substack{\ell=1 \\ \ell \neq i}}^N a_{i\ell}, & i = j, \quad i, j \leq N. \end{cases}$$

By assumption, the graph  $G$  contains a spanning tree. If, in addition, it is strongly connected, the results follows along the lines of the proof of [2, Proposition 2]. If the graph is not strongly connected, the result follows by observing that by reordering the network's states, the Laplacian  $L$  may be transformed into that of a connected network that consists in a spanning-tree of strongly-connected sub-graphs. Hence, the transformed Laplacian matrix possesses a convenient lower-block-triangular form (see Lemma 2 below). Then, the statement follows after a cascades argument, from the fact that the trajectories of each strongly-connected sub-graph are bounded and remain bounded under the effect of the interconnections (see Lemma 3).

\*A. Lazri is with L2S, CNRS, Univ Paris-Saclay, France (e-mail: anes.lazri@centralesupelec.fr) E. Panteley and A. Loria are with L2S, CNRS, (e-mail: {elena.panteley,antonio.loria}@cnrs.fr).

<sup>1</sup>For clarity of exposition, and without loss of generality, we assume that  $x_i \in \mathbb{R}$ . However, all the statements remain true if  $x_i \in \mathbb{R}^n$  for any  $n > 1$ .

**Lemma 2** Let  $L \in \mathbb{R}^{N \times N}$  be the Laplacian matrix associated to a directed connected graph  $G$  that contains a directed spanning tree, but is not strongly connected. Then, there exists a permutation matrix  $T$  and a number  $m \in \{2, 3, \dots, N\}$ , such that

$$T^\top L T = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ -A_{21} & A_{22} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ -A_{m1} & \cdots & -A_{m,m-1} & A_{m,m} \end{bmatrix}, \quad (4)$$

where for each  $i \in \{2, 3, \dots, m\}$ ,  $A_{ii} \in \mathbb{R}^{n_i \times n_i}$ ,  $A_{ii} = L_{ii} + D_i$ , where  $L_{ii} \in \mathbb{R}^{n_i \times n_i}$  corresponds to the Laplacian of a strongly-connected directed graph,  $D_i$  is a diagonal matrix of non-negative entries, and  $A_{ij} \in \mathbb{R}^{n_i \times n_j}$  is such that  $D_i \mathbf{1}_{n_i} = \sum_{j=1}^{i-1} A_{ij} \mathbf{1}_{n_j}$ , with  $\mathbf{1}_{n_i} := [1 \cdots 1]^\top \in \mathbb{R}^{n_i}$ , and  $D_1 := 0$ .  $\square$

Let Lemma 2 generate a permutation matrix  $T$  and define  $z := T^\top x$ . Since  $T$  is a permutation matrix it is invertible with  $T^{-1} := T^\top$ . In turn,  $t \mapsto x(t)$  of (3) is globally bounded if and only if so is  $t \mapsto z(t)$ , solution to

$$\dot{z} = F_z(z) - T^\top L T z, \quad (5)$$

where  $F_z(z) := T^\top F(Tz)$ .

**Remark 1** Since  $T$  is a permutation matrix the  $i$ th element in the vector  $F_z(z)$  depends only on  $z_i$ , i.e.,  $F_z(z) := [f_1(z_1) \cdots f_N(z_N)]^\top$ .

It is only left to show that  $t \mapsto z(t)$  is globally bounded. To that end, we use the lower block-triangular structure of  $T^\top L T$ . Consider the first  $n_1$  equations in (5), that is, let  $\bar{z}_1 := [z_1 \cdots z_{n_1}]^\top$ ,  $\bar{F}_{z_1}(\bar{z}_1) := [f_1(z_1) \cdots f_{n_1}(z_{n_1})]^\top$ . Then,

$$\dot{\bar{z}}_1 = \bar{F}_{z_1}(\bar{z}_1) - A_{11} \bar{z}_1, \quad (6)$$

where, after Lemma 2,  $A_{11}$  is the Laplacian of a strongly connected graph (since  $D_1 := 0$ ). It follows that the equation (6) corresponds to the dynamics of a strongly connected network, whose solutions are globally bounded. The latter follows from the proof of [2, Proposition 2]<sup>2</sup>.

