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► **To cite this version:**

Florian Galliot, Sylvain Gravier, Isabelle Sivignon. Maker-Breaker is solved in polynomial time on hypergraphs of rank 3. 2023. hal-04298262

**HAL Id: hal-04298262**

**<https://hal.science/hal-04298262v1>**

Preprint submitted on 21 Nov 2023

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# Maker-Breaker is solved in polynomial time on hypergraphs of rank 3

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## Abstract

In the Maker-Breaker positional game, Maker and Breaker take turns picking vertices of a hypergraph  $H$ , and Maker wins if and only if he claims all the vertices of some edge of  $H$ . This paper provides a general framework to study Maker-Breaker games, centered on the notion of *danger at a vertex  $x$* , which is a subhypergraph representing an urgent threat that Breaker must hit with his next pick if Maker picks  $x$ . We then apply this concept in hypergraphs of rank 3, providing a structural characterization of the winner with perfect play as well as optimal strategies for both players based on danger intersections. We construct a family  $\mathcal{F}$  of dangers such that a hypergraph  $H$  of rank 3 is a Breaker win if and only if the  $\mathcal{F}$ -dangers at  $x$  in  $H$  intersect for all  $x$ . By construction of  $\mathcal{F}$ , this will mean that  $H$  is a Maker win if and only if Maker can guarantee the appearance, within the first three rounds of play, of a very specific elementary subhypergraph (on which Maker easily wins). This last result has a consequence on the algorithmic complexity of deciding which player has a winning strategy on a given hypergraph: this problem, which is known to be PSPACE-complete on 6-uniform hypergraphs [15], is in polynomial time on hypergraphs of rank 3. This validates a conjecture by Rahman and Watson [14]. Another corollary of our result is that, if Maker has a winning strategy on a hypergraph of rank 3, then he can ensure to claim an edge in a number of rounds that is logarithmic in the number of vertices.

*Note: The present updated version of this deposit provides a counterexample to a similar result which was incorrectly claimed recently [1].*

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# Introduction

A *positional game* is a combinatorial game played on a hypergraph  $H$ , where two players take turns picking vertices of  $H$  and the outcome is decided by one of several conventions. Let us mention two major ones:

- In the *Maker-Maker* convention, the winner is the player who first claims all vertices of some edge of  $H$ , or the game ends in a draw if no player achieves this. The first general formulation of this convention seemingly goes back to Hales and Jewett [9], while the first general results are due to Erdős and Selfridge [6]. The game of tic-tac-toe and its generalizations [9] [3] are the most famous examples of Maker-Maker positional games.
- In the *Maker-Breaker* convention, one player (Maker) wins if he claims all vertices of some edge of  $H$ , while the other (Breaker) wins if he can prevent this from happening. No draw is possible here. The first general formulation of this convention is due to Beck and Csirmaz [4]. The game of Hex and the Shannon switching game [7] [13] [5] are the most famous examples of Maker-Breaker positional games.

This paper deals with the Maker-Breaker convention, which is the most studied in the literature.

We always assume that Maker plays first: if Breaker plays first, then we can consider all possibilities of his first pick to reduce to the case where Maker plays first. Given a hypergraph  $H$ , either Maker or Breaker has a winning strategy on  $H$ , and we say  $H$  is a *Maker win* or a *Breaker win* accordingly. Research on the Maker-Breaker game mainly consists in finding criteria for  $H$  to be a Maker win or a Breaker win, as well as evaluating the algorithmic complexity on various hypergraph classes of the MAKERBREAKER decision problem, which takes  $H$  as an input and decides whether  $H$  is a Maker win.

On the algorithmic front, the founding result by Schaefer [16] states that MAKERBREAKER is PSPACE-complete on hypergraphs of rank 11. An improvement of the PSPACE-completeness result to 6-uniform hypergraphs has since been obtained by Rahman and Watson [15]. What about tractability results? A polynomial-time algorithm for MAKERBREAKER on a given class of hypergraphs normally follows from some characterization of Maker wins (or Breaker wins, equivalently) in said class that can be verified efficiently. Counting-type results, based on numerical formulas involving quantities such as the number of edges and their size for example, can provide conditions for a Maker win that are either necessary (like the Erdős-Selfridge theorem [6][2]) or sufficient (like Beck's criterion involving the 2-degree [2]) but usually not both. Instead, we are looking for structural results, corresponding to strategies where the player's picks are based on the existence and the interdependence of certain subhypergraphs. These evolve during the game, hence why we introduce *marked hypergraphs*, which are hypergraphs where some vertices are marked (with the game in mind, these would be the vertices already owned by Maker). The following notion explains which subhypergraphs both players will base their strategy upon. If  $x$  is a non-picked vertex, we define a *danger at  $x$*  as a subhypergraph  $D$  containing  $x$  which would be a Maker win if  $x$  were picked by Maker already. Dangers at  $x$  thus represent urgent threats for Breaker, who must play in their intersection if Maker picks  $x$ . This remains true throughout the game, hence the necessity to study intersection properties not only of present dangers but also of subhypergraphs that could become dangers as the game progresses.

Since the 6-uniform case is PSPACE-complete, and since smaller edges means less structural complexity, it makes sense to first address hypergraphs of small rank. In 2-uniform hypergraphs i.e. graphs, there exists a trivial structural characterization: a graph is a Breaker win if and only if it is a matching. Therefore, hypergraphs of rank 3 constitute the first interesting case.

In this paper, we present a structural characterization of Breaker wins on the class of hypergraphs of rank 3, and we explain how it implies that MAKERBREAKER is tractable on

this class, as conjectured by Rahman and Watson [14].

Previous work on hypergraphs of rank 3 had been done by Kutz [12] in the *linear* case, meaning that any two distinct edges intersect on at most one vertex. He first gives a polynomial reduction to the class of all linear hypergraphs of rank 3 that: are connected, have no articulation vertex, and contain exactly one edge of size 2. He then gives a precise structural characterization of Breaker wins on that class, from which he derives a polynomial-time algorithm for MAKERBREAKER on linear hypergraphs of rank 3. The central substructure at play here is what is called a *linear path* (or simply a *path*, as Kutz calls it and ourselves also will), which is a hypergraph defined by a sequence of edges where any two consecutive edges intersect on exactly one vertex and any two non-consecutive edges do not intersect. Things are different in general hypergraphs of rank 3, where intersections of size 2 somehow may hamper connections between vertices. Indeed, the main difficulty in the non-linear case is that the union of two linear paths, the first between  $x$  and  $y$  and the second between  $y$  and  $z$ , does not necessarily contain a linear path between  $x$  and  $z$ . In particular, Kutz's structural result seems difficult to generalize.

Instead, we apply our danger-based approach in 3-uniform marked hypergraphs, which are equivalent to hypergraphs of rank 3. We construct a family  $\mathcal{F}$  of dangers, whose structure is of reasonable complexity and can be described precisely. Our main result is that, apart from some trivial cases:  $H$  is a Breaker win if and only if, for every non-marked  $x$ , the  $\mathcal{F}$ -dangers at  $x$  in  $H$  intersect ( $x$  itself being excluded from the intersection). In any given position, a winning move for Breaker is then to play anywhere inside the intersection of the  $\mathcal{F}$ -dangers at the vertex that Maker has just played. This can also be expressed from the point of view of Maker. Indeed, because of the way that  $\mathcal{F}$  is built, a corollary of this result is that  $H$  is a Maker win if and only if Maker can ensure the appearance of a *nunchaku* (or a *necklace*, which is sensibly the same thing) within the first three rounds of play. A nunchaku is defined as a linear path between two marked vertices, and it is a Maker win since Maker can force all of Breaker's picks along the path, starting from one end until Breaker is trapped at the other. Coupled with the fact that the linear path existence problem is solvable in polynomial time [8], this last result yields tractability for MAKERBREAKER on all hypergraphs of rank 3.

A result similar to the aforementioned corollary concerning Maker was claimed very recently in a preprint [1]. The author states that, in the case of a Maker win, two rounds are sufficient for Maker instead of three, albeit with a slightly looser definition of nunchaku/necklace. However, our Figure 16 will be a counterexample to this claim. On the subject, one of our results consists in exhibiting a significant subclass of hypergraphs in which two rounds do suffice, which contains the class that Kutz reduces to.

There exists a more general game, which is played on a CNF formula instead of a hypergraph. Two players take turns picking variables and assigning them a truth value of their choice: the first player (False) wants the formula to be false while the second player (True) wants the formula to be true. If the formula is positive i.e. all its literals are positive, then False (resp. True) always assigns the value 0 (resp. 1) to the variable he picks, and the game is equivalent to the Maker-Breaker game: False is Maker, True is Breaker, and clauses correspond to edges. Rahman and Watson [14] have studied this game played on a 3-CNF formula, i.e. all clauses are of size at most 3, with the added constraint that each clause must possess a "spare variable" which appears in no other clause. They define some "obstacles" for True, which are elementary subformulas on which False wins, the main one being called a *manriki*. The authors show that, in all non-trivial cases, True wins if and only if he can break any manriki that appears during the first three rounds of play. This yields a polynomial-time algorithm deciding the winner of the game. They conjecture that these results remain true for general 3-CNF formulas, without

the spare variable constraint, except for the needed number of rounds which might be more than three. Since manrikis with positive literals correspond exactly to nunchakus and necklaces, our result proves Rahman and Watson’s conjecture for positive 3-CNFs, with a number of rounds equal to three.

Finally, another studied subject in positional games is the duration of the game when players try to win as fast as possible. For the Maker-Breaker convention, the question is: given a hypergraph  $H$  that is a Maker win, what is the minimum number of rounds in which Maker can ensure to complete an edge? A corollary of our structural result is that, if a hypergraph  $H$  of rank 3 is a Maker win, then Maker can ensure to complete an edge in  $O(\log(|V(H)|))$  rounds.

The outline of this paper is as follows. In Section 1, we start by introducing marked hypergraphs and recalling rudiments about the Maker-Breaker game. We then define dangers, and we explore what it means for Breaker to survive a given family of dangers during  $r$  rounds, in terms of some intersection properties of subhypergraph collections. In Section 2, we present elementary 3-uniform hypergraphs such as paths and cycles, of which we give some basic structural properties. These hypergraphs play a major role in the Maker-Breaker game, as demonstrated in Section 3, which also features the statement of our main results on 3-uniform marked hypergraphs. The proofs of the structural results are technical, and require a preliminary study of the more complex elements in our family of dangers  $\mathcal{F}$ : this analysis is carried out in Section 4. Section 5 then completes the proofs of the main results. Finally, we conclude with some perspectives for future research.

# 1 Marked hypergraphs and the Maker-Breaker game

## 1.1 First definitions and notations

**Definition 1.1.** A *marked hypergraph*  $H$  is defined by:

- a finite nonempty *vertex set*  $V(H)$ ;
- an *edge set*  $E(H)$  consisting of nonempty subsets of  $V(H)$ ;
- a set of *marked vertices*  $M(H) \subseteq V(H)$ .

**Notation 1.2.** A marked hypergraph consisting of a single edge  $e$  may be simply denoted by  $e$ .

*Remark.* A hypergraph may be seen as a marked hypergraph with no marked vertex, so that all definitions and notations associated with marked hypergraphs apply to hypergraphs as well.

**Definition 1.3.** Let  $H$  be a marked hypergraph.

- The *rank* of  $H$  is defined as the size of its biggest edge.
- We say  $H$  is  $k$ -uniform if all its edges are of size exactly  $k$ .

**Definition 1.4.** Let  $H$  be a marked hypergraph. A *subhypergraph* of  $H$  is a marked hypergraph  $X$  such that:  $V(X) \subseteq V(H)$ ,  $E(X) \subseteq E(H)$  and  $M(X) = V(X) \cap M(H)$ . The notation  $X \subseteq H$  means that  $X$  is a subhypergraph of  $H$ .

**Definition 1.5.** Let  $\mathcal{X} = \{X_1, \dots, X_t\}$  be a finite collection of marked hypergraphs. The *union* of  $\mathcal{X}$ , denoted by  $\langle \mathcal{X} \rangle$ , is the marked hypergraph defined by:  $V(\langle \mathcal{X} \rangle) = \bigcup_{X \in \mathcal{X}} V(X)$ ,  $E(\langle \mathcal{X} \rangle) = \bigcup_{X \in \mathcal{X}} E(X)$  and  $M(\langle \mathcal{X} \rangle) = \bigcup_{X \in \mathcal{X}} M(X)$ . We may also use the notation  $\langle \mathcal{X} \rangle = X_1 \cup \dots \cup X_t$ .

*Remark.* It is possible for two elements of  $\mathcal{X}$  to share a vertex that is marked in one and non-marked in the other, in which case that vertex is marked in the union. However this will not happen in practice, since we will always consider collections whose elements are all subhypergraphs of some common marked hypergraph.

**Notation 1.6.** Let  $H$  be a marked hypergraph, and let  $x, y \in V(H) \setminus M(H)$ .

- We denote by  $H^{+x}$  the marked hypergraph obtained from  $H$  by marking  $x$ , i.e.:  $V(H^{+x}) = V(H)$ ,  $E(H^{+x}) = E(H)$ ,  $M(H^{+x}) = M(H) \cup \{x\}$ .  
By convention, if  $X \subseteq H$  does not contain  $x$  then we define  $X^{+x} = X$ .  
If  $\mathcal{X}$  is a collection of subhypergraphs of  $H$ , then we define  $\mathcal{X}^{+x} = \{X^{+x}, X \in \mathcal{X}\}$  which is a collection of subhypergraphs of  $H^{+x}$ .
- We denote by  $H^{-y}$  the marked hypergraph obtained from  $H$  by removing  $y$ , assuming  $V(H) \neq \{y\}$ , i.e.:  $V(H^{-y}) = V(H) \setminus \{y\}$ ,  $E(H^{-y}) = \{e \in E(H), y \notin e\}$ ,  $M(H^{-y}) = M(H)$ .  
By convention, if  $X \subseteq H$  does not contain  $y$  then we define  $X^{-y} = X$ .
- We may combine these notations, as in  $H^{+x-y} = (H^{+x})^{-y} = (H^{-y})^{+x}$  if  $x \neq y$  for instance.

*Remark.* It should be noted that  $H^{-y}$  is a subhypergraph of  $H$ , while  $H^{+x}$  is not because of the additional marked vertex.

**Notation 1.7.** Let  $\mathcal{X}$  be a collection of marked hypergraphs. Let  $H$  be a marked hypergraph, and let  $y \in V(H)$ . We define  $\mathcal{X} - y = \{X \in \mathcal{X}, y \notin V(X)\}$ .

**Definition 1.8.** A *pointed marked hypergraph* is a pair  $(H, x)$  where  $H$  is a marked hypergraph and  $x \in V(H) \setminus M(H)$ .

**Definition 1.9.** We say two pointed marked hypergraphs  $(H, x)$  and  $(H', x')$  are *isomorphic* if there exists a bijection  $\varphi : V(H) \rightarrow V(H')$  such that:

- For all  $e \subseteq V(H)$ :  $e \in E(H) \iff \varphi(e) \in E(H')$ .
- For all  $v \in V(H)$ :  $v \in M(H) \iff \varphi(v) \in M(H')$ .
- $\varphi(x) = x'$ .

**Notation 1.10.** Let  $\mathcal{F}$  be a family of pointed marked hypergraphs. Let  $H$  be a marked hypergraph and  $x \in V(H) \setminus M(H)$ . We denote by  $x\mathcal{F}(H)$  the collection of all subhypergraphs  $X$  of  $H$  such that  $x \in V(X)$  and  $(X, x)$  is isomorphic to an element of  $\mathcal{F}$ .

## 1.2 Intersecting collections

**Definition 1.11.** Let  $\mathcal{X}$  be a collection of marked hypergraphs and let  $H$  be a marked hypergraph. We define the *intersection of  $\mathcal{X}$  in  $H$*  as:

$$\begin{aligned} I_H(\mathcal{X}) &= \{y \in V(H) \setminus M(H), y \in V(X) \text{ for all } X \in \mathcal{X}\} \\ &= \{y \in V(H) \setminus M(H), \mathcal{X} - y = \emptyset\}. \end{aligned}$$

We may say  $\mathcal{X}$  is *intersecting in  $H$*  if  $I_H(\mathcal{X}) \neq \emptyset$ .

Note that we have the following property:

**Proposition 1.12.** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be collections of marked hypergraphs, and let  $H$  be a marked hypergraph. If  $\mathcal{X} \subseteq \mathcal{Y}$ , then  $I_H(\mathcal{Y}) \subseteq I_H(\mathcal{X})$ .*

**Definition 1.13.** Let  $\mathcal{X}$  be a collection of marked hypergraphs and let  $H$  be a marked hypergraph. An *obstruction of  $\mathcal{X}$  in  $H$*  is a subcollection  $\mathcal{O} \subseteq \mathcal{X}$  such that  $I_H(\mathcal{O}) = \emptyset$ . The set of all obstructions of  $\mathcal{X}$  in  $H$  is denoted by  $\mathfrak{O}_H(\mathcal{X})$ .

The following characterization of intersecting collections is trivial:

**Proposition 1.14.** *Let  $\mathcal{X}$  be a collection of marked hypergraphs and let  $H$  be a marked hypergraph. Then  $I_H(\mathcal{X}) = \emptyset$  if and only if  $\mathfrak{O}_H(\mathcal{X}) \neq \emptyset$ .*

*Proof.* If  $I_H(\mathcal{X}) = \emptyset$ , then  $\mathcal{X} \in \mathfrak{O}_H(\mathcal{X})$  hence  $\mathfrak{O}_H(\mathcal{X}) \neq \emptyset$ . Conversely, if  $\mathcal{X}$  is intersecting in  $H$ , then so are all of its subcollections hence  $\mathfrak{O}_H(\mathcal{X}) = \emptyset$ . ■

If a collection is not intersecting, when is it possible to make it intersecting by removing a non-marked vertex? This question can also be answered in terms of obstructions. By Proposition 1.14, the fact that a collection  $\mathcal{X}$  is not intersecting in  $H$  equates to  $\mathcal{X}$  admitting obstructions in  $H$ . We now show that  $\mathcal{X}$  can be made intersecting in  $H$ , by removing a non marked-vertex, if and only if the unions of its obstructions in  $H$  form an intersecting collection in  $H$ . More precisely:

**Proposition 1.15.** *Let  $\mathcal{X}$  be a finite collection of marked hypergraphs and let  $H$  be a marked hypergraph. Let  $y \in V(H) \setminus M(H)$ . Then  $I_H(\mathcal{X} - y) \neq \emptyset$  if and only if  $y \in I_H(\{\langle \mathcal{O} \rangle, \mathcal{O} \in \mathfrak{O}_H(\mathcal{X})\})$ .*

*Proof.* If  $I_H(\mathcal{X} - y) = \emptyset$ , then define  $\mathcal{O} := \mathcal{X} - y \subseteq \mathcal{X}$ : we have  $\mathcal{O} \in \mathfrak{O}_H(\mathcal{X})$  and  $y \notin V(\langle \mathcal{O} \rangle)$ , so  $y \notin I_H(\{\langle \mathcal{O} \rangle, \mathcal{O} \in \mathfrak{O}_H(\mathcal{X})\})$ . Conversely, if  $y \notin I_H(\{\langle \mathcal{O} \rangle, \mathcal{O} \in \mathfrak{O}_H(\mathcal{X})\})$ , then let  $\mathcal{O} \in \mathfrak{O}_H(\mathcal{X})$  such that  $y \notin V(\langle \mathcal{O} \rangle)$ : we have  $\mathcal{O} \subseteq \mathcal{X} - y$ , so  $I_H(\mathcal{X} - y) \subseteq I_H(\mathcal{O}) = \emptyset$ . ■

*Remark.* The fact that the collection  $\mathcal{X}$  is finite ensures that its obstructions also are, so that their unions are well defined. In practice, as mentioned before, we will only consider collections whose elements are all subhypergraphs of some common marked hypergraph, and such collections are obviously finite.

## 1.3 The Maker-Breaker game on marked hypergraphs

### 1.3.1 Description and elementary properties

In the literature, the Maker-Breaker game is played on a standard hypergraph  $H$  rather than a marked hypergraph. Maker and Breaker take turns coloring vertices of  $H$  in red and blue respectively (with Maker playing first), and Maker wins if and only if he manages to color an entire edge of  $H$  in red (i.e. Breaker wins if and only if he manages to color a full transversal in blue). The actions of both players can be seen as follows: Maker *marks* vertices, Breaker *removes* vertices. Indeed, Breaker coloring some vertex  $y$  in blue means the edges containing  $y$  can no longer be colored entirely in red and are thus rendered useless.

For this reason, it is natural to consider the game as played on marked hypergraphs. On each turn, Maker selects a non-marked vertex and marks it, then Breaker selects a non-marked vertex and removes it (meaning the vertex is removed as well as all edges containing it). Note that some vertices may be marked already before the game starts. Maker wins if and only if he manages to get an edge whose vertices are all marked. An example on the "tic-tac-toe hypergraph" is given in Figure 1: here, we see that Maker wins by completing the middle row of the hypergraph.

The operators  $+x$  and  $-y$  can be interpreted as the effect of Maker picking  $x$  and Breaker picking  $y$  respectively, so that after a round of play on a marked hypergraph  $H$  where Maker marks  $x$  and Breaker removes  $y$ , a new game effectively starts on the marked hypergraph  $H^{+x-y}$ . Since

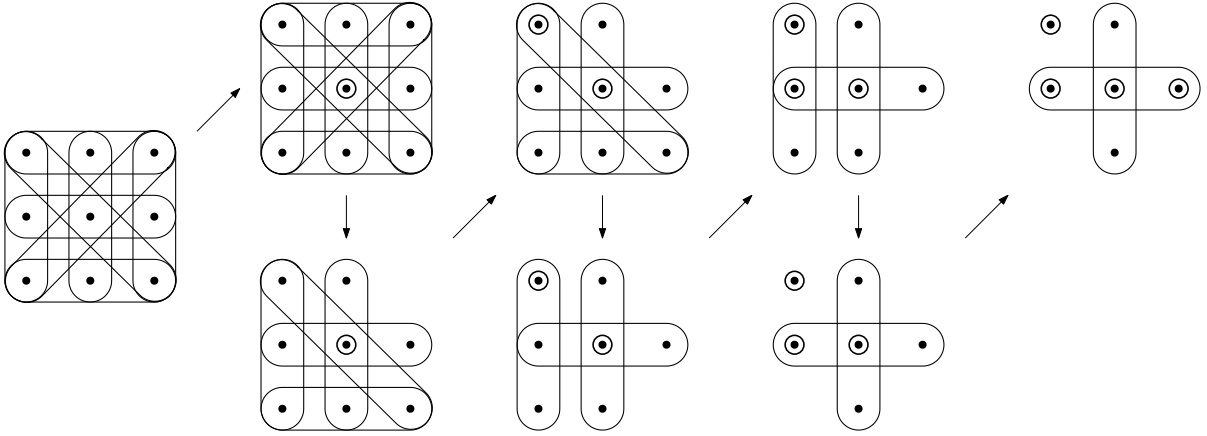


Figure 1: Evolution of the marked hypergraph during a game. The marked vertices are circled, as they will be in all figures. The top (resp. bottom) row corresponds to positions after Maker's (resp. Breaker's) moves.

we can assume that the game continues (even if Maker has already won) until all vertices are taken, the winner of the game with perfect play can be defined in the following way:

**Definition 1.16.** Let  $H$  be a marked hypergraph. We say  $H$  is a *trivial Maker win* if some edge  $e \in E(H)$  satisfies  $|e \setminus M(H)| \leq 1$ .

**Definition 1.17.** Let  $H$  be a marked hypergraph. The fact that  $H$  is a *Maker win* is defined recursively as follows:

- (1) If  $|V(H) \setminus M(H)| \leq 1$ , then  $H$  is a Maker win if and only if  $H$  is a trivial Maker win.
- (2) If  $|V(H) \setminus M(H)| \geq 2$ , then  $H$  is a Maker win if and only if there exists  $x \in V(H) \setminus M(H)$  such that, for all  $y \in V(H^{+x}) \setminus M(H^{+x})$ ,  $H^{+x-y}$  is a Maker win.

Otherwise, we say  $H$  is a *Breaker win*.

Recall that Maker plays first, hence why what we call a trivial Maker win should indeed be a Maker win: if there exists some edge  $e \in E(H)$  such that  $e \setminus M(H) = \{x\}$ , then Maker can win instantly on  $H$  by picking  $x$ .

**Notation 1.18.** Let **MAKERBREAKER** be the decision problem that takes as input a marked hypergraph  $H$  and outputs "yes" if and only if  $H$  is a Maker win.

It can also be interesting to consider a version of the game where Maker tries to win as quickly as possible. The following notation is introduced in [10], and we adapt it to marked hypergraphs:

**Notation 1.19.** Let  $H$  be a marked hypergraph. We define  $\tau_M(H)$  as the minimum number of rounds in which Maker can guarantee to get a fully marked edge when playing the Maker-Breaker game on  $H$ , with  $\tau_M(H) = \infty$  by convention if  $H$  is a Breaker win. Equivalently,  $\tau_M(H)$  may be defined recursively as follows:

- (0) If  $H$  is a trivial Maker win, then define  $\tau_M(H) \in \{0, 1\}$  as the minimum number of non-marked vertices in an edge of  $H$ .
- (1) If  $H$  is not a trivial Maker win and  $|V(H) \setminus M(H)| \leq 1$ , then define  $\tau_M(H) = \infty$ .
- (2) If  $H$  is not a trivial Maker win and  $|V(H) \setminus M(H)| \geq 2$ , then define

$$\tau_M(H) = 1 + \min_{x \in V(H) \setminus M(H)} \max_{y \in V(H^{+x}) \setminus M(H^{+x})} \tau_M(H^{+x-y}).$$



The study of the Maker-Breaker game revolves around considering certain classes of (marked) hypergraphs for which we try to:

- identify criteria ensuring a Maker win or a Breaker win;
- evaluate  $\tau_M(\cdot)$  in the case of a Maker win i.e. find fast-winning strategies for Maker;
- determine the algorithmic complexity of MAKERBREAKER.

About that last problem, notice that we can always limit ourselves to the uniform case:

**Proposition 1.20.** *For any  $k \geq 2$ , the following three decision problems all reduce polynomially to one another:*

- (a) MAKERBREAKER on hypergraphs of rank  $k$ ;
- (b) MAKERBREAKER on marked hypergraphs of rank  $k$ ;
- (c) MAKERBREAKER on  $k$ -uniform marked hypergraphs.

*Proof.* The reduction from (a) to (b) is trivial since hypergraphs are special cases of marked hypergraphs. For the reduction from (b) to (c), let  $H$  be a marked hypergraph of rank  $k$  and define the  $k$ -uniform marked hypergraph  $H_0$  obtained from  $H$  as follows: for each edge  $e$  of  $H$ , we create  $k - |e|$  new marked vertices, and we add them to  $e$ . It is clear that  $H$  is a Maker win if and only if  $H_0$  is a Maker win. For the reduction from (c) to (a), we reverse this idea. Let  $H$  be a  $k$ -uniform marked hypergraph, and let  $H_0$  be the hypergraph of rank  $k$  obtained from  $H$  by removing all marked vertices and replacing each edge  $e$  of  $H$  by  $e \setminus M(H)$ . It is clear that  $H$  is a Maker win if and only if  $H_0$  is a Maker win. ■

The next property is essential. The idea is simple: if Maker can win within  $t$  rounds on some subhypergraph  $X \subseteq H$ , then he can do the same on  $H$ , by simply playing all his picks inside  $X$  and following his strategy on  $X$  (note that Breaker might pick vertices outside  $X$ , but this can only benefit Maker if he focuses on  $X$ ). We now give a rigorous proof that is adapted to our recursive definitions.

