



HAL
open science

Global Uniform Ultimate Boundedness of Semi-Passive Systems Interconnected over Directed Graphs

Anes Lazri, Mohamed Maghenem, Elena Panteley, Antonio Loria

► To cite this version:

Anes Lazri, Mohamed Maghenem, Elena Panteley, Antonio Loria. Global Uniform Ultimate Boundedness of Semi-Passive Systems Interconnected over Directed Graphs. CNRS - Laboratoire des signaux et systèmes; CNRS - GIPSA-Lab. 2023, 9 p. hal-04298308

HAL Id: hal-04298308

<https://hal.science/hal-04298308>

Submitted on 21 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Global Uniform Ultimate Boundedness of Semi-Passive Systems Interconnected over Directed Graphs

Anes Lazri Mohamed Maghenem Elena Panteley Antonio Loria *

September 21, 2023

Abstract

We analyse the solutions of networked heterogeneous nonlinear systems¹

$$\dot{x}_i = f_i(x_i) + u_i \quad x_i \in \mathbb{R}, \quad i \in \{1, 2, \dots, n\}, \quad (1)$$

where $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is continuous for all $i \in \{1, 2, \dots, n\}$ and the control inputs are set to

$$u_i := -\gamma \sum_{j=1}^n a_{ij}(x_i - x_j) \quad \forall i \in \{1, 2, \dots, n\}, \quad (2)$$

where $\gamma > 0$ is a coupling gain and $a_{ij} \geq 0$ are interconnection weights. We assume that the closed-loop interconnected systems form a network with an underlying connected directed graph that contains a directed spanning tree. For these systems, we establish global uniform ultimate boundedness of the solutions, under the assumption that each system (1) defines a semi-passive [5] map $u_i \mapsto x_i$. As a corollary, we also establish global uniform global boundedness of the solutions.

1 Preliminaries

Notations. For $x \in \mathbb{R}^n$, x^\top denotes its transpose, $|x|$ denotes its Euclidean norm, $\text{blkdiag}\{x\} \in \mathbb{R}^{n \times n}$ denotes the diagonal matrix whose i th diagonal element is the i th element of x . For a set $K \subset \mathbb{R}^n$, $|x|_K := \min\{|x - y| : y \in K\}$ denotes the distance of x to the set K . For a symmetric matrix $Q \in \mathbb{R}^{n \times n}$, $\lambda_i(Q)$ denotes the i th smallest eigenvalue of Q . For an invertible matrix $M \in \mathbb{R}^{n \times n}$, M^- or M^{-1} denotes its inverse. Given $N \in \mathbb{R}^{n \times n}$, $\text{Ker}(N) := \{v : Nv = 0\}$ denotes the kernel of N . A class \mathcal{K}^∞ function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is continuous, strictly increasing, unbounded, and $\alpha(0) = 0$. Furthermore α^- denotes the inverse function of α .

1.1 On Some Classes of Matrices

A matrix $M := [m_{ij}]$, $(i, j) \in \{1, 2, \dots, n\}^2$, is a \mathcal{Z} -matrix if $m_{ij} \leq 0$ whenever $i \neq j$. It is an M -matrix if it is a \mathcal{Z} -matrix and its eigenvalues have non-negative real parts. Equivalently, $M := \lambda I_n - B$, where B is a non-negative matrix and $\lambda \geq \rho(B)$, where $\rho(B) := \max\{|\lambda_i(B)| : i \in \{1, 2, \dots, n\}\}$ is the spectral radius of B . M is a non-singular M -matrix if it is a \mathcal{Z} -matrix and its eigenvalues have positive real parts. Equivalently, $M := \lambda I_n - B$, where B is a non-negative matrix and $\lambda > \rho(B) > 0$; see [2, 3] for more details.

*M. Maghenem is with University of Grenoble Alpes, CNRS, Grenoble INP, GIPSA-Lab, France. E-mail: mohamed.maghenem@cns.fr; E. Panteley and A. Loria are with L2S, CNRS, 91192 Gif-sur-Yvette, France. E-mail: elena.panteley@cns.fr and antonio.loria@cns.fr A. Lazri is with L2S, CNRS, Univ Paris-Saclay, France (e-mail: anes.lazri@centralesupelec.fr)

¹For simplicity, but without loss of generality, we assume that $x \in \mathbb{R}$; all statements hold after pertinent changes in the notation, if $x \in \mathbb{R}^p$, with $p > 1$.

1.2 Graph Notions

A directed graph or a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is characterized by the set of nodes $\mathcal{V} = \{1, 2, \dots, n\}$, and the set of directed edges \mathcal{E} . The edge set \mathcal{E} consists of ordered pairs, of the form (k, i) , that indicate a directed link from node k to node i . Given a directed edge $(k, i) \in \mathcal{E}$, then node k is called an *in-neighbor* of node i . We assign a positive weight a_{ik} to each edge (k, i) . That is, $a_{ik} = 0$ if (k, i) is not an edge. The Laplacian matrix of a digraph is given by

$$L := \begin{bmatrix} d_1 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & d_2 & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n-11} & \cdots & d_{n-1} & -a_{n-1n} \\ -a_{n1} & \cdots & -a_{nn-1} & d_n \end{bmatrix} =: D - A, \quad (3)$$

where $d_i := \sum_{j=1}^n a_{ij}$ for all $i \in \{1, 2, \dots, n\}$, D is the diagonal part of L and A is called the adjacency matrix.