Now, the second set of equations in (5) corresponds to

$$\dot{\bar{z}}_2 = \bar{F}_{z_2}(\bar{z}_2) - L_{22} \bar{z}_2 - D_2 \bar{z}_2 + A_{21} \bar{z}_1, \quad \bar{z}_2 \in \mathbb{R}^{n_2} \quad (7)$$

where  $L_{22}$  is the Laplacian of a strongly connected graph and  $D_2 \mathbf{1}_{n_2} = A_{21} \mathbf{1}_{n_1}$ . Note that (7) corresponds to the dynamics model of a strongly-connected network of semi-passive systems of the form (1)-(2), of dimension  $N = n_2$ , with an additional stabilizing term  $-D_2 \bar{z}_2$ , and perturbed by an input  $v_1 := \bar{z}_1$ . For such systems, we have the following (see the proof below).

**Lemma 3** Consider a group of  $N$  semi-passive systems (1) with input (2), interconnected over a strongly connected directed graph with associated Laplacian  $L$ . Let  $B_j \in \mathbb{R}^{N \times M_j}$ , with  $j \leq p$ , and  $D \in \mathbb{R}^{N \times N}$ ,  $D := \text{diag}[d_k]$  be matrices whose entries are non-negative, and such that for any  $k \leq N$ ,  $d_k := \sum_{j=1}^p \sum_{\ell=1}^{M_j} [B_j]_{k\ell}$ . Let  $\bar{x} := [x_1 \cdots x_N]^\top$ ,  $F(\bar{x}) := [f_1(x_1) \cdots f_N(x_N)]^\top$  and  $v_j \in \mathbb{R}^{M_j}$  be external bounded inputs. Then, the trajectories of the perturbed networked system

$$\dot{\bar{x}} = F(\bar{x}) - L \bar{x} - D \bar{x} + \sum_{j \leq p} B_j v_j, \quad (8)$$

$t \mapsto \bar{x}(t)$ , are globally bounded.  $\square$

<sup>2</sup>We invoke the proof of [2, Proposition 2] and not the statement since it is therein inappropriately assumed that the graph is undirected, but the proof of the statement applies to connected-and-balanced graphs, as well as to strongly-connected ones.

Now, Lemma 3 applies to Eq. (7) with  $p = 1$ ,  $M_1 = n_1$ ,  $B_1 := A_{21}$ , and the input  $v_1 := \bar{z}_1$ , which we established to be bounded. In addition, the  $k$ th element of  $D_2$ , denoted  $d_{2k}$  satisfies  $d_{2k} = \sum_{\ell=1}^{n_1} [A_{21}]_{k\ell}$ , where  $[A_{21}]_{k\ell}$  denotes the  $\ell$ th element of the  $k$ th row of  $A_{21}$ . It follows, from Lemma 3 that the solutions  $t \mapsto \bar{z}_2(t)$  are globally bounded. In turn, for any  $i \leq m$ , the  $i$ th set of equations in (5) reads

$$\dot{\bar{z}}_i = \bar{F}_{z_i}(\bar{z}_i) - L_{ii}\bar{z}_i - D_i\bar{z}_i + \sum_{j=1}^{i-1} A_{ij}\bar{z}_j. \quad (9)$$

Equation (9) is of the form (8), with  $p = i - 1$ ,  $v_j := \bar{z}_j$ ,  $B_j := A_{ij}$ , and  $L_{ii}$  corresponds to the Laplacian of a strongly connected network. For each  $k \leq n_i$ , the  $k$ th element in the diagonal of  $D_i$  satisfies, by definition,  $d_{i_k} = \sum_{j=1}^{i-1} \sum_{\ell=1}^{n_j} [A_{ij}]_{k\ell}$ , where  $[A_{ij}]_{k\ell}$  corresponds to the  $\ell$ th element in the  $k$ th row of  $A_{ij}$ . Therefore, Invoking Lemma 3, with  $\bar{x} := \bar{z}_i$ , it follows that  $t \mapsto \bar{z}_i(t)$  is globally bounded. The statement of Lemma 1 follows by applying the previous arguments, sequentially, for each  $i \in \{3, 4, \dots, m\}$ .  $\blacksquare$

*Proof of Lemma 2:* Consider the following.

