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# Global Uniform Ultimate Boundedness of Semi-Passive Systems Interconnected over Directed Graphs

Anes Lazri   Mohamed Maghenem   Elena Panteley   Antonio Loria \*

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## Abstract

We analyse the solutions of networked heterogeneous nonlinear systems<sup>1</sup>

$$\dot{x}_i = f_i(x_i) + u_i \quad x_i \in \mathbb{R}, \quad i \in \{1, 2, \dots, n\}, \quad (1)$$

where  $f_i : \mathbb{R} \rightarrow \mathbb{R}$  is continuous for all  $i \in \{1, 2, \dots, n\}$  and the control inputs are set to

$$u_i := -\gamma \sum_{j=1}^n a_{ij}(x_i - x_j) \quad \forall i \in \{1, 2, \dots, n\}, \quad (2)$$

where  $\gamma > 0$  is a coupling gain and  $a_{ij} \geq 0$  are interconnection weights. We assume that the closed-loop interconnected systems form a network with an underlying connected directed graph that contains a directed spanning tree. For these systems, we establish global uniform ultimate boundedness of the solutions, under the assumption that each system (1) defines a semi-passive [5] map  $u_i \mapsto x_i$ . As a corollary, we also establish global uniform global boundedness of the solutions.

## 1 Preliminaries

**Notations.** For  $x \in \mathbb{R}^n$ ,  $x^\top$  denotes its transpose,  $|x|$  denotes its Euclidean norm,  $\text{blkdiag}\{x\} \in \mathbb{R}^{n \times n}$  denotes the diagonal matrix whose  $i$ th diagonal element is the  $i$ th element of  $x$ . For a set  $K \subset \mathbb{R}^n$ ,  $|x|_K := \min\{|x - y| : y \in K\}$  denotes the distance of  $x$  to the set  $K$ . For a symmetric matrix  $Q \in \mathbb{R}^{n \times n}$ ,  $\lambda_i(Q)$  denotes the  $i$ th smallest eigenvalue of  $Q$ . For an invertible matrix  $M \in \mathbb{R}^{n \times n}$ ,  $M^-$  or  $M^{-1}$  denotes its inverse. Given  $N \in \mathbb{R}^{n \times n}$ ,  $\text{Ker}(N) := \{v : Nv = 0\}$  denotes the kernel of  $N$ . A class  $\mathcal{K}^\infty$  function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is continuous, strictly increasing, unbounded, and  $\alpha(0) = 0$ . Furthermore  $\alpha^-$  denotes the inverse function of  $\alpha$ .

### 1.1 On Some Classes of Matrices

A matrix  $M := [m_{ij}]$ ,  $(i, j) \in \{1, 2, \dots, n\}^2$ , is a  $\mathcal{Z}$ -matrix if  $m_{ij} \leq 0$  whenever  $i \neq j$ . It is an  $M$ -matrix if it is a  $\mathcal{Z}$ -matrix and its eigenvalues have non-negative real parts. Equivalently,  $M := \lambda I_n - B$ , where  $B$  is a non-negative matrix and  $\lambda \geq \rho(B)$ , where  $\rho(B) := \max\{|\lambda_i(B)| : i \in \{1, 2, \dots, n\}\}$  is the spectral radius of  $B$ .  $M$  is a non-singular  $M$ -matrix if it is a  $\mathcal{Z}$ -matrix and its eigenvalues have positive real parts. Equivalently,  $M := \lambda I_n - B$ , where  $B$  is a non-negative matrix and  $\lambda > \rho(B) > 0$ ; see [2, 3] for more details.

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<sup>1</sup>For simplicity, but without loss of generality, we assume that  $x \in \mathbb{R}$ ; all statements hold after pertinent changes in the notation, if  $x \in \mathbb{R}^p$ , with  $p > 1$ .

## 1.2 Graph Notions

A directed graph or a digraph  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  is characterized by the set of nodes  $\mathcal{V} = \{1, 2, \dots, n\}$ , and the set of directed edges  $\mathcal{E}$ . The edge set  $\mathcal{E}$  consists of ordered pairs, of the form  $(k, i)$ , that indicate a directed link from node  $k$  to node  $i$ . Given a directed edge  $(k, i) \in \mathcal{E}$ , then node  $k$  is called an *in-neighbor* of node  $i$ . We assign a positive weight  $a_{ik}$  to each edge  $(k, i)$ . That is,  $a_{ik} = 0$  if  $(k, i)$  is not an edge. The Laplacian matrix of a digraph is given by

$$L := \begin{bmatrix} d_1 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & d_2 & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n-11} & \cdots & d_{n-1} & -a_{n-1n} \\ -a_{n1} & \cdots & -a_{nn-1} & d_n \end{bmatrix} =: D - A, \quad (3)$$

where  $d_i := \sum_{j=1}^n a_{ij}$  for all  $i \in \{1, 2, \dots, n\}$ ,  $D$  is the diagonal part of  $L$  and  $A$  is called the adjacency matrix.