A digraph is *strongly connected* if, for any two distinct nodes i and j , there is a path from i to j . The Laplacian matrix of a strongly connected graph admits $\lambda_1(L) = 0$ as an eigenvalue with the corresponding right and left eigenvectors $\mathbf{1}_n = [1 \ 1 \ \cdots \ 1]^\top$ and $\mathbf{v}_o := [v_1 \ v_2 \ \cdots \ v_n]^\top$, respectively, where $v_i > 0$ for all $i \leq n$.

1.3 Graph and Matrix Decomposition

Suppose that the digraph \mathcal{G} is connected and contains a spanning tree. Then, it admits a decomposition into a leading strongly connected subgraph $\mathcal{G}_\ell \neq \emptyset$ and a subgraph $\mathcal{G}_f := \mathcal{G} \setminus \mathcal{G}_\ell$ of followers; namely, the agents that do not belong to the leading component, and which we call the follower agents. In this case, up to a permutation, the Laplacian L admits the lower-block decomposition

$$L = \begin{bmatrix} L_\ell & 0 \\ -A_{\ell f} & M_f \end{bmatrix}, \quad (4)$$

where $L_\ell := D_\ell - A_\ell \in \mathbb{R}^{n_\ell \times n_\ell}$ is the Laplacian matrix of the strongly connected component \mathcal{G}_ℓ , the lower-left block $A_{\ell f} \in \mathbb{R}^{n_\ell \times n - n_\ell}$, $n_f := n - n_\ell$, is a non-negative matrix, and the lower-right block $M_f \in \mathbb{R}^{n_f \times n_f}$ is a non-singular M-matrix. The block M_f can be seen as the sum of the Laplacian matrix L_f corresponding to \mathcal{G}_f and a diagonal matrix $D_{\ell f}$ gathering the weights of the interconnections between nodes in \mathcal{G}_ℓ and the nodes in \mathcal{G}_f . That is, $M_f = L_f + D_{\ell f}$, where $L_f = D_f - A_f$.

1.4 Lyapunov Analysis of a Directed Graph

Consider a network of n single integrators of the form $\dot{x}_i = u_i$ interconnected according to the classical consensus protocol

$$u_i := - \sum_{j=1}^n a_{ij}(x_i - x_j) \quad \forall i \in \{1, 2, \dots, n\}.$$

In closed loop, the network is governed by the linear system $\dot{x} = -Lx$, where $L \in \mathbb{R}^n$ is the Laplacian matrix of a connected di-graph \mathcal{G} that contains a directed spanning tree. According to Section 1, we can decompose the state x into $x^\top := [x_\ell^\top \ x_f^\top]$, where $x_\ell \in \mathbb{R}^{n_\ell}$ gathers the states of the leading component and is governed by

$$\Sigma_\ell : \dot{x}_\ell = -L_\ell x_\ell,$$

and the non-leading component whose state is $x_f \in \mathbb{R}^{n_f}$, are governed by

$$\Sigma_f : \dot{x}_f = -M_f x_f,$$

on the manifold $\{x_f = 0\}$. In the rest of this section, we overview some Lyapunov-function constructions allowing to prove uniform exponential stability of \mathcal{A} for Σ_ℓ , where

$$\mathcal{A} := \{x_\ell \in \mathbb{R}^{n_\ell} : x_{\ell 1} = x_{\ell 2} = \dots = x_{\ell n_\ell}\}, \quad (5)$$

and exponential stability of the origin for Σ_f .

1.4.1 Proof of uniform exponential stability of \mathcal{A} for Σ_ℓ

let $v_o := [v_1 \ v_2 \ \dots \ v_{n_\ell}]^\top$ be a left eigenvector associated to $\lambda_1(L_\ell) = 0$ and $V_o := \text{blkdiag}\{v_o\}$. Based on Lemma 1 in the Appendix, $Q_o := L_\ell^\top V_o + V_o L_\ell$ is symmetric and positive semi-definite, and its kernel is spanned by $\mathbf{1}_{n_\ell}$. Then, the derivative of the Lyapunov function candidate $W(x_\ell) := x_\ell^\top V_o x_\ell$, along the solutions to Σ_ℓ , satisfies

$$\dot{W}(x_\ell) = -x_\ell^\top (L_\ell^\top V_o + V_o L_\ell) x_\ell \leq -\lambda_2(Q_o) |x_\ell|_{\mathcal{A}_\ell}^2.$$