**Fact 1** *If a graph  $G$ , with Laplacian  $L_G$ , contains a directed spanning tree and is not strongly connected, then there exists a permutation matrix  $P_G$  such that  $P_G^\top P_G = I$  and*

$$P_G^\top L_G P_G = \begin{bmatrix} Q_{G'} & 0 \\ -R_{\bar{G}'} & S_{\bar{G}'} \end{bmatrix}, \quad (10)$$

where  $Q_{G'} \in \mathbb{R}^{n' \times n'}$ , with  $n' < N$ , is the Laplacian matrix of the largest strongly-connected sub-graph  $G' \subset G$ , containing  $n'$  nodes, including all the root nodes in  $G$ . The matrix  $S_{\bar{G}'}$  satisfies  $S_{\bar{G}'} = L_{\bar{G}'} + D_{\bar{G}'}$ , where  $L_{\bar{G}'}$  is a Laplacian matrix associated to the graph  $\bar{G}' := G \setminus G'$  and  $D_{\bar{G}'}$  is the degree matrix, which is diagonal and contains the weights of the links from  $G'$  to  $\bar{G}'$ .  $\square$

The previous fact is true because if  $G$  contains only one spanning tree, say  $\mathcal{T}_G$  with root node  $\nu_0$ , then  $\nu_0$  has no incoming link. Therefore, we can set  $G' := (\{\nu_0\}, \emptyset)$ . If  $\nu_0$  has incoming links, it necessarily forms part of a strongly-connected graph containing at least two nodes including  $\nu_0$  and a bidirectional link, thereby forming a strongly connected set. The same reasoning holds if  $G$  has several spanning trees, in which case the respective roots also make part of  $G'$ .

Thus, since by assumption, the graph  $G$  contains a directed spanning tree  $\mathcal{T}_G$ , let Fact 1 generate the largest strongly connected sub-graph of  $G$ , which containing all the roots of  $G$  and  $n_1$  nodes in total and we call  $G_1 \subset G$ . Then, let  $Q_{G_1} \in \mathbb{R}^{n_1 \times n_1}$  denote the Laplacian associated to  $G_1$ . Then, for the block  $A_{11}$  in (4) we set  $A_{11} := Q_{G_1}$ . That is,  $A_{11} \in \mathbb{R}^{n_1 \times n_1}$  is the Laplacian of a strongly connected graph, as desired. Let  $\bar{G}_1 := G \setminus G_1$  denote the complement of  $G_1$ . Fact 1 also generates the matrices  $R_{\bar{G}_1}$  and  $S_{\bar{G}_1}$ . That is,

$$P_G^\top L_G P_G = \begin{bmatrix} A_{11} & 0 \\ -R_{\bar{G}_1} & S_{\bar{G}_1} \end{bmatrix}, \quad (11)$$

The off-diagonal entries of the matrix  $S_{\bar{G}_1} \in \mathbb{R}^{(N-n_1) \times (N-n_1)}$  are non-positive. They represent edges belonging to the graph  $\bar{G}_1$ . Indeed, The matrix  $S_{\bar{G}_1} = L_{\bar{G}_1} + D_{\bar{G}_1}$ , where  $L_{\bar{G}_1}$  corresponds to the Laplacian associated to the graph  $\bar{G}_1$  and  $D_{\bar{G}_1}$  is the degree matrix, which is diagonal positive semidefinite and contains the weights of the links from  $G_1$  to  $\bar{G}_1$ . The entries in the matrix  $R_{\bar{G}_1} \in \mathbb{R}^{(N-n_1) \times n_1}$  represent the outgoing links emanating from nodes belonging to  $G_1$  towards nodes in the rest of the graph, *i.e.*,  $\bar{G}_1$ . If  $\bar{G}_1$  is strongly connected, the matrix in (11) has the desired structure in (4) and the proof ends.

If  $\bar{G}_1$  is not strongly connected, we look for a permutation matrix  $P_{\bar{G}_1} \in \mathbb{R}^{(N-n_1) \times (N-n_1)}$  such that  $P_{\bar{G}_1}^\top L_{\bar{G}_1} P_{\bar{G}_1}$  has a block-triangular form as in (10). To that end, we consider two possibilities depending on whether  $\bar{G}_1$  contains or not a spanning tree.