A digraph is *strongly connected* if, for any two distinct nodes  $i$  and  $j$ , there is a path from  $i$  to  $j$ . The Laplacian matrix of a strongly connected graph admits  $\lambda_1(L) = 0$  as an eigenvalue with the corresponding right and left eigenvectors  $\mathbf{1}_n = [1 \ 1 \ \cdots \ 1]^\top$  and  $\mathbf{v}_o := [v_1 \ v_2 \ \cdots \ v_n]^\top$ , respectively, where  $v_i > 0$  for all  $i \leq n$ .

## 1.3 Graph and Matrix Decomposition

Suppose that the digraph  $\mathcal{G}$  is connected and contains a spanning tree. Then, it admits a decomposition into a leading strongly connected subgraph  $\mathcal{G}_\ell \neq \emptyset$  and a subgraph  $\mathcal{G}_f := \mathcal{G} \setminus \mathcal{G}_\ell$  of followers; namely, the agents that do not belong to the leading component, and which we call the follower agents. In this case, up to a permutation, the Laplacian  $L$  admits the lower-block decomposition

$$L = \begin{bmatrix} L_\ell & 0 \\ -A_{\ell f} & M_f \end{bmatrix}, \quad (4)$$

where  $L_\ell := D_\ell - A_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$  is the Laplacian matrix of the strongly connected component  $\mathcal{G}_\ell$ , the lower-left block  $A_{\ell f} \in \mathbb{R}^{n_\ell \times n - n_\ell}$ ,  $n_f := n - n_\ell$ , is a non-negative matrix, and the lower-right block  $M_f \in \mathbb{R}^{n_f \times n_f}$  is a non-singular M-matrix. The block  $M_f$  can be seen as the sum of the Laplacian matrix  $L_f$  corresponding to  $\mathcal{G}_f$  and a diagonal matrix  $D_{\ell f}$  gathering the weights of the interconnections between nodes in  $\mathcal{G}_\ell$  and the nodes in  $\mathcal{G}_f$ . That is,  $M_f = L_f + D_{\ell f}$ , where  $L_f = D_f - A_f$ .

## 1.4 Lyapunov Analysis of a Directed Graph

Consider a network of  $n$  single integrators of the form  $\dot{x}_i = u_i$  interconnected according to the classical consensus protocol

$$u_i := - \sum_{j=1}^n a_{ij}(x_i - x_j) \quad \forall i \in \{1, 2, \dots, n\}.$$

In closed loop, the network is governed by the linear system  $\dot{x} = -Lx$ , where  $L \in \mathbb{R}^n$  is the Laplacian matrix of a connected di-graph  $\mathcal{G}$  that contains a directed spanning tree. According to Section 1, we can decompose the state  $x$  into  $x^\top := [x_\ell^\top \ x_f^\top]$ , where  $x_\ell \in \mathbb{R}^{n_\ell}$  gathers the states of the leading component and is governed by

$$\Sigma_\ell : \dot{x}_\ell = -L_\ell x_\ell,$$

and the non-leading component whose state is  $x_f \in \mathbb{R}^{n_f}$ , are governed by

$$\Sigma_f : \dot{x}_f = -M_f x_f,$$

on the manifold  $\{x_f = 0\}$ . In the rest of this section, we overview some Lyapunov-function constructions allowing to prove uniform exponential stability of  $\mathcal{A}$  for  $\Sigma_\ell$ , where

$$\mathcal{A} := \{x_\ell \in \mathbb{R}^{n_\ell} : x_{\ell 1} = x_{\ell 2} = \dots = x_{\ell n_\ell}\}, \quad (5)$$

and exponential stability of the origin for  $\Sigma_f$ .

#### 1.4.1 Proof of uniform exponential stability of $\mathcal{A}$ for $\Sigma_\ell$

let  $v_o := [v_1 \ v_2 \ \dots \ v_{n_\ell}]^\top$  be a left eigenvector associated to  $\lambda_1(L_\ell) = 0$  and  $V_o := \text{blkdiag}\{v_o\}$ . Based on Lemma 1 in the Appendix,  $Q_o := L_\ell^\top V_o + V_o L_\ell$  is symmetric and positive semi-definite, and its kernel is spanned by  $\mathbf{1}_{n_\ell}$ . Then, the derivative of the Lyapunov function candidate  $W(x_\ell) := x_\ell^\top V_o x_\ell$ , along the solutions to  $\Sigma_\ell$ , satisfies

$$\dot{W}(x_\ell) = -x_\ell^\top (L_\ell^\top V_o + V_o L_\ell) x_\ell \leq -\lambda_2(Q_o) |x_\ell|_{\mathcal{A}_\ell}^2.$$