Now, we let

$$Z(x_\ell) := (x_\ell - \mathbf{1}_{n_\ell} v_o^\top x_\ell)^\top V_o (x_\ell - \mathbf{1}_{n_\ell} v_o^\top x_\ell),$$

which is positive definite. Its derivative along the solutions of $\dot{x}_\ell = -L_\ell x_\ell$ satisfies

$$\dot{Z}(x_\ell) = -x_\ell^\top Q_o x_\ell \leq -\lambda_2(Q_o) |x_\ell|_{\mathcal{A}_\ell}^2. \quad (6)$$

To obtain the previous expression we used $v_o^\top L = 0$, $v_1^\top \mathbf{1}_{n_\ell} = 1$ and that $\mathbf{1}_{n_\ell}$ is in the kernel of $I_{n_\ell} - \mathbf{1}_{n_\ell} v_o^\top$. Moreover, $I_{n_\ell} - \mathbf{1}_{n_\ell} v_o^\top$ is the Laplacian matrix of an all-to-all graph; hence, $\mathbf{1}_{n_\ell}$ spans the kernel of $I_{n_\ell} - \mathbf{1}_{n_\ell} v_o^\top$. Therefore, there exist $\bar{z}, \underline{z} > 0$ such that

$$\underline{z} |x_\ell|_{\mathcal{A}_\ell}^2 \leq Z(x_\ell) \leq \bar{z} |x_\ell|_{\mathcal{A}_\ell}^2 \quad \forall x_\ell \in \mathbb{R}^{n_\ell}. \quad (7)$$

Uniform exponential stability of \mathcal{A}_ℓ from (6) and (7) and standard Lyapunov-stability theory.

1.4.2 Proof of exponential Stability of the Origin for Σ_f

based on Lemma 2, since M_f is a non-singular M -matrix, we can use the Lyapunov function candidate $Y(x_f) := x_f^\top R_f x_f$, where $R_f := \text{blkdiag}\{M_f^{-\top} \mathbf{1}_{n_f}\} \left(\text{blkdiag}\{M_f^{-1} \mathbf{1}_{n_f}\} \right)^{-1}$, which is positive definite. Furthermore, along the solutions to Σ_f , we have

$$\dot{Y}(x_f) = -x_f^\top [M_f^\top R_f + R_f M_f] x_f.$$

Now, since $(M_f^\top R_f + R_f M_f)$ is positive definite, exponential stability of the origin for Σ_f follows.

2 Problem formulation

Consider the systems (1)-(2), with $\gamma > 0$ and $a_{ij} \geq 0$. Then, defining $x := [x_1 \ \dots \ x_n]^\top$, and $F(x) := [f_1(x_1), f_2(x_2), \dots, f_n(x_n)]^\top$, we may write the closed-loop system in compact form as

$$\dot{x} = F(x) - \gamma L x, \quad (8)$$

where L is defined as in (3). This is a networked system with an underlying topology that may be represented by a graph \mathcal{G} .

Assumption 1 *The graph \mathcal{G} is connected and contains a directed spanning tree.* •

We are interested in verifying the following two boundedness properties for (8).

(P1) *Global Uniform Boundedness (GUB).* The solutions $t \rightarrow x(t)$ to (8) are globally bounded, uniformly in γ , if, for every $r_o > 0$ and $\gamma_o > 0$, there exists $R = R(r_o, \gamma_o) \geq r_o$ such that, for all $\gamma \geq \gamma_o$,

$$|x(t_o)| \leq r_o \Rightarrow |x(t)| \leq R \quad \forall t \geq 0.$$

(P2) *Global Uniform Ultimate Boundedness (GUUB).* The solutions $t \rightarrow x(t)$ to (8) are ultimately bounded, uniformly in γ , if given $\gamma_o > 0$, there exists $r = r(\gamma_o) > 0$ such that, for all $r_o > 0$, there exists $T = T(r_o, \gamma_o) \geq 0$ such that, for all $\gamma \geq \gamma_o$,

$$|x(t_o)| \leq r_o \Rightarrow |x(t)| \leq r \quad \forall t \geq T.$$

To verify the latter two properties, we make the following assumption on the individual nodes' dynamics in (1).

Assumption 2 (State strict semi-passivity) *For each $i \in \{1, 2, \dots, n\}$, the input-output map $u_i \mapsto x_i$ defined by the dynamics (1) is state strict semipassive [1]. Furthermore, there exists a continuously differentiable storage function $V_i : \mathbb{R}^n \rightarrow \mathbb{R}^+$, a class \mathcal{K}_∞ function $\underline{\alpha}_i$, a constant $\rho_i > 0$, a continuous function $H_i : \mathbb{R} \rightarrow \mathbb{R}$, and a continuous function $\psi_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$, such that*

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i), \quad \dot{V}_i(x_i) \leq 2u_i x_i - H_i(x_i), \quad (9)$$

and $H_i(x_i) \geq \psi_i(|x_i|)$ for all $|x_i| \geq \rho_i$. •

Remark 1 The property described in Assumption 2 is called strict *quasipassivity* in [4]. In [5] the authors define a similar concept named strict semi-passivity, but radial unboundedness of the storage function is not imposed. See also [1]. •