*Case 1:* Assume that  $\bar{G}_1$  contains a spanning tree, or several. Necessarily, the root of at least one of the trees has an incoming link from  $G_1$ . Then, let Fact 1 generate the largest strongly-connected graph  $G_2 \subset \bar{G}_1$ , containing  $n_2$  nodes, including all the roots in  $\bar{G}_1$ . Also after Fact 1 there exists a permutation matrix  $P_{\bar{G}_1}$  such that

$$P_{\bar{G}_1}^\top L_{\bar{G}_1} P_{\bar{G}_1} = \begin{bmatrix} Q_{G_2} & 0 \\ -R_{\bar{G}_2} & S_{\bar{G}_2} \end{bmatrix}, \quad (12)$$

where  $Q_{G_2} \in \mathbb{R}^{n_2 \times n_2}$  is the Laplacian matrix associated to  $G_2$ ,  $S_{\bar{G}_2} := L_{\bar{G}_2} + D_{\bar{G}_2}$ . Also, we define  $\bar{G}_2 := \bar{G}_1 \setminus G_2 = G \setminus \{G_1 \cup G_2\}$ , *i.e.*,  $\bar{G}_2$  contains all the nodes in  $G$ , but which are not contained in  $G_1$  nor in  $G_2$ .

Then, we apply the permutation blockdiag $[I_{n_1} P_{\bar{G}_1}^\top]$  on the matrix on the right-hand side of (11). We obtain the matrix

$$\begin{bmatrix} A_{11} & 0 \\ -P_{\bar{G}_1}^\top R_{\bar{G}_1} & P_{\bar{G}_1}^\top S_{\bar{G}_1} P_{\bar{G}_1} \end{bmatrix}, \quad (13)$$

which has a lower-block-triangular structure and  $A_{11}$  corresponds to the Laplacian associated to a strongly connected graph, as desired. Furthermore, the block  $P_{\bar{G}_1}^\top R_{\bar{G}_1} \in \mathbb{R}^{(N-n_1) \times n_1}$  may be split into two stacked sub-blocks. The upper one is of dimension  $n_2 \times n_1$  and contains the links that connect the nodes in  $G_1$  to nodes in  $G_2$ ; for the purpose of constructing (4), we name this sub-block  $A_{21}$ . On the other hand, by the definition of  $S_{\bar{G}_1}$  and (12),

$$P_{\bar{G}_1}^\top S_{\bar{G}_1} P_{\bar{G}_1} = \begin{bmatrix} Q_{G_2} & 0 \\ -R_{\bar{G}_2} & S_{\bar{G}_2} \end{bmatrix} + P_{\bar{G}_1}^\top D_{\bar{G}_1} P_{\bar{G}_1}. \quad (14)$$

The last term on the right-hand side of (14) is diagonal and may be split into two diagonal sub-blocks, *i.e.*,  $P_{\bar{G}_1}^\top D_{\bar{G}_1} P_{\bar{G}_1} =: \text{blockdiag}[D'_{\bar{G}_1}, D''_{\bar{G}_1}]$ . Then, we set  $A_{22}$  in (4) to  $A_{22} := Q_{G_2} + D'_{\bar{G}_1}$  and we redefine  $S_{\bar{G}_2} := L_{\bar{G}_2} + D_{\bar{G}_2} + D''_{\bar{G}_1}$ . Thus, after (13) and (14), and the previous definitions, we have

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & P_{\bar{G}_1}^\top \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ -R_{\bar{G}_1} & S_{\bar{G}_1} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & P_{\bar{G}_1} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 \\ -A_{21} & A_{22} & 0 \\ [*] & -R_{\bar{G}_2} & S_{\bar{G}_2} \end{bmatrix}. \quad (15)$$

In the matrix on the right-hand side of (15), the entries of  $R_{\bar{G}_2}$  represent the edges connecting the nodes from  $G_2 \subset \bar{G}_1$  to the rest of the sub-graph  $\bar{G}_1$ , *i.e.*,  $\bar{G}_2$ . Now, as previously remarked for  $\bar{G}_1$ ,  $\bar{G}_2$  may or may not contain a spanning tree. If it does, and  $\bar{G}_2$  is strongly connected, the matrix on the right-hand side of (15) qualifies as the sought matrix in (4). If  $\bar{G}_2$  contains a spanning tree, but is not strongly connected, Fact 1 applies to  $\bar{G}_2$  and generates a strongly connected graph  $G_3 \subset \bar{G}_2$  and its complement  $\bar{G}_3 := \bar{G}_2 \setminus G_3 = G \setminus \{G_1 \cup G_2 \cup G_3\}$ . Then, we repeat the procedure above with the pertinent changes in the notation, *etc.* The process repeats as long as Fact 1 applies, thereby generating a finite sequence of subgraphs  $\{G_k\}$ , with  $k \in \{1, 2, \dots, m\}$  and  $m \leq N$ , such that  $G_k$  is the largest strongly connected sub-graph having incoming links *only* from subgraphs  $G_\ell$  with  $\ell \leq k-1$ . For any such  $k$ , we obtain