Now, we let

$$Z(x_\ell) := (x_\ell - \mathbf{1}_{n_\ell} v_o^\top x_\ell)^\top V_o (x_\ell - \mathbf{1}_{n_\ell} v_o^\top x_\ell),$$

which is positive definite. Its derivative along the solutions of  $\dot{x}_\ell = -L_\ell x_\ell$  satisfies

$$\dot{Z}(x_\ell) = -x_\ell^\top Q_o x_\ell \leq -\lambda_2(Q_o) |x_\ell|_{\mathcal{A}_\ell}^2. \quad (6)$$

To obtain the previous expression we used  $v_o^\top L = 0$ ,  $v_1^\top \mathbf{1}_{n_\ell} = 1$  and that  $\mathbf{1}_{n_\ell}$  is in the kernel of  $I_{n_\ell} - \mathbf{1}_{n_\ell} v_o^\top$ . Moreover,  $I_{n_\ell} - \mathbf{1}_{n_\ell} v_o^\top$  is the Laplacian matrix of an all-to-all graph; hence,  $\mathbf{1}_{n_\ell}$  spans the kernel of  $I_{n_\ell} - \mathbf{1}_{n_\ell} v_o^\top$ . Therefore, there exist  $\bar{z}, \underline{z} > 0$  such that

$$\underline{z} |x_\ell|_{\mathcal{A}_\ell}^2 \leq Z(x_\ell) \leq \bar{z} |x_\ell|_{\mathcal{A}_\ell}^2 \quad \forall x_\ell \in \mathbb{R}^{n_\ell}. \quad (7)$$

Uniform exponential stability of  $\mathcal{A}_\ell$  from (6) and (7) and standard Lyapunov-stability theory.

#### 1.4.2 Proof of exponential Stability of the Origin for $\Sigma_f$

based on Lemma 2, since  $M_f$  is a non-singular  $M$ -matrix, we can use the Lyapunov function candidate  $Y(x_f) := x_f^\top R_f x_f$ , where  $R_f := \text{blkdiag}\{M_f^{-\top} \mathbf{1}_{n_f}\} \left( \text{blkdiag}\{M_f^{-1} \mathbf{1}_{n_f}\} \right)^{-1}$ , which is positive definite. Furthermore, along the solutions to  $\Sigma_f$ , we have

$$\dot{Y}(x_f) = -x_f^\top [M_f^\top R_f + R_f M_f] x_f.$$

Now, since  $(M_f^\top R_f + R_f M_f)$  is positive definite, exponential stability of the origin for  $\Sigma_f$  follows.

## 2 Problem formulation

Consider the systems (1)-(2), with  $\gamma > 0$  and  $a_{ij} \geq 0$ . Then, defining  $x := [x_1 \ \dots \ x_n]^\top$ , and  $F(x) := [f_1(x_1), f_2(x_2), \dots, f_n(x_n)]^\top$ , we may write the closed-loop system in compact form as

$$\dot{x} = F(x) - \gamma L x, \quad (8)$$

where  $L$  is defined as in (3). This is a networked system with an underlying topology that may be represented by a graph  $\mathcal{G}$ .

**Assumption 1** *The graph  $\mathcal{G}$  is connected and contains a directed spanning tree.* •

We are interested in verifying the following two boundedness properties for (8).

(P1) *Global Uniform Boundedness (GUB).* The solutions  $t \rightarrow x(t)$  to (8) are globally bounded, uniformly in  $\gamma$ , if, for every  $r_o > 0$  and  $\gamma_o > 0$ , there exists  $R = R(r_o, \gamma_o) \geq r_o$  such that, for all  $\gamma \geq \gamma_o$ ,

$$|x(t_o)| \leq r_o \Rightarrow |x(t)| \leq R \quad \forall t \geq 0.$$

(P2) *Global Uniform Ultimate Boundedness (GUUB).* The solutions  $t \rightarrow x(t)$  to (8) are ultimately bounded, uniformly in  $\gamma$ , if given  $\gamma_o > 0$ , there exists  $r = r(\gamma_o) > 0$  such that, for all  $r_o > 0$ , there exists  $T = T(r_o, \gamma_o) \geq 0$  such that, for all  $\gamma \geq \gamma_o$ ,

$$|x(t_o)| \leq r_o \Rightarrow |x(t)| \leq r \quad \forall t \geq T.$$

To verify the latter two properties, we make the following assumption on the individual nodes' dynamics in (1).