3 Main result

Theorem 1 (Uniform ultimate boundedness) *The solutions of the networked system (1)-(2) are globally uniformly ultimately bounded, i.e., Property (P2) holds, if Assumptions 1 and 2 are satisfied.* □

Proof: Under Assumption 1, the Laplacian matrix L admits a permutation, such that (4) holds. Therefore, the state x may be decomposed into $x := [x_\ell^\top \ x_f^\top]^\top$ and the system (8) takes the cascaded form

$$\dot{x}_\ell = f_\ell(x_\ell) - \gamma L_\ell x_\ell, \quad f_\ell(x_\ell) := [f_1(x_{\ell_1}) \ \cdots \ f_{n_\ell}(x_{\ell_{n_\ell}})]^\top \quad (10a)$$

$$\dot{x}_f = f_f(x_f) + \gamma A_{\ell f} x_\ell - \gamma M_f x_f, \quad f_f(x_f) := [f_{n_\ell+1}(x_{f_1}) \ \cdots \ f_{n_\ell+n_f}(x_{f_{n_f}})]^\top. \quad (10b)$$

Equation (10a) corresponds to the dynamics of a *leading* component, a networked system with an underlying strongly connected graph \mathcal{G}_ℓ , and a *follower* component, with dynamics (10b). The proof of the statement is constructed using a cascades argument and proving, firstly, global uniform ultimate boundedness for the solutions of (10a) and, consequently, the same property for (10b).

To that end, let $r_o > 0$ be arbitrarily fixed and let $|x(0)| \leq r_o$. Then, $|x_\ell(0)| \leq r_o$ and $|x_f(0)| \leq r_o$.

1) *Uniform ultimate boundedness for the leading component:* after Assumption 2, for each $i \in \{1, 2, \dots, n_\ell\}$, there exists a storage function V_i such that its total derivative along the trajectories of (1) satisfies

$$\dot{V}_i(x_{\ell i}) \leq 2u_i^\top x_{\ell i} - H_i(x_{\ell i}), \quad H_i(x_{\ell i}) \geq \psi_i(|x_{\ell i}|) \quad \forall |x_{\ell i}| \geq \rho_i. \quad (11)$$

Next, let $W(x_\ell) := \sum_{i=1}^{n_\ell} v_i V_i(x_{\ell i})$, where v_i corresponds to the i th element of v_o , which is the left eigenvector associated to the zero eigenvalue of L_ℓ . Since the graph \mathcal{G}_ℓ is strongly connected, then $v_i > 0$ for all $i \leq n_\ell$, so W is positive definite and radially unbounded. Now, from (11), we obtain

$$\dot{W}(x_\ell) \leq 2 \sum_{i=1}^{n_\ell} v_i u_i^\top x_{\ell i} - \sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}), \quad \forall x_\ell \in \mathbb{R}^{n_\ell}. \quad (12)$$

The first term on the right-hand side of (12) satisfies

$$\sum_{i=1}^{n_\ell} v_i u_i^\top x_{\ell i} = u^\top V_o x_\ell, \quad (13)$$

where $V_o := \text{blkdiag}\{v_o\}$ and, since $u = -\gamma L_\ell x_\ell$, it follows that

$$\begin{aligned} \dot{W}(x_\ell) &\leq - \sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}) - \gamma x_\ell^\top [L_\ell^\top V_o + V_o L_\ell] x_\ell \\ &\leq - \sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}) - \gamma x_\ell^\top Q_o x_\ell, \end{aligned} \quad (14)$$

with $Q_o := V_o L_\ell + L_\ell^\top V_o$, which is positive semi-definite—see Lemma 1 in the Appendix. Furthermore, we note that

$$-x_\ell^\top Q_o x_\ell = - [x_\ell - \mathbf{1}_{n_\ell} \mathbf{1}_{n_\ell}^\top x_\ell / n_\ell]^\top Q_o [x_\ell - \mathbf{1}_{n_\ell} \mathbf{1}_{n_\ell}^\top x_\ell / n_\ell] \leq -\lambda_2(Q_o) |x_\ell|_{\mathcal{A}}^2,$$

where $|x_\ell|_{\mathcal{A}}$ denotes the distance of x_ℓ to the set \mathcal{A} and $\lambda_2(Q_o)$ is the second smallest eigenvalue of Q_o .