**Proposition 1.21.** *Let  $H$  be a marked hypergraph, and let  $X$  be a subhypergraph of  $H$ . Then  $\tau_M(H) \leq \tau_M(X)$ . In particular, if  $X$  is a Maker win then  $H$  is a Maker win.*

*Proof.* Let us first address the case where  $H$  is a trivial Maker win:

Claim 1. *If  $H$  is a trivial Maker win, then  $\tau_M(H) \leq \tau_M(X)$ .*

*Proof of Claim 1.* By definition,  $\tau_M(H) \in \{0, 1\}$  is the minimum number of non-marked vertices in an edge of  $H$ . If  $\tau_M(H) = 0$  then  $\tau_M(H) \leq \tau_M(X)$  trivially. If  $\tau_M(H) = 1$  i.e. there is no fully marked edge in  $H$ , then there is none in  $X$  either hence  $\tau_M(X) \geq 1 = \tau_M(H)$ . □

We now prove the proposition by induction on  $|V(X) \setminus M(X)|$ .

- Suppose that  $|V(X) \setminus M(X)| \leq 1$ . Assume  $H$  is not a trivial Maker win, otherwise Claim 1 concludes. Then  $X$  is not a trivial Maker win either, so  $\tau_M(X) = \infty$  hence  $\tau_M(H) \leq \tau_M(X)$  trivially.

- Suppose that  $|V(X) \setminus M(X)| \geq 2$  (note that it implies  $|V(H) \setminus M(H)| \geq 2$ ) and that the result holds for all subhypergraphs with less non-marked vertices than  $X$ . Again, assume  $H$  is not a trivial Maker win, otherwise Claim 1 concludes. Then  $X$  is not a trivial Maker win either. We thus have  $\tau_M(H) = 1 + \min_{x \in V(H) \setminus M(H)} \max_{y \in V(H^{+x}) \setminus M(H^{+x})} \tau_M(H^{+x-y})$

and  $\tau_M(X) = 1 + \min_{x \in V(X) \setminus M(X)} \max_{y \in V(X^{+x}) \setminus M(X^{+x})} \tau_M(X^{+x-y})$ .

Let  $x_0 \in \arg \min_{x \in V(X) \setminus M(X)} \max_{y \in V(X^{+x}) \setminus M(X^{+x})} \tau_M(X^{+x-y})$  and  $y_0 \in \arg \max_{y \in V(H^{+x_0}) \setminus M(H^{+x_0})} \tau_M(H^{+x_0-y})$ .

Since  $x_0 \in V(X) \setminus M(X) \subseteq V(H) \setminus M(H)$ , we have:

$$\tau_M(H) \leq 1 + \max_{y \in V(H^{+x_0}) \setminus M(H^{+x_0})} \tau_M(H^{+x_0-y}) = 1 + \tau_M(H^{+x_0-y_0}),$$

where the last equality holds by definition of  $y_0$ . If  $y_0 \in V(X)$  then let  $y_1 := y_0$ , otherwise let  $y_1 \in V(X^{+x}) \setminus M(X^{+x})$  be arbitrary. In both cases, we have  $X^{+x_0-y_1} \subseteq H^{+x_0-y_0}$ : indeed, if  $y_0 \notin V(X)$  i.e.  $X \subseteq H^{-y_0}$  then  $X^{-y_1} \subseteq H^{-y_0}$  hence  $X^{+x_0-y_1} \subseteq H^{+x_0-y_0}$ . Therefore, by the induction hypothesis, the previous inequality yields:

$$\tau_M(H) \leq 1 + \tau_M(X^{+x_0-y_1}) \leq 1 + \max_{y \in V(X^{+x_0}) \setminus M(X^{+x_0})} \tau_M(X^{+x_0-y}) = \tau_M(X),$$

where the last equality holds by definition of  $x_0$ . ■

### 1.3.2 Dangers

From now on, we adopt Breaker's point of view. The idea is to design strategies for Breaker that consist, on each turn, in focusing solely on some identified immediate threats and picking a vertex that eliminates all these specific threats (if possible).

**Definition 1.22.** A *danger* is a pointed marked hypergraph  $(D, x)$  such that  $D^{+x}$  is a Maker win.

*Example.* A *trivial danger of size  $k$*  is of the form  $(D, x)$  where  $D$  consists of a single edge with  $k$  vertices that are all marked except for  $x$  and one other vertex. It is obviously a danger because  $D^{+x}$  is a trivial Maker win. See Figure 2.



Figure 2: A trivial danger  $(D, x)$  of size 5.

**Definition 1.23.** Let  $H$  be a marked hypergraph and  $x \in V(H) \setminus M(H)$ . A *danger at  $x$  in  $H$*  is a subhypergraph  $D$  of  $H$  containing  $x$  such that  $(D, x)$  is a danger i.e. such that  $D^{+x}$  is a Maker win.

Dangers at  $x$  constitute urgent threats for Breaker in the case Maker picks  $x$ . Indeed, if Maker picks  $x$  then any danger  $D$  at  $x$  must be immediately destroyed i.e. Breaker must pick some  $y \in V(D)$  next, otherwise the resulting marked hypergraph would contain  $D^{+x}$  and thus be a Maker win according to Proposition 1.21. Therefore, if  $\mathcal{X}_x$  is any collection of dangers at  $x$  in  $H$  and Maker picks  $x$ , then Breaker is forced to "destroy" all elements of  $\mathcal{X}_x$  i.e. answer some  $y$  belonging to the intersection of  $\mathcal{X}_x$  in  $H^{+x}$  (we have to take the intersection in  $H^{+x}$ , because  $x$  is no longer pickable for Breaker after Maker has picked it). We thus introduce the following key property, which is necessary for Breaker to win:

**Notation 1.24.** Let  $H$  be a marked hypergraph such that  $|V(H) \setminus M(H)| \geq 2$ . For all  $x \in V(H) \setminus M(H)$ , let  $\mathcal{X}_x$  be a collection of dangers at  $x$  in  $H$ . We denote by  $J((\mathcal{X}_x)_{x \in V(H) \setminus M(H)}, H)$  the following property:

$$\forall x \in V(H) \setminus M(H), I_{H^{+x}}(\mathcal{X}_x) \neq \emptyset.$$

*Remark.* Dangers are not relevant when there is less than one full round of play left, hence the assumption that  $|V(H) \setminus M(H)| \geq 2$ . This also avoids some dull cases where the property would fail on a technicality, by ensuring that if  $\mathcal{X}_x = \emptyset$  then  $I_{H+x}(\mathcal{X}_x) = V(H^{+x}) \setminus M(H^{+x}) \neq \emptyset$ .

**Proposition 1.25.** *Let  $H$  be a marked hypergraph such that  $|V(H) \setminus M(H)| \geq 2$ . For all  $x \in V(H) \setminus M(H)$ , let  $\mathcal{X}_x$  be a collection of dangers at  $x$  in  $H$ . Then, for all  $x \in V(H) \setminus M(H)$  and for all  $y \in V(H^{+x}) \setminus M(H^{+x})$  such that  $y \notin I_{H+x}(\mathcal{X}_x)$ ,  $H^{+x-y}$  is a Maker win.*

*Proof.* Since  $y \in V(H^{+x}) \setminus M(H^{+x})$ , the fact that  $y \notin I_{H+x}(\mathcal{X}_x)$  means there exists  $D \in \mathcal{X}_x$  such that  $y \notin V(D)$ . By definition of a danger at  $x$ ,  $D^{+x}$  is a Maker win, and it is a subhypergraph of  $H^{+x-y}$  because  $y \notin V(D)$ . Therefore,  $H^{+x-y}$  is a Maker win by Proposition 1.21. ■

**Corollary 1.26.** *Let  $H$  be a marked hypergraph such that  $|V(H) \setminus M(H)| \geq 2$ . For all  $x \in V(H) \setminus M(H)$ , let  $\mathcal{X}_x$  be a collection of dangers at  $x$  in  $H$ . If  $H$  is a Breaker win, then  $J((\mathcal{X}_x)_{x \in V(H) \setminus M(H)}, H)$  holds.*

*Proof.* Suppose  $J((\mathcal{X}_x)_{x \in V(H) \setminus M(H)}, H)$  does not hold. Maker can then choose  $x$  such that  $I_{H+x}(\mathcal{X}_x) = \emptyset$ , so that Breaker's answer  $y$  cannot be in  $I_{H+x}(\mathcal{X}_x)$ , thus ensuring that  $H^{+x-y}$  is a Maker win by Proposition 1.25. Therefore,  $H$  is a Maker win. ■

When considering all possible dangers at each non-marked vertex, this condition is also sufficient:

**Theorem 1.27.** *Let  $H$  be a marked hypergraph such that  $|V(H) \setminus M(H)| \geq 2$ . For all  $x \in V(H) \setminus M(H)$ , let  $\mathcal{X}_x$  be the collection of **all** dangers at  $x$  in  $H$ . Then  $H$  is a Breaker win if and only if  $J((\mathcal{X}_x)_{x \in V(H) \setminus M(H)}, H)$  holds.*

*Proof.* The "only if" direction is given by Corollary 1.26, so we show the "if" direction. Suppose  $J((\mathcal{X}_x)_{x \in V(H) \setminus M(H)}, H)$  holds. Maker picks some  $x \in V(H) \setminus M(H)$ , and Breaker answers with some  $y \in I_{H+x}(\mathcal{X}_x)$ . Since  $y \notin V(H^{-y})$ , we have  $H^{-y} \notin \mathcal{X}_x$ . By definition of a danger at  $x$ , this means  $(H^{-y})^{+x} = H^{+x-y}$  is a Breaker win, so  $H$  is a Breaker win. ■

### 1.3.3 Considering a fixed family of dangers

Theorem 1.27 is unlikely to be useful from an algorithmic point of view, since identifying general dangers at a given  $x$  is as difficult as identifying Maker wins. We would like the same equivalence to hold for smaller collections  $\mathcal{X}_x$  so that property  $J()$  is easier to check. A natural idea is to consider dangers at  $x$  of the same type for all  $x$ , belonging to some fixed family of dangers  $\mathcal{F}$  that would be independent of  $x$  and easy to recognize:

**Definition 1.28.** Let  $\mathcal{F}$  be a family of dangers. An element of  $\mathcal{F}$  may be referred to as an  $\mathcal{F}$ -danger. If  $H$  is a marked hypergraph and  $x \in V(H) \setminus M(H)$ , then an element of the collection  $x\mathcal{F}(H)$  (recall Notation 1.10) is called an  $\mathcal{F}$ -danger at  $x$  in  $H$ .

For any family of dangers  $\mathcal{F}$ , Breaker needs the ability to destroy all  $\mathcal{F}$ -dangers at whatever vertex  $x$  that Maker picks in the first round, according to Corollary 1.26. Actually, this remains true for all subsequent rounds, hence the following notation and necessary condition for a Breaker win:

**Notation 1.29.** Let  $\mathcal{F}$  be a family of dangers. Let  $r \geq 1$  be an integer, and let  $H$  be a marked hypergraph such that  $|V(H) \setminus M(H)| \geq 2r$ . We recursively define the following properties:

- Property  $J_1(\mathcal{F}, H)$  refers to property  $J((x\mathcal{F}(H))_{x \in V(H) \setminus M(H)}, H)$  i.e.:

$$\forall x \in V(H) \setminus M(H), I_{H+x}(x\mathcal{F}(H)) \neq \emptyset.$$

- Property  $J_r(\mathcal{F}, H)$ , for  $r \geq 2$ , means that:

$$\forall x \in V(H) \setminus M(H), \exists y \in I_{H+x}(x\mathcal{F}(H)), J_{r-1}(\mathcal{F}, H^{+x-y}) \text{ holds.}$$

For any  $r \geq 1$ , property  $J_r(\mathcal{F}, H)$  should be understood as: "In each of the first  $r$  rounds of the Maker-Breaker game played on  $H$ , Breaker will be able to destroy all  $\mathcal{F}$ -dangers at the vertex that Maker has just picked".

**Proposition 1.30.** *Let  $\mathcal{F}$  be a family of dangers. Let  $r \geq 1$  be an integer, and let  $H$  be a marked hypergraph such that  $|V(H) \setminus M(H)| \geq 2r$ . If  $H$  is a Breaker win, then  $J_r(\mathcal{F}, H)$  holds.*

*Proof.* We proceed by induction on  $r$ . For  $r = 1$ , this is simply Corollary 1.26 with  $\mathcal{X}_x = x\mathcal{F}(H)$ . Now let  $r \geq 2$  such that property  $J_{r-1}(\mathcal{F}, \cdot)$  is necessary for a Breaker win. Let  $x \in V(H) \setminus M(H)$ : the condition  $y \in I_{H+x}(x\mathcal{F}(H))$  is necessary by Proposition 1.25, and the condition that  $J_{r-1}(\mathcal{F}, H^{+x-y})$  holds is necessary by the induction hypothesis, which concludes. ■

We can make some observations:

**Proposition 1.31.** *Let  $\mathcal{F}$  be a family of dangers. Let  $r \geq 1$  be an integer, and let  $H$  be a marked hypergraph such that  $|V(H) \setminus M(H)| \geq 2r$ .*

- (i) *For any integer  $1 \leq s \leq r$ :  $J_r(\mathcal{F}, H) \implies J_s(\mathcal{F}, H)$ .*
- (ii) *For any family of dangers  $\mathcal{G} \subseteq \mathcal{F}$ :  $J_r(\mathcal{F}, H) \implies J_r(\mathcal{G}, H)$ .*
- (iii) *For any subhypergraph  $X \subseteq H$  such that  $|V(X) \setminus M(X)| \geq 2r$ :  $J_r(\mathcal{F}, H) \implies J_r(\mathcal{F}, X)$ .*

*Proof.* Item (i) is straightforward. Item (ii) comes from the fact that  $x\mathcal{G}(H) \subseteq x\mathcal{F}(H)$  hence  $I_{H+x}(x\mathcal{F}(H)) \subseteq I_{H+x}(x\mathcal{G}(H))$ . Let us now prove item (iii) by induction on  $r$ .

- Let us first show the implication for  $r = 1$ . Suppose  $J_1(\mathcal{F}, H)$  holds. Let  $x \in V(X) \setminus M(X)$ : we want to show that there exists  $y \in I_{X+x}(x\mathcal{F}(X))$ . By  $J_1(\mathcal{F}, H)$ , there exists  $y' \in I_{H+x}(x\mathcal{F}(H))$ . If  $y' \in V(X)$ , then  $y := y'$  is suitable since  $x\mathcal{F}(X) \subseteq x\mathcal{F}(H)$ . If  $y' \notin V(X)$ , then in particular  $x\mathcal{F}(X) = \emptyset$  (indeed, if there existed  $D_0 \in x\mathcal{F}(X) \subseteq x\mathcal{F}(H)$  then we would have  $y' \in V(D_0) \subseteq V(X)$ ), therefore any  $y \in V(X^{+x}) \setminus M(X^{+x})$  is suitable.
- Now, let  $r \geq 2$  such that the implication is true for  $J_{r-1}(\mathcal{F}, \cdot)$ . Suppose  $J_r(\mathcal{F}, H)$  holds. Let  $x \in V(X) \setminus M(X)$ : we want to show that there exists  $y \in I_{X+x}(x\mathcal{F}(X))$  such that  $J_{r-1}(\mathcal{F}, X^{+x-y})$  holds. By  $J_r(\mathcal{F}, H)$ , there exists  $y' \in I_{H+x}(x\mathcal{F}(H))$  such that  $J_{r-1}(\mathcal{F}, H^{+x-y'})$  holds. If  $y' \in V(X)$ , then  $y := y'$  is suitable: indeed, the fact that  $J_{r-1}(\mathcal{F}, H^{+x-y'})$  holds implies that  $J_{r-1}(\mathcal{F}, X^{+x-y'})$  holds by the induction hypothesis. If  $y' \notin V(X)$ , then in particular  $x\mathcal{F}(X) = \emptyset$  (as above), therefore any  $y \in V(X^{+x}) \setminus M(X^{+x})$  is suitable: indeed, we have  $X \subseteq H^{-y'}$  hence  $X^{+x-y} \subseteq H^{+x-y'}$ , so the fact that  $J_{r-1}(\mathcal{F}, H^{+x-y'})$  holds implies that  $J_{r-1}(\mathcal{F}, X^{+x-y})$  holds by the induction hypothesis. ■

In general,  $J_r(\mathcal{F}, H)$  is stronger than  $J_{r-1}(\mathcal{F}, H)$ , because dangers can appear during the game: every time Maker picks a vertex  $x$ , that might create new  $\mathcal{F}$ -dangers at other vertices since  $x$  is now marked. Of course, dangers can also disappear during the game: every time Breaker picks a vertex  $y$ , that removes all  $\mathcal{F}$ -dangers containing  $y$  at all vertices.

*Example.* The hypergraph  $H$  from Figure 3 illustrates the difference between properties  $J_1(\mathcal{F}, H)$  and  $J_2(\mathcal{F}, H)$ . The  $\mathcal{F}$ -dangers in  $H$  are as follows:  $x\mathcal{F}(H) = \{D_1, D_2\}$ ,  $z\mathcal{F}(H) = \{D_3, D_4\}$ , and  $a\mathcal{F}(H) = \emptyset$  for all  $a \in V(H) \setminus \{x, z\}$ . Since  $I_{H+x}(x\mathcal{F}(H)) = \{y\} \neq \emptyset$  and  $I_{H+z}(z\mathcal{F}(H)) =$

$\{u, v\} \neq \emptyset$ , property  $J_1(\mathcal{F}, H)$  holds. Suppose that Maker picks  $x$ : Breaker has to pick  $y$  to destroy the  $\mathcal{F}$ -dangers at  $x$ . Now suppose that, though  $C$  was not an  $\mathcal{F}$ -danger at  $z$ ,  $C^{+x}$  is one: this means that, by picking  $x$  in the first round, Maker has created a third  $\mathcal{F}$ -danger at  $z$  in addition to the already existing ones  $D_3$  and  $D_4$ . Since  $y \notin V(D_3) \cup V(D_4) \cup V(C^{+x})$ , we have  $z\mathcal{F}(H^{+x-y}) = \{D_3, D_4, C^{+x}\}$  hence  $I_{H^{+x-y+z}}(z\mathcal{F}(H^{+x-y})) = \emptyset$ , so  $J_1(\mathcal{F}, H^{+x-y})$  does not hold and neither does  $J_2(\mathcal{F}, H)$ . After the first round, Maker can simply pick  $z$  and go on to win.

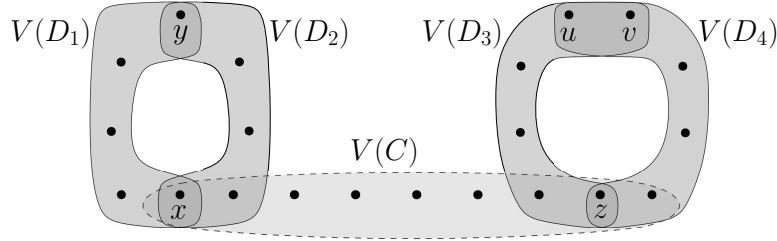


Figure 3: Some vertex subsets in a hypergraph (no vertex is marked). The edges are not represented.

Given a class  $\mathcal{H}$  of marked hypergraphs, which we assume to be stable under the operators  $^{+x}$  and  $^{-y}$ , our idea is to find a family of dangers  $\mathcal{F}$  as simple as possible and a constant  $r$  as small as possible such that the necessary condition from Proposition 1.30 is actually sufficient on  $\mathcal{H}$ , that is:

For all  $H \in \mathcal{H}$  :  $H$  is a Breaker win if and only if  $J_r(\mathcal{F}, H)$  holds. (\*)

Since property  $J_r(\mathcal{F}, H)$  does not seem to guarantee anything after the first  $r$  rounds, a statement such as (\*) is very strong. In particular, if  $\mathcal{F}$  is efficiently identifiable i.e. deciding whether there exists an  $\mathcal{F}$ -danger at a given  $x$  in a marked hypergraph  $H \in \mathcal{H}$  on  $n$  vertices can be done in polynomial time  $P(n)$ , then (\*) would yield a  $O(n^{2r}P(n))$  polynomial-time algorithm for MAKERBREAKER on the class  $\mathcal{H}$ .

By Proposition 1.30,  $H$  is a Breaker win only if  $J_s(\mathcal{F}, H)$  holds for all  $s \geq 1$ . If  $\mathcal{F}$  contains the trivial dangers and  $H$  is not a trivial Maker win, then this condition is also sufficient: indeed, Maker is unable to complete an edge in the first round because  $H$  is not a trivial Maker win, or in any subsequent round because Breaker destroys the trivial dangers each time. Therefore, it is natural to look for a family  $\mathcal{F}$  containing the trivial dangers, and to exclude the trivial Maker wins from the considered class  $\mathcal{H}$ . If, additionally, there exists  $r$  such that  $J_r(\mathcal{F}, H)$  implies  $J_{r+1}(\mathcal{F}, H)$  (and thus implies  $J_s(\mathcal{F}, H)$  for all  $s \geq 1$  by induction), then we get (\*).

An example of a very simple class  $\mathcal{H}$  is that of graphs, which can be seen as 2-uniform marked hypergraphs with no marked vertex. The Maker-Breaker game on graphs is trivial: the graphs that are Breaker wins are exactly matchings. Therefore, on the class of graphs, the trivial danger of size 2 alone is enough to get (\*) with  $r = 1$ :

**Theorem 1.32.** *Let  $\mathcal{F}$  be the singleton family consisting of the trivial danger of size 2. Let  $G$  be a graph on at least two vertices. Then  $G$  is a Breaker win if and only if  $J_1(\mathcal{F}, G)$  holds.*

*Proof.* For all  $x \in V(G)$ ,  $x\mathcal{F}(G)$  is the collection of all individual edges of  $G$  that are incident to  $x$  and  $I_{G^{+x}}(x\mathcal{F}(G))$  is the intersection of these edges minus  $x$ , therefore  $I_{G^{+x}}(x\mathcal{F}(G)) \neq \emptyset$  if and only if  $x$  is of degree at most 1. In conclusion,  $J_1(\mathcal{F}, G)$  holds if and only if  $G$  is a matching, which is equivalent to  $G$  being a Breaker win. ■

When playing on a graph, the naive strategy for Breaker that consists in picking a vertex that is the only non-marked vertex of some edge (if there exists one, or an arbitrary vertex otherwise)

is actually optimal. With edges of size 3, this strategy no longer works, but one of our main results will be to exhibit a simple family of dangers  $\mathcal{F}$  such that we get (\*) with  $r = 3$  on the class of 3-uniform marked hypergraphs.

### 1.3.4 Danger prevention

The goal of this segment is to show that  $J_r(\mathcal{F}, H)$  is equivalent to  $J_1(\mathcal{F}^{*(r-1)}, H)$  for some family of dangers  $\mathcal{F}^{*(r-1)}$  that we are going to introduce. In other words, preventing issues with the  $\mathcal{F}$ -dangers that could arise in the first  $r$  rounds comes down to dealing with a larger family of dangers as soon as the first round.

The idea is the following. Say Breaker wants to be able to manage the  $\mathcal{F}$ -dangers in the second round. Maker now picks  $x$ . As Breaker ponders his answer  $y$ , he needs to already think about the (yet unknown) vertex  $z$  that Maker is going to pick next. Now that  $x$  is marked, the collection of  $\mathcal{F}$ -dangers at  $z$  is  $z\mathcal{F}(H^{+x})$ : Breaker must choose a vertex  $y$  such that  $z\mathcal{F}(H^{+x}) - y$  is intersecting, so as to be able to destroy all the remaining  $\mathcal{F}$ -dangers at  $z$  in the next round. By Proposition 1.15, this means  $y$  must hit all unions of obstructions of the collection  $z\mathcal{F}(H^{+x})$ , or equivalently (from the viewpoint of  $H$  rather than  $H^{+x}$ ) of the collection  $\{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\}$ . This must hold for all possibilities of Maker's next pick  $z$ . Here is the rigorous result:

**Proposition 1.33.** *Let  $\mathcal{F}$  be a family of dangers. Let  $H$  be a marked hypergraph such that  $|V(H) \setminus M(H)| \geq 4$ , and let  $x \in V(H) \setminus M(H)$  and  $y \in V(H^{+x}) \setminus M(H^{+x})$ . Then the following two assertions are equivalent:*

(a)  $J_1(\mathcal{F}, H^{+x-y})$  holds.

(b)  $y \in I_{H^{+x}} \left( \bigcup_{z \in V(H^{+x}) \setminus M(H^{+x})} \{\langle \mathcal{O} \rangle, \mathcal{O} \in \mathfrak{O}_{H^{+x+z}}(\{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\})\} \right)$ .

*Proof.* We make a series of innocuous rewritings before applying Proposition 1.15. First of all, recall that by definition:

$$(a) \iff \forall z \in V(H^{+x-y}) \setminus M(H^{+x-y}), I_{H^{+x-y+z}}(z\mathcal{F}(H^{+x-y})) \neq \emptyset.$$

The subhypergraphs of  $H^{+x-y}$  are exactly the subhypergraphs of  $H^{+x}$  that do not contain  $y$ , so:

$$(a) \iff \forall z \in V(H^{+x-y}) \setminus M(H^{+x-y}), I_{H^{+x-y+z}}(z\mathcal{F}(H^{+x}) - y) \neq \emptyset.$$

Consider the collection  $z\mathcal{F}(H^{+x}) - y$ : since its elements do not contain  $y$ , if it is nonempty then its intersection in  $H^{+x-y+z}$  is the same as in  $H^{+x+z}$ . Therefore:

$$(a) \iff \forall z \in V(H^{+x-y}) \setminus M(H^{+x-y}), I_{H^{+x+z}}(z\mathcal{F}(H^{+x}) - y) \neq \emptyset.$$

Since the intersection of a collection does not depend on the marked vertices of its elements, this can be reformulated in terms of subhypergraphs of  $H$  rather than  $H^{+x}$ :

$$(a) \iff \forall z \in V(H^{+x-y}) \setminus M(H^{+x-y}), I_{H^{+x+z}}(\{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\} - y) \neq \emptyset.$$

We now use Proposition 1.15, which yields:

$$(a) \iff \forall z \in V(H^{+x-y}) \setminus M(H^{+x-y}), y \in I_{H^{+x+z}}(\{\langle \mathcal{O} \rangle, \mathcal{O} \in \mathfrak{O}_{H^{+x+z}}(\{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\})\}).$$

Since  $y \neq z$ , this can be rewritten as:

$$(a) \iff \forall z \in V(H^{+x-y}) \setminus M(H^{+x-y}), \\ y \in I_{H^{+x}}(\{\langle \mathcal{O} \rangle, \mathcal{O} \in \mathfrak{O}_{H^{+x+z}}(\{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\})\}).$$

Finally, the assertion " $y \in I_{H^{+x}}(\{\langle \mathcal{O} \rangle, \mathcal{O} \in \mathfrak{O}_{H^{+x+z}}(\{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\})\})$ " would trivially be true for  $z = y$  since all elements of  $\{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\}$  contain  $z$ . Therefore:

$$(a) \iff \forall z \in V(H^{+x}) \setminus M(H^{+x}), \\ y \in I_{H^{+x}}(\{\langle \mathcal{O} \rangle, \mathcal{O} \in \mathfrak{O}_{H^{+x+z}}(\{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\})\}) \\ \iff y \in I_{H^{+x}}\left(\bigcup_{z \in V(H^{+x}) \setminus M(H^{+x})} \{\langle \mathcal{O} \rangle, \mathcal{O} \in \mathfrak{O}_{H^{+x+z}}(\{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\})\}\right). \blacksquare$$

The subhypergraphs  $\langle \mathcal{O} \rangle$  from Proposition 1.33 that contain  $x$  may thus be interpreted as dangers at  $x$ , since Breaker has to destroy them. We will call them  $\mathcal{F}^\mathfrak{O}$ -dangers at  $x$ :

**Notation 1.34.** Let  $\mathcal{F}$  be a family of dangers. We denote by  $\mathcal{F}^\mathfrak{O}$  the family of all pointed marked hypergraphs  $(D, x)$  such that, for some non-marked  $z \neq x$  which we call an  $\mathcal{F}$ -dangerous vertex in  $(D, x)$ , we can write  $D = \langle \mathcal{O} \rangle$  where the collection  $\mathcal{O}$  satisfies the following properties:

- each  $X \in \mathcal{O}$  containing  $x$  is such that  $X^{+x}$  is an  $\mathcal{F}$ -danger at  $z$ ;
- each  $X \in \mathcal{O}$  not containing  $x$  is already an  $\mathcal{F}$ -danger at  $z$ ;
- $I_{D^{+x+z}}(\mathcal{O}) = \emptyset$ .