**Assumption 2 (State strict semi-passivity)** *For each  $i \in \{1, 2, \dots, n\}$ , the input-output map  $u_i \mapsto x_i$  defined by the dynamics (1) is state strict semipassive [1]. Furthermore, there exists a continuously differentiable storage function  $V_i : \mathbb{R}^n \rightarrow \mathbb{R}^+$ , a class  $\mathcal{K}_\infty$  function  $\underline{\alpha}_i$ , a constant  $\rho_i > 0$ , a continuous function  $H_i : \mathbb{R} \rightarrow \mathbb{R}$ , and a continuous function  $\psi_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$ , such that*

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i), \quad \dot{V}_i(x_i) \leq 2u_i x_i - H_i(x_i), \quad (9)$$

and  $H_i(x_i) \geq \psi_i(|x_i|)$  for all  $|x_i| \geq \rho_i$ . •

**Remark 1** The property described in Assumption 2 is called strict *quasipassivity* in [4]. In [5] the authors define a similar concept named strict semi-passivity, but radial unboundedness of the storage function is not imposed. See also [1]. •

### 3 Main result

**Theorem 1 (Uniform ultimate boundedness)** *The solutions of the networked system (1)-(2) are globally uniformly ultimately bounded, i.e., Property (P2) holds, if Assumptions 1 and 2 are satisfied.* □

*Proof:* Under Assumption 1, the Laplacian matrix  $L$  admits a permutation, such that (4) holds. Therefore, the state  $x$  may be decomposed into  $x := [x_\ell^\top \ x_f^\top]^\top$  and the system (8) takes the cascaded form

$$\dot{x}_\ell = f_\ell(x_\ell) - \gamma L_\ell x_\ell, \quad f_\ell(x_\ell) := [f_1(x_{\ell_1}) \ \cdots \ f_{n_\ell}(x_{\ell_{n_\ell}})]^\top \quad (10a)$$

$$\dot{x}_f = f_f(x_f) + \gamma A_{\ell f} x_\ell - \gamma M_f x_f, \quad f_f(x_f) := [f_{n_\ell+1}(x_{f_1}) \ \cdots \ f_{n_\ell+n_f}(x_{f_{n_f}})]^\top. \quad (10b)$$

Equation (10a) corresponds to the dynamics of a *leading* component, a networked system with an underlying strongly connected graph  $\mathcal{G}_\ell$ , and a *follower* component, with dynamics (10b). The proof of the statement is constructed using a cascades argument and proving, firstly, global uniform ultimate boundedness for the solutions of (10a) and, consequently, the same property for (10b).

To that end, let  $r_o > 0$  be arbitrarily fixed and let  $|x(0)| \leq r_o$ . Then,  $|x_\ell(0)| \leq r_o$  and  $|x_f(0)| \leq r_o$ .

1) *Uniform ultimate boundedness for the leading component:* after Assumption 2, for each  $i \in \{1, 2, \dots, n_\ell\}$ , there exists a storage function  $V_i$  such that its total derivative along the trajectories of (1) satisfies

$$\dot{V}_i(x_{\ell i}) \leq 2u_i^\top x_{\ell i} - H_i(x_{\ell i}), \quad H_i(x_{\ell i}) \geq \psi_i(|x_{\ell i}|) \quad \forall |x_{\ell i}| \geq \rho_i. \quad (11)$$

Next, let  $W(x_\ell) := \sum_{i=1}^{n_\ell} v_i V_i(x_{\ell i})$ , where  $v_i$  corresponds to the  $i$ th element of  $v_o$ , which is the left eigenvector associated to the zero eigenvalue of  $L_\ell$ . Since the graph  $\mathcal{G}_\ell$  is strongly connected, then  $v_i > 0$  for all  $i \leq n_\ell$ , so  $W$  is positive definite and radially unbounded. Now, from (11), we obtain

$$\dot{W}(x_\ell) \leq 2 \sum_{i=1}^{n_\ell} v_i u_i^\top x_{\ell i} - \sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}), \quad \forall x_\ell \in \mathbb{R}^{n_\ell}. \quad (12)$$

The first term on the right-hand side of (12) satisfies

$$\sum_{i=1}^{n_\ell} v_i u_i^\top x_{\ell i} = u^\top V_o x_\ell, \quad (13)$$

where  $V_o := \text{blkdiag}\{v_o\}$  and, since  $u = -\gamma L_\ell x_\ell$ , it follows that

$$\begin{aligned} \dot{W}(x_\ell) &\leq - \sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}) - \gamma x_\ell^\top [L_\ell^\top V_o + V_o L_\ell] x_\ell \\ &\leq - \sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}) - \gamma x_\ell^\top Q_o x_\ell, \end{aligned} \quad (14)$$

with  $Q_o := V_o L_\ell + L_\ell^\top V_o$ , which is positive semi-definite—see Lemma 1 in the Appendix. Furthermore, we note that

$$-x_\ell^\top Q_o x_\ell = - [x_\ell - \mathbf{1}_{n_\ell} \mathbf{1}_{n_\ell}^\top x_\ell / n_\ell]^\top Q_o [x_\ell - \mathbf{1}_{n_\ell} \mathbf{1}_{n_\ell}^\top x_\ell / n_\ell] \leq -\lambda_2(Q_o) |x_\ell|_{\mathcal{A}}^2,$$

where  $|x_\ell|_{\mathcal{A}}$  denotes the distance of  $x_\ell$  to the set  $\mathcal{A}$  and  $\lambda_2(Q_o)$  is the second smallest eigenvalue of  $Q_o$ .