Now, on one hand, we have that $v_i > 0$ for all $i \in \{1, 2, \dots, n\}$ and, on the other, $-H_i(x_{\ell i}) > 0$ only if $|x_{\ell i}| \leq \rho_i$. Therefore, the constant $H_\ell := -\sum_{i=1}^{n_\ell} \max_{|x_i| \leq \rho_i} \{v_i H_i(x_{\ell i})\} > 0$. Therefore, after (14), we get

$$\dot{W}(x_\ell) \leq H_\ell - \gamma \lambda_2(Q_o) |x_\ell|_{\mathcal{A}}^2 \quad \forall x_\ell \in \mathbb{R}^{n_\ell}. \quad (15)$$

In turn, given $\gamma_o > 0$ and $\epsilon > 0$, for all $\gamma \geq \gamma_o$, we have

$$\dot{W}(x_\ell) \leq H_\ell - \gamma_o \lambda_2(Q_o) |x_\ell|_{\mathcal{A}}^2 \leq -\epsilon \quad \forall x_\ell \notin \mathcal{C}, \quad (16)$$

where

$$\mathcal{C} := \left\{ x_\ell \in \mathbb{R}^{n_\ell} : |x_\ell|_{\mathcal{A}} \leq \sqrt{n_\ell} R_e := \sqrt{\frac{\epsilon + H_\ell}{\gamma_o \lambda_2(Q_o)}} \right\}.$$

Next, let $\bar{\rho} := \arg \max_{i \in \{1, 2, \dots, n_\ell\}} \rho_i$ and

$$\mathcal{B}_\beta := \{x_\ell \in \mathbb{R}^{n_\ell} : |x_\ell| \leq \beta := \sqrt{n_\ell}(\bar{\rho} + 2R_e)\}.$$

Note that for all $x_\ell \notin \mathcal{B}_\beta$, we have $|x_\ell| > \sqrt{n_\ell}(\bar{\rho} + 2R_e)$ and, for all $x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta$,

$$|x_\ell| > \sqrt{n_\ell}(\bar{\rho} + 2R_e) \quad \text{and} \quad |x_\ell|_{\mathcal{A}} \leq \sqrt{n_\ell} R_e. \quad (17)$$

Furthermore, we use the fact that $x_\ell = \mathbf{1}_{n_\ell}(\mathbf{1}_{n_\ell}^\top x_\ell)/n_\ell + [x_\ell - \mathbf{1}_{n_\ell}(\mathbf{1}_{n_\ell}^\top x_\ell)/n_\ell]$, and the fact that $|x_\ell|_{\mathcal{A}} = |x_\ell - \mathbf{1}_{n_\ell}(\mathbf{1}_{n_\ell}^\top x_\ell)/n_\ell|$, to conclude that

$$|x_\ell| \leq |x_\ell|_{\mathcal{A}} + |\mathbf{1}_{n_\ell}^\top x_\ell|/\sqrt{n_\ell}. \quad (18)$$

Now, combining (17) and (18), we conclude that for all $x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta$,

$$\sqrt{n_\ell}(\bar{\rho} + 2R_e) < |x_\ell| \leq |x_\ell|_{\mathcal{A}} + |\mathbf{1}_{n_\ell}^\top x_\ell|/\sqrt{n_\ell} \leq \sqrt{n_\ell}R_e + |\mathbf{1}_{n_\ell}^\top x_\ell|/\sqrt{n_\ell}. \quad (19)$$

So, for all $x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta$, $|\mathbf{1}_{n_\ell}^\top x_\ell|/n_\ell > \bar{\rho} + R_e$. Next, we use the fact that

$$x_{\ell i} = \mathbf{1}_{n_\ell}^\top x_\ell/n_\ell + (x_{\ell i} - \mathbf{1}_{n_\ell}^\top x_\ell/n_\ell) \quad \forall i \in \{1, 2, \dots, n_\ell\}$$

to conclude that $|x_{\ell i}| > |\mathbf{1}_{n_\ell}^\top x_\ell|/n_\ell - |(x_{\ell i} - \mathbf{1}_{n_\ell}^\top x_\ell/n_\ell)|$. Hence,

$$|x_{\ell i}| > \bar{\rho} + R_e - \sqrt{n_\ell}R_e > \bar{\rho} \quad \forall i \in \{1, 2, \dots, n_\ell\}$$

for all $x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta$. The latter, under Assumption 2, implies that

$$-\sum_{i=1}^{n_\ell} v_i H_i(x_{\ell i}) \leq -\sum_{i=1}^{n_\ell} v_i \psi_i(|x_{\ell i}|) \leq 0 \quad \forall x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta.$$

As a result, setting $\Psi(x_\ell) := \sum_{i=1}^{n_\ell} v_i \psi_i(|x_{\ell i}|)$ —note that Ψ is continuous and positive—we conclude that

$$\dot{W}(x_\ell) \leq -\Psi(x_\ell) \quad \forall x_\ell \in \mathcal{C} \setminus \mathcal{B}_\beta.$$

Combining the latter inequality to (16), we conclude that

$$\dot{W}(x_\ell) \leq -\min\{\Psi(x_\ell), \epsilon\} \quad \forall x_\ell \in \mathbb{R}^{n_\ell} \setminus \mathcal{B}_\beta.$$