In other words, given a marked hypergraph  $H$  and a vertex  $x \in V(H) \setminus M(H)$ , we have:

$$x\mathcal{F}^\mathfrak{O}(H) = \bigcup_{z \in V(H^{+x}) \setminus M(H^{+x})} \{\langle \mathcal{O} \rangle, \mathcal{O} \in \mathfrak{O}_{H^{+x+z}}(\{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\}), x \in V(\langle \mathcal{O} \rangle)\}.$$

*Example.* Going back to the example in Figure 3, we have  $\{D_3, D_4, C^{+x}\} \subseteq z\mathcal{F}(H^{+x})$  i.e.  $\mathcal{O} := \{D_3, D_4, C\} \subseteq \{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\}$ , moreover  $I_{H^{+x+z}}(\mathcal{O}) = \emptyset$  so  $D := \langle \mathcal{O} \rangle = D_3 \cup D_4 \cup C \in x\mathcal{F}^\mathfrak{O}(H)$ .

**Proposition 1.35.** Let  $\mathcal{F}$  and  $\mathcal{G}$  be families of dangers. If  $\mathcal{G} \subseteq \mathcal{F}$ , then  $\mathcal{G}^\mathfrak{O} \subseteq \mathcal{F}^\mathfrak{O}$ .

*Proof.* This is clear since a  $\mathcal{G}$ -danger at  $z$  is also an  $\mathcal{F}$ -danger at  $z$ . ■

**Proposition 1.36.** Let  $\mathcal{F}$  be a family of dangers. Then  $\mathcal{F}^\mathfrak{O}$  is a family of dangers.

*More precisely:* for all  $(D, x) \in \mathcal{F}^\mathfrak{O}$ , if  $|V(D^{+x}) \setminus M(D^{+x})| \geq 2$  then  $J_1(\mathcal{F}, D^{+x})$  does not hold so  $D^{+x}$  is a Maker win, otherwise  $D^{+x}$  is a trivial Maker win.

*Proof.* Let  $(D, x) \in \mathcal{F}^\mathfrak{O}$ , and write  $D = \langle \mathcal{O} \rangle$  as in the definition with  $z$  an  $\mathcal{F}$ -dangerous vertex in  $(D, x)$ . If  $|V(D^{+x}) \setminus M(D^{+x})| \geq 2$ , then we can apply Proposition 1.30: since  $\mathcal{O}^{+x} \subseteq z\mathcal{F}(D^{+x})$ , we have  $I_{D^{+x+z}}(z\mathcal{F}(D^{+x})) \subseteq I_{D^{+x+z}}(\mathcal{O}^{+x}) = I_{D^{+x+z}}(\mathcal{O}) = \emptyset$ , therefore  $J_1(\mathcal{F}, D^{+x})$  does not hold so  $D^{+x}$  is a Maker win. If  $|V(D^{+x}) \setminus M(D^{+x})| \leq 1$  i.e.  $V(D^{+x}) \setminus M(D^{+x}) = \{z\}$ , then let  $X \in \mathcal{O}$ :  $(X^{+x})^{+z}$  is a Maker win whose vertices are all marked, so  $(X^{+x})^{+z}$  has a fully marked edge, therefore  $X^{+x}$  is a trivial Maker win and so is  $D^{+x} \supseteq X^{+x}$ . ■

Proposition 1.33 instantly yields the following result:

**Proposition 1.37.** Let  $\mathcal{F}$  be a family of dangers. Let  $H$  be a marked hypergraph such that  $|V(H) \setminus M(H)| \geq 4$ , and let  $x \in V(H) \setminus M(H)$  and  $y \in V(H^{+x}) \setminus M(H^{+x})$ . Moreover, suppose that  $J_1(\mathcal{F}, H)$  holds. Then the following two assertions are equivalent:

- (a)  $J_1(\mathcal{F}, H^{+x-y})$  holds.
- (b)  $y \in I_{H+x}(x\mathcal{F}^\mathfrak{O}(H))$ .

*Proof.* Given the characterization of  $x\mathcal{F}^\mathfrak{O}(H)$  from Notation 1.34, the only difference with Proposition 1.33 is that the subhypergraphs  $\langle \mathcal{O} \rangle$  from Proposition 1.33 do not necessarily contain  $x$ , whereas  $\mathcal{F}^\mathfrak{O}$ -dangers at  $x$  do. This is where we use the additional assumption that  $J_1(\mathcal{F}, H)$  holds. It is impossible that  $x \notin V(\langle \mathcal{O} \rangle)$  for some  $\mathcal{O} \in \mathfrak{O}_{H+x+z}(\{X \subseteq H, X^{+x} \in z\mathcal{F}(H^{+x})\})$ : indeed, we would then have  $\mathcal{O} \subseteq z\mathcal{F}(H)$  hence  $I_{H+z}(z\mathcal{F}(H)) \subseteq I_{H+z}(\mathcal{O}) = \emptyset$ , contradicting  $J_1(\mathcal{F}, H)$ . Therefore, under property  $J_1(\mathcal{F}, H)$ , the collection from item (b) in Proposition 1.33 coincides exactly with  $x\mathcal{F}^\mathfrak{O}(H)$ .  $\blacksquare$

Let us now introduce the families of dangers that correspond to the multiple-round prevention of intersection issues with  $\mathcal{F}$ -dangers.

**Notation 1.38.** Let  $\mathcal{F}$  be a family of dangers. For all  $r \in \mathbb{N}$ , we define a family of dangers  $\mathcal{F}^{*r}$ , recursively as follows:

- $\mathcal{F}^{*0} := \mathcal{F}$ .
- For  $r \geq 1$ :  $\mathcal{F}^{*r} := \mathcal{F} \cup (\mathcal{F}^{*(r-1)})^\mathfrak{O}$ . The family  $\mathcal{F}^{*1} = \mathcal{F} \cup \mathcal{F}^\mathfrak{O}$  may be denoted as  $\mathcal{F}^*$ .

**Proposition 1.39.** Let  $\mathcal{F}$  be a family of dangers, and let  $r \in \mathbb{N}$ .

- (i) For any family of dangers  $\mathcal{G} \subseteq \mathcal{F}$ :  $\mathcal{G}^{*r} \subseteq \mathcal{F}^{*r}$ .
- (ii) For any integer  $0 \leq s \leq r$ :  $\mathcal{F}^{*s} \subseteq \mathcal{F}^{*r}$ .
- (iii)  $\mathcal{F}^{*r} = (\mathcal{F}^{*(r-1)})^*$ .

*Proof.* (i) We proceed by induction on  $r$ . For  $r = 0$ , there is nothing to show. Now suppose that  $r \geq 1$  and that the result holds for  $r - 1$ . Let  $\mathcal{G} \subseteq \mathcal{F}$ . By definition:  $\mathcal{G}^{*r} = \mathcal{G} \cup (\mathcal{G}^{*(r-1)})^\mathfrak{O}$ . We have  $\mathcal{G} \subseteq \mathcal{F}$ , moreover  $\mathcal{G}^{*(r-1)} \subseteq \mathcal{F}^{*(r-1)}$  by the induction hypothesis hence  $(\mathcal{G}^{*(r-1)})^\mathfrak{O} \subseteq (\mathcal{F}^{*(r-1)})^\mathfrak{O}$ , so in conclusion  $\mathcal{G}^{*r} \subseteq \mathcal{F} \cup (\mathcal{F}^{*(r-1)})^\mathfrak{O} = \mathcal{F}^{*r}$ .

(ii) Again, we proceed by induction on  $r$ . For  $r = 0$ , there is nothing to show. Now suppose that  $r \geq 1$  and that the result holds for  $r - 1$ . Let  $0 \leq s \leq r$  be an integer: we have  $\mathcal{F}^{*s} = \mathcal{F} \cup (\mathcal{F}^{*(s-1)})^\mathfrak{O}$  by definition. Moreover, the induction hypothesis ensures that  $\mathcal{F}^{*(s-1)} \subseteq \mathcal{F}^{*(r-1)}$  hence  $(\mathcal{F}^{*(s-1)})^\mathfrak{O} \subseteq (\mathcal{F}^{*(r-1)})^\mathfrak{O}$ , therefore  $\mathcal{F}^{*s} \subseteq \mathcal{F} \cup (\mathcal{F}^{*(r-1)})^\mathfrak{O} = \mathcal{F}^{*r}$ .

(iii) Since  $\mathcal{F} \subseteq \mathcal{F}^{*(r-1)}$ , we have  $\mathcal{F}^{*r} = \mathcal{F} \cup (\mathcal{F}^{*(r-1)})^\mathfrak{O} \subseteq \mathcal{F}^{*(r-1)} \cup (\mathcal{F}^{*(r-1)})^\mathfrak{O} = (\mathcal{F}^{*(r-1)})^*$ . On the other hand, we have  $\mathcal{F}^{*(r-1)} \subseteq \mathcal{F}^{*r}$  by item (ii) and  $(\mathcal{F}^{*(r-1)})^\mathfrak{O} \subseteq \mathcal{F}^{*r}$  by definition of  $\mathcal{F}^{*r}$ , therefore  $(\mathcal{F}^{*(r-1)})^* = \mathcal{F}^{*(r-1)} \cup (\mathcal{F}^{*(r-1)})^\mathfrak{O} \subseteq \mathcal{F}^{*r}$ .  $\blacksquare$

We can now rephrase  $J_r(\cdot)$  in terms of dangers in the first round only:

**Proposition 1.40.** Let  $\mathcal{F}$  be a family of dangers and let  $r \geq 1$  be an integer. Then, for all marked hypergraph  $H$  such that  $|V(H) \setminus M(H)| \geq 2r$ , the properties  $J_r(\mathcal{F}, H)$  and  $J_1(\mathcal{F}^{*(r-1)}, H)$  are equivalent.

*Proof.* We proceed by induction on  $r$ . For  $r = 1$ , this statement is a tautology. Let  $r \geq 2$  such



that the equivalence holds for  $r - 1$ :

$$\begin{aligned}
J_r(\mathcal{F}, H) &\iff \forall x \in V(H) \setminus M(H), \exists y \in I_{H+x}(x\mathcal{F}(H)), J_{r-1}(\mathcal{F}, H^{+x-y}) \text{ holds} \\
&\stackrel{\text{ind. hyp.}}{\iff} \forall x \in V(H) \setminus M(H), \exists y \in I_{H+x}(x\mathcal{F}(H)), J_1(\mathcal{F}^{*(r-2)}, H^{+x-y}) \text{ holds} \\
&\stackrel{\text{Prop. 1.37}}{\iff} \forall x \in V(H) \setminus M(H), \exists y \in I_{H+x}(x\mathcal{F}(H)), y \in I_{H+x}(x(\mathcal{F}^{*(r-2)})^\ominus(H)) \\
&\iff \forall x \in V(H) \setminus M(H), \exists y \in I_{H+x}(x(\mathcal{F} \cup (\mathcal{F}^{*(r-2)})^\ominus)(H)) \\
&\iff J_1(\mathcal{F} \cup (\mathcal{F}^{*(r-2)})^\ominus, H) \\
&\iff J_1(\mathcal{F}^{*(r-1)}, H).
\end{aligned}$$

The use of Proposition 1.37 is justified by the fact that both  $J_r(\mathcal{F}, H)$  and  $J_1(\mathcal{F}^{*(r-1)}, H)$  imply  $J_1(\mathcal{F}^{*(r-2)}, H)$ : indeed,  $J_r(\mathcal{F}, H)$  implies  $J_{r-1}(\mathcal{F}, H)$  which is equivalent to  $J_1(\mathcal{F}^{*(r-2)}, H)$  by the induction hypothesis, while  $J_1(\mathcal{F}^{*(r-1)}, H)$  implies  $J_1(\mathcal{F}^{*(r-2)}, H)$  because  $\mathcal{F}^{*(r-1)} \supseteq \mathcal{F}^{*(r-2)}$ . ■

The advantage of  $J_1(\mathcal{F}^{*(r-1)}, H)$  over the equivalent property  $J_r(\mathcal{F}, H)$  is that we study a single fixed hypergraph  $H$ , instead of having to consider all hypothetical evolutions of  $H$  during  $r$  rounds. However, this is done at the cost of a bigger and possibly much more complex family of dangers. If  $\mathcal{F}^{*(r-1)}$  is somewhat manageable, then we will prefer to work with property  $J_1(\mathcal{F}^{*(r-1)}, H)$ .

### 1.3.5 Restricted obstructions

We conclude this section with a trivial remark. There can be redundancies in the family  $\mathcal{F}^* = \mathcal{F} \cup \mathcal{F}^\ominus$ , in the sense that an  $\mathcal{F}^\ominus$ -danger at  $x$  might contain an  $\mathcal{F}$ -danger at  $x$ . Such  $\mathcal{F}^\ominus$ -dangers may be ignored:

**Notation 1.41.** Let  $\mathcal{F}$  be a family of dangers. We denote by  $\mathcal{F}^{\ominus, \text{rest}} \subseteq \mathcal{F}^\ominus$  the family of all  $(D, x) \in \mathcal{F}^\ominus$  such that  $D$  contains no  $\mathcal{F}$ -danger at  $x$ .

**Proposition 1.42.** *Let  $\mathcal{F}$  is a family of dangers. Let  $H$  be a marked hypergraph and let  $x \in V(H) \setminus M(H)$ . Then  $I_{H+x}(x\mathcal{F}^*(H)) = I_{H+x}(x(\mathcal{F} \cup \mathcal{F}^{\ominus, \text{rest}})(H))$ .*

*Proof.* Obviously  $I_{H+x}(x\mathcal{F}^*(H)) \subseteq I_{H+x}(x(\mathcal{F} \cup \mathcal{F}^{\ominus, \text{rest}})(H))$  since  $\mathcal{F} \cup \mathcal{F}^{\ominus, \text{rest}} \subseteq \mathcal{F}^*$ . Moreover, let  $y \in I_{H+x}(x(\mathcal{F} \cup \mathcal{F}^{\ominus, \text{rest}})(H))$ : for all  $D \in x\mathcal{F}^*(H)$ , either  $D$  contains an  $\mathcal{F}$ -danger  $D'$  at  $x$  hence  $y \in V(D') \subseteq V(D)$ , or by definition  $D \in x\mathcal{F}^{\ominus, \text{rest}}(H)$  hence  $y \in V(D)$ . Therefore  $I_{H+x}(x(\mathcal{F} \cup \mathcal{F}^{\ominus, \text{rest}})(H)) \subseteq I_{H+x}(x\mathcal{F}^*(H))$ , which concludes. ■

## 2 Basic structures in 3-uniform (marked) hypergraphs

In this section, we define a few types of elementary (marked) hypergraphs that are going to arise in our study of the Maker-Breaker game on 3-uniform marked hypergraphs, and we study some of their properties. This section is not about the game, it is purely about the structure of these objects.

### 2.1 In 3-uniform hypergraphs

We can ignore the marked vertices at first, and work in the context of standard hypergraphs.

### 2.1.1 Sequences

We will use sequences of edges as a way to describe hypergraphs and navigate through them.

**Definition 2.1.** A *sequence* is some  $\vec{U} = (U_1, \dots, U_l)$  where  $U_1, \dots, U_l$  are subsets of a common set.

**Notation 2.2.** In a sequence, a singleton  $U_i = \{x\}$  might be simply denoted as  $x$ .

**Definition 2.3.** Two sequences are said to be *equivalent* if they are the same when removing all their singleton elements.

**Notation 2.4.** Let  $\vec{U} = (U_1, \dots, U_l)$  be a sequence.

- We define  $V(\vec{U}) = \bigcup_{1 \leq i \leq l} U_i$  and  $E(\vec{U}) = \{U_i, 1 \leq i \leq l \text{ and } |U_i| \geq 2\}$ .
- Provided  $U_1, \dots, U_l$  are not all singletons, we denote by  $\text{start}(\vec{U})$  (resp.  $\text{end}(\vec{U})$ ) the first (resp. last) element of  $\vec{U}$  that is not a singleton.
- We define the reverse sequence  $\overleftarrow{U} = (U_l, \dots, U_1)$ .
- If  $\vec{U}' = (U'_1, \dots, U'_l')$  is another sequence, then we define the concatenated sequence  $\vec{U} \oplus \vec{U}' = (U_1, \dots, U_l, U'_1, \dots, U'_l')$ .
- Given a set  $W$  such that  $W \cap V(\vec{U}) \neq \emptyset$ , we define  $\vec{U}|_W = (U_1, \dots, U_j)$  where  $j = \min\{1 \leq i \leq l, W \cap U_i \neq \emptyset\}$ .

**Definition 2.5.** The hypergraph *induced* by a sequence  $\vec{U}$  is the hypergraph, denoted by  $[\vec{U}]$ , defined by  $V([\vec{U}]) = V(\vec{U})$  and  $E([\vec{U}]) = E(\vec{U})$ .

**Definition 2.6.** Let  $\vec{U} = (U_1, \dots, U_l)$  be a sequence.

- We say  $\vec{U}$  is *connected* if  $U_i \cap U_{i+1} \neq \emptyset$  for all  $1 \leq i \leq l-1$ .
- Let  $x \in V(\vec{U})$ . We say  $x$  is a *repeated vertex* in  $\vec{U}$  if there exist indices  $i, j$  such that  $|i - j| \geq 2$  and  $x \in U_i \cap U_j$ .

**Definition 2.7.** We say a sequence  $\vec{U} = (U_1, \dots, U_l)$  is *linear* if  $|U_i \cap U_{i+1}| \leq 1$  for all  $1 \leq i \leq l-1$ .

Our definition of linearity for sequences is consistent with the global notion of linearity for hypergraphs, which is defined as follows in the literature:

**Definition 2.8.** A hypergraph  $H$  is said to be *linear* (sometimes: *almost-disjoint*) if  $|e \cap e'| \leq 1$  for all distinct  $e, e' \in E(H)$ .

Almost all of the hypergraphs that we are now going to define are linear.

### 2.1.2 Paths

The following definition corresponds to what is usually called a *linear path* or a *loose path* in the literature.

**Definition 2.9.** An *ab-path* is a 3-uniform hypergraph  $P$  such that there exists a sequence  $\vec{U} = (a, e_1, \dots, e_L, b)$  inducing  $P$  where:

- $a$  and  $b$  are the only singletons;
- $L = 0$  if and only if  $a = b$ ;
- $\vec{U}$  is linear connected;
- $\vec{U}$  has no repeated vertex.

Any such sequence, or any sequence that is equivalent to a such sequence, is said to *represent*  $P$ . The  $ab$ -path  $P$  may also be referred to as an  $a$ -path, a  $b$ -path or simply a *path*. Finally, we say  $L = |E(P)|$  is the *length* of  $P$ . See Figure 4.

*Remark.* An  $ab$ -path is the same as a  $ba$ -path.

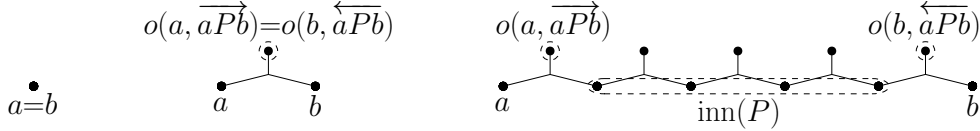


Figure 4: An  $ab$ -path  $P$  of length 0 (left), length 1 (middle), length 5 (right).

**Notation 2.10.** Let  $P$  be an  $ab$ -path. For fixed  $a$  and  $b$ , there is a unique sequence satisfying the definition: we denote it by  $\overrightarrow{aPb} = (a, e_1, \dots, e_L, b)$ . Similarly, for fixed  $a$  (resp. fixed  $b$ ), the sequence  $\overrightarrow{aP} = (a, e_1, \dots, e_L)$  (resp.  $\overrightarrow{bP} = (b, e_L, \dots, e_1)$ ) is well defined. Note that the sequences  $\overrightarrow{aPb}$ ,  $\overrightarrow{bPa} = \overleftarrow{aPb}$ ,  $\overrightarrow{aP}$ ,  $\overrightarrow{bP}$  all represent  $P$ .

**Definition 2.11.** Let  $P$  be an  $ab$ -path. We define  $\text{inn}(P) = \bigcup_{e, e' \in E(P), e \neq e'} (e \cap e')$ , which corresponds to the set of vertices of degree 2 in  $P$ . An element of  $\text{inn}(P)$  is called an *inner vertex* of  $P$ . See Figure 4.

**Notation 2.12.** Let  $P$  be an  $ab$ -path of positive length. We denote by  $o(a, \overrightarrow{aPb})$  the only vertex in  $\text{start}(\overrightarrow{aPb}) \setminus (\text{inn}(P) \cup \{a, b\})$ . See Figure 4.

**Notation 2.13.** Let  $H$  be a hypergraph, and let  $a, b \in V(H)$ . We denote by  $\text{dist}_H(a, b)$  the length of a shortest  $ab$ -path in  $H$ . If there exists none, then  $\text{dist}_H(a, b) = \infty$  by convention.

### 2.1.3 Cycles and tadpoles

**Definition 2.14.** An  $a$ -cycle is a 3-uniform hypergraph  $C$  such that there exists a sequence  $\vec{U} = (a, e_1, \dots, e_L, a)$  inducing  $C$  where:

- $a$  is the only singleton;
- $L \geq 2$ ;
- if  $L \geq 3$  then  $\vec{U}$  is linear connected, and if  $L = 2$  then  $|e_1 \cap e_2| = 2$ ;
- $a$  is the only repeated vertex in  $\vec{U}$ , and  $\{1 \leq i \leq L, a \in e_i\} = \{1, L\}$ .

Any such sequence, or any sequence that is equivalent to a such sequence, is said to *represent*  $C$ . We may simply say  $C$  is a *cycle*. Finally, we say  $L = |E(C)|$  is the *length* of  $C$ . See Figure 5.

*Remark.* Note that a cycle is a linear hypergraph except if it is of length 2.

**Notation 2.15.** Let  $C$  be an  $a$ -cycle. For fixed  $a$ , there are exactly two sequences satisfying Definition 2.14: if the first one is written as  $(a, e_1, \dots, e_L, a)$ , then the second one is  $(a, e_L, \dots, e_1, a)$ .

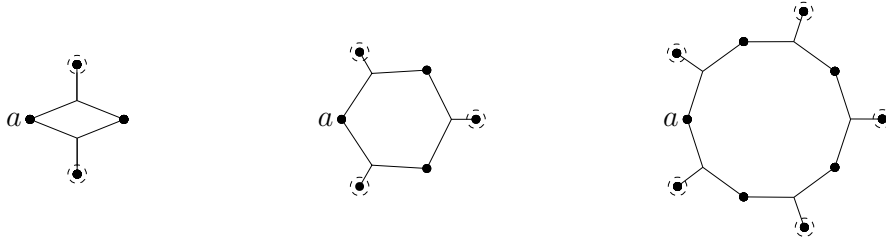


Figure 5: An  $a$ -cycle  $C$  of length 2 (left), length 3 (middle), length 5 (right). The outer vertices are highlighted, the others are inner vertices.

We denote the former by  $\overrightarrow{(a - e_1)C}$  and the latter by  $\overrightarrow{(a - e_L)C}$ . When wishing to consider one of the two arbitrarily, we may use the notation  $\overrightarrow{aC}$ .

**Definition 2.16.** Let  $C$  be an  $a$ -cycle.

- We define  $\text{inn}(C) = \bigcup_{e, e' \in E(C), e \neq e'} (e \cap e')$ , which corresponds to the set of vertices of degree 2 in  $C$ . An element of  $\text{inn}(C)$  is called an *inner vertex* of  $C$ .
- We define  $\text{out}(C) = V(C) \setminus \text{inn}(C)$ , which corresponds to the set of vertices of degree 1 in  $C$ . An element of  $\text{out}(C)$  is called an *outer vertex* of  $C$ .

See Figure 5.

*Remark.* An  $a$ -cycle  $C$  is also a  $b$ -cycle for any  $b \in \text{inn}(C)$  (note that  $a \in \text{inn}(C)$  for instance), however it is not a  $b$ -cycle if  $b \in \text{out}(C)$ .

**Definition 2.17.** An  $a$ -tadpole is a 3-uniform hypergraph  $T$  such that there exists a sequence  $\overrightarrow{U} = (a, e_1, \dots, e_s, b, e_{s+1}, \dots, e_t, b)$  inducing  $T$  where:

- $a$  and  $b$  are the only singletons;
- $(a, e_1, \dots, e_s, b)$  represents an  $ab$ -path  $P_T$ ;
- $(b, e_{s+1}, \dots, e_t, b)$  represents a  $b$ -cycle  $C_T$ ;
- $V(P_T) \cap V(C_T) = \{b\}$ .

Any such sequence, or any sequence that is equivalent to a such sequence, is said to *represent*  $T$ . We may simply say  $T$  is a *tadpole*. The  $ab$ -path  $P_T$  and the  $b$ -cycle  $C_T$  are clearly unique, so we may keep these notations. It is important to note that an  $a$ -cycle is a particular case of an  $a$ -tadpole, where  $s = 0$  i.e.  $a = b$ . See Figure 6.

*Remark.* In other words, an  $a$ -tadpole is the union, for some vertex  $b$ , of an  $ab$ -path and a  $b$ -cycle whose only common vertex is  $b$ . Also note that a tadpole  $T$  is a linear hypergraph except if  $C_T$  is of length 2.

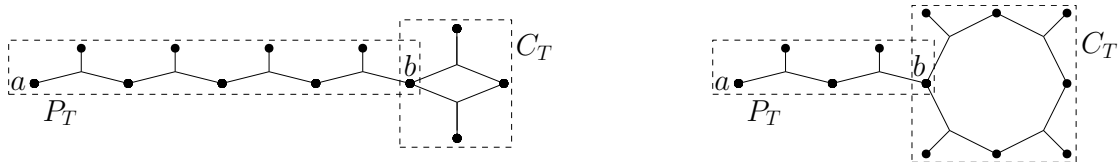


Figure 6: An  $a$ -tadpole  $T$  (that is not an  $a$ -cycle), two examples.

**Notation 2.18.** Let  $T$  be an  $a$ -tadpole. For fixed  $a$ , there are exactly two sequences satisfying Definition 2.17: if the first one is written as  $(a, e_1, \dots, e_s, b, e_{s+1}, \dots, e_t, b)$ , then the second one is  $(a, e_1, \dots, e_s, b, e_t, e_{t-1}, \dots, e_{s+1}, b)$ . The notation  $\overrightarrow{aT}$  refers to any of the two arbitrarily.

### 2.1.4 Substructures inside paths and tadpoles

We now address the existence, and sometimes unicity, of paths and tadpoles inside other paths and tadpoles. These results are easy and intuitive, but we give rigorous proofs using sequences.

**Proposition 2.19.** *Let  $P$  be a path and let  $u, v \in V(P)$ . Then there exists a unique  $uv$ -path in  $P$ .*

*Proof.* Let  $a, b$  such that  $P$  is an  $ab$ -path, and write  $\overrightarrow{aPb} = (a, e_1, \dots, e_L, b)$ .