Now, on one hand, we have that  $v_i > 0$  for all  $i \in \{1, 2, \dots, n\}$  and, on the other,  $-H_i(x_{\ell i}) > 0$  only if  $|x_{\ell i}| \leq \rho_i$ . Therefore, the constant  $H_\ell := -\sum_{i=1}^{n_\ell} \max_{|x_i| \leq \rho_i} \{v_i H_i(x_{\ell i})\} > 0$ . Therefore, after (14), we get

$$\dot{W}(x_\ell) \leq H_\ell - \gamma \lambda_2(Q_o) |x_\ell|_{\mathcal{A}}^2 \quad \forall x_\ell \in \mathbb{R}^{n_\ell}. \quad (15)$$

In turn, given  $\gamma_o > 0$  and  $\epsilon > 0$ , for all  $\gamma \geq \gamma_o$ , we have

$$\dot{W}(x_\ell) \leq H_\ell - \gamma_o \lambda_2(Q_o) |x_\ell|_{\mathcal{A}}^2 \leq -\epsilon \quad \forall x_\ell \notin \mathcal{C}, \quad (16)$$

where

$$\mathcal{C} := \left\{ x_\ell \in \mathbb{R}^{n_\ell} : |x_\ell|_{\mathcal{A}} \leq \sqrt{n_\ell} R_e := \sqrt{\frac{\epsilon + H_\ell}{\gamma_o \lambda_2(Q_o)}} \right\}.$$

Next, let  $\bar{\rho} := \arg \max_{i \in \{1, 2, \dots, n_\ell\}} \rho_i$  and

$$\mathcal{B}_\beta := \{x_\ell \in \mathbb{R}^{n_\ell} : |x_\ell| \leq \beta := \sqrt{n_\ell}(\bar{\rho} + 2R_e)\}.$$

Note that for all  $x_\ell \notin \mathcal{B}_\beta$ , we have  $|x_\ell| > \sqrt{n_\ell}(\bar{\rho} + 2R_e)$  and, for all  $x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta$ ,

$$|x_\ell| > \sqrt{n_\ell}(\bar{\rho} + 2R_e) \quad \text{and} \quad |x_\ell|_{\mathcal{A}} \leq \sqrt{n_\ell} R_e. \quad (17)$$

Furthermore, we use the fact that  $x_\ell = \mathbf{1}_{n_\ell}(\mathbf{1}_{n_\ell}^\top x_\ell)/n_\ell + [x_\ell - \mathbf{1}_{n_\ell}(\mathbf{1}_{n_\ell}^\top x_\ell)/n_\ell]$ , and the fact that  $|x_\ell|_{\mathcal{A}} = |x_\ell - \mathbf{1}_{n_\ell}(\mathbf{1}_{n_\ell}^\top x_\ell)/n_\ell|$ , to conclude that

$$|x_\ell| \leq |x_\ell|_{\mathcal{A}} + |\mathbf{1}_{n_\ell}^\top x_\ell|/\sqrt{n_\ell}. \quad (18)$$

Now, combining (17) and (18), we conclude that for all  $x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta$ ,

$$\sqrt{n_\ell}(\bar{\rho} + 2R_e) < |x_\ell| \leq |x_\ell|_{\mathcal{A}} + |\mathbf{1}_{n_\ell}^\top x_\ell|/\sqrt{n_\ell} \leq \sqrt{n_\ell}R_e + |\mathbf{1}_{n_\ell}^\top x_\ell|/\sqrt{n_\ell}. \quad (19)$$

So, for all  $x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta$ ,  $|\mathbf{1}_{n_\ell}^\top x_\ell|/n_\ell > \bar{\rho} + R_e$ . Next, we use the fact that

$$x_{\ell i} = \mathbf{1}_{n_\ell}^\top x_\ell/n_\ell + (x_{\ell i} - \mathbf{1}_{n_\ell}^\top x_\ell/n_\ell) \quad \forall i \in \{1, 2, \dots, n_\ell\}$$

to conclude that  $|x_{\ell i}| > |\mathbf{1}_{n_\ell}^\top x_\ell|/n_\ell - |(x_{\ell i} - \mathbf{1}_{n_\ell}^\top x_\ell/n_\ell)|$ . Hence,

$$|x_{\ell i}| > \bar{\rho} + R_e - \sqrt{n_\ell}R_e > \bar{\rho} \quad \forall i \in \{1, 2, \dots, n_\ell\}$$

for all  $x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta$ . The latter, under Assumption 2, implies that

$$-\sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}) \leq -\sum_{i=1}^{n_\ell} v_i \psi_i(|x_{\ell i}|) \leq 0 \quad \forall x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta.$$

As a result, setting  $\Psi(x_\ell) := \sum_{i=1}^{n_\ell} v_i \psi_i(|x_{\ell i}|)$ —note that  $\Psi$  is continuous and positive—we conclude that

$$\dot{W}(x_\ell) \leq -\Psi(x_\ell) \quad \forall x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta.$$