$$\begin{bmatrix} A_{11} & 0 & \cdots & \cdots & 0 \\ -A_{21} & A_{22} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \\ -A_{k-1,1} & \cdots & & A_{kk} & 0 \\ [*] & \cdots & [*] & -R_{\bar{G}_k} & S_{\bar{G}_k} \end{bmatrix}. \quad (16)$$

By construction, the lowest-rightest sub-block in the matrix on the right-hand side of (16) may be decomposed as  $S_{\bar{G}_k} = L_{\bar{G}_k} + D_{\bar{G}_k}$ , where  $D_{\bar{G}_k}$  contains the weights of the links from the

graphs  $G_\ell$  with  $\ell \leq k$  to  $\bar{G}_k$  and the previous arguments apply if  $\bar{G}_k$  contains a spanning tree. On the contrary, if  $\bar{G}_k$ , for any  $k \geq 1$ , does not contain a spanning tree, the following applies.

*Case 2:* Assume that  $\bar{G}_k$ , with  $k \geq 1$ , does not contain a spanning tree. It follows that the associated Laplacian  $L_{\bar{G}_k}$  has  $\mu_k > 1$  null eigenvalues. After [3, Theorem 3.2]—cf. [4, Proposition 3], it follows that there exists a permutation matrix  $T$  such that

$$T^\top L_{\bar{G}_k} T = \begin{bmatrix} L_{\bar{G}_k}^1 & 0 & \cdots & 0 \\ 0 & L_{\bar{G}_k}^2 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ -M_{\mu_k+1,1} & \cdots & -M_{\mu_k+1,\mu_k} & M_{\mu_k+1,\mu_k+1} \end{bmatrix}, \quad (17)$$

where each block  $L_{\bar{G}_k}^i$ , with  $i \leq \mu_k$  corresponds to a Laplacian matrix associated to a sub-graph of  $\bar{G}_k$ , that contains a spanning tree and that we denote  $G_{k+i}$ , for all  $i \in \{1, 2, \dots, \mu_k\}$ . Therefore, each  $L_{\bar{G}_k}^i$  corresponding to a strongly connected graph  $G_{k+i}$  may be placed in the appropriate order in the block diagonal of a block-triangular matrix of the form (4), hence renamed  $A_{jj}$ . On the other hand, for each  $G_{k+i}$  that is not strongly connected, Fact 1 above applies, so we proceed as in Case 1. The sub-block  $M_{\mu_k+1,\mu_k+1}$  may be decomposed into  $M_{\mu_k+1,\mu_k+1} := L_{\mu_k+1,\mu_k+1} + D_{\mu_k+1,\mu_k+1}$ , where  $L_{\mu_k+1,\mu_k+1}$  is a Laplacian and  $D_{\mu_k+1,\mu_k+1}$  is a degree (diagonal semi-positive definite) matrix.  $L_{\mu_k+1,\mu_k+1}$  corresponds to a graph that may or may not have a spanning tree, so either Case above applies.

Since the graph  $G$  has a finite number of nodes  $N$ , the processes described in Cases 1 and 2 above finish when either  $\bar{G}_k$  in Case 1 or the graph with Laplacian  $L_{\mu_k+1,\mu_k+1}$  in Case 2 is strongly connected, so we set either  $A_{m,m} := S_{\bar{G}_k}$  or  $A_{m,m} := M_{\mu_k+1,\mu_k+1}$ . This event will surely occur because after sufficiently many iterations, either of those graphs may contain only one leaf node, which constitutes a strongly connected (trivial) graph with incoming edges. ■