- Firstly, suppose  $u = v$ . Then that single vertex forms the only  $uv$ -path in  $P$ .
- Secondly, suppose  $u \neq v$  and there exists some  $1 \leq i \leq L$  such that  $\{u, v\} \subseteq e_i$  (note that  $i$  is unique since two distinct edges of a path cannot intersect on two vertices). Then  $(u, e_i, v)$  represents a  $uv$ -path. Moreover, if some sequence  $\overrightarrow{U'}$  represents a  $uv$ -path in  $P$ , then we have  $u \in \text{start}(\overrightarrow{U'})$  and  $v \in \text{end}(\overrightarrow{U'})$ , so  $\text{start}(\overrightarrow{U'}) = \text{end}(\overrightarrow{U'}) = e_i$  hence the unicity.
- Finally, suppose  $u \neq v$  and no edge of  $P$  contains both  $u$  and  $v$ . For  $x \in \{u, v\}$ , define  $j(x) = \min\{1 \leq i \leq L, x \in e_i\}$  and  $j'(x) = \max\{1 \leq i \leq L, x \in e_i\}$ : note that  $j'(x) = j(x) + 1$  if  $x \in \text{inn}(P)$  and  $j'(x) = j(x)$  otherwise. Up to swapping the roles of  $u$  and  $v$ , assume  $j(u) \leq j(v)$ : we actually have  $j(u) < j(v)$ , otherwise  $e_{j(u)} = e_{j(v)}$  would contain both  $u$  and  $v$ . Since  $j'(u) \in \{j(u), j(u) + 1\}$ , this yields  $j'(u) \leq j(v)$  hence  $j'(u) < j(v)$  for the same reason. We claim that  $\overrightarrow{U} := (u, e_{j'(u)}, e_{j'(u)+1}, \dots, e_{j(v)}, v)$  represents a  $uv$ -path. Indeed:
  - The fact that  $\overrightarrow{aPb}$  is a linear connected sequence by definition of a path, coupled with the fact that  $u \in e_{j'(u)}$  and  $v \in e_{j(v)}$ , implies that  $\overrightarrow{U}$  is a linear connected sequence.
  - The fact that there is no repeated vertex in  $\overrightarrow{aPb}$  by definition of a path, coupled with the maximality of  $j'(u)$  and the minimality of  $j(v)$ , implies that there is no repeated vertex in  $\overrightarrow{U}$ .

Let us now address the unicity. Let  $\overrightarrow{U'} := (u, e_{i_1}, e_{i_2}, \dots, e_{i_t}, v)$  be a sequence representing a  $uv$ -path in  $P$ , where  $i_1, \dots, i_t$  are pairwise distinct indices in  $\{1, \dots, L\}$ . Since  $u \in e_{i_1}$  and  $v \in e_{i_t}$ , we have  $i_1 \in \{j(u), j'(u)\}$  and  $i_t \in \{j(v), j'(v)\}$ . We have seen that  $j'(u) < j(v)$ , so  $i_1 < i_t$ . For all  $1 \leq s \leq t - 1$ , we have  $|e_{i_s} \cap e_{i_{s+1}}| = 1$  by definition of a path hence  $|i_s - i_{s+1}| = 1$ . Since  $i_1 < i_t$  and the indices  $i_1, \dots, i_t$  are pairwise distinct, this implies  $i_{s+1} = i_s + 1$  for all  $1 \leq s \leq t - 1$ . To conclude that  $\overrightarrow{U'} = \overrightarrow{U}$ , it only remains to show that  $i_1 = j'(u)$  and  $i_t = j(v)$ . We have mentioned that  $i_1 \in \{j(u), j'(u)\}$ : if  $i_1 = j(u) = j'(u) - 1$ , then  $e_{i_2} = e_{j'(u)} \ni u$ , hence a repetition in  $\overrightarrow{U'}$  which contradicts the definition of a path. Therefore  $i_1 = j'(u)$ , and an analogous reasoning yields  $i_t = j(v)$ . ■

We are also interested in the existence of paths inside cycles. First of all, we need to describe what happens when we remove a vertex from a cycle:

**Proposition 2.20.** *Let  $C$  be a cycle and let  $w \in V(C)$ . Let  $w_1, w_2$  be the two inner vertices of  $C$  that are adjacent to  $w$  in  $C$  (if  $C$  is of length 2 and  $w \in \text{inn}(C)$  then  $w_1 = w_2$ ).*

- If  $w \in \text{out}(C)$  then  $C^{-w}$  is a  $w_1w_2$ -path.
- If  $w \in \text{inn}(C)$  then  $C^{-w}$  is the union of a  $w_1w_2$ -path and two isolated vertices which are the two outer vertices of  $C$  that are adjacent to  $w$  in  $C$ .

*Proof.* Let us first address the case where  $C$  is of length 2. If  $w \in \text{out}(C)$ , then write  $E(C) = \{\{w_1, w, w_2\}, \{w_1, u, w_2\}\}$ :  $C^{-w}$  consists of the edge  $\{w_1, u, w_2\}$ , which forms a  $w_1w_2$ -path. If  $w \in \text{inn}(C)$ , then write  $E(C) = \{\{w, u_1, w_1\}, \{w, u_2, w_1\}\}$ :  $C^{-w}$  consist of the three isolated

vertices  $w_1 = w_2$ ,  $u_1$  and  $u_2$ .

Now assume that  $C$  is of length at least 3. Let  $e$  be the edge of  $C$  containing both  $w$  and  $w_1$ , and write  $\overrightarrow{(w_1 - e)C} = (w_1, e = e_1, e_2, \dots, e_L, w_1)$ . We have  $e_1 \cap e_L = \{w_1\}$ . If  $w \in \text{out}(C)$ , then  $e_1 = \{w_1, w, w_2\}$  so  $e_1 \cap e_2 = \{w_2\}$ . If  $w \in \text{inn}(C)$ , then  $e_1 \cap e_2 = \{w\}$  hence  $e_2 \cap e_3 = \{w_2\}$  since  $w_2$  is adjacent to  $w$ . Therefore, defining  $i = 2$  if  $w \in \text{out}(C)$  and  $i = 3$  if  $w \in \text{inn}(C)$ , the only edges of  $C$  containing  $w_2$  are  $e_{i-1}$  and  $e_i$ . We claim that  $\vec{U} := (w_2, e_i, \dots, e_L, w_1)$  is linear connected and has no repeated vertex. Indeed:

- By definition of a cycle, the sequence  $\overrightarrow{(w_1 - e)C}$  is linear connected since  $C$  is of length at least 3, and  $w_1$  is its only repeated vertex with  $\{1 \leq i \leq L, w_1 \in e_i\} = \{1, L\}$ . Therefore, its subsequence  $(e_i, \dots, e_L, w_1)$  is also linear connected, and has no repeated vertex since it does not contain the edge  $e_1$ .
- The addition of  $w_2$  at the beginning of the sequence  $(e_i, \dots, e_L, w_1)$  preserves the linearity and the connectedness (since  $w_2 \in e_i$ ) and the absence of a repeated vertex (since  $w_2 \notin e_j$  for all  $j > i$ ).

Therefore, by definition,  $\vec{U} := (w_2, e_i, \dots, e_L, w_1)$  represents a  $w_2w_1$ -path. We can now conclude:

- If  $w \in \text{out}(C)$ , then  $V(C^{-w}) = V(C) \setminus \{w\} = e_2 \cup \dots \cup e_L = V(\vec{U})$  and  $E(C^{-w}) = E(C) \setminus \{e_1\} = \{e_2, \dots, e_L\} = E(\vec{U})$ , so  $C^{-w}$  is the  $w_1w_2$ -path represented by  $\vec{U}$ .
- If  $w \in \text{inn}(C)$ , then let  $u_1$  and  $u_2$  be the outer vertices of  $C$  in  $e_1$  and  $e_2$  respectively: we have  $V(C^{-w}) = V(C) \setminus \{w\} = (e_3 \cup \dots \cup e_L) \cup \{u_1, u_2\} = V(\vec{U}) \cup \{u_1, u_2\}$  and  $E(C^{-w}) = E(C) \setminus \{e_1, e_2\} = \{e_3, \dots, e_L\} = E(\vec{U})$ , so  $C^{-w}$  is the union of the  $w_1w_2$ -path represented by  $\vec{U}$  and the two isolated vertices  $u_1$  and  $u_2$ . ■

We can now conclude about the existence of paths between two given vertices of a cycle, first when trying to avoid a third vertex, then in general.

**Proposition 2.21.** *Let  $C$  be a cycle and let  $u, v, w \in V(C)$  with  $w \neq u, v$ . Then there exists a unique  $uv$ -path in  $C$  that does not contain  $w$ , unless all the following hold:  $w \in \text{inn}(C)$ ,  $u \neq v$ , and  $u$  or  $v$  is an outer vertex of  $C$  that is adjacent to  $w$  (in which case there exists none).*

*Proof.* First of all, note that a  $uv$ -path in  $C$  that does not contain  $w$  is exactly a  $uv$ -path in  $C^{-w}$ . Assume  $u \neq v$ , otherwise the result is trivial. If  $w \in \text{out}(C)$ , then  $C^{-w}$  is a path according to Proposition 2.20, which contains a unique  $uv$ -path by Proposition 2.19. Now assume  $w \in \text{inn}(C)$ : then  $C^{-w}$  is the union of a path  $P$  and two isolated vertices  $u_1, u_2$  that are the two outer vertices of  $C$  adjacent to  $w$  according to Proposition 2.20. If  $u \in \{u_1, u_2\}$  or  $v \in \{u_1, u_2\}$ , then there obviously cannot exist a  $uv$ -path in  $C^{-w}$ . Otherwise  $u, v \in V(P)$ , so there exists a unique  $uv$ -path in  $P$  (and in  $C^{-w}$  as a result) by Proposition 2.19. ■

**Proposition 2.22.** *Let  $C$  be a cycle and let  $u, v \in V(C)$ . Then there exists a  $uv$ -path in  $C$ , unless  $C$  is of length 2 and  $\text{out}(C) = \{u, v\}$ .*

*Proof.* If  $C$  is of length 2 and  $\text{out}(C) = \{u, v\}$ , then there is no  $uv$ -path in  $C$ , because  $|e_u \cap e_v| = 2$  where  $e_u$  (resp.  $e_v$ ) denotes the only edge of  $C$  containing  $u$  (resp.  $v$ ). Otherwise, there exists  $w \in \text{out}(C) \setminus \{u, v\}$ : by Proposition 2.21, there exists a unique  $uv$ -path in  $C$  that does not contain  $w$ , so in particular  $C$  contains a  $uv$ -path. ■

We now give analogous results for tadpoles.

**Proposition 2.23.** *Let  $T$  be a tadpole and let  $u, v, w \in V(T)$ . If  $w \in \text{out}(C_T) \setminus \{u, v\}$ , then there exists a  $uv$ -path in  $T$  that does not contain  $w$ .*

*Proof.* Note that  $w \notin V(P_T)$ , so that Proposition 2.19 concludes if  $u, v \in V(P_T)$ . If  $u, v \in V(C_T)$ , then Proposition 2.21 concludes. Therefore, assume  $u \in V(P_T)$  and  $v \in V(C_T)$ . Let  $b$  be the only vertex in  $V(P_T) \cap V(C_T)$ . By Proposition 2.19, there exists a  $ub$ -path  $P_{ub}$  in  $P_T$ , that does not contain  $w$  since  $w \notin V(P_T)$ . By Proposition 2.21, there exists a  $bv$ -path  $P_{bv}$  in  $C_T$  that does not contain  $w$ . Since  $V(P_{ub}) \cap V(P_{bv}) = \{b\}$ , it is clear that  $\overrightarrow{uP_{ub}b} \oplus \overrightarrow{bP_{bv}v}$  represents a  $uv$ -path in  $T$  that does not contain  $w$ . ■

**Proposition 2.24.** *Let  $T$  be a tadpole and let  $u, v \in V(T)$ . Then there exists a  $uv$ -path in  $T$ , unless  $C_T$  is of length 2 and  $\text{out}(C_T) = \{u, v\}$ .*

*Proof.* If  $C_T$  is of length 2 and  $\text{out}(C_T) = \{u, v\}$ , then there is no  $uv$ -path in  $T$ , because  $|e_u \cap e_v| = 2$  where  $e_u$  (resp.  $e_v$ ) denotes the only edge of  $T$  containing  $u$  (resp.  $v$ ). Otherwise, there exists  $w \in \text{out}(C) \setminus \{u, v\}$ : by Proposition 2.23, there exists a  $uv$ -path in  $T$  that does not contain  $w$ , so in particular  $T$  contains a  $uv$ -path. ■

On the subject of tadpoles, let us make one final remark:

**Proposition 2.25.** *Let  $T$  be a tadpole and let  $u \in V(T) \setminus \text{out}(C_T)$ . Then  $T$  contains a  $u$ -tadpole.*

*Proof.* Let  $b$  be the only vertex in  $V(P_T) \cap V(C_T)$ . Since  $u \notin \text{out}(C_T)$ , we have  $u \in \text{inn}(C_T)$  or  $u \in V(P_T)$ . If  $u \in \text{inn}(C_T)$ , then  $C_T$  is a  $u$ -cycle. If  $u \in V(P_T)$ , then there exists a  $ub$ -path  $P_{ub}$  in  $P_T$  by Proposition 2.19, so  $\overrightarrow{uP_{ub}b} \oplus \overrightarrow{bC_T}$  represents a  $u$ -tadpole. ■

### 2.1.5 Projections

One of the most common tools that we will use is, inside a path or a tadpole, to follow a subpath starting from some vertex  $u$  until reaching some vertex set  $W$ , as made possible by the previous results:

**Proposition 2.26.** *Let  $H$  be a hypergraph. Let  $X$  be a path or a tadpole in  $H$ , let  $u \in V(X)$ , and let  $W \subseteq V(H)$  such that  $W \cap V(X) \neq \emptyset$ . In the case where  $X$  is a tadpole with  $C_X$  of length 2 and  $u \in \text{out}(C_X)$ , also suppose that  $W \cap V(X) \neq \text{out}(C_X) \setminus \{u\}$ . Then there exists a  $u$ -path  $\mathbf{P}_W(u, X)$  in  $X$  such that:*

- *If  $u \in W$ , then  $\mathbf{P}_W(u, X)$  is of length 0.*
- *If  $u \notin W$ , then  $\mathbf{P}_W(u, X)$  is of positive length and its only edge intersecting  $W$  is  $\text{end}(\overrightarrow{u\mathbf{P}_W(u, X)})$ , with  $|\text{end}(\overrightarrow{u\mathbf{P}_W(u, X)}) \cap W| \in \{1, 2\}$ .*

*Proof.* Let us start by showing the existence of  $w \in W \cap V(X)$  such that there exists a  $uw$ -path in  $X$ . If  $X$  is a path, then any  $w \in W \cap V(X)$  is suitable by Proposition 2.19. If  $X$  is a tadpole, then any  $w \in W \cap V(X)$  is suitable by Proposition 2.24, unless  $C_X$  is of length 2 and  $u \in \text{out}(C_X)$  in which case we choose  $w \in W \cap V(X) \setminus (\text{out}(C_X) \setminus \{u\})$  as allowed by the assumption.

Let  $w \in W \cap V(X)$  minimizing the length of a shortest  $uw$ -path in  $X$ , and let  $P$  be a shortest  $uw$ -path in  $X$ . We claim that  $\mathbf{P}_W(u, X) := P$  has the desired properties. Clearly,  $P$  is of positive length if and only if  $u \notin W$ . Assume  $u \notin W$ . By definition, the sequence  $\overrightarrow{uP}|_W$  only has one edge intersecting  $W$ , which is  $\text{end}(\overrightarrow{uP}|_W)$ , so in particular  $|\text{end}(\overrightarrow{uP}|_W) \cap W| \in \{1, 2\}$ . Therefore, it suffices to show that  $\overrightarrow{uP}|_W = \overrightarrow{uP}$  to finish the proof. Let  $w' \in \text{end}(\overrightarrow{uP}|_W)$ . The sequence  $\overrightarrow{uP}|_W$  induces a  $uw'$ -path, which cannot be shorter than  $P$  by minimality of  $w$ , hence why  $\overrightarrow{uP}|_W = \overrightarrow{uP}$ . ■

*Remark.* There is not necessarily unicity, even if  $X$  is a path: indeed, it is possible that there are vertices of  $W$  on both sides of  $u$  in the path.

**Definition 2.27.** For  $X, u, W$  satisfying the required conditions, a  $u$ -path  $\mathbf{P}_W(u, X)$  from Proposition 2.26 is called a *projection of  $u$  onto  $W$  in  $X$* . As there is no unicity in general, we will consider that the notation  $\mathbf{P}_W(u, X)$  always refers to the same path for given  $X, u, W$ .

### 2.1.6 Union lemmas

We now look at some structures that appear in unions of paths and tadpoles. The following three lemmas are immediately deduced from the concatenation of the sequences representing the paths, cycles and tadpoles involved in their statements. We will use them often without necessarily referencing them.

**Lemma 2.28.** *If  $P$  is an  $ab$ -path and  $P'$  is a  $bc$ -path such that  $V(P) \cap V(P') = \{b\}$ , then  $P \cup P'$  is an  $ac$ -path.*

**Lemma 2.29.** *If  $P$  and  $P'$  are  $ab$ -paths such that  $V(P) \cap V(P') = \{a, b\}$ , then  $P \cup P'$  is an  $a$ -cycle and a  $b$ -cycle.*

**Lemma 2.30.** *If  $P$  is an  $ab$ -path and  $T$  is a  $b$ -tadpole such that  $V(P) \cap V(T) = \{b\}$ , then  $P \cup T$  is an  $a$ -tadpole.*

However, when the intersection of the two objects is more complex, it is less clear what their union contains. Let us first consider the union of an  $ab$ -path  $P$  of positive length and an edge  $e^*$  such that  $e^* \cap V(P) \neq \emptyset$  and there exists  $u \in e^* \setminus V(P)$ . When is it possible to prolong a subpath of  $P$  with the edge  $e^*$  to get an  $au$ -path and/or a  $bu$ -path?

If  $|e^* \cap V(P)| = 1$ , then we get both an  $au$ -path and a  $bu$ -path, represented by the sequences  $\overrightarrow{aPb}|_{e^*} \oplus (e^*, u)$  and  $\overleftarrow{aPb}|_{e^*} \oplus (e^*, u)$  respectively, as illustrated in Table 1.

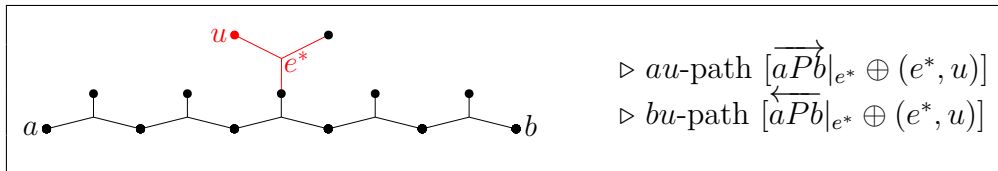


Table 1: An edge  $e^*$  intersecting an  $ab$ -path  $P$  on one vertex.

If  $|e^* \cap V(P)| = 2$  though, then the sequence  $\overrightarrow{aPb}|_{e^*} \oplus (e^*, u)$  does not necessarily represent an  $au$ -path (same for  $b$ ). If  $a \in e^*$  i.e.  $\overrightarrow{aPb}|_{e^*} = (a)$ , then it obviously does. But if  $a \notin e^*$  i.e.  $\overrightarrow{aPb}|_{e^*}$  represents a path of positive length, then it does if and only if  $|e^* \cap \text{end}(\overrightarrow{aPb}|_{e^*})| = 1$ . We see a key notion appearing here:

**Notation 2.31.** Let  $P$  be an  $ab$ -path of positive length and let  $e^*$  be an edge. Write  $\overrightarrow{aPb} = (a, e_1, \dots, e_L, b)$ . The notation  $e^* \perp \overrightarrow{aPb}$  (or  $e^* \perp \overrightarrow{aP}$  equivalently) means that either  $e_1 \setminus \{a\} \subseteq e^*$  or  $e_i \setminus e_{i-1} \subseteq e^*$  for some  $2 \leq i \leq L$ . See Figure 7.

*Remark.* Note that it is technically possible to have both  $e^* \perp \overrightarrow{aPb}$  and  $e^* \perp \overleftarrow{aPb}$ . This is the case if, for some  $j$ , we have  $e^* = e_j$  or  $e^* = \{o_j, o_{j+1}\} \cup (e_j \cap e_{j+1})$  where  $o_i$  denotes the only vertex in  $e_i \setminus (\{a, b\} \cup \text{inn}(P))$ . However this will never happen for us, as in practice we will always have either  $e^* \not\subseteq V(P)$  or  $a \in e^*$ .



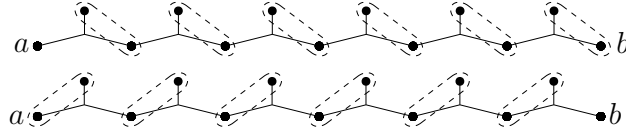


Figure 7: We have  $e^* \perp \overrightarrow{aPb}$  (resp.  $e^* \perp \overleftarrow{aPb}$ ) if and only if  $e^*$  contains one of the pairs of vertices highlighted at the top (resp. at the bottom).

In the case at hand  $|e^* \cap V(P)| = 2$ , we can see that  $e^* \perp \overrightarrow{aPb}$  if and only if  $a \notin e^*$  and  $|e^* \cap \text{end}(\overrightarrow{aPb}|_{e^*})| = 2$ . Therefore, the sequence  $\overrightarrow{aPb}|_{e^*} \oplus (e^*, u)$  represents an  $au$ -path if and only if  $e^* \not\perp \overrightarrow{aPb}$ , and similarly the sequence  $\overleftarrow{aPb}|_{e^*} \oplus (e^*, u)$  represents a  $bu$ -path if and only if  $e^* \not\perp \overleftarrow{aPb}$ . All of this is illustrated in Table 2: note that, if no  $au$ -path (resp. no  $bu$ -path) appears, then we get a  $b$ -tadpole (resp. an  $a$ -tadpole).

	$e^* \not\perp \overleftarrow{aPb}$	$e^* \perp \overleftarrow{aPb}$
$e^* \not\perp \overrightarrow{aPb}$	<p> <math>\triangleright au\text{-path } [\overrightarrow{aPb} _{e^*} \oplus (e^*, u)]</math>  <math>\triangleright bu\text{-path } [\overleftarrow{aPb} _{e^*} \oplus (e^*, u)]</math> </p>	<p> <math>\triangleright au\text{-path } [\overrightarrow{aPb} _{e^*} \oplus (e^*, u)]</math>  <math>\triangleright a\text{-tadpole } [\overrightarrow{aPb} _{e^*} \oplus (e^*, \text{end}(\overleftarrow{aPb} _{e^*}))]</math> </p>
$e^* \perp \overrightarrow{aPb}$	<p> <math>\triangleright bu\text{-path } [\overleftarrow{aPb} _{e^*} \oplus (e^*, u)]</math>  <math>\triangleright b\text{-tadpole } [\overleftarrow{aPb} _{e^*} \oplus (e^*, \text{end}(\overrightarrow{aPb} _{e^*}))]</math> </p>	impossible

Table 2: An edge  $e^*$  intersecting an  $ab$ -path  $P$  on two vertices: all cases. The  $a$ -tadpole or  $b$ -tadpole, when one appears, is highlighted.

Let us now consider the union of an  $ab$ -path  $P$  of positive length and some edge  $e^* \neq \text{start}(\overrightarrow{aPb})$  that intersects  $P$  on at least two vertices including  $a$ : do we get an  $a$ -cycle? If  $|e^* \cap V(P)| = 2$  then the answer is yes, as illustrated in Table 3. If  $|e^* \cap V(P)| = 3$  then Table 4 shows that it is possible that no  $a$ -cycle appears, in which case we get a  $b$ -tadpole.

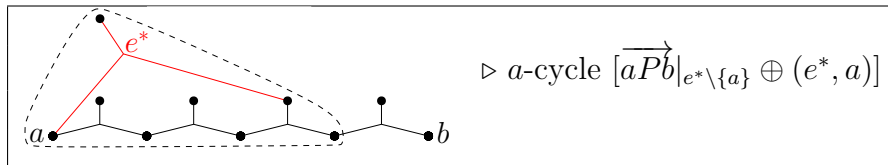


Table 3: An edge  $e^*$  intersecting an  $ab$ -path  $P$  on two vertices including  $a$ . The  $a$ -cycle is highlighted.

Using these tables, we get the following three union lemmas, which are fundamental in our structural study of 3-uniform hypergraphs. They give us some basic information about the union of two paths or the union of a path and a tadpole.

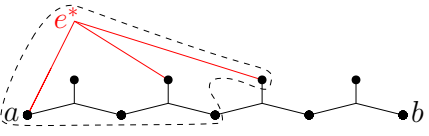
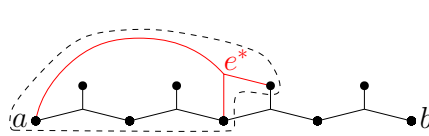
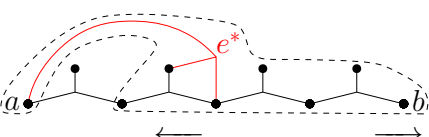
	$e^* \not\perp \overleftarrow{aPb}$	$e^* \perp \overleftarrow{aPb}$
$e^* \not\perp \overrightarrow{aPb}$	 <p><math>\triangleright a\text{-cycle } [\overrightarrow{aPb} _{e^* \setminus \{a\}} \oplus (e^*, a)]</math></p>	 <p><math>\triangleright a\text{-cycle } [\overrightarrow{aPb} _{e^* \setminus \{a\}} \oplus (e^*, a)]</math></p>
$e^* \perp \overrightarrow{aPb}$	 <p><math>\triangleright b\text{-tadpole } [\overleftarrow{aPb} _{e^*} \oplus (e^*, \text{end}(\overrightarrow{aPb} _{e^* \setminus \{a\}}))]</math></p>	impossible

Table 4: An edge  $e^*$  intersecting an  $ab$ -path  $P$  on three vertices including  $a$  (and  $e^* \neq \text{start}(\overrightarrow{aPb})$ ): all cases. The  $a$ -cycle or  $b$ -tadpole is highlighted.

**Lemma 2.32.** *Let  $a, b, c$  be distinct vertices. Let  $P_{ab}$  be an  $ab$ -path, and let  $P_c$  be a  $c$ -path such that  $c \notin V(P_{ab})$  and  $V(P_c) \cap V(P_{ab}) \neq \emptyset$ . In particular,  $e^* := \text{end}(c\overrightarrow{P}_{V(P_{ab})}(c, P_c))$  is well defined. Suppose there is no  $ca$ -path in  $P_{ab} \cup P_c$ . Then  $|e^* \cap V(P_{ab})| = 2$  and  $e^* \perp \overrightarrow{aPb}$ , moreover there is a  $cb$ -path in  $P_{ab} \cup P_c$  and a  $b$ -tadpole in  $P_{ab} \cup e^* \subseteq P_{ab} \cup P_c$ . See Figure 8.*

*Proof.* By definition of a projection, we have  $|e^* \cap V(P_{ab})| \in \{1, 2\}$ . Let  $u \in e^* \setminus V(P_{ab})$ . All ways that  $e^*$  might intersect  $P_{ab}$  are summarized in Tables 1 and 2. There is no  $au$ -path  $P_{au}$  in  $P_{ab} \cup e^*$ , otherwise the sequence  $c\overrightarrow{P}_c|_{\{u\}} \oplus u\overrightarrow{P}_{au}a$  would represent a  $ca$ -path in  $P_{ab} \cup P_c$ , contradicting the assumption of the lemma. Therefore, we are necessarily in the bottom-left case of Table 2, which means that:  $|e^* \cap V(P_{ab})| = 2$ ,  $e^* \perp \overrightarrow{aPb}$ , there is a  $b$ -tadpole in  $P_{ab} \cup e^*$ , and there is a  $bu$ -path  $P_{bu}$  in  $P_{ab} \cup e^*$ . The sequence  $c\overrightarrow{P}_c|_{\{u\}} \oplus u\overrightarrow{P}_{bu}b$  represents a  $cb$ -path in  $P_{ab} \cup P_c$ . ■

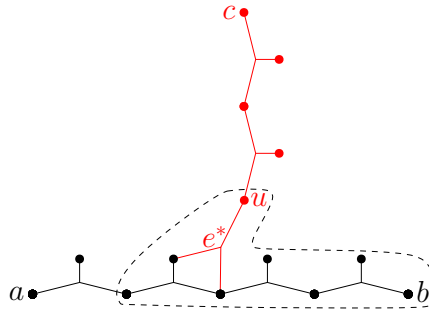


Figure 8: Illustration of Lemma 2.32. The represented paths are  $P_{ab}$  and  $\overrightarrow{P}_{V(P_{ab})}(c, P_c)$ . The  $b$ -tadpole is highlighted.