Combining the latter inequality to (16), we conclude that

$$\dot{W}(x_\ell) \leq -\min\{\Psi(x_\ell), \epsilon\} \quad \forall x_\ell \in \mathbb{R}^{n_\ell} \setminus \mathcal{B}_\beta.$$

The latter is enough to conclude global attractivity and forward invariance of the set

$$\mathcal{S}_\sigma := \{x_\ell \in \mathbb{R}^{n_\ell} : W(x_\ell) \leq \sigma\}, \quad \sigma := \max\{W(y) : y \in \mathcal{B}_\beta\}.$$

Furthermore, since  $W : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is continuous and  $\mathcal{B}_\beta$  is bounded, we conclude that  $\sigma$  is well defined. Consequently, the ultimate bound is

$$r_\ell := \left[ \min_i \{\underline{\alpha}_i\} \right]^- (\sigma),$$

where, with an abuse of notation,  $\min_i \{\underline{\alpha}_i\}$  corresponds to the function  $s \mapsto \psi(s)$  defined as  $\psi(s) := \min_i \{\underline{\alpha}_i(s)\}$  for each  $s \geq 0$  and  $\underline{\alpha}_i$  is defined in Assumption 2, so  $\psi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is strictly increasing and radially unbounded, hence, globally invertible. Thus,  $W(x_\ell) \leq \sigma$  implies that  $|x_\ell| \leq r_\ell$ .

Next, we compute an upperbound  $T_\ell(r_o, \gamma_o)$  on the time that the solutions to (10a), with  $\gamma \geq \gamma_o$  and starting from  $\mathcal{B}_{r_o} := \{x_\ell \in \mathbb{R}^{n_\ell} : |x_\ell| \leq r_o\}$ , take to reach the compact set  $\mathcal{B}_\beta \subset \mathcal{S}_\sigma$ . For this, we assume without loss of generality that  $r_o \geq \beta$ , and we define

$$\epsilon_{r_o} := \min\{\min\{\Psi(x_\ell), \epsilon\} : |x_\ell| \geq \beta, x_\ell \in \mathcal{S}_{\sigma_o}\},$$

where

$$\mathcal{S}_{\sigma_o} := \{x_\ell \in \mathbb{R}^{n_\ell} : W(x_\ell) \leq \sigma_o\}, \quad \sigma_o := \max\{W(y) : y \in \mathcal{B}_{r_o}\}. \quad (20)$$

Clearly,  $\mathcal{S}_{\sigma_o}$  is compact; hence,  $\epsilon_{r_o}$  is positive.

Therefore, along every solution  $t \mapsto x_\ell(t)$  to (10a) starting from  $x_\ell(0) \in \mathcal{B}_{r_o} \setminus \mathcal{B}_\beta$ , we have  $\dot{W}(x_\ell(t)) \leq -\epsilon_{r_o}$ , up to the earliest time when  $x_\ell$  reaches  $\mathcal{B}_\beta$ . For any earlier time, we have

$$W(x_\ell(t)) \leq -\epsilon_{r_o} t + W(x_\ell(0)), \quad (21)$$

so we can take  $T_\ell(r_o, \gamma_o) = \sigma_o/\epsilon_{r_o}$ . Clearly,  $T_\ell$  depends only on  $(r_o, \gamma_o)$  and  $r_\ell$  depends only on  $\gamma_o$ . Thus, the ultimate boundedness guaranteed for the solutions of (10a) is uniform in  $\gamma$ .

2) *Uniform ultimate boundedness for the follower dynamics*: following up the previous computations and arguments, establish global uniform ultimate boundedness for the non-leading component, determined by (10b).

Using Lemma 2, we conclude that the matrices

$$S := PM_f + M_f^\top P \quad \text{and} \quad P := \text{blkdiag} \left\{ M_f^{\top-1} 1_n \right\} \left( \text{blkdiag} \left\{ M_f^- 1_n \right\} \right)^-$$

are symmetric and positive definite. We also note that  $P$  is diagonal. Then, let  $p_i$ , for  $i \in \{1, 2, \dots, n_f\}$ , be the  $i$ th diagonal element of  $P$ . In addition, let  $Z(x_f) := \sum_{i=1}^{n_f} p_i V_i(x_{fi})$ . Its total derivative along the trajectories of (10b) satisfies

$$\dot{Z}(x_f) \leq - \sum_{i=1}^{n_f} p_i H_i(x_{fi}) - \gamma x_f^\top [PM_f + M_f^\top P] x_f + 2\gamma x_f^\top [PA_{\ell f}] x_\ell. \quad (22)$$