The latter is enough to conclude global attractivity and forward invariance of the set

$$\mathcal{S}_\sigma := \{x_\ell \in \mathbb{R}^{n_\ell} : W(x_\ell) \leq \sigma\}, \quad \sigma := \max\{W(y) : y \in \mathcal{B}_\beta\}.$$

Furthermore, since $W : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is continuous and \mathcal{B}_β is bounded, we conclude that σ is well defined. Consequently, the ultimate bound is

$$r_\ell := \left[\min_i \{\underline{\alpha}_i\} \right]^- (\sigma),$$

where, with an abuse of notation, $\min_i \{\underline{\alpha}_i\}$ corresponds to the function $s \mapsto \psi(s)$ defined as $\psi(s) := \min_i \{\underline{\alpha}_i(s)\}$ for each $s \geq 0$ and $\underline{\alpha}_i$ is defined in Assumption 2, so $\psi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is strictly increasing and radially unbounded, hence, globally invertible. Thus, $W(x_\ell) \leq \sigma$ implies that $|x_\ell| \leq r_\ell$.

Next, we compute an upperbound $T_\ell(r_o, \gamma_o)$ on the time that the solutions to (10a), with $\gamma \geq \gamma_o$ and starting from $\mathcal{B}_{r_o} := \{x_\ell \in \mathbb{R}^{n_\ell} : |x_\ell| \leq r_o\}$, take to reach the compact set $\mathcal{B}_\beta \subset \mathcal{S}_\sigma$. For this, we assume without loss of generality that $r_o \geq \beta$, and we define

$$\epsilon_{r_o} := \min\{\min\{\Psi(x_\ell), \epsilon\} : |x_\ell| \geq \beta, x_\ell \in \mathcal{S}_{\sigma_o}\},$$

where

$$\mathcal{S}_{\sigma_o} := \{x_\ell \in \mathbb{R}^{n_\ell} : W(x_\ell) \leq \sigma_o\}, \quad \sigma_o := \max\{W(y) : y \in \mathcal{B}_{r_o}\}. \quad (20)$$

Clearly, \mathcal{S}_{σ_o} is compact; hence, ϵ_{r_o} is positive.

Therefore, along every solution $t \mapsto x_\ell(t)$ to (10a) starting from $x_\ell(0) \in \mathcal{B}_{r_o} \setminus \mathcal{B}_\beta$, we have $\dot{W}(x_\ell(t)) \leq -\epsilon_{r_o}$, up to the earliest time when x_ℓ reaches \mathcal{B}_β . For any earlier time, we have

$$W(x_\ell(t)) \leq -\epsilon_{r_o} t + W(x_\ell(0)), \quad (21)$$

so we can take $T_\ell(r_o, \gamma_o) = \sigma_o/\epsilon_{r_o}$. Clearly, T_ℓ depends only on (r_o, γ_o) and r_ℓ depends only on γ_o . Thus, the ultimate boundedness guaranteed for the solutions of (10a) is uniform in γ .

2) *Uniform ultimate boundedness for the follower dynamics*: following up the previous computations and arguments, establish global uniform ultimate boundedness for the non-leading component, determined by (10b).

Using Lemma 2, we conclude that the matrices

$$S := PM_f + M_f^\top P \quad \text{and} \quad P := \text{blkdiag} \left\{ M_f^{\top-1} \mathbf{1}_n \right\} \left(\text{blkdiag} \left\{ M_f^- \mathbf{1}_n \right\} \right)^-$$

are symmetric and positive definite. We also note that P is diagonal. Then, let p_i , for $i \in \{1, 2, \dots, n_f\}$, be the i th diagonal element of P . In addition, let $Z(x_f) := \sum_{i=1}^{n_f} p_i V_i(x_{fi})$. Its total derivative along the trajectories of (10b) satisfies

$$\dot{Z}(x_f) \leq - \sum_{i=1}^{n_f} p_i H_i(x_{fi}) - \gamma x_f^\top [PM_f + M_f^\top P] x_f + 2\gamma x_f^\top [PA_{\ell f}] x_\ell. \quad (22)$$

On one hand, we already established the existence of $r_\ell(\gamma_o) > 0$ and $T_\ell(\gamma_o, r_o)$ such that

$$|x_\ell(t)| < r_\ell \quad \forall t \geq T_\ell.$$

On the other, for all $|x_\ell| \leq r_\ell$,

$$\dot{Z}(x_f) \leq H_f - \gamma \lambda_1(S) |x_f|^2 + 2\gamma \bar{p} r_\ell |x_f|, \quad (23)$$

where $\bar{p} := |PA_{\ell f}|$ and $H_f := \sum_{i=1}^{n_f} \max_{|x_i| \leq \rho_i} \{v_i H_i(x_{fi})\}$. Now, from this and (22), we obtain

$$\begin{aligned} \dot{Z}(x_f) &\leq H_f - \gamma x_f^\top S x_f + 2\gamma x_f^\top [PA_{\ell f}] x_\ell \\ &\leq H_f - \gamma [x_f^\top S x_f / 2 - 2x_\ell^\top A_{\ell f}^\top P^\top S^- PA_{\ell f} x_\ell]. \end{aligned}$$