**Lemma 2.33.** *Let  $a, b$  be distinct vertices. Let  $P_{ab}$  be an  $ab$ -path, and let  $P_a$  be an  $a$ -path such that  $\text{start}(\overrightarrow{aP}_a) \neq \text{start}(\overrightarrow{aP}_{ab})$  and  $V(P_a) \cap (V(P_{ab}) \setminus \{a\}) \neq \emptyset$ . In particular,  $e^* := \text{end}(\overrightarrow{aP}_{V(P_{ab}) \setminus \{a\}}(a, P_a))$  is well defined. Suppose there is no  $a$ -cycle in  $P_{ab} \cup P_a$ . Then  $e^* \perp \overrightarrow{aPb}$  and there is a  $b$ -tadpole in  $P_{ab} \cup e^* \subseteq P_{ab} \cup P_a$ . See Figure 9.*

*Proof.* We distinguish between two cases:

- First suppose  $a \in e^*$ . Since  $e^* \neq \overrightarrow{aP_{ab}b}$ , all ways that  $e^*$  might intersect  $P_{ab}$  are summarized in Tables 3 and 4. Since there is no  $a$ -cycle in  $P_{ab} \cup P_a \supseteq P_{ab} \cup e^*$  by assumption, we are necessarily in the bottom-left case of Table 4, so  $e^* \perp \overrightarrow{aPb}$  and there is a  $b$ -tadpole in  $P_{ab} \cup e^*$ .
- Now suppose  $a \notin e^*$ , meaning the projection  $\mathbf{P}_{V(P_{ab}) \setminus \{a\}}(a, P_a)$  is of length at least 2. Write  $\text{start}(\overrightarrow{aP_a}) = \{a, c, c'\}$  where  $c \in \text{inn}(P_a)$ , as in Figure 9. Define the  $c$ -path  $P_c := P_a^{-a-c'}$ : we have  $c \notin V(P_{ab})$  and  $e^* = \text{end}(\overrightarrow{c\mathbf{P}_{V(P_{ab})}(c, P_c)})$ , so the idea is to apply Lemma 2.32 to  $P_{ab}$  and  $P_c$ . If there was a  $ca$ -path  $P_{ca}$  in  $P_{ab} \cup P_c$ , then  $(a, \text{start}(\overrightarrow{aP_a}), c) \oplus \overrightarrow{cP_{ca}a}$  would represent an  $a$ -cycle in  $P_{ab} \cup P_a$ , contradicting the assumption of the lemma. Therefore, there is no  $ca$ -path in  $P_{ab} \cup P_c$ , so Lemma 2.32 ensures that  $e^* \perp \overrightarrow{aPb}$  and that there is a  $b$ -tadpole in  $P_{ab} \cup e^*$ .  $\blacksquare$

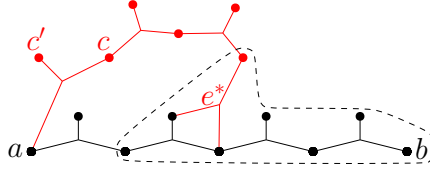


Figure 9: Illustration of Lemma 2.33. The represented paths are  $P_{ab}$  and  $\mathbf{P}_{V(P_{ab}) \setminus \{a\}}(a, P_a)$ . The  $b$ -tadpole is highlighted.

**Lemma 2.34.** *Let  $a, c$  be distinct vertices. Let  $T$  be an  $a$ -tadpole, and let  $P_c$  be a  $c$ -path such that  $c \notin V(T)$  and  $V(P_c) \cap V(T) \neq \emptyset$ . In particular,  $e^* := \text{end}(\overrightarrow{c\mathbf{P}_{V(T)}(c, P_c)})$  is well defined. Suppose there is no  $ca$ -path in  $T \cup P_c$ . Then  $T$  is not a cycle,  $|e^* \cap V(T)| = 2$  and  $e^* \perp \overrightarrow{aP_T}$ , moreover there is a  $c$ -tadpole in  $T \cup P_c$ . See Figure 10.*

*Proof.* Up to replacing  $P_c$  by the projection  $\mathbf{P}_{V(T)}(c, P_c)$ , assume that  $e^*$  is the only edge of  $P_c$  intersecting  $T$ . Let  $b$  be the only vertex in  $V(P_T) \cap V(P_c)$ .

Claim 2.  $e^* \cap (V(P_T) \setminus \{a\}) \neq \emptyset$ .

*Proof of Claim 2.* We already know  $a \notin e^*$ , otherwise  $P_c$  would be a  $ca$ -path, contradicting the assumption of the lemma. Therefore we must show that  $e^* \cap V(P_T) \neq \emptyset$ . Suppose for a contradiction that  $e^* \cap V(P_T) = \emptyset$ . There are two possibilities:

- Suppose  $|e^* \cap V(C_T)| = 1$ , and write  $e^* \cap V(C_T) = \{v\}$ . Note that  $P_c$  is a  $cv$ -path. By Proposition 2.22 (with  $u = b$ ), there exists a  $bv$ -path  $P_{bv}$  in  $C_T$ . The sequence  $\overrightarrow{cP_c v} \oplus \overrightarrow{vP_{bv} b} \oplus \overrightarrow{bP_T a}$  represents a  $ca$ -path in  $T \cup P_c$ , contradicting the assumption of the lemma.
- Suppose  $|e^* \cap V(C_T)| = 2$ , and write  $e^* \cap V(C_T) = \{v, w\}$ . Note that  $P_c$  is both a  $cv$ -path and a  $cw$ -path. Up to swapping the roles of  $v$  and  $w$ , we can assume that  $w \in \text{out}(C_T)$  or  $v \in \text{inn}(C_T)$ . Since  $b \in \text{inn}(C_T)$  and  $b \neq w$  (indeed  $b \in V(P_T)$  whereas  $e^* \cap V(P_T) = \emptyset$ ), Proposition 2.21 (with  $u = b$ ) thus ensures that there exists a  $bv$ -path  $P_{bv}$  in  $C_T$  that does not contain  $w$ . The fact that  $w \notin V(P_{bv})$  implies that  $V(P_c) \cap V(P_{bv}) = \{v\}$ . The sequence  $\overrightarrow{cP_c v} \oplus \overrightarrow{vP_{bv} b} \oplus \overrightarrow{bP_T a}$  thus represents a  $ca$ -path in  $T \cup P_c$ , contradicting the assumption of the lemma.  $\square$

Claim 2 implies that  $V(P_T) \setminus \{a\} \neq \emptyset$  i.e.  $P_T$  is of positive length i.e.  $T$  is not a cycle. It also implies that  $V(P_c) \cap V(P_T) \neq \emptyset$ , so we can apply Lemma 2.32 with  $P_c$  and the  $ab$ -path  $P_{ab} = P_T$ . Since there is no  $ca$ -path in  $T \cup P_c \supseteq P_T \cup P_c$  by assumption, Lemma 2.32 tells us

that:  $|e^* \cap V(P_T)| = 2$ ,  $e^* \perp \overrightarrow{aP_T b}$ , and there is a  $cb$ -path  $P_{cb}$  in  $P_T \cup P_c$ . Since  $|e^* \cap V(P_T)| = 2$ , we have  $e^* \cap (V(C_T) \setminus \{b\}) = \emptyset$ , hence  $V(P_{cb}) \cap V(C_T) = \{b\}$ . Therefore  $P_{cb} \cup C_T$  is a  $c$ -tadpole in  $T \cup P_c$ , which concludes. ■

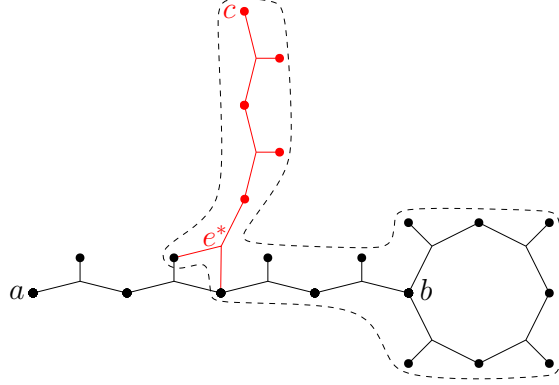


Figure 10: Illustration of Lemma 2.34. The represented objects are  $T$  and  $\mathbf{P}_{V(T)}(c, P_c)$ . The  $c$ -tadpole is highlighted.

## 2.2 In 3-uniform marked hypergraphs

**Definition 2.35.** A marked hypergraph is called an  $ab$ -path (resp. an  $a$ -cycle, resp. an  $a$ -tadpole) if its underlying hypergraph is an  $ab$ -path (resp. an  $a$ -cycle, resp. an  $a$ -tadpole).

All previous results from this section obviously hold for marked paths/cycles/tadpoles as well. We now introduce a type of path that will be central in this paper:

**Definition 2.36.** An  $x$ -snake, or  $xm$ -snake, is an  $xm$ -path of positive length where the vertex  $m$  is marked.

*Remark.* A snake might have more than one marked vertex.

If the  $ab$ -path is an  $ab$ -snake, Lemma 2.32 can be reformulated as follows:

**Lemma 2.37.** Let  $a, b, c$  be distinct vertices, where  $b$  is marked. Let  $S_{ab}$  be an  $ab$ -snake, and let  $P_c$  be a  $c$ -path such that  $c \notin V(S_{ab})$  and  $V(P_c) \cap V(S_{ab}) \neq \emptyset$ .

- Suppose there is no  $c$ -snake in  $S_{ab} \cup P_c$ . Then there is both a  $ca$ -path and an  $a$ -tadpole in  $S_{ab} \cup P_c$ .
- Suppose there is no  $ca$ -path in  $S_{ab} \cup P_c$ . Then there is both a  $cb$ -snake and a  $b$ -tadpole in  $S_{ab} \cup P_c$ .

*Proof.* The second item is exactly Lemma 2.32. The first item is Lemma 2.32 where the roles of  $a$  and  $b$  are reversed. ■

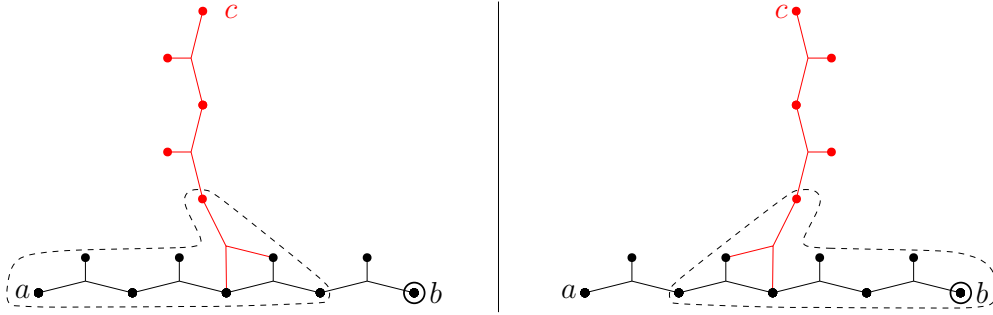


Figure 11: Illustration of Lemma 2.37 (first item on the left, second item on the right). The represented paths are  $S_{ab}$  and  $\mathbf{P}_{V(S_{ab})}(c, P_c)$ .

### 3 The Maker-Breaker game on 3-uniform marked hypergraphs

#### 3.1 Nunchakus and the forcing principle

The following definition is the equivalent for marked hypergraphs of what is called a *manriki* in [14].

**Definition 3.1.** An *ab-nunchaku* is an *ab*-path  $N$  of positive length such that  $M(N) = \{a, b\}$ . An *ab*-nunchaku may also be referred to as an *a-nunchaku*, a *b-nunchaku* or simply a *nunchaku*.

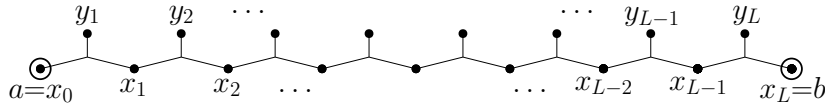


Figure 12: An *ab*-nunchaku.

An easy remark that one can make about the Maker-Breaker game on 3-uniform marked hypergraphs is that Maker has a "forcing" strategy to win on a nunchaku:

**Proposition 3.2.** Any marked hypergraph containing a nunchaku is a Maker win.

*Proof.* By Proposition 1.21, it suffices to show that any nunchaku is a Maker win. Let  $N$  be an *ab*-nunchaku of length  $L$ . Assume  $L \geq 2$  (otherwise  $N$  is a trivial Maker win) and define  $a = x_0, y_1, x_1, y_2, \dots, x_{L-1}, y_L, x_L = b$  as in Figure 12. Maker picks  $x_1$ , threatening to complete the edge  $\{a, y_1, x_1\}$  on his next go: Breaker is forced to pick  $y_1$ . Maker continues to force all of Breaker's picks along the path, by picking  $x_2, x_3, \dots, x_{L-2}$  successively which forces Breaker to pick  $y_2, y_3, \dots, y_{L-2}$  successively. Maker now picks  $x_{L-1}$ , threatening to pick either  $y_{L-1}$  or  $y_L$  on his next go to complete the edge  $\{x_{L-2}, y_{L-1}, x_{L-1}\}$  or  $\{x_{L-1}, y_L, b\}$  respectively. Breaker will lose in the next round as he cannot address both threats at once. ■

Note that, in a marked hypergraph that has an *ab*-path as a strict subhypergraph, the forcing technique might also be useful if  $a$  is marked but not  $b$ : it will not be enough to win the game, but it is a way for Maker to get all of  $x_1, x_2, \dots, x_L$  while making sure that Breaker gets exactly  $y_1, y_2, \dots, y_L$  in the meantime.

## 3.2 The Maker-Breaker game on 3-uniform marked hyperforests

**Definition 3.3.** A 3-uniform marked hyperforest is a 3-uniform marked hypergraph that contains no cycle.

### 3.2.1 Solving the game

The class of 3-uniform marked hyperforests is one for which there exists a simple criterion characterizing the winner of the Maker-Breaker game, as it is one for which the converse of Proposition 3.2 holds. We can even give the exact value of  $\tau_M(\cdot)$  for Maker wins in that case: recall that this is defined as the minimum number of rounds in which Maker can ensure to get a fully marked edge. In particular, we observe that the forcing strategy is not the most efficient way for Maker to win on a nunchaku, as it requires a number of rounds that is linear in the number of vertices rather than logarithmic.

**Notation 3.4.** Let  $H$  be a marked hypergraph. We denote by  $L(H)$  the length of a shortest nunchaku in  $H$ . If  $H$  contains no nunchaku, then  $L(H) = \infty$  by convention.

**Theorem 3.5.** *Let  $H$  be a 3-uniform marked hyperforest with no fully marked edge. Then  $H$  is a Maker win if and only if  $H$  contains a nunchaku. Moreover, if  $H$  is a Maker win then  $\tau_M(H) = 1 + \lceil \log_2(L(H)) \rceil$ .*

*Proof.* The case where  $H$  contains a nunchaku of length 1 is obvious:

Claim 3. *Let  $H$  be a 3-uniform marked hyperforest with no fully marked edge, and suppose  $L(H) = 1$ . Then  $H$  is a trivial Maker win and  $\tau_M(H) = 1 = 1 + \lceil \log_2(L(H)) \rceil$ .*

*Proof of Claim 3.* This is obvious since a nunchaku of length 1 consists of a single edge, which contains exactly one non-marked vertex.  $\square$

When there exists a nunchaku of length at least 2, Maker can use a "dichotomy strategy" to halve the length of a shortest nunchaku each round, until he gets one of length 1 (which is a trivial Maker win):

Claim 4. *Let  $H$  be a 3-uniform marked hypergraph such that  $2 \leq L(H) < \infty$ . Then there exists  $x \in V(H) \setminus M(H)$  such that, for all  $y \in V(H^{+x}) \setminus M(H^{+x})$ , we have  $L(H^{+x-y}) \leq \lceil \frac{L(H)}{2} \rceil$ .*

*Proof of Claim 4.* Let  $N$  be a shortest nunchaku in  $H$ . Let  $x \in \text{inn}(N)$  be in the exact middle of  $N$  if  $N$  is of even length, or as close to the middle as possible if  $N$  is of odd length. By picking  $x$ , Maker creates two nunchakus of length at most  $\lceil \frac{L(H)}{2} \rceil$  whose sole common vertex is  $x$ , so Breaker's answer  $y$  cannot be contained in both of them at once. Therefore, at least one of these two nunchakus will be present in  $H^{+x-y}$ .  $\square$

Meanwhile, Breaker has a strategy ensuring that, if there exists a nunchaku at the beginning of a round then the length of a shortest nunchaku has not been more than halved after the round, and if there is no nunchaku before a round then there is still none after the round:

Claim 5. *Let  $H$  be a 3-uniform marked hyperforest. Then, for all  $x \in V(H) \setminus M(H)$ , there exists  $y \in V(H^{+x}) \setminus M(H^{+x})$  such that  $L(H^{+x-y}) \geq \lceil \frac{L(H)}{2} \rceil$  ( $= \infty$  if  $L(H) = \infty$ ).*

*Proof of Claim 5.* Let  $x \in V(H) \setminus M(H)$ . Note that, for any  $y$ , the nunchakus in  $H^{+x-y}$  are exactly the nunchakus in  $H^{+x}$  that do not contain  $y$ . Therefore, let  $\mathcal{N}$  be the collection of all nunchakus in  $H^{+x}$  whose length is less than  $\lceil \frac{L(H)}{2} \rceil$ : proving the claim comes down to showing the existence of some  $y \in V(H^{+x}) \setminus M(H^{+x})$  such that all elements of  $\mathcal{N}$  contain  $y$ . We can assume  $\mathcal{N} \neq \emptyset$ , otherwise there is nothing to show.

First of all, notice that all elements of  $\mathcal{N}$  are  $x$ -nunchakus. Indeed, if some element of  $\mathcal{N}$  was not an  $x$ -nunchaku i.e. did not contain  $x$ , then it would be a nunchaku in  $H$ , which is impossible since it is of length less than  $\lceil \frac{L(H)}{2} \rceil \leq L(H)$ . Therefore, let  $N_x \in \mathcal{N}$ : we know  $N_x$  is an  $xm$ -nunchaku for some  $m \in M(H)$ . We now show that all elements of  $\mathcal{N}$  contain  $y := o(x, \overrightarrow{xN_x m})$ , which is non-marked since  $M(N_x) = \{x, m\}$ . Suppose for a contradiction that there exists  $N'_x \in \mathcal{N}_x$  such that  $y \notin V(N'_x)$ : we know  $N'_x$  is an  $xm'$ -nunchaku for some  $m' \in M(H)$ .

- Suppose  $V(N_x) \cap V(N'_x) \neq \{x\}$ . Since  $y \notin V(N'_x)$ , we have  $\text{start}(\overrightarrow{xN_x m}) \neq \text{start}(\overrightarrow{xN'_x m'})$ , therefore Lemma 2.33 ensures that  $N_x \cup N'_x$  contains an  $x$ -cycle or an  $m$ -tadpole. Both possibilities contradict the fact that  $H$  is a hyperforest.
- Suppose  $V(N_x) \cap V(N'_x) = \{x\}$ . Then  $N_x \cup N'_x$  is an  $mm'$ -path in  $H^{+x}$  and  $M(N_x \cup N'_x) = \{m, m', x\}$ . Let  $N$  be the same as  $N_x \cup N'_x$  except that  $x$  is non-marked: since  $N_x \cup N'_x$  is a subhypergraph of  $H^{+x}$ ,  $N$  is a subhypergraph of  $H$ . Therefore  $N$  is an  $mm'$ -nunchaku in  $H$ , of length equal to the sum of the lengths of  $N_x$  and  $N'_x$ . By definition of  $\mathcal{N}$ ,  $N_x$  and  $N'_x$  are both of length less than  $\lceil \frac{L(H)}{2} \rceil$ , therefore  $N$  is of length less than  $L(H)$ , contradicting the definition of  $L(H)$ .  $\square$

We now have all the elements to prove the theorem by induction on  $|V(H) \setminus M(H)|$ .

For the base case, let  $H$  be a 3-uniform marked hyperforest with no fully marked edge such that  $|V(H) \setminus M(H)| \leq 1$ . If  $H$  contains a nunchaku, then it is necessarily of length 1, hence the result by Claim 3. If  $H$  contains no nunchaku, then in particular  $H$  contains no nunchaku of length 1: since  $H$  has no fully marked edge, this means  $H$  is not a trivial Maker win, so  $H$  is a Breaker win.

For the induction step, let  $H$  be a 3-uniform marked hyperforest with no fully marked edge such that  $|V(H) \setminus M(H)| \geq 2$ , and assume that the theorem is true for all 3-uniform marked hyperforests with less than  $|V(H) \setminus M(H)|$  non-marked vertices. Since Claim 3 concludes if  $L(H) = 1$ , also assume  $L(H) > 1$ : this ensures that  $H$  is not a trivial Maker win and that there cannot be a fully marked edge after one round. Therefore, by the induction hypothesis:

$$\tau_M(H) = 1 + \min_{x \in V(H) \setminus M(H)} \max_{y \in V(H^{+x}) \setminus M(H^{+x})} (1 + \lceil \log_2(L(H^{+x-y})) \rceil),$$

with the convention that  $1 + \lceil \log_2(L(H^{+x-y})) \rceil = \infty$  if  $L(H^{+x-y}) = \infty$ .

- Firstly, suppose  $H$  contains a nunchaku i.e.  $2 \leq L(H) < \infty$ . By Claims 4 and 5 respectively, the above equality yields  $\tau_M(H) \leq 1 + \left(1 + \lceil \log_2\left(\lceil \frac{L(H)}{2} \rceil\right) \rceil\right)$  and  $\tau_M(H) \geq 1 + \left(1 + \lceil \log_2\left(\lceil \frac{L(H)}{2} \rceil\right) \rceil\right)$ . Therefore  $\tau_M(H) = 1 + \left(1 + \lceil \log_2\left(\lceil \frac{L(H)}{2} \rceil\right) \rceil\right) = 1 + \lceil \log_2(L(H)) \rceil < \infty$ , so  $H$  is a Maker win.
- Finally, suppose  $H$  contains no nunchaku i.e.  $L(H) = \infty$ . By Claim 5: for all  $x \in V(H) \setminus M(H)$ , there exists  $y \in V(H^{+x}) \setminus M(H^{+x})$  such that  $L(H^{+x-y}) = \infty$ . Therefore  $\tau_M(H) = \infty$ , so  $H$  is a Breaker win. This ends the proof.  $\blacksquare$

### 3.2.2 Interpretation in terms of the family of dangers $\mathcal{S}$

**Notation 3.6.** We define the family  $\mathcal{S}$  of all pointed marked hypergraphs  $(S, x)$  such that  $S$  is an  $x$ -snake and  $|M(S)| = 1$ .

From Breaker's point of view, the fact that a nunchaku is a Maker win may be reformulated as follows:

**Proposition 3.7.**  $\mathcal{S}$  is a family of dangers.

*Proof.* Let  $(S, x) \in \mathcal{S}$ :  $S^{+x}$  is a nunchaku, therefore it is a Maker win by Proposition 3.2. ■

Moreover, we get the following results:

**Proposition 3.8.** Let  $N$  be a nunchaku of length at least 2. Then  $J_1(\mathcal{S}, N)$  does not hold.

*Proof.* Since  $N$  is of length at least 2, we have  $\text{inn}(N) \neq \emptyset$ . Let  $x \in \text{inn}(N)$ :  $N$  is the union of two  $x$ -snakes  $S_1$  and  $S_2$  such that  $V(S_1) \cap V(S_2) = \{x\}$ . We have  $S_1, S_2 \in x\mathcal{S}(N)$  hence  $I_{N^{+x}}(x\mathcal{S}(N)) = \emptyset$ , so  $J_1(\mathcal{S}, N)$  does not hold. ■

**Theorem 3.9.** Let  $H$  be a 3-uniform marked hyperforest that is not a trivial Maker win, with  $|V(H) \setminus M(H)| \geq 2$ . Then  $H$  is a Breaker win if and only if  $J_1(\mathcal{S}, H)$  holds.

*Proof.* Recall that the "only if" direction is automatic by Proposition 1.30. Suppose that  $H$  is a Maker win: since  $H$  is not a trivial Maker win,  $H$  has no fully marked edge, therefore Theorem 3.5 ensures that  $H$  contains a nunchaku  $N$ . Again, since  $H$  is not a trivial Maker win,  $N$  is of length at least 2. By Proposition 3.8,  $J_1(\mathcal{S}, N)$  does not hold, so neither does  $J_1(\mathcal{S}, H)$ . ■

### 3.3 The families of dangers $\mathcal{C}$ and $\mathcal{D}_0$

In general 3-uniform marked hypergraphs, with cycles allowed, the equivalence from Theorem 3.9 does not hold. Indeed, there exist 3-uniform Maker wins that have no marked vertex, like the one in Figure 13 (which is actually a smallest one in terms of number of edges).

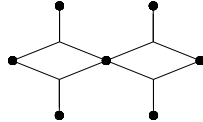


Figure 13: A 3-uniform Maker win with no marked vertex.

We can see that nunchakus have a cycle counterpart:

**Definition 3.10.** An  $a$ -necklace is an  $a$ -cycle  $C$  such that  $M(C) = \{a\}$ . An  $a$ -necklace may simply be referred to as a necklace.

**Proposition 3.11.** Let  $C$  be a necklace. Then  $C$  is a Maker win and satisfies  $\tau_M(C) = 1 + \lceil \log_2(L) \rceil$  where  $L$  is the length of  $C$ . Moreover,  $J_1(\mathcal{S}, C)$  does not hold.

*Proof.* When applying the reduction from Proposition 1.20, where marked vertices are deleted and removed from each edge, a necklace and a nunchaku of same length have the same equivalent non-marked hypergraph which is pictured in Figure 14 (this is the manriki from [14]). Since this reduction clearly preserves  $\tau_M(\cdot)$ , the first assertion follows from Theorem 3.5. As for the final assertion, let  $x$  be the only marked vertex of  $C$  and let  $z \in \text{inn}(C) \setminus \{x\}$ . Since  $x$  and  $z$  are distinct inner vertices of  $C$ , we can write  $C = S_1 \cup S_2$  where  $V(S_1) \cap V(S_2) = \{z, x\}$ , as in Figure 15. Since  $M(C) = \{x\}$ , we have  $S_1, S_2 \in z\mathcal{S}(C)$  hence  $I_{C^{+z}}(z\mathcal{S}(C)) = \emptyset$ , so  $J_1(\mathcal{S}, C)$  does not hold. ■



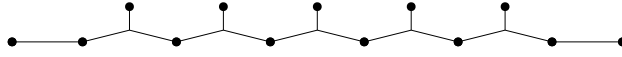


Figure 14: The hypergraph of rank 3 obtained by applying the reduction from Proposition 1.20 to a nunchaku or a necklace.

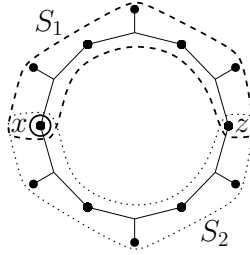


Figure 15: A necklace as a union of two snakes.