*Proof of Lemma 3:* We follow the proof-lines of [2, Proposition 2]. Under the assumption that  $u_i \mapsto x_i$  defines a semi-passive map, for any  $i \leq N$  there exists a radially unbounded storage function  $V_i : \mathbb{R} \rightarrow \mathbb{R}_+$ , a continuous function  $H_i$ , a positive continuous function  $\psi_i$ , and a positive constant  $\rho_i$ , such that the total derivative along the trajectories of (1) yields

$$\dot{V}_i(x_i) \leq u_i^\top x_i - H_i(x_i), \quad (18)$$

where  $H_i(x_i) \geq \psi_i(|x_i|)$  for all  $|x_i| \geq \rho_i$ . Next, let  $V_\Sigma(\bar{x}) := \sum_{i=1}^N \mu_i V_i(x_i)$ , where  $\mu_i$  corresponds to the  $i$ th element of the left eigen-vector associated to the zero eigen-value of  $L$ . Since by assumption the network is strongly connected,  $\mu_i > 0$  for all  $i \leq N$ . Then, using (18) we see that

$$\dot{V}_\Sigma(\bar{x}) \leq \sum_{i=1}^N \mu_i u_i^\top x_i - \sum_{i=1}^N \mu_i H_i(x_i). \quad (19)$$

The first term on the right-hand side of (19) gives

$$\sum_{i=1}^N \mu_i u_i^\top x_i = u^\top \mathcal{M} \bar{x}, \quad (20)$$

where  $\mathcal{M} := \text{diag}[\mu_i]$ . Then, setting

$$u = -L\bar{x} - D\bar{x} + \sum_{j \leq p} B_j v_j,$$

it follows that the derivative of  $V_\Sigma(\bar{x})$  along the trajectories of (8) satisfies

$$\dot{V}_\Sigma(\bar{x}) \leq - \sum_{i=1}^N \mu_i H_i(x_i) - \bar{x}^\top L^\top \mathcal{M} \bar{x}$$

$$-\bar{x}^\top D\mathcal{M}\bar{x} + \left[ \sum_{j \leq p} B_j v_j \right]^\top \mathcal{M}\bar{x}. \quad (21)$$

Now, since the units are semi-passive, for each  $i \leq N$ , there exists  $\rho_i > 0$  such that  $H_i(x_i) \geq \psi_i(|x_i|)$  for all  $|x_i| \geq \rho_i$ . Then, let  $\bar{\rho} := \max_i \{\rho_i\}$ ; it follows that

$$-\sum_{i=1}^N \mu_i H_i(x_i) \leq -\sum_{i=1}^N \mu_i \psi_i(x_i), \quad (22)$$

for all all  $|x_i| \geq \bar{\rho}$ .

Furthermore, since the graph is strongly connected,  $L^\top \mathcal{M} + \mathcal{M}L$  is positive semi-definite—*cf.* [2, Proof of Proposition 2]. Hence, the second term on the right-hand side of (21) is non-positive.

For the last two terms on the right-hand side of (21) we observe that by the definition of  $d_k$  and the fact that all the elements of any  $B_j$  are non-negative, we have  $d_k = 0$  if and only if  $[B_j]_{k\ell} = 0$  for all  $\ell \leq M_j$  and for all  $j \leq p$ . Therefore, the third and fourth terms on the right-hand side of (21) satisfy

$$-\bar{x}^\top D\mathcal{M}\bar{x} + \left[ \sum_{j \leq p} B_j v_j \right]^\top \mathcal{M}\bar{x} \leq -\sum_{i=1}^N [c_{1i}x_i^2 - c_{2i}|x_i|], \quad (23)$$

where  $c_{1i}, c_{2i} \geq 0$  and  $c_{1i} = 0$  if and only if  $c_{2i} = 0$ . Therefore, for any  $i \leq N$  there exists  $\eta_i \geq 0$  such that for all  $|x_i| \geq \eta_i$ ,  $c_{1i}x_i^2 \geq c_{2i}|x_i|$ . Thus,

$$\dot{V}_\Sigma(x) \leq -\sum_{i=1}^N \mu_i \psi_i(|x_i|) \leq 0 \quad (24)$$

for all  $|x_i| \geq \max\{\rho_i, \eta_i\}$ . We conclude that if for any  $i \leq N$ ,  $|x_i(t)| \rightarrow \infty$  then there exists  $T > 0$  such that for all  $t \geq T$ , we have  $\dot{V}_\Sigma(x(t)) \leq 0$  for all  $t \geq T$ . The statement follows.  $\blacksquare$

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