At the same time, integrating (15), we obtain that, for each $t \in [0, T_\ell]$,

$$W(x_\ell(t)) \leq H_\ell T_\ell + W(x_\ell(0)) \leq H_\ell T_\ell + \sigma_o,$$

where σ_o comes from (20). Defining

$$R_\ell := \left[\min_i \{\underline{\alpha}_i\} \right]^- (H_\ell T_\ell + \sigma_o),$$

we have

$$\dot{Z}(x_f) \leq H_f - \gamma [\lambda_1(S) |x_f|^2 / 2 - 2|A_{\ell f}^\top P^\top S^- PA_{\ell f}| R_\ell^2],$$

for all $|x_\ell| \leq R_\ell$.

Note that, for all x_f such that

$$|x_f|^2 > d_f^2 := \frac{4|A_{\ell f}^\top P^\top S^- PA_{\ell f}| R_\ell^2}{\lambda_1(S)} + \frac{42H_f}{\lambda_1(S)\gamma_o},$$

$\dot{Z}(x_f) \leq 0$. This implies that, for all $t \in [0, T_\ell]$,

$$Z(x_f(t)) \leq \max \{ \sigma_{f_o}, \sigma_f \}, \quad \sigma_f := \max \{ Z(x_f) : |x_f| \leq d_f \} \quad \sigma_{f_o} := \max \{ Z(x_f) : |x_f| \leq r_o \}. \quad (24)$$

In turn, for each $t \in [0, T_\ell]$,

$$|x_f(t)| \leq \bar{r}_o := \left[\left(\sum_{i=1}^{n_f} p_i \right) \min_i \{\underline{\alpha}_i\} \right]^- (\max \{ \sigma_{f_o}, \sigma_f \}). \quad (25)$$

Clearly, the previous upper bound is uniform in $\gamma \geq \gamma_o$.

Next, we focus on the solutions' behaviour after T_ℓ (i.e., once $|x_\ell| \leq r_\ell$). Given $\epsilon > 0$, we see that, for all $\gamma \geq \gamma_o$ and for all x_f and x_ℓ such that

$$|x_f| > \beta_1 := 1 + \frac{2\bar{p}r_\ell}{\lambda_1(S)} + \sqrt{\frac{\epsilon + H_f}{\gamma_o \lambda_1(S)}} \quad \text{and} \quad |x_\ell| \leq r_\ell,$$

after (23), we conclude that $\dot{Z}(x_f) \leq -\epsilon$. Furthermore, $|x_\ell(t)| \leq r_\ell$ for all $t \geq T_\ell$, then the set

$$\mathcal{S}_{\sigma_1} := \{x_f \in \mathbb{R}^{n_f} : Z(x_f) \leq \sigma_1\}, \quad \sigma_1 := \max\{Z(y) : y \in \mathcal{B}_{\beta_1}\}, \quad \mathcal{B}_{\beta_1} := \{x_f \in \mathbb{R}^{n_f} : |x_f| \leq \beta_1\},$$

is attractive and becomes forward invariant after time T_ℓ .

Since $Z : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is continuous and \mathcal{B}_{β_1} is bounded, we conclude that σ_1 is well defined. As a result, the ultimate bound for $x_f(t)$ is

$$r_f = \left[\left(\sum_{i=1}^{n_f} p_i \right) \min_i \{\underline{\alpha}_i\} \right]^- (\sigma_1).$$

Indeed, $Z(x_f) \leq \sigma_1$ implies $|x_f| \leq r_f$.

Finally, as for $t \mapsto x_\ell(t)$ we give next an upperbound, denoted by $T_f(r_o, \gamma_o)$, on the time that the solutions to (10b), with $\gamma \geq \gamma_o$ and starting from $\mathcal{B}_{r_o} := \{x_f \in \mathbb{R}^{n_f} : |x_f| \leq r_o\}$, take to reach $\mathcal{B}_{\beta_1} \subset \mathcal{S}_{\sigma_1}$.

Let a solution $t \mapsto x_f(t)$ to (10b) starting from $x_f(0) \in \mathcal{B}_{r_o}$. Now, we use the fact $|x_f(T_\ell)| \leq \bar{r}_o$ with \bar{r}_o coming from (25) and \bar{r}_o is uniform in γ . As a result, along the solution $t \mapsto x_f(t)$, we have $\dot{Z}(x_f(t)) \leq -\epsilon$ from T_ℓ and up to when it reaches \mathcal{B}_{β_1} for the first time after T_ℓ . Hence, before reaching \mathcal{B}_{β_1} , we have

$$Z(x_f(t)) \leq -\epsilon t + Z(x_f(T_\ell)) \quad (26)$$

and, thus, using (24), we can take $T_f = T_\ell + \max\{\sigma_{f_o}, \sigma_f\}/\epsilon$. Clearly, T_f and r_f depend only on (r_o, γ_o) . Thus, the ultimate bounded guaranteed for the solutions of (10b) is also uniform in γ . ■