Therefore, instead of  $\mathcal{S}$ , it makes sense to consider the family of dangers  $\mathcal{D}_0$ :

**Notation 3.12.** We define the family  $\mathcal{C}$  of all pointed marked hypergraphs  $(C, x)$  such that  $C$  is an  $x$ -cycle and  $M(C) = \emptyset$ . We also define  $\mathcal{D}_0 := \mathcal{S} \cup \mathcal{C}$ .

**Proposition 3.13.**  $\mathcal{C}$  and  $\mathcal{D}_0$  are families of dangers.

*Proof.* If  $(C, x) \in \mathcal{C}$  then  $C^{+x}$  is a necklace, so  $C^{+x}$  is a Maker win by Proposition 3.11 i.e.  $(C, x)$  is a danger. ■

As explained in Section 1, we seek characterizations of Breaker wins in terms of dangers, of the form (\*). The family  $\mathcal{F} = \mathcal{D}_0$  fulfills the requirements in the realm of 3-uniform marked hypergraphs, as it contains the trivial danger of size 3 (snake of length 1) and is identifiable in polynomial time thanks to a result from [8] which will be cited in Section 5. For 3-uniform marked hyperforests, Theorem 3.9 gives the desired characterization with  $\mathcal{F} = \mathcal{D}_0$  (even:  $\mathcal{F} = \mathcal{S}$ ) and  $r = 1$ . Could this be true for all 3-uniform marked hypergraphs? Unfortunately, the answer is no. In fact, not only is  $J_1(\mathcal{D}_0, H)$  not sufficient for  $H$  to be a Breaker win in general, but even  $J_2(\mathcal{D}_0, H)$  is not, contrary to what is basically stated in [1, Theorem 22]. Figure 16 (left) features an instance of a Maker win  $H$  such that it can be checked that  $J_2(\mathcal{D}_0, H)$  holds but not  $J_3(\mathcal{D}_0, H)$ .

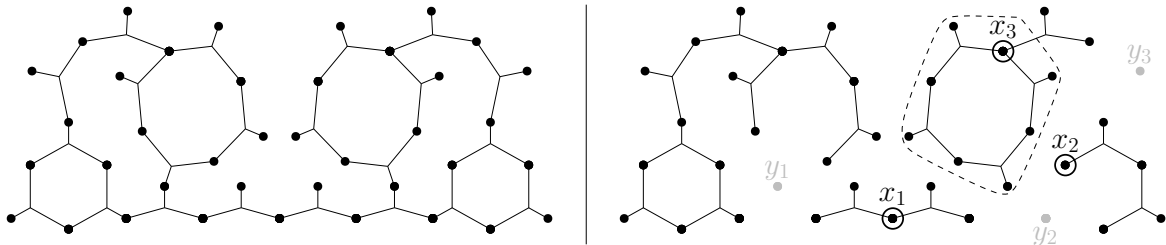


Figure 16:  $H$  is on the left,  $H^{+x_1-y_1+x_2-y_2+x_3-y_3}$  is on the right (the necklace is highlighted).

From Maker's point of view, property  $J_r(\mathcal{D}_0, H)$  not holding means that Maker can force the appearance of a nunchaku or a necklace after at most  $r$  rounds of play (we are talking about full rounds of play, i.e. the marked hypergraph updated **after** Breaker has played contains a nunchaku or a necklace):

**Proposition 3.14.** *Let  $r \geq 1$  be an integer. Let  $H$  be a 3-uniform marked hypergraph that is not a trivial Maker win, with  $|V(H) \setminus M(H)| \geq 2r$ . Then  $J_r(\mathcal{D}_0, H)$  does not hold if and only if Maker has a strategy ensuring that, after  $r$  rounds of play on  $H$  with successive picks  $x_1, y_1, \dots, x_r, y_r$ , the updated marked hypergraph  $H^{+x_1-y_1+\dots+x_r-y_r}$  contains a fully marked edge, a nunchaku or a necklace.*

*(We make the harmless assumption that the players complete  $r$  rounds of play even in the case where Maker effectively wins during the first  $r - 1$  rounds.)*

*Proof.* Suppose  $J_r(\mathcal{D}_0, H)$  holds. In particular  $J_1(\mathcal{S}, H)$  holds, so  $H$  contains no necklace by Proposition 3.11 and no nunchaku of length at least 2 by Proposition 3.8. Since  $H$  is not a trivial Maker win, this means  $H$  contains no nunchaku at all (and no fully marked edge). When Maker picks  $x_i$ , the nunchakus and necklaces that he creates are exactly all the  $D^{+x_i}$  where  $D$  is a  $\mathcal{D}_0$ -danger at  $x_i$ . By definition of  $J_r(\mathcal{D}_0, H)$ , Breaker is thus able, in each of the first  $r$  rounds, to destroy all the nunchakus and necklaces that Maker has just created. For the nunchakus of length 1, this means Maker never gets a fully marked edge.

Conversely, suppose  $J_r(\mathcal{D}_0, H)$  does not hold. Then Maker can ensure that the updated hypergraph at the end of one of the first  $r$  rounds will contain a nunchaku or a necklace. If it happens before the  $r$ -th round, then Maker may for instance use the dichotomy strategy to get a nunchaku (or, eventually, a fully marked edge) at the end of each subsequent round as well, until  $r$  rounds are played. ■

In the hypergraph from Figure 16 (left), Maker needs exactly three rounds to guarantee the appearance of a nunchaku or a necklace: an example of the first three picks by both players is shown on the right. At the end of this section, we will explain how this hypergraph has been built. Just before that, let us state the main results that will be proved in this paper.

### 3.4 Statement of the main results

We have now introduced all concepts and notations needed to state our four main results about the Maker-Breaker game on 3-uniform marked hypergraphs, which we will prove in Section 5. As we have just seen, property  $J_2(\mathcal{D}_0, H)$  is not equivalent to  $H$  being a Breaker win in general. However, the central result of this paper certifies that property  $J_3(\mathcal{D}_0, H)$  is, and we can even give optimal strategies for both players based on the intersection of the  $\mathcal{D}_0^{*2}$ -dangers:

**Theorem 3.15.** *Let  $H$  be a 3-uniform marked hypergraph that is not a trivial Maker win, with  $|V(H) \setminus M(H)| \geq 6$ . Then  $H$  is a Breaker win if and only if  $J_3(\mathcal{D}_0, H)$  holds i.e.  $J_1(\mathcal{D}_0^{*2}, H)$  holds. More precisely:*

- (i) *If  $J_1(\mathcal{D}_0^{*2}, H)$  does not hold, then  $H$  is a Maker win and: any  $x_1 \in V(H) \setminus M(H)$  such that  $I_{H+x_1}(x_1\mathcal{D}_0^{*2}(H)) = \emptyset$  is a winning first pick for Maker.*
- (ii) *If  $J_1(\mathcal{D}_0^{*2}, H)$  holds then  $H$  is a Breaker win and: for any first pick  $x_1 \in V(H) \setminus M(H)$  of Maker, any  $y_1 \in I_{H+x_1}(x_1\mathcal{D}_0^{*2}(H))$  is a winning answer for Breaker.*

*Therefore,  $H$  is a Maker win if and only if Maker has a strategy ensuring that, after three rounds of play on  $H$  with successive picks  $x_1, y_1, x_2, y_2, x_3, y_3$ , the updated marked hypergraph  $H^{+x_1-y_1+x_2-y_2+x_3-y_3}$  contains a fully marked edge, a nunchaku or a necklace.*

This proves the conjecture from Rahman and Watson [14] for positive 3-CNF formulas, given the equivalence between a nunchaku/necklace and what the authors call a manriki.

Previous work had been made by Martin Kutz on linear hypergraphs of rank 3 [11][12]. When translated in our language of marked hypergraphs, the author's result is a structural characterization of Breaker wins for the class of linear 3-uniform marked hypergraphs with at least one marked vertex that are connected and have no articulation vertex. One can notice an interesting

thing in the proof from [11]. The author shows that, if  $H$  does not have said structure, then there is always some simple subhypergraph  $X \subseteq H$  which is a Maker win. It can actually be checked in all cases that, not only is  $X$  a Maker win, but in fact  $J_2(\mathcal{D}_0, X)$  does not hold. Therefore, it can be derived from [11] that, for all  $H$  in the considered class (apart from some trivial cases),  $H$  is a Breaker win if and only if  $J_2(\mathcal{D}_0, H)$  holds. We give a new independent proof of this with our second main result, whose statement is actually slightly stronger:

**Theorem 3.16.** *Let  $H$  be a 3-uniform marked hypergraph that is not a trivial Maker win, with  $|V(H) \setminus M(H)| \geq 4$ . Suppose that, for any  $x \in V(H) \setminus M(H)$ , there exists an  $x$ -snake in  $H$ . Then  $H$  is a Breaker win if and only if  $J_2(\mathcal{D}_0, H)$  holds i.e.  $J_1(\mathcal{D}_0^*, H)$  holds. Therefore,  $H$  is a Maker win if and only if Maker has a strategy ensuring that, after two rounds of play on  $H$  with successive picks  $x_1, y_1, x_2, y_2$ , the updated marked hypergraph  $H^{+x_1-y_1+x_2-y_2}$  contains a fully marked edge, a nunchaku or a necklace.*

Our third main result states that MAKERBREAKER is in polynomial time on 3-uniform marked hypergraphs (and on hypergraphs of rank 3 as a consequence). This improves on [11], which showed the same in the linear case only, and validates the conjecture by Rahman and Watson [14] for positive 3-CNF formulas. The proof relies on the fact that Theorem 3.15 yields an immediate reduction to the path existence problem, which is in polynomial time according to a separate paper [8].

**Theorem 3.17.** *There exists a polynomial-time algorithm that decides whether a 3-uniform marked hypergraph  $H$  is a Maker win.*

Finally, our fourth main result is an easy consequence of Theorem 3.15. It states that, if Maker has a winning strategy on a 3-uniform marked hypergraph, then he can ensure that the game does not last more than a logarithmic number of rounds. From what Theorem 3.5 tells us about nunchakus, the bound is optimal in general up to an additive three rounds at most.

**Theorem 3.18.** *Let  $H$  be a 3-uniform marked hypergraph such that  $|V(H) \setminus M(H)| \geq 6$ . If  $H$  is a Maker win, then  $\tau_M(H) \leq 3 + \lceil \log_2(|V(H) \setminus M(H)| - 5) \rceil$ .*

### 3.5 Approximating $\mathcal{D}_0^*$ and $\mathcal{D}_0^{*2}$

In order to tackle Theorem 3.15, we can choose which property to consider between  $J_3(\mathcal{D}_0, H)$  and  $J_1(\mathcal{D}_0^{*2}, H)$ , which are equivalent according to Proposition 1.40. As explained in Section 1,  $J_1(\mathcal{D}_0^{*2}, H)$  is preferable as long as we get a reasonable understanding of the family  $\mathcal{D}_0^{*2}$ . In this subsection, we exhibit subfamilies  $\mathcal{D}_1 \subseteq \mathcal{D}_0^*$  and  $\mathcal{D}_2 \subseteq \mathcal{D}_0^{*2}$  which will be sufficient approximations, in the sense that Theorems 3.15 and 3.16 will actually hold in a stronger version where  $\mathcal{D}_0^{*2}$  and  $\mathcal{D}_0^*$  are replaced by their respective approximations  $\mathcal{D}_2$  and  $\mathcal{D}_1$ :

**Theorem 3.19.** *Let  $H$  be a 3-uniform marked hypergraph that is not a trivial Maker win, with  $|V(H) \setminus M(H)| \geq 2$ . Then  $H$  is a Breaker win if and only if  $J_1(\mathcal{D}_2, H)$  holds. More precisely:*

- (i) *If  $J_1(\mathcal{D}_2, H)$  does not hold, then  $H$  is a Maker win and: any  $x_1 \in V(H) \setminus M(H)$  such that  $I_{H^{+x_1}}(x_1\mathcal{D}_2(H)) = \emptyset$  is a winning first pick for Maker.*
- (ii) *If  $J_1(\mathcal{D}_2, H)$  holds, then  $H$  is a Breaker win and: for any first pick  $x_1 \in V(H) \setminus M(H)$  of Maker, any  $y_1 \in I_{H^{+x_1}}(x_1\mathcal{D}_2(H))$  is a winning answer for Breaker.*

**Theorem 3.20.** *Let  $H$  be a 3-uniform marked hypergraph that is not a trivial Maker win, with  $|V(H) \setminus M(H)| \geq 2$ . Suppose that, for any  $x \in V(H) \setminus M(H)$ , there exists an  $x$ -snake in  $H$ .*

Then  $H$  is a Breaker win if and only if  $J_1(\mathcal{D}_1, H)$  holds.

Let us now define  $\mathcal{D}_1$  and  $\mathcal{D}_2$ . Recall that  $\mathcal{D}_0 \subseteq \mathcal{D}_0^* \subseteq \mathcal{D}_0^{*2}$ . To build  $\mathcal{D}_1$  from  $\mathcal{D}_0$ , we are only going to add the most elementary new dangers that appear in the jump from  $\mathcal{D}_0$  to  $\mathcal{D}_0^*$ , which are tadpoles.

### 3.5.1 The families of dangers $\mathcal{T}$ and $\mathcal{D}_1$

Recall that  $\mathcal{D}_0^* = \mathcal{D}_0 \cup \mathcal{D}_0^\heartsuit$  by definition. When looking for basic examples of  $\mathcal{D}_0^\heartsuit$ -dangers, we can see tadpoles appear:

**Notation 3.21.** We define the family  $\mathcal{T} \supseteq \mathcal{C}$  of all pointed marked hypergraphs  $(T, x)$  such that  $T$  is an  $x$ -tadpole and  $M(T) = \emptyset$ . We also define  $\mathcal{D}_1 := \mathcal{D}_0 \cup \mathcal{T} = \mathcal{S} \cup \mathcal{T}$ .

**Proposition 3.22.** We have  $\mathcal{T} \setminus \mathcal{C} \subseteq \mathcal{D}_0^\heartsuit$ . In particular:  $\mathcal{D}_1 \subseteq \mathcal{D}_0^*$ .

*Proof.* Let  $(T, x) \in \mathcal{T} \setminus \mathcal{C}$ , and let  $z$  be the only vertex in  $V(P_T) \cap V(C_T)$ . Note that  $z \neq x$  since  $T$  is not a cycle. We can write  $T = \langle \mathcal{O} \rangle$  with  $\mathcal{O} := \{P_T, C_T\}$ . Since  $V(P_T) \cap V(C_T) = \{z\}$ , we have  $I_{T^{+x+z}}(\mathcal{O}) = \emptyset$ . Moreover, since  $M(P_T) = M(C_T) = \emptyset$ , we have  $P_T^{+x} \in z\mathcal{S}(T^{+x}) \subseteq z\mathcal{D}_0(T^{+x})$  and  $C_T^{+x} = C_T \in z\mathcal{C}(T^{+x}) \subseteq z\mathcal{D}_0(T^{+x})$  i.e.  $\mathcal{O} \subseteq \{X \subseteq T, X^{+x} \in z\mathcal{D}_0(T^{+x})\}$ . See Figure 17. In conclusion, we get  $\mathcal{O} \in \heartsuit_{T^{+x+z}}(\{X \subseteq T, X^{+x} \in z\mathcal{D}_0(T^{+x})\})$ .

This proves that  $\mathcal{T} \setminus \mathcal{C} \subseteq \mathcal{D}_0^\heartsuit \subseteq \mathcal{D}_0^*$ . Since  $\mathcal{D}_0 \subseteq \mathcal{D}_0^*$ , this yields  $\mathcal{D}_1 \subseteq \mathcal{D}_0^*$ . ■

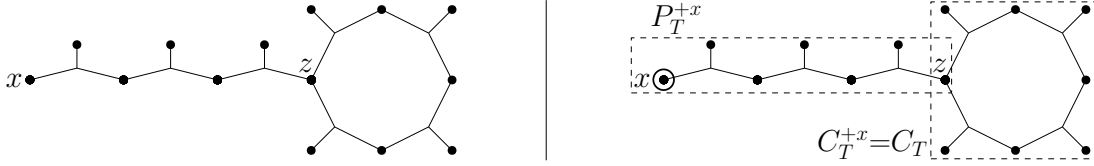


Figure 17: Left:  $(T, x) \in \mathcal{T} \setminus \mathcal{C}$ . Right:  $T^{+x}$  is the union of two  $\mathcal{D}_0$ -dangers at  $z$  whose intersection in  $T^{+x+z}$  is empty.

Although the tadpoles from  $\mathcal{T}$  (resp. the snakes from  $\mathcal{S}$ ) are required to have exactly 0 (resp. 1) marked vertex by definition, the next proposition ensures that in practice we will never have to worry about the number of marked vertices in a tadpole or a snake.

**Proposition 3.23.** Let  $H$  be a marked hypergraph that is not a trivial Maker win, and let  $u \in V(H) \setminus M(H)$ . Then any  $u$ -tadpole or  $u$ -snake in  $H$  contains a  $\mathcal{D}_1$ -danger at  $u$ . In particular, if  $u' \in I_{H+u}(u\mathcal{D}_1(H))$ , then any  $u$ -tadpole or  $u$ -snake in  $H$  contains  $u'$ .

*Proof.* Let  $X$  be a  $u$ -tadpole or  $u$ -snake in  $H$ . If  $M(X) = \emptyset$ , then  $X$  is necessarily a  $u$ -tadpole, and  $X \in u\mathcal{T}(H) \subseteq u\mathcal{D}_1(H)$ . Therefore, assume that  $M(X) \neq \emptyset$ , so that the path  $S := \mathbf{P}_{M(X)}(u, X) \subseteq X$  is well defined. By definition of a projection, the only edge of  $S$  that intersects  $M(X)$  is  $\text{end}(\overrightarrow{uS})$ . Moreover, since  $H$  is not a trivial Maker win, that edge contains exactly one marked vertex hence  $S \in u\mathcal{S}(H) \subseteq u\mathcal{D}_1(H)$ .

The final assertion of this proposition ensues immediately: all  $u$ -snakes and  $u$ -tadpoles contain some  $D \in u\mathcal{D}_1(H)$ , and  $u' \in V(D)$  since  $u' \in I_{H+u}(u\mathcal{D}_1(H))$ . ■

Let us also mention another useful property:

**Proposition 3.24.** *Let  $H$  be a marked hypergraph that is not a trivial Maker win, with  $|(V(H) \setminus M(H))| \geq 2$ , and suppose  $J_1(\mathcal{D}_0, H)$  holds. Then, for any  $m \in M(H)$ , there is no  $m$ -tadpole and no  $m$ -snake in  $H$ .*

*Proof.* Suppose for a contradiction that there exists a subhypergraph  $X$  of  $H$  that is an  $m$ -tadpole or an  $m$ -snake for some marked vertex  $m$ . Let  $H_0$  (resp.  $X_0$ ) be the same as  $H$  (resp.  $X$ ) except that  $m$  is non-marked. By Proposition 3.23 applied to  $X_0$  with  $u = m$ , there exists some  $D \subseteq X_0$  such that  $(D, m) \in \mathcal{D}_1$ . We have  $D^{+m} \subseteq X \subseteq H$ .

- First suppose  $(D, m) \in \mathcal{S}$  i.e.  $D^{+m}$  is a nunchaku. Since  $H$  is not a trivial Maker win,  $D^{+m}$  is of length at least 2, so  $J_1(\mathcal{S}, D^{+m})$  does not hold according to Proposition 3.8. Therefore,  $J_1(\mathcal{S}, H)$  and  $J_1(\mathcal{D}_0, H)$  do not hold either, a contradiction.
- Now suppose  $(D, m) \in \mathcal{C}$ . By Proposition 3.11,  $J_1(\mathcal{S}, D^{+m})$  does not hold. Therefore,  $J_1(\mathcal{S}, H)$  and  $J_1(\mathcal{D}_0, H)$  do not hold either, a contradiction.
- Finally, suppose  $(D, m) \in \mathcal{T} \setminus \mathcal{C}$ . By Proposition 3.22, we have  $(D, m) \in \mathcal{D}_0^\heartsuit$ . Moreover, since  $D$  is a tadpole that is not a cycle, we have  $|V(D)| \geq 6$  hence  $|V(D^{+m}) \setminus M(D^{+m})| = |V(D)| - 1 \geq 5 \geq 2$ , so Proposition 1.36 ensures that  $J_1(\mathcal{D}_0, D^{+m})$  does not hold. Therefore,  $J_1(\mathcal{D}_0, H)$  does not hold either, a contradiction. ■

### 3.5.2 The families of dangers $\mathcal{D}_1^\heartsuit$ and $\mathcal{D}_2$

We want to define  $\mathcal{D}_2 \subseteq \mathcal{D}_0^{*2}$  such that property  $J_1(\mathcal{D}_2, H)$  is sufficient for a 3-uniform marked hypergraph  $H$  to be a Breaker win. The idea is to prove this sufficiency result by induction, as follows:

1. Assume  $J_1(\mathcal{D}_2, H)$  holds. Maker picks some  $x$ , Breaker picks some  $y \in I_{H+x}(x\mathcal{D}_2(H))$ .
2. If we have chosen  $\mathcal{D}_2 \supseteq \mathcal{D}_1$  in such a way that  $I_{H+x}(x\mathcal{D}_2(H)) \subseteq I_{H+x}(x\mathcal{D}_1^\heartsuit(H))$ , then  $y \in I_{H+x}(x\mathcal{D}_1^\heartsuit(H))$ , therefore  $J_1(\mathcal{D}_1, H^{+x-y})$  holds by Proposition 1.37.
3. To complete the induction step, it remains to show that  $J_1(\mathcal{D}_1, H^{+x-y})$  implies  $J_1(\mathcal{D}_2, H^{+x-y})$ , which will be the difficult part of the proof.

Therefore, for step 2 above, we must define  $\mathcal{D}_2 \supseteq \mathcal{D}_1$  so that destroying the  $\mathcal{D}_2$ -dangers destroys the  $\mathcal{D}_1^\heartsuit$ -dangers as well. This will force us to include (almost) all of  $\mathcal{D}_1^\heartsuit$  inside of  $\mathcal{D}_2$ . As a consequence, we need to understand the structure of the  $\mathcal{D}_1^\heartsuit$ -dangers in all generality.

A  $\mathcal{D}_1^\heartsuit$ -danger at  $x$  is by definition a union of subhypergraphs having the common property that they will be  $\mathcal{D}_1$ -dangers at some common  $z$  after  $x$  is marked. Which subhypergraphs have this property? The following result is elementary and answers this question: those not containing  $x$  are  $\mathcal{D}_1$ -dangers at  $z$  already, while those containing  $x$  are  $zx$ -paths (which will become  $zx$ -snakes after  $x$  is marked).

**Notation 3.25.** Let  $H$  be a marked hypergraph and  $u, v \in V(H)$ . We denote by  $\mathcal{P}_{uv}(H)$  the set of all  $uv$ -paths  $P$  in  $H$  such that  $M(P) = \emptyset$ .

**Proposition 3.26.** *Let  $H$  be a marked hypergraph and let  $x, z \in V(H) \setminus M(H)$  be distinct. We have  $\{X \subseteq H, X^{+x} \in z\mathcal{D}_1(H^{+x})\} = z\mathcal{D}_1(H^{-x}) \cup \mathcal{P}_{zx}(H)$ .*

*Proof.* Let  $X \subseteq H$  such that  $X^{+x} \in z\mathcal{D}_1(H^{+x})$ . There are two possibilities:

- Suppose  $x \notin V(X)$ : then  $X = X^{+x}$  is a  $\mathcal{D}_1$ -danger at  $z$  in  $H^{+x}$ , moreover  $X \subseteq H^{-x}$  so  $X \in z\mathcal{D}_1(H^{-x})$ .
- Suppose  $x \in V(X)$ . By definition of  $\mathcal{D}_1$ , the only  $\mathcal{D}_1$ -dangers containing a marked vertex are the  $\mathcal{S}$ -dangers, and they contain exactly one marked vertex. Therefore  $X^{+x}$  is a  $zx$ -snake whose only marked vertex is  $x$ , so  $X$  is a  $zx$ -path with no marked vertex i.e.  $X \in \mathcal{P}_{zx}(H)$ . ■

*Remark.* The notation  $z\mathcal{D}_1(H^{-x})$  is just a compact way to refer to the collection of all  $\mathcal{D}_1$ -dangers at  $z$  in  $H$  that do not contain  $x$ .

From this, we deduce the structural characterization of the  $\mathcal{D}_1^\emptyset$ -dangers.

**Notation 3.27.** Let  $D$  be a marked hypergraph and let  $x, z \in V(D) \setminus M(D)$  be distinct. We define the collection  $\mathcal{O}_{x,z}(D) := z\mathcal{D}_1(D^{-x}) \cup \mathcal{P}_{zx}(D) = z\mathcal{T}(D^{-x}) \cup z\mathcal{S}(D^{-x}) \cup \mathcal{P}_{zx}(D)$ .

**Proposition 3.28.** *A pointed marked hypergraph  $(D, x)$  is in  $\mathcal{D}_1^\emptyset$ , with  $\mathcal{D}_1$ -dangerous vertex  $z$ , if and only if  $D = \langle \mathcal{O}_{x,z}(D) \rangle$  and  $I_{D^{+x+z}}(\mathcal{O}_{x,z}(D)) = \emptyset$ .*

*Proof.* By definition of the family  $\mathcal{D}_1^\emptyset$ : a pointed marked hypergraph  $(D, x)$  is in  $\mathcal{D}_1^\emptyset$ , with  $\mathcal{D}_1$ -dangerous vertex  $z$ , if and only if  $D = \langle \mathcal{O} \rangle$  for some  $\mathcal{O} \subseteq \{X \subseteq D, X^{+x} \in z\mathcal{D}_1(D^{+x})\}$  such that  $I_{D^{+x+z}}(\mathcal{O}) = \emptyset$ . Moreover, we have  $\{X \subseteq D, X^{+x} \in z\mathcal{D}_1(D^{+x})\} = \mathcal{O}_{x,z}(D)$  by Proposition 3.26. Therefore: a pointed marked hypergraph  $(D, x)$  is in  $\mathcal{D}_1^\emptyset$ , with  $\mathcal{D}_1$ -dangerous vertex  $z$ , if and only if  $D = \langle \mathcal{O} \rangle$  for some  $\mathcal{O} \subseteq \mathcal{O}_{x,z}(D)$  such that  $I_{D^{+x+z}}(\mathcal{O}) = \emptyset$ . Finally, since  $\mathcal{O}_{x,z}(D)$  is a collection of subhypergraphs of  $D$ , we always have  $D \supseteq \langle \mathcal{O}_{x,z}(D) \rangle$ , so saying that  $D = \langle \mathcal{O} \rangle$  for some  $\mathcal{O} \subseteq \mathcal{O}_{x,z}(D)$  is obviously equivalent to saying that  $D = \langle \mathcal{O}_{x,z}(D) \rangle$ . ■

*Example.* Figure 18 features some examples of  $\mathcal{D}_1^\emptyset$ -dangers. The middle one was actually used to build the example  $H$  from Figure 16: it has been "duplicated" at  $x = x_1$  so that  $I_{H^{+x_1}}(x_1\mathcal{D}_1^\emptyset(H)) = \emptyset$ , ensuring that  $J_1(\mathcal{D}_1^\emptyset, H)$  does not hold from which  $J_1(\mathcal{D}_0^{*2}, H)$  i.e.  $J_3(\mathcal{D}_0, H)$  does not hold either.