On one hand, we already established the existence of  $r_\ell(\gamma_o) > 0$  and  $T_\ell(\gamma_o, r_o)$  such that

$$|x_\ell(t)| < r_\ell \quad \forall t \geq T_\ell.$$

On the other, for all  $|x_\ell| \leq r_\ell$ ,

$$\dot{Z}(x_f) \leq H_f - \gamma \lambda_1(S) |x_f|^2 + 2\gamma \bar{p} r_\ell |x_f|, \quad (23)$$

where  $\bar{p} := |PA_{\ell f}|$  and  $H_f := \sum_{i=1}^{n_f} \max_{|x_i| \leq \rho_i} \{v_i H_i(x_{fi})\}$ . Now, from this and (22), we obtain

$$\begin{aligned} \dot{Z}(x_f) &\leq H_f - \gamma x_f^\top S x_f + 2\gamma x_f^\top [PA_{\ell f}] x_\ell \\ &\leq H_f - \gamma [x_f^\top S x_f / 2 - 2x_\ell^\top A_{\ell f}^\top P^\top S^- PA_{\ell f} x_\ell]. \end{aligned}$$

At the same time, integrating (15), we obtain that, for each  $t \in [0, T_\ell]$ ,

$$W(x_\ell(t)) \leq H_\ell T_\ell + W(x_\ell(0)) \leq H_\ell T_\ell + \sigma_o,$$

where  $\sigma_o$  comes from (20). Defining

$$R_\ell := \left[ \min_i \{\underline{\alpha}_i\} \right]^- (H_\ell T_\ell + \sigma_o),$$

we have

$$\dot{Z}(x_f) \leq H_f - \gamma [\lambda_1(S) |x_f|^2 / 2 - 2|A_{\ell f}^\top P^\top S^- PA_{\ell f}| R_\ell^2],$$

for all  $|x_\ell| \leq R_\ell$ .

Note that, for all  $x_f$  such that

$$|x_f|^2 > d_f^2 := \frac{4|A_{\ell f}^\top P^\top S^- PA_{\ell f}| R_\ell^2}{\lambda_1(S)} + \frac{42H_f}{\lambda_1(S)\gamma_o},$$

$\dot{Z}(x_f) \leq 0$ . This implies that, for all  $t \in [0, T_\ell]$ ,

$$Z(x_f(t)) \leq \max \{ \sigma_{f_o}, \sigma_f \}, \quad \sigma_f := \max \{ Z(x_f) : |x_f| \leq d_f \} \quad \sigma_{f_o} := \max \{ Z(x_f) : |x_f| \leq r_o \}. \quad (24)$$

In turn, for each  $t \in [0, T_\ell]$ ,

$$|x_f(t)| \leq \bar{r}_o := \left[ \left( \sum_{i=1}^{n_f} p_i \right) \min_i \{ \underline{\alpha}_i \} \right]^- (\max \{ \sigma_{f_o}, \sigma_f \}). \quad (25)$$



Clearly, the previous upper bound is uniform in  $\gamma \geq \gamma_o$ .

Next, we focus on the solutions' behaviour after  $T_\ell$  (i.e., once  $|x_\ell| \leq r_\ell$ ). Given  $\epsilon > 0$ , we see that, for all  $\gamma \geq \gamma_o$  and for all  $x_f$  and  $x_\ell$  such that

$$|x_f| > \beta_1 := 1 + \frac{2\bar{p}r_\ell}{\lambda_1(S)} + \sqrt{\frac{\epsilon + H_f}{\gamma_o \lambda_1(S)}} \quad \text{and} \quad |x_\ell| \leq r_\ell,$$

after (23), we conclude that  $\dot{Z}(x_f) \leq -\epsilon$ . Furthermore,  $|x_\ell(t)| \leq r_\ell$  for all  $t \geq T_\ell$ , then the set

$$\mathcal{S}_{\sigma_1} := \{x_f \in \mathbb{R}^{n_f} : Z(x_f) \leq \sigma_1\}, \quad \sigma_1 := \max\{Z(y) : y \in \mathcal{B}_{\beta_1}\}, \quad \mathcal{B}_{\beta_1} := \{x_f \in \mathbb{R}^{n_f} : |x_f| \leq \beta_1\},$$

is attractive and becomes forward invariant after time  $T_\ell$ .

Since  $Z : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is continuous and  $\mathcal{B}_{\beta_1}$  is bounded, we conclude that  $\sigma_1$  is well defined. As a result, the ultimate bound for  $x_f(t)$  is

$$r_f = \left[ \left( \sum_{i=1}^{n_f} p_i \right) \min_i \{\underline{\alpha}_i\} \right]^- (\sigma_1).$$

Indeed,  $Z(x_f) \leq \sigma_1$  implies  $|x_f| \leq r_f$ .

Finally, as for  $t \mapsto x_\ell(t)$  we give next an upperbound, denoted by  $T_f(r_o, \gamma_o)$ , on the time that the solutions to (10b), with  $\gamma \geq \gamma_o$  and starting from  $\mathcal{B}_{r_o} := \{x_f \in \mathbb{R}^{n_\ell} : |x_f| \leq r_o\}$ , take to reach  $\mathcal{B}_{\beta_1} \subset \mathcal{S}_{\sigma_1}$ .