Corollary 1 (Uniform boundedness) *Under Assumptions 1 and 2 the solutions of the closed-loop system in (8) are globally uniformly bounded, i.e., Property (P1) holds. □*

Proof: The statement of Theorem 1 holds, therefore, given $r_o > 0$ and $\gamma_o > 0$, for all $\gamma \geq \gamma_o$, we have

$$|x_\ell(0)| \leq r_o \implies |x_\ell(t)| \leq r_\ell(\gamma_o) \quad \forall t \geq T_\ell(r_o, \gamma_o).$$

Furthermore, we were able to show that on the interval $[0, T_\ell(r_o, \gamma_o)]$, we have

$$W(x_\ell(t)) \leq H_\ell T_\ell + W(x_\ell(0)).$$

Hence, if we let $\sigma_\ell := \max\{W(y) : |y| \leq r_o\}$, it follows that

$$|x_\ell(t)| \leq R_\ell := \left[\min_i \{\underline{\alpha}_i\} \right]^- (\sigma_\ell + H_\ell T_\ell + r_\ell) \quad \forall t \geq 0.$$

Next, for the solutions to (10b), for any $\gamma > \gamma_o$ and $|x_f(0)| \leq r_o$, we know that

$$|x_f(t)| \leq r_f \quad \forall t \geq T_f(\gamma_o, r_o).$$

At the same time, from the previous proof, we know that

$$\dot{Z}(x_f) \leq H_f - \gamma [\lambda_1(S)|x_f|^2/2 - 2|A_{\ell f}^\top P^\top S^- P A_{\ell f}| R_\ell^2].$$

As a result, when

$$|x_f|^2 > d_f^2 := \frac{4|A_{\ell f}^\top P^\top S^- P A_{\ell f}| R_\ell^2}{\lambda_1(S)} + \frac{42H_f}{\lambda_1(S)\gamma_o},$$

then $\dot{Z}(x_f) \leq 0$. Hence, for each $t \geq 0$,

$$Z(x_f(t)) \leq \max\{\sigma_{f_o}, \sigma_f\}, \quad \sigma_f := \max\{Z(x_f) : |x_f| \leq d_f\}, \quad \sigma_{f_o} := \max\{Z(x_f) : |x_f| \leq r_o\}. \quad (27)$$

In turn, for each $t \geq 0$, we have

$$|x_f(t)| \leq R_f := \left[\left(\sum_{i=1}^{n_f} p_i \right) \min_i \{\alpha_i\} \right]^- (\max\{\sigma_{f_o}, \sigma_f\}). \quad (28)$$

■

Appendix

The following lemma is proposed in [3], see also [6].

Lemma 1 *Let $L \in \mathbb{R}^{n \times n}$ be the Laplacian matrix of a directed and strongly connected graph. Let $v_o := [v_1, v_2, \dots, v_n]^\top \in \mathbb{R}^n$ be the left eigenvector of L associated to the null eigenvalue of L .*

Then, the vector v has strictly positive entries and, for $V_o := \text{blkdiag}(v_o)$, we have $\text{Ker}(V_o L + L^\top V_o) = \text{Span}(1_n)$ and $V_o L + L^\top V_o \geq 0$. □

The next result can be deduced from [6, Section 4.3.5].

Lemma 2 *Let $M \in \mathbb{R}^{n \times n}$ be a non-singular M -matrix. Then, the matrices*

$$S := RM + M^\top R \quad \text{and} \quad R := \text{blkdiag}\{M^{\top-} 1_n\} (\text{blkdiag}\{M^{-} 1_n\})^-$$

are positive definite. □

□

References

- [1] A. Pogromsky. Passivity-based design of synchronizing systems. *International Journal of Bifurcation and Chaos*, 8, 02 1998.
- [2] X. Chen, B. Xudong, M-A. Belabbas, and T. Basar. Controllability of formations over directed time-varying graphs. *IEEE Transactions on Control of Network Systems*, 4(3):407–416, 2017.
- [3] M. U. Javed, J. I. Poveda, and X. Chen. Excitation conditions for uniform exponential stability of the cooperative gradient algorithm over weakly connected digraphs. *IEEE Control Systems Letters*, 6:67–72, 2021.
- [4] I. G. Polushin, D. Hill, and A. L. Fradkov. Strict quasipassivity and ultimate boundedness for nonlinear control systems. *IFAC Proceedings Volumes*, 31(17):505–510, 1998. 4th IFAC Symposium on Nonlinear Control Systems Design 1998 (NOLCOS'98), Enschede, The Netherlands, 1-3 July.
- [5] A. Y. Pogromsky, T. Glad, and H. Nijmeijer. On diffusion driven oscillations in coupled dynamical systems. *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, 9(4):629–644, 1999.
- [6] Z. Qu. *Cooperative control of dynamical systems: applications to autonomous vehicles*. Springer Verlag, London, UK, 2009.