Instead of defining  $\mathcal{D}_2$  as  $\mathcal{D}_1 \cup \mathcal{D}_1^\emptyset = \mathcal{D}_1^*$ , we have seen at the very end of Section 1 that we can actually avoid some redundancies by defining it as follows:

**Notation 3.29.** We define  $\mathcal{D}_2 := \mathcal{D}_1 \cup \mathcal{D}_1^{\emptyset, \text{rest}}$ .

Indeed, destroying the  $\mathcal{D}_1$ -dangers automatically destroys the  $(\mathcal{D}_1^\emptyset \setminus \mathcal{D}_1^{\emptyset, \text{rest}})$ -dangers as well. As a concrete example, take  $\mathcal{T} \setminus \mathcal{C}$  for instance: we have  $\mathcal{T} \setminus \mathcal{C} \subseteq \mathcal{D}_0^\emptyset \subseteq \mathcal{D}_1^\emptyset$ , however we already have  $\mathcal{T} \setminus \mathcal{C} \subseteq \mathcal{D}_1$ , so defining  $\mathcal{D}_2 = \mathcal{D}_1 \cup \mathcal{D}_1^{\emptyset, \text{rest}}$  instead of  $\mathcal{D}_2 = \mathcal{D}_1 \cup \mathcal{D}_1^\emptyset$  means we do not consider the  $(\mathcal{T} \setminus \mathcal{C})$ -dangers twice in a way.

**Proposition 3.30.** *We have  $\mathcal{D}_2 \subseteq \mathcal{D}_0^{*2}$ .*

*Proof.* We have  $\mathcal{D}_2 = \mathcal{D}_1 \cup \mathcal{D}_1^{\emptyset, \text{rest}} \subseteq \mathcal{D}_1 \cup \mathcal{D}_1^\emptyset = \mathcal{D}_1^* \subseteq (\mathcal{D}_0^*)^* = \mathcal{D}_0^{*2}$ . ■

## 4 Structural properties of the $\mathcal{D}_1^{\emptyset, \text{rest}}$ -dangers

We have just approximated the pivotal family  $\mathcal{D}_0^{*2}$  with the family  $\mathcal{D}_2 = \mathcal{D}_1 \cup \mathcal{D}_1^{\emptyset, \text{rest}} \subseteq \mathcal{D}_0^{*2}$ . While the  $\mathcal{D}_1$ -dangers are very basic objects, the  $\mathcal{D}_1^{\emptyset, \text{rest}}$ -dangers must be studied as to better understand their shape and structural behavior. At the end of Section 3, we have given the general structure of the elements of  $\mathcal{D}_1^\emptyset \supseteq \mathcal{D}_1^{\emptyset, \text{rest}}$ . The restriction that defines the family  $\mathcal{D}_1^{\emptyset, \text{rest}}$  compared to the family  $\mathcal{D}_1^\emptyset$  comes with added structural properties, which are the subject of this section.

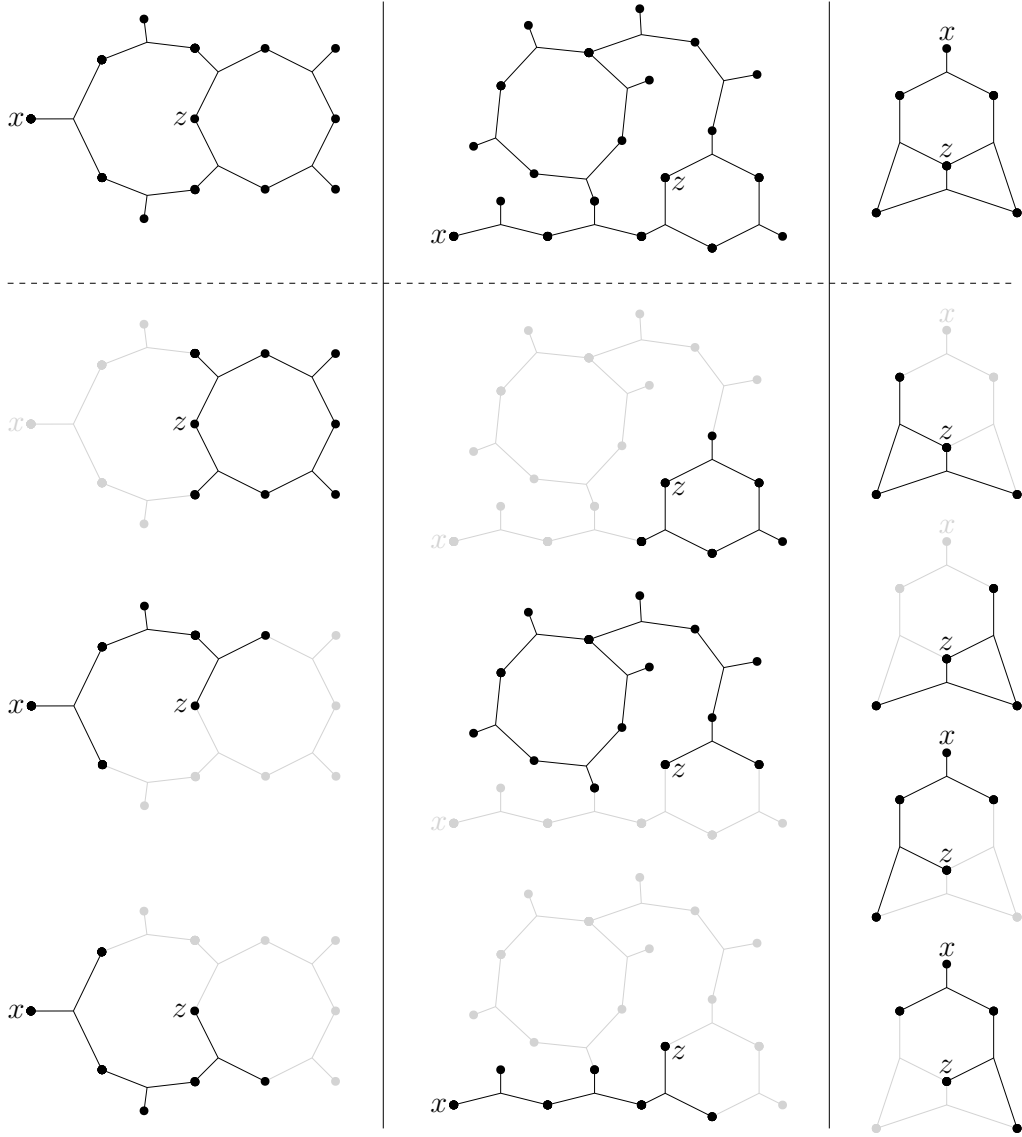


Figure 18: Three examples of  $\mathcal{D}_1^{\mathcal{O}}$ -dangers  $(D, x)$ . Each one is the union of the subhypergraphs highlighted below it, which only intersect at  $z$ .

## 4.1 Structural properties in general

**Proposition 4.1.** *Let  $(D, x) \in \mathcal{D}_1^{\mathcal{O}, \text{rest}}$ , with  $\mathcal{D}_1$ -dangerous vertex  $z$ . We have the following properties:*

- (a)  $D = \langle \mathcal{O}_{x,z}(D) \rangle = \langle z\mathcal{D}_1(D^{-x}) \cup \mathcal{P}_{zx}(D) \rangle$ .
- (b)  $I_{D+z}(\mathcal{O}_{x,z}(D)) = \emptyset$ .
- (c)  $z\mathcal{D}_1(D^{-x}) \neq \emptyset$ .
- (d)  $\mathcal{P}_{zx}(D) \neq \emptyset$ .
- (e)  $D$  is not a trivial Maker win.
- (f) There is no  $x$ -tadpole and no  $x$ -snake in  $D$ .
- (g) There exists a  $z$ -cycle in  $D$ .

*Proof.* Let us start with items (a), (d) and (e), which actually hold for general  $\mathcal{D}_1^{\mathcal{O}}$ -dangers. Proposition 3.28 gives us item (a). As for item (d), it is impossible that  $\mathcal{P}_{zx}(D) = \emptyset$ , because we would get  $D = \langle z\mathcal{D}_1(D^{-x}) \rangle$ , contradicting the fact that  $D$  contains  $x$  while the subhypergraphs in the collection  $z\mathcal{D}_1(D^{-x})$  do not. Finally, the elements of the collection  $z\mathcal{D}_1(D^{-x})$  have no edge with more than one marked vertex by definition of  $\mathcal{D}_1$ , and the elements of the collection

$\mathcal{P}_{zx}(D)$  have no marked vertex by definition, hence item (e).

We now check the remaining properties. Before this, using item (d), let  $P_{zx} \in \mathcal{P}_{zx}(D)$  be shortest, and define  $v := o(x, \overrightarrow{xP_{zx}z})$  and  $w := o(z, \overleftarrow{xP_{zx}z})$ : this path will be useful. Note that  $M(P_{zx}) = \emptyset$  by definition of the collection  $\mathcal{P}_{zx}(D)$ .

- Item (f) is straightforward. Since  $D$  is not a trivial Maker win by item (e), Proposition 3.23 ensures that if there was an  $x$ -tadpole or an  $x$ -snake in  $D$  then  $D$  would contain a  $\mathcal{D}_1$ -danger at  $x$ , contradicting the definition of the restricted family  $\mathcal{D}_1^{\mathcal{O}, \text{rest}}$ .
- Let us prove item (c). Suppose for a contradiction that  $z\mathcal{D}_1(D^{-x}) = \emptyset$  hence  $D = \langle \mathcal{P}_{zx}(D) \rangle$ . We are going to use the path  $P_{zx}$ . We know  $I_{D+x+z}(\mathcal{O}_{x,z}(D)) = \emptyset$  by Proposition 3.28, so  $v \notin I_{D+x+z}(\mathcal{O}_{x,z}(D))$ . Since  $v \notin M(D^{+x+z})$ , this means some element of the collection  $\mathcal{O}_{x,z}(D) = \mathcal{P}_{zx}(D)$  does not contain  $v$ : let  $P^{\bar{v}} \in \mathcal{P}_{zx}(D)$  such that  $v \notin V(P^{\bar{v}})$ . We have  $\text{start}(\overrightarrow{xP^{\bar{v}}z}) \neq \text{start}(\overrightarrow{xP_{zx}z}) \ni v$  and  $V(P^{\bar{v}}) \cap (V(P_{zx}) \setminus \{x\}) \supseteq \{z\} \neq \emptyset$ , so we can apply Lemma 2.33: since  $P_{zx} \cup P^{\bar{v}} \subseteq D$  contains no  $x$ -cycle by item (f), it contains a  $z$ -tadpole  $T$ . If  $x \notin V(T)$  as on the left of Figure 19, then  $T \in z\mathcal{T}(D^{-x})$ , contradicting the fact that  $z\mathcal{D}_1(D^{-x}) = \emptyset$ . If  $x \in V(T)$ , then the only possibility is that the projection  $\mathbf{P}_{V(P_{zx}) \setminus \{x\}}(x, P^{\bar{v}})$  consists of a single edge  $e$  as illustrated on the right of Figure 19: since  $v \notin e$ , we get a  $zx$ -path  $P'_{zx}$  that is strictly shorter than  $P_{zx}$ , a contradiction.

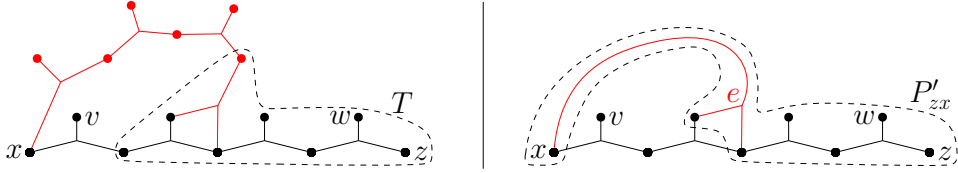


Figure 19: The contradiction that yields item (c), if  $x \notin V(T)$  (left) or if  $x \in V(T)$  (right). The represented paths are  $P_{zx}$  (bottom) and  $\mathbf{P}_{V(P_{zx}) \setminus \{x\}}(x, P^{\bar{v}})$ .

- Item (b) directly ensues from item (c). Indeed, we already know that  $I_{D+x+z}(\mathcal{O}_{x,z}(D)) = \emptyset$  by Proposition 3.28, hence  $I_{D+z}(\mathcal{O}_{x,z}(D)) \subseteq \{x\}$ . Since the collection  $z\mathcal{D}_1(D^{-x})$  is nonempty and none of its elements contain  $x$ , we have  $x \notin I_{D+z}(z\mathcal{D}_1(D^{-x})) \supseteq I_{D+z}(\mathcal{O}_{x,z}(D))$ , so in conclusion  $I_{D+z}(\mathcal{O}_{x,z}(D)) = \emptyset$ .
- Finally, let us prove item (g). We are going to use the path  $P_{zx}$  again. We know  $I_{D+z}(\mathcal{O}_{x,z}(D)) = \emptyset$  by item (b), so  $w \notin I_{D+z}(\mathcal{O}_{x,z}(D))$ . Since  $w \notin M(D^{+z})$ , this means some element of the collection  $\mathcal{O}_{x,z}(D)$  does not contain  $w$ : let  $X^{\bar{w}} \in \mathcal{O}_{x,z}(D)$  such that  $w \notin V(X^{\bar{w}})$ . By definition  $\mathcal{O}_{x,z}(D) = z\mathcal{T}(D^{-x}) \cup z\mathcal{S}(D^{-x}) \cup \mathcal{P}_{zx}(D)$ , so there are three possibilities for  $X^{\bar{w}}$ , and we claim that  $V(X^{\bar{w}}) \cap (V(P_{zx}) \setminus \{z\}) \neq \emptyset$  for all of them:
  - If  $X^{\bar{w}} \in \mathcal{P}_{zx}(D)$ , then this is obvious because  $x \in V(X^{\bar{w}})$ .
  - If  $X^{\bar{w}} \in z\mathcal{T}(D^{-x})$ , then this is true because otherwise  $P_{zx} \cup X^{\bar{w}}$  would be an  $x$ -tadpole in  $D$ , contradicting item (f).
  - If  $X^{\bar{w}} \in z\mathcal{S}(D^{-x})$ , then this is true because otherwise  $P_{zx} \cup X^{\bar{w}}$  would be an  $x$ -snake in  $D$ , contradicting item (f).

Therefore, the projection  $P := \mathbf{P}_{V(P_{zx}) \setminus \{z\}}(z, X^{\bar{w}})$  is well defined. Since  $w \notin V(P)$  and  $w \in \text{start}(\overleftarrow{xP_{zx}z})$ , we have  $\text{start}(\overrightarrow{zP}) \neq \text{start}(\overleftarrow{xP_{zx}z})$  so we can apply Lemma 2.33: since  $P_{zx} \cup P \subseteq D$  contains no  $x$ -tadpole by item (f), it contains a  $z$ -cycle.  $\blacksquare$

The proofs of items (c) and (g) are typical of the methods that we will use extensively. The key is that, thanks to item (b), for any non-marked vertex  $u \neq z$  there exists some element of  $\mathcal{O}_{x,z}(D)$  that does not contain  $u$ . Therefore, item (b) is a powerful existence tool, providing us with subhypergraphs of  $D$  which we can use to partially reconstruct  $D$  and establish structural properties.



Beyond these basic characteristics, it is difficult to say much about the structure of  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -dangers in general. However, we now give additional properties that hold in all interesting cases.

## 4.2 Structural properties when $I_{H+z}(z\mathcal{D}_1(H)) \neq \emptyset$

In practice, we will always consider  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -dangers in some hypergraph  $H$  such that  $J_1(\mathcal{D}_1, H)$  is satisfied. Given some  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -danger  $D$  at  $x$  in  $H$ , with  $z$  a  $\mathcal{D}_1$ -dangerous vertex in  $(D, x)$ , this implies that  $I_{H+z}(z\mathcal{D}_1(H)) \neq \emptyset$ . In other words, even though the intersection in  $H+z$  of  $z\mathcal{D}_1(H) \cup \mathcal{P}_{zx}(H)$  is empty by Proposition 4.1, the intersection in  $H+z$  of  $z\mathcal{D}_1(H)$  **alone** is not: it contains some  $s$ . This vertex  $s$  will often be useful.

**Proposition 4.2.** *Let  $H$  be a marked hypergraph that is not a trivial Maker win. Let  $D$  be a  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -danger at some  $x$  in  $H$ , and let  $z$  be a  $\mathcal{D}_1$ -dangerous vertex in  $(D, x)$ . Suppose  $I_{H+z}(z\mathcal{D}_1(H)) \neq \emptyset$ , and let  $s \in I_{H+z}(z\mathcal{D}_1(H))$ . Then:*

- Any  $z$ -tadpole or  $z$ -snake in  $H$  contains  $s$ .
- $s \in V(D) \setminus (M(D) \cup \{x, z\})$ .
- There exists  $P^{\bar{s}} \in \mathcal{P}_{zx}(D)$  such that  $s \notin V(P^{\bar{s}})$ . Moreover, the edges  $\overrightarrow{\text{start}(xP^{\bar{s}}z)}$  and  $\overleftarrow{\text{end}(xP^{\bar{s}}z)}$  are the same for any choice of  $P^{\bar{s}}$ .

*Proof.* We prove all three assertions separately:

- Since  $s \in I_{H+z}(z\mathcal{D}_1(H))$  and  $H$  is not a trivial Maker win by assumption, Proposition 3.23 applies with  $u = z$  and  $u' = s$ , hence the first assertion.
- By definition of  $I_{H+z}(\cdot)$ , we have  $s \notin M(H+z) = M(H) \cup \{z\}$ . Let  $X \in z\mathcal{D}_1(D^{-x})$ , which exists by Proposition 4.1(c): since  $z\mathcal{D}_1(D^{-x}) \subseteq z\mathcal{D}_1(H)$  and  $s \in I_{H+z}(z\mathcal{D}_1(H))$ , we have  $s \in V(X) \subseteq V(D^{-x}) = V(D) \setminus \{x\}$ . All in all, we get  $s \in V(D) \setminus (M(D) \cup \{x, z\})$ .
- Since  $s \notin M(H+z)$ , Proposition 4.1(b) ensures the existence of some  $X^{\bar{s}} \in \mathcal{O}_{x,z}(D)$  such that  $s \notin V(X^{\bar{s}})$ . Since  $s \in I_{H+z}(z\mathcal{D}_1(H)) \subseteq I_{H+z}(z\mathcal{D}_1(D^{-x}))$ , it is impossible that  $X^{\bar{s}} \in z\mathcal{D}_1(D^{-x})$ , so necessarily  $X^{\bar{s}} \in \mathcal{P}_{zx}(D)$ . Finally, let  $P_1^{\bar{s}}, P_2^{\bar{s}} \in \mathcal{P}_{zx}(D)$  such that  $s \notin V(P_1^{\bar{s}})$  and  $s \notin V(P_2^{\bar{s}})$ . Suppose for a contradiction that  $\overrightarrow{\text{start}(xP_1^{\bar{s}}z)} \neq \overrightarrow{\text{start}(xP_2^{\bar{s}}z)}$ : by Lemma 2.33,  $P_1^{\bar{s}} \cup P_2^{\bar{s}} \subseteq D$  contains an  $x$ -cycle (contradicting Proposition 4.1(f)) or a  $z$ -tadpole (which does not contain  $s$ , also a contradiction). Similarly, suppose for a contradiction that  $\overleftarrow{\text{end}(xP_1^{\bar{s}}z)} \neq \overleftarrow{\text{end}(xP_2^{\bar{s}}z)}$  i.e.  $\overleftarrow{\text{start}(xP_1^{\bar{s}}z)} \neq \overleftarrow{\text{start}(xP_2^{\bar{s}}z)}$ : by Lemma 2.33,  $P_1^{\bar{s}} \cup P_2^{\bar{s}} \subseteq D$  contains an  $x$ -tadpole (contradicting Proposition 4.1(f)) or a  $z$ -cycle (which does not contain  $s$ , also a contradiction). ■

We now establish some important properties of the  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -dangers in an ambient hypergraph  $H$  where  $I_{H+z}(z\mathcal{D}_1(H)) \neq \emptyset$ , or sometimes under the stronger assumption that  $J_1(\mathcal{D}_1, H)$  holds. We will also make the costless assumption that  $H$  is not a trivial Maker win, as we have already done in Proposition 4.2.

### 4.2.1 Union lemmas

The next two lemmas are the analog for  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -dangers of the union lemmas from Section 2. We look at what happens in the union of a  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -danger and a path.

**Lemma 4.3.** *Let  $H$  be a marked hypergraph that is not a trivial Maker win, and let  $x \in V(H) \setminus M(H)$ . Let  $D$  be a  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -danger at  $x$  in  $H$ , with  $z$  a  $\mathcal{D}_1$ -dangerous vertex in  $(D, x)$ . Let  $c \in V(H) \setminus V(D)$ , and let  $P_c$  be a  $c$ -path such that  $V(P_c) \cap V(D) \neq \emptyset$ .*

- (i) If  $I_{H+z}(z\mathcal{D}_1(H)) \neq \emptyset$ , then there is a  $c$ -tadpole, a  $c$ -snake or a  $cx$ -path in  $D \cup P_c$ .
- (ii) If  $J_1(\mathcal{D}_1, H)$  holds, then there is a  $c$ -tadpole or a  $cx$ -path in  $D \cup P_c$ .

*Proof.* First of all, we can assume that  $P_c$  consists of a single edge  $e$ . Indeed, let  $e := \overrightarrow{\text{end}(c\mathbf{P}_{V(D)}(c, P_c))}$  and  $c' \in e \setminus V(D)$ :

- If there is a  $c'x$ -path  $P$  in  $D \cup e$ , then  $\overrightarrow{cP_c}|_{\{c'\}} \oplus \overrightarrow{c'Px}$  represents a  $cx$ -path in  $D \cup P_c$ .
- If there is a  $c'$ -snake  $S$  in  $D \cup e$ , then  $\overrightarrow{cP_c}|_{\{c'\}} \oplus \overrightarrow{c'S}$  represents a  $c$ -snake in  $D \cup P_c$ .
- If there is a  $c'$ -tadpole  $T$  in  $D \cup e$ , then  $\overrightarrow{cP_c}|_{\{c'\}} \oplus \overrightarrow{c'T}$  represents a  $c$ -tadpole in  $D \cup P_c$ .

Therefore, we are working in  $D \cup P_c = D \cup e$ . Now suppose for a contradiction that:

$D \cup e$  contains no  $c$ -tadpole and no  $cx$ -path, and also no  $c$ -snake in the case of item (i). (C)

Since  $I_{H+z}(z\mathcal{D}_1(H)) \neq \emptyset$  by assumption, let  $s \in I_{H+z}(z\mathcal{D}_1(H))$ , and let  $P^{\bar{s}} \in \mathcal{P}_{zx}(D)$  such that  $s \notin V(P^{\bar{s}})$  as per Proposition 4.2. Define  $w := o(z, xP^{\bar{s}}z)$ . These notations are summed up in Figure 20.

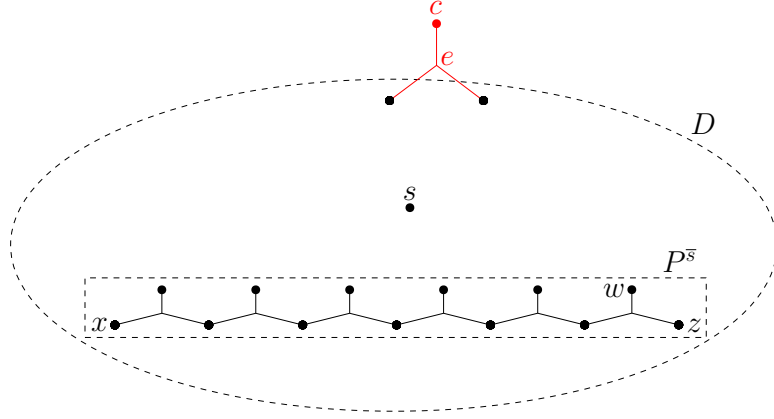


Figure 20:  $D$  is only partially represented. In this picture we have  $|e \cap V(D)| = 2$ , but it is also possible that  $|e \cap V(D)| = 1$ .

The key to the proof is the fact that every  $z$ -tadpole contains  $s$ , whereas  $P^{\bar{s}}$  does not. For example, we can start by making a simple observation:

**Claim 6.** *Let  $P_{cz}$  be a  $cz$ -path in  $D \cup e$ , and write  $\overrightarrow{c\mathbf{P}_{V(P^{\bar{s}})}(c, P_{cz})} = (c, e_1, \dots, e_j)$ . Then:  $j > 1$ ,  $e_j \perp xP^{\bar{s}}z$ , and  $e_{j-1} \cap e_j = \{s\}$ . In particular, the  $cs$ -path in  $P_{cz}$  is disjoint from  $P^{\bar{s}}$ .*

*Proof of Claim 6.* Since there is no  $cx$ -path in  $D \cup e$  by (C), we apply Lemma 2.32 with  $a = x$ ,  $b = z$ ,  $P_{ab} = P^{\bar{s}}$ . Note that  $e_j$  is precisely the edge  $e^* := \overrightarrow{\text{end}(c\mathbf{P}_{V(P^{\bar{s}})}(c, P_{cz}))}$  from Lemma 2.32.

We get that:  $|e_j \cap V(P^{\bar{s}})| = 2$ ,  $e_j \perp xP^{\bar{s}}z$ , and there is a  $z$ -tadpole  $T$  in  $P^{\bar{s}} \cup e_j$ . Since  $|e_j \cap V(P^{\bar{s}})| = 2$ , there is exactly one vertex of  $T$  that is not in  $P^{\bar{s}}$ . That vertex is necessarily  $s$ , as pictured on the left of Figure 21: indeed, we know  $s \in V(T)$  by definition of  $s$ , and  $s \notin V(P^{\bar{s}})$  by definition of  $P^{\bar{s}}$ . In particular, since  $c \neq s$  ( $s \in V(D)$  whereas  $c \notin V(D)$ ), we get  $j > 1$ . We know  $(e_1 \cup \dots \cup e_{j-1}) \cap V(P^{\bar{s}}) = \emptyset$  by definition of a projection, therefore  $e_{j-1} \cap e_j = \{s\}$  and  $(e_1, \dots, e_{j-1})$  represents the unique  $cs$ -path in  $P_{cz}$ , which is disjoint from  $P^{\bar{s}}$ .  $\square$

Therefore, the idea of the proof is the following, which is illustrated on the right of Figure 21. We want to show that there exists a  $cz$ -path  $P_{cz}^{\bar{w}}$  in  $D \cup e$  that does not contain  $w$ . Indeed, suppose we manage to exhibit one. On the one hand, following  $P_{cz}^{\bar{w}}$  starting from  $c$  until touching

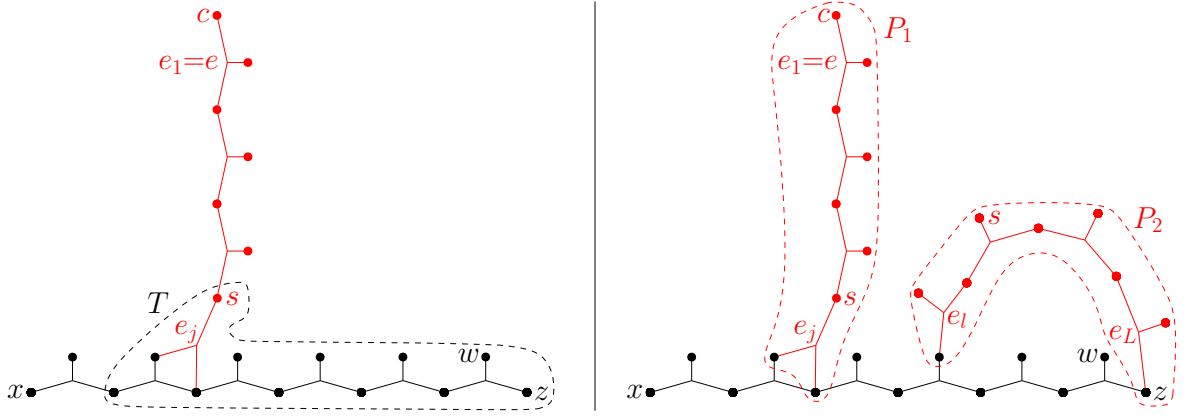


Figure 21: Left: illustration of Claim 6. Right: the desired contradiction, with  $s$  having two different locations at once.