Let a solution  $t \mapsto x_f(t)$  to (10b) starting from  $x_f(0) \in \mathcal{B}_{r_o}$ . Now, we use the fact  $|x_f(T_\ell)| \leq \bar{r}_o$  with  $\bar{r}_o$  coming from (25) and  $\bar{r}_o$  is uniform in  $\gamma$ . As a result, along the solution  $t \mapsto x_f(t)$ , we have  $\dot{Z}(x_f(t)) \leq -\epsilon$  from  $T_\ell$  and up to when it reaches  $\mathcal{B}_{\beta_1}$  for the first time after  $T_\ell$ . Hence, before reaching  $\mathcal{B}_{\beta_1}$ , we have

$$Z(x_f(t)) \leq -\epsilon t + Z(x_f(T_\ell)) \quad (26)$$

and, thus, using (24), we can take  $T_f = T_\ell + \max\{\sigma_{f_o}, \sigma_f\}/\epsilon$ . Clearly,  $T_f$  and  $r_f$  depend only on  $(r_o, \gamma_o)$ . Thus, the ultimate bounded guaranteed for the solutions of (10b) is also uniform in  $\gamma$ . ■

**Corollary 1 (Uniform boundedness)** *Under Assumptions 1 and 2 the solutions of the closed-loop system in (8) are globally uniformly bounded, i.e., Property (P1) holds. □*

*Proof:* The statement of Theorem 1 holds, therefore, given  $r_o > 0$  and  $\gamma_o > 0$ , for all  $\gamma \geq \gamma_o$ , we have

$$|x_\ell(0)| \leq r_o \implies |x_\ell(t)| \leq r_\ell(\gamma_o) \quad \forall t \geq T_\ell(r_o, \gamma_o).$$

Furthermore, we were able to show that on the interval  $[0, T_\ell(r_o, \gamma_o)]$ , we have

$$W(x_\ell(t)) \leq H_\ell T_\ell + W(x_\ell(0)).$$

Hence, if we let  $\sigma_\ell := \max\{W(y) : |y| \leq r_o\}$ , it follows that

$$|x_\ell(t)| \leq R_\ell := \left[ \min_i \{\underline{\alpha}_i\} \right]^- (\sigma_\ell + H_\ell T_\ell + r_\ell) \quad \forall t \geq 0.$$

Next, for the solutions to (10b), for any  $\gamma > \gamma_o$  and  $|x_f(0)| \leq r_o$ , we know that

$$|x_f(t)| \leq r_f \quad \forall t \geq T_f(\gamma_o, r_o).$$

At the same time, from the previous proof, we know that

$$\dot{Z}(x_f) \leq H_f - \gamma [\lambda_1(S)|x_f|^2/2 - 2|A_{\ell f}^\top P^\top S^- P A_{\ell f}| R_\ell^2].$$

As a result, when

$$|x_f|^2 > d_f^2 := \frac{4|A_{\ell f}^\top P^\top S^- P A_{\ell f}| R_\ell^2}{\lambda_1(S)} + \frac{42H_f}{\lambda_1(S)\gamma_o},$$

then  $\dot{Z}(x_f) \leq 0$ . Hence, for each  $t \geq 0$ ,

$$Z(x_f(t)) \leq \max\{\sigma_{f_o}, \sigma_f\}, \quad \sigma_f := \max\{Z(x_f) : |x_f| \leq d_f\}, \quad \sigma_{f_o} := \max\{Z(x_f) : |x_f| \leq r_o\}. \quad (27)$$

In turn, for each  $t \geq 0$ , we have

$$|x_f(t)| \leq R_f := \left[ \left( \sum_{i=1}^{n_f} p_i \right) \min_i \{\alpha_i\} \right]^- (\max\{\sigma_{f_o}, \sigma_f\}). \quad (28)$$

■

## Appendix

The following lemma is proposed in [3], see also [6].

**Lemma 1** *Let  $L \in \mathbb{R}^{n \times n}$  be the Laplacian matrix of a directed and strongly connected graph. Let  $v_o := [v_1, v_2, \dots, v_n]^\top \in \mathbb{R}^n$  be the left eigenvector of  $L$  associated to the null eigenvalue of  $L$ .*

*Then, the vector  $v$  has strictly positive entries and, for  $V_o := \text{blkdiag}(v_o)$ , we have  $\text{Ker}(V_o L + L^\top V_o) = \text{Span}(1_n)$  and  $V_o L + L^\top V_o \geq 0$ . □*

The next result can be deduced from [6, Section 4.3.5].

**Lemma 2** *Let  $M \in \mathbb{R}^{n \times n}$  be a non-singular  $M$ -matrix. Then, the matrices*

$$S := RM + M^\top R \quad \text{and} \quad R := \text{blkdiag}\{M^{\top-} 1_n\} (\text{blkdiag}\{M^{-} 1_n\})^-$$

*are positive definite.* □

□

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