$P^{\bar{s}}$ , we get a path  $P_1$  which contains  $s$  as in Claim 6. On the other hand, following  $P_{cz}^{\bar{w}}$  starting from  $z$  until touching  $P^{\bar{s}}$  again, we get a path  $P_2$  which creates a  $z$ -cycle and thus must also contain  $s$ . This is a contradiction about the location of  $s$ . We now proceed with the proof, in three steps. We prove items (i) and (ii) jointly: there are only two times during the proof where we will have to differentiate the two very briefly to make separate arguments.

1) Firstly: we show there exists a  $cz$ -path  $P_{cz}$  in  $D \cup e$ .

Since  $e \cap V(D) \neq \emptyset$ , there exists  $X \in \mathcal{O}_{x,z}(D)$  such that  $e \cap V(X) \neq \emptyset$ . By definition of  $\mathcal{O}_{x,z}(D)$ , there are three possibilities for  $X$ , and for each of them we can use an adequate union lemma from Section 2:

- Suppose  $X =: T \in z\mathcal{T}(D^{-x})$ . Since there is no  $c$ -tadpole in  $D \cup e \supseteq T \cup e$  by (C), Lemma 2.34 ensures that there is a  $cz$ -path in  $T \cup e$ .
- Suppose  $X =: S \in z\mathcal{S}(D^{-x})$ . We address items (i) and (ii) separately. For (i), there is no  $c$ -snake in  $D \cup e \supseteq S \cup e$  by (C). For (ii), let  $m$  be the marked vertex such that  $S$  is a  $zm$ -snake: since  $J_1(\mathcal{D}_1, H)$  holds, Proposition 3.24 tells us there is no  $m$ -tadpole in  $D \cup e \supseteq S \cup e$ . In both cases, Lemma 2.37 ensures that there is a  $cz$ -path in  $S \cup e$ .
- Suppose  $X =: P \in \mathcal{P}_{zx}(D)$ . Since there is no  $cx$ -path in  $D \cup e \supseteq P \cup e$  by (C), Lemma 2.32 ensures that there is a  $cz$ -path in  $P \cup e$ .

In all cases, we get a  $cz$ -path  $P_{cz}$  in  $D \cup e$ .

2) Secondly: we show there exists a  $cz$ -path  $P_{cz}^{\bar{w}}$  in  $D \cup e$  that does not contain  $w$ .

Recall that, by Claim 6,  $P_{cz}$  contains a  $cs$ -path  $P_{cs}$  such that  $V(P_{cs}) \cap V(P^{\bar{s}}) = \emptyset$ : in particular  $w \notin V(P_{cs})$ . Moreover, since  $w$  is non-marked (otherwise  $P^{\bar{s}}$  would contain an  $x$ -snake, contradicting Proposition 4.1(f)), Proposition 4.1(b) ensures that there exists  $X^{\bar{w}} \in \mathcal{O}_{x,z}(D)$  such that  $w \notin V(X^{\bar{w}})$ . We thus find  $P_{cz}^{\bar{w}}$  inside  $P_{cs} \cup X^{\bar{w}}$ :

- Suppose  $X^{\bar{w}} =: T \in z\mathcal{T}(D^{-x})$ . In particular  $s \in V(T)$ , so  $V(P_{cs}) \cap V(T) \neq \emptyset$ . Since  $D \cup e \supseteq P_{cs} \cup T$  does not contain a  $c$ -tadpole by (C), Lemma 2.34 ensures that  $P_{cs} \cup T$  contains a  $cz$ -path.
- Suppose  $X^{\bar{w}} =: S \in z\mathcal{S}(D^{-x})$ . In particular  $s \in V(S)$ , so  $V(P_{cs}) \cap V(S) \neq \emptyset$ . For the second and last time in this proof, we address items (i) and (ii) separately. For (i), there is no  $c$ -snake in  $D \cup e \supseteq P_{cs} \cup S$  by (C). For (ii), let  $m$  be the marked vertex such that  $S$  is a  $zm$ -snake: Proposition 3.24 tells us there is no  $m$ -tadpole in  $D \cup e \supseteq P_{cs} \cup S$ . In both cases, Lemma 2.37 ensures that there is a  $cz$ -path in  $P_{cs} \cup S$ .

- Suppose  $X^{\bar{w}} =: P \in \mathcal{P}_{zx}(D)$ . Since  $w \notin V(P)$ , we have  $w \notin \text{start}(\overleftarrow{xPz})$  hence  $\text{start}(\overleftarrow{xPz}) \neq \text{start}(\overleftarrow{xP^{\bar{s}}z})$ . By the final assertion of Proposition 4.2, this implies  $s \in V(P)$ , so  $V(P_{cs}) \cap V(P) \neq \emptyset$ . Since  $D \cup e \supseteq P_{cs} \cup P$  does not contain a  $cx$ -path by (C), Lemma 2.32 ensures that  $P_{cs} \cup P$  contains a  $cz$ -path.

In all cases, we get a  $cz$ -path  $P_{cz}^{\bar{w}}$  in  $P_{cs} \cup X^{\bar{w}} \subseteq D \cup e$ , that does not contain  $w$  since neither  $P_{cs}$  nor  $X^{\bar{w}}$  does.

- 3) Finally: we conclude by getting the desired contradiction illustrated on the right of Figure 21. We now work exclusively inside  $P_{cz}^{\bar{w}} \cup P^{\bar{s}}$ .

We start by defining the paths  $P_1$  and  $P_2$  pictured on the right of Figure 21. Define the projection  $P_1 := \mathbf{P}_{V(P^{\bar{s}})}(c, P_{cz}^{\bar{w}})$ . By (C), it is impossible that  $V(P_{cz}^{\bar{w}}) \cap V(P^{\bar{s}}) = \{z\}$ , because  $P_{cz}^{\bar{w}} \cup P^{\bar{s}}$  would then be a  $cx$ -path. Therefore, the projection  $P_2 := \mathbf{P}_{V(P^{\bar{s}}) \setminus \{z\}}(z, P_{cz}^{\bar{w}})$  is also well defined. Write  $\overrightarrow{cP_{cz}^{\bar{w}}z} = (c, e_1, \dots, e_L, z)$ ,  $\overrightarrow{cP_1} = (c, e_1, \dots, e_j)$ , and  $\overrightarrow{zP_2} = (z, e_L, e_{L-1}, \dots, e_l)$ , i.e.  $j = \min\{1 \leq i \leq L, e_i \cap V(P^{\bar{s}}) \neq \emptyset\}$  and  $l = \max\{1 \leq i \leq L, e_i \cap (V(P^{\bar{s}}) \setminus \{z\}) \neq \emptyset\}$ . Note that necessarily  $e_1 = e$ , since  $e$  is the only edge incident to  $c$ .

- First of all, we show that  $1 < j < l$  and that  $s \in e_{j-1}$ . By Claim 6, we have:  $j > 1$ ,  $e_j \perp \overrightarrow{xP^{\bar{s}}z}$ , and  $e_{j-1} \cap e_j = \{s\}$ . Moreover, since  $w \notin V(P_{cz}^{\bar{w}})$ , we have  $w \notin e_j$ : since  $e_j \perp \overrightarrow{xP^{\bar{s}}z}$ , this implies  $z \notin e_j$ . Therefore  $j < L$ , so we can consider the edge  $e_{j+1}$ . Since  $s \in e_{j-1} \cap e_j$ , we have  $s \notin e_{j+1}$ , so  $e_j \cap e_{j+1} \subseteq e_j \setminus \{s\} \subseteq V(P^{\bar{s}}) \setminus \{z\}$ : in particular  $j < l$  by maximality of  $l$ .
- Finally, we show that  $s \in e_i$  for some  $l \leq i \leq L$  i.e.  $s \in V(P_2)$ . Note that  $P_2 \subseteq D$ : indeed, we have  $P_2 \subseteq P_{cz}^{\bar{w}} \subseteq D \cup e$ , and  $e = e_1$  is not an edge of  $P_2$  because  $l \geq 2$ . Since  $P_2 \subseteq P_{cz}^{\bar{w}}$  does not contain  $w$ , we have  $\text{start}(\overrightarrow{zP_2}) \neq \text{start}(\overrightarrow{zP^{\bar{s}}})$ , so we can apply Lemma 2.33. There is no  $x$ -tadpole in  $P_2 \cup P^{\bar{s}} \subseteq D$  by Proposition 4.1(f), so we get a  $z$ -cycle  $C$  in  $P_2 \cup P^{\bar{s}}$ . Since  $C$  must contain  $s$ , we have  $s \in V(P_2) \cup V(P^{\bar{s}})$  hence  $s \in V(P_2)$ .

Since  $j < l$ ,  $e_{j-1}$  is disjoint from  $e_l, \dots, e_L$  by definition of a path. However, we have just shown that  $s \in e_{j-1}$  and  $s \in e_i$  for some  $l \leq i \leq L$ . This is a contradiction.  $\blacksquare$

**Lemma 4.4.** *Let  $H$  be a marked hypergraph that is not a trivial Maker win, and let  $x \in V(H) \setminus M(H)$ . Let  $D$  be a  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -danger at  $x$  in  $H$ , with  $z$  a  $\mathcal{D}_1$ -dangerous vertex in  $(D, x)$  such that  $I_{H+z}(z\mathcal{D}_1(H)) \neq \emptyset$ . Then there is a unique edge  $e_x$  in  $D$  that is incident to  $x$ . Moreover, let  $P_x$  be an  $x$ -path in  $H$  such that  $V(P_x) \cap (V(D) \setminus \{x\}) \neq \emptyset$  and  $\text{start}(\overrightarrow{xP_x}) \neq e_x$ : then  $D \cup P_x$  contains an  $x$ -snake or an  $x$ -tadpole.*

*Proof.* Let  $s \in I_{H+z}(z\mathcal{D}_1(H))$ , and let  $P^{\bar{s}} \in \mathcal{P}_{zx}(D)$  such that  $s \notin V(P^{\bar{s}})$  as per Proposition 4.2. We define  $e_x := \text{start}(\overrightarrow{xP^{\bar{s}}z})$ . We will show at the end of the proof that  $e_x$  is the unique edge of  $D$  containing  $x$ .

For now, let  $P_x$  be an  $x$ -path in  $H$  such that  $V(P_x) \cap (V(D) \setminus \{x\}) \neq \emptyset$  and  $\text{start}(\overrightarrow{xP_x}) \neq e_x$ . Up to replacing  $P_x$  by the projection  $\mathbf{P}_{V(D) \setminus \{x\}}(x, P_x)$ , assume that  $\text{end}(\overrightarrow{xP_x})$  is the only edge of  $P_x$  that intersects  $V(D) \setminus \{x\}$ . Suppose for a contradiction that:

There is no  $x$ -snake and no  $x$ -tadpole in  $D \cup P_x$ . (C)

Let  $e := \text{start}(\overrightarrow{xP_x})$ . We distinguish between two cases.

- (1) Case 1:  $P_x$  is of length at least 2, i.e.  $e \cap V(D) = \{x\}$ .

Write  $e = \{x, a, c\}$  where  $c$  is the only vertex in  $\text{inn}(P_x) \cap e$ , and let  $P_c$  be the  $c$ -path defined

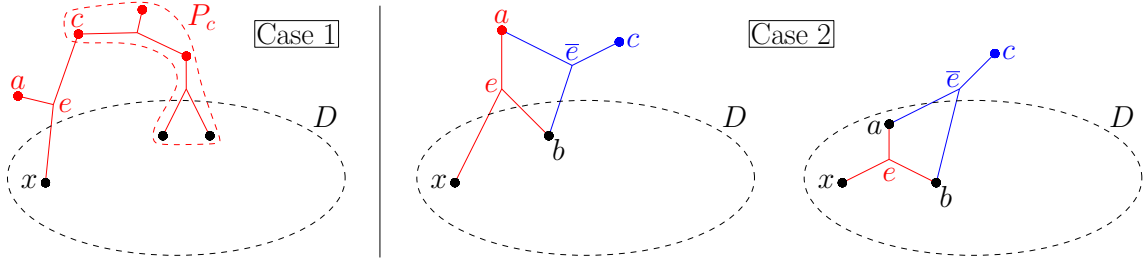


Figure 22: Case 1:  $e \cap V(D) = \{x\}$ . Case 2:  $|e \cap V(D)| = 2$  or  $|e \cap V(D)| = 3$ .

as  $P_c = P_x^{-x-a}$  (see Figure 22). Since  $H$  is not a trivial Maker win and  $I_{H+z}(z\mathcal{D}_1(H)) \neq \emptyset$ , we can apply item (i) of Lemma 4.3 in  $H$  to  $D$  and  $P_c$ , which tells us that  $D \cup P_c$  contains one of the following:

- a  $cx$ -path  $P$ . Then the sequence  $(x, e, c) \oplus \overrightarrow{cPx}$  represents an  $x$ -cycle in  $D \cup P_x$ .
- a  $c$ -tadpole  $T$ . If  $x \in V(T)$ , then  $T$  contains a  $cx$ -path so we simply go back to that case. If  $x \notin V(T)$ , then the sequence  $(x, e, c) \oplus \overrightarrow{cT}$  represents an  $x$ -tadpole in  $D \cup P_x$ .
- a  $c$ -snake  $S$ . If  $x \in V(S)$ , then  $S$  contains a  $cx$ -path so we simply go back to that case. If  $x \notin V(S)$ , then the sequence  $(x, e, c) \oplus \overrightarrow{cS}$  represents an  $x$ -snake in  $D \cup P_x$ .

All three possibilities contradict (C).

(2) Case 2:  $P_x = e$  is of length 1, i.e.  $|e \cap V(D)| \geq 2$ .

Write  $e = \{x, a, b\}$ . As a gadget, we create a new non-marked vertex  $c$  and an edge  $\bar{e} = \{a, b, c\}$  (see Figure 22).

*Claim 7. Let  $X$  be a subhypergraph of  $D \cup \bar{e}$  such that  $\bar{e} \in E(X)$  and  $x \notin V(X)$ , and define the subhypergraph  $\varphi(X)$  of  $D \cup e$  obtained from  $X$  by replacing  $c$  by  $x$  and  $\bar{e}$  by  $e$ . Then we have the isomorphisms of pointed marked hypergraphs:  $(X, c) \sim (\varphi(X), x)$  and  $(X, v) \sim (\varphi(X), v)$  for all  $v \in V(X) \setminus (M(X) \cup \{c\})$ .*

*Proof of Claim 7.* This is straightforward. □

The idea is to apply Lemma 4.3 in  $D \cup \bar{e}$  to  $D$  and  $P_c := \bar{e}$ , and then contradict (C) through replacing  $\bar{e}$  by  $e$  in the obtained subhypergraph as per Claim 7. To do so, we need to check that  $D \cup \bar{e}$  is not a trivial Maker win and that  $I_{(D \cup \bar{e})+z}(z\mathcal{D}_1(D \cup \bar{e})) \neq \emptyset$ . The former is clear: we know  $D \subseteq H$  is not a trivial Maker win, moreover there is no  $x$ -snake in  $P_x$  by (C) so  $M(e) = \emptyset$  hence  $M(\bar{e}) = \emptyset$ , so  $D \cup \bar{e}$  is not a trivial Maker win either.

The latter is more difficult, because the addition of  $\bar{e}$  may create new  $\mathcal{D}_1$ -dangers at  $z$ . However, we now show that they all contain  $s$  i.e.  $s \in I_{(D \cup \bar{e})+z}(z\mathcal{D}_1(D \cup \bar{e}))$ . Indeed, let  $X$  be a  $\mathcal{D}_1$ -danger at  $z$  in  $D \cup \bar{e}$ : we want to show that  $s \in V(X)$ .

- Suppose  $\bar{e} \notin E(X)$ . Then  $X \in z\mathcal{D}_1(H)$ , hence  $s \in V(X)$  by definition of  $s$ .
- Suppose  $\bar{e} \in E(X)$  and  $x \notin V(X)$ . By Claim 7, we have  $(X, z) \sim (\varphi(X), z)$ , therefore  $\varphi(X)$  is a  $\mathcal{D}_1$ -danger at  $z$  in  $D \cup e$  hence  $s \in V(\varphi(X))$ . Since  $s \neq x$  by Proposition 4.2, this yields  $s \in V(X)$ .
- Finally, suppose  $\bar{e} \in E(X)$  and  $x \in V(X)$ . In particular, we have  $c, x \in V(X)$ .
  - If there exists a  $cx$ -path  $P$  in  $X$ , then necessarily  $\text{start}(\overrightarrow{cPx}) = \bar{e}$  since  $\bar{e}$  is the only edge incident to  $c$  in  $D \cup \bar{e}$ . Either  $a$  or  $b$ , say  $b$ , is an inner vertex of  $P$ , so that  $P^{-c-a}$  is a  $bx$ -path in  $D$  that does not contain  $a$ . This means that  $P^{-c-a} \cup e$  is an  $x$ -cycle in  $D \cup e$ , contradicting (C).

- If there is no  $cx$ -path in  $X$ , then the only possibility according to Propositions 2.19 and 2.24 is that  $X =: T$  is a  $z$ -tadpole such that  $C_T$  is of length 2 and  $\text{out}(C_T) = \{c, x\}$  as in Figure 23. Therefore, the edges incident respectively to  $c$  and  $x$  in  $T$  intersect on two vertices. Since the edge incident to  $c$  in  $T$  is necessarily  $\bar{e} = \{a, b, c\}$ , the edge incident to  $x$  in  $T$  is precisely  $\{a, b, x\} = e$ . Define the  $zx$ -path  $P := T^{-c}$ , as in Figure 23. Since  $T \subseteq D \cup \bar{e}$ , we have  $P \subseteq D$  i.e.  $P \in \mathcal{P}_{zx}(D)$ . Moreover  $\text{start}(\overrightarrow{xPz}) = e \neq e_x = \text{start}(\overrightarrow{xP^s z})$  by assumption, so the last assertion of Proposition 4.2 ensures that  $s \in V(P) \subseteq V(X)$ .

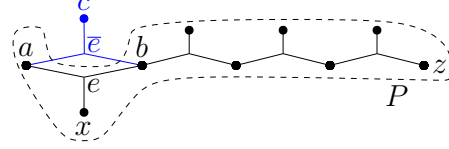


Figure 23: Illustration of  $X = T$  if there is no  $cx$ -path in  $X$ .

Now that we have shown that  $I_{(D \cup \bar{e})+z}(z\mathcal{D}_1(D \cup \bar{e})) \neq \emptyset$ , we can apply Lemma 4.3 in  $D \cup \bar{e}$  to  $D$  and  $P_c = \bar{e}$ , which tells us that  $D \cup \bar{e}$  contains one of the following:

- a  $cx$ -path  $P$ . In this case, taking  $P$  and replacing  $\bar{e}$  by  $e$  yields an  $x$ -cycle. Indeed, write  $\overrightarrow{cPx} = (c, e_1, \dots, e_L, x)$ : since  $\bar{e}$  is the only edge incident to  $c$ , we have  $e_1 = \bar{e}$ , so  $(x, e, e_2, \dots, e_L, x)$  represents an  $x$ -cycle in  $D \cup e$ .
- a  $c$ -tadpole  $T$ . Since  $\bar{e}$  is the only edge incident to  $c$ , we have  $\bar{e} \in E(T)$ . If  $x \in V(T)$ , then  $T$  contains a  $cx$ -path so we simply go back to that case. If  $x \notin V(T)$ , then by Claim 7 we have  $(T, c) \sim (\varphi(T), x)$ , so  $\varphi(T)$  is an  $x$ -tadpole in  $D \cup e$ .
- a  $c$ -snake  $S$ . Since  $\bar{e}$  is the only edge incident to  $c$ , we have  $\bar{e} \in E(S)$ . If  $x \in V(S)$ , then  $S$  contains a  $cx$ -path so we simply go back to that case. If  $x \notin V(S)$ , then by Claim 7 we have  $(S, c) \sim (\varphi(S), x)$ , so  $\varphi(S)$  is an  $x$ -snake in  $D \cup e$ .

All three possibilities thus contradict (C), which concludes the proof of the final assertion of this lemma.

Finally, we prove that  $e_x$  is the only edge of  $D$  that is incident to  $x$ : suppose for a contradiction that there exists  $e'_x \in E(D)$  such that  $x \in e'_x$  and  $e'_x \neq e_x$ . Define  $P_x := e'_x$ : we have  $V(P_x) \cap (V(D) \setminus \{x\}) = e'_x \setminus \{x\} \neq \emptyset$  and  $\text{start}(\overrightarrow{xP_x}) = e'_x \neq e_x$ . Therefore, we can apply what we have shown above to the path  $P_x$ : we get an  $x$ -snake or an  $x$ -tadpole in  $D \cup P_x = D$ , contradicting Proposition 4.1(f). ■

#### 4.2.2 Inside structure

The two previous lemmas are about the union of a  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -danger and a path. We now look at a  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -danger alone. In Figure 18, all featured examples were unions of  $z$ -tadpoles and  $zx$ -paths only, no  $z$ -snakes. Also,  $x$  was of degree 1 in all of them. We can now show these properties hold in all interesting cases:

**Proposition 4.5.** *Let  $(D, x) \in \mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ , with  $\mathcal{D}_1$ -dangerous vertex  $z$ . If  $J_1(\mathcal{D}_1, D)$  holds, then  $M(D) = \emptyset$ . In particular, we have  $\mathcal{O}_{x,z}(D) = z\mathcal{T}(D^{-x}) \cup \mathcal{P}_{zx}(D)$ .*

*Proof.* Suppose for a contradiction that there exists some  $m \in M(D)$ . As a gadget, we add two new non-marked vertices  $a$  and  $c$  as well as a new edge  $\bar{e} = \{a, c, m\}$ . This does not create any new  $\mathcal{D}_1$ -danger at  $z$ : indeed, it is obvious that a  $z$ -snake or a  $z$ -tadpole cannot contain an edge with two non-marked vertices of degree 1 other than  $z$ . For that reason, the fact that  $J_1(\mathcal{D}_1, D)$  holds implies that  $J_1(\mathcal{D}_1, D \cup \bar{e})$  holds as well. Moreover, since  $D$  is not a trivial Maker win

by Proposition 4.1(e),  $D \cup \bar{e}$  is not either. Therefore, item (ii) of Lemma 4.3 applied to  $D$  and  $P_c := \bar{e}$  ensures that  $D \cup \bar{e}$  contains a  $cx$ -path or a  $c$ -tadpole. Since  $\bar{e}$  is the only edge containing  $c$ , it is easy to see by removing  $\bar{e}$  that  $D$  contains an  $m$ -snake or an  $m$ -tadpole respectively (see Figure 24). The former is impossible because an  $m$ -snake in  $D$  is an  $x$ -snake in  $D$ , which cannot exist by Proposition 4.1(f). The latter is impossible by Proposition 3.24. We can conclude that  $M(D) = \emptyset$ , which implies  $z\mathcal{S}(D^{-x}) = \emptyset$  hence the last assertion.  $\blacksquare$

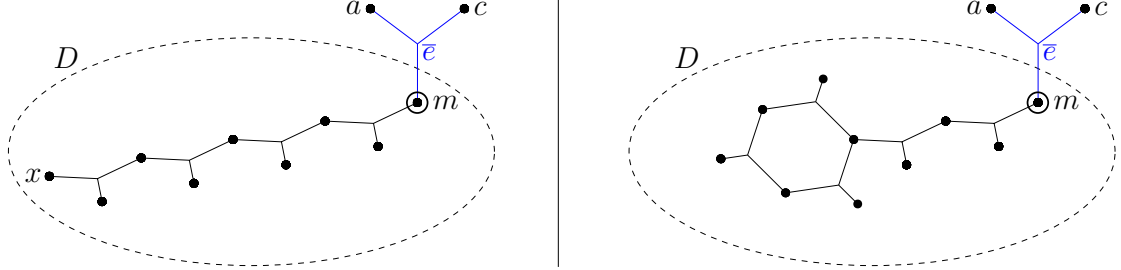


Figure 24: Left: a  $cx$ -path yields an  $xm$ -snake. Right: a  $c$ -tadpole yields an  $m$ -tadpole.

**Proposition 4.6.** *Let  $(D, x) \in \mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ , with  $\mathcal{D}_1$ -dangerous vertex  $z$ . If  $I_{D+z}(z\mathcal{D}_1(D)) \neq \emptyset$ , then  $x$  is of degree 1 in  $D$ .*

*Proof.* This is the first assertion of Lemma 4.4 applied in  $H = D$ .  $\blacksquare$

The next result delves into the inside structure of the  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -dangers with much more precision.

**Proposition 4.7.** *Let  $(D, x) \in \mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ , with  $\mathcal{D}_1$ -dangerous vertex  $z$ . Suppose that  $J_1(\mathcal{D}_1, D)$  holds. Then  $D$  is of at least one of the two following types (see Figure 25):*

- (1)  $D$  contains:
  - a  $z$ -cycle  $C$  such that  $x \notin V(C)$ ;
  - an  $xw$ -path  $P_{xw}$  for some  $w \in \text{out}(C)$  such that  $V(P_{xw}) \cap V(C) = \{w\}$ ;
  - some  $X \in \mathcal{O}_{x,z}(D)$  such that  $V(X) \cap V(P_{xw}) \neq \emptyset$  and  $e \setminus \{w\} \not\subseteq V(X)$  where  $e$  denotes the unique edge of  $C$  containing  $w$ .
- (2)  $D$  contains:
  - a  $z$ -cycle  $C$  such that  $x \notin V(C)$ ;
  - an  $xw$ -path  $\overleftarrow{P_{xw}}$  for some  $w \in V(C)$  such that  $V(P_{xw}) \cap V(C) = \{w, w'\}$  where  $w' := o(w, \overleftarrow{P_{xw}}w)$ .

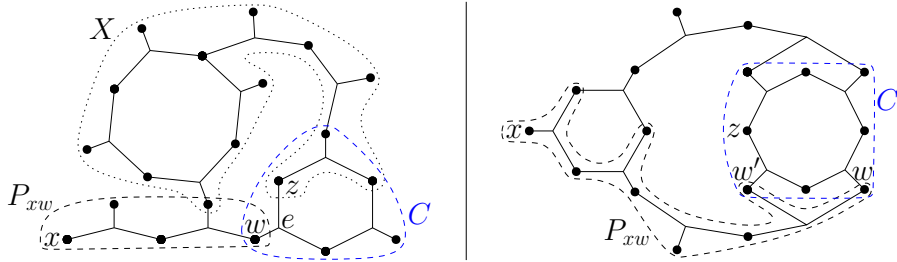


Figure 25: Two  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -dangers. The left one is of type (1) only (same for the other two from Figure 18). The right one is of type (2) only.

*Proof.* Assume that  $D$  is not of type (2): we show that  $D$  is of type (1).

Claim 8. *There exists a pair  $(C, P_{xw})$  where  $C$  is a  $z$ -cycle and  $P_{xw}$  is an  $xw$ -path for some  $w \in \text{out}(C)$  such that  $V(P_{xw}) \cap V(C) = \{w\}$ .*

*Proof of Claim 8.* The existence of  $C$  is given by Proposition 4.1(g). The existence of  $P_{xw}$  is also straightforward:

- Suppose  $x \in V(C)$ . Necessarily  $x \in \text{out}(C)$ , otherwise  $C$  would be an  $x$ -cycle, contradicting Proposition 4.1(f). Therefore, simply take  $w := x$  and  $P_{xw}$  of length 0.
- Suppose  $x \notin V(C)$ . Let  $P \in \mathcal{P}_{zx}(D)$  and define  $P_x := \mathbf{P}_{V(C)}(x, P)$ . By definition of a projection:  $|\text{end}(\overrightarrow{xP_x}) \cap V(C)| \in \{1, 2\}$ . We cannot have  $|\text{end}(\overrightarrow{xP_x}) \cap V(C)| = 2$  because  $D$  would be of type (2), therefore  $|\text{end}(\overrightarrow{xP_x}) \cap V(C)| = 1$ . Let  $w$  be the only vertex in  $\text{end}(\overrightarrow{xP_x}) \cap V(C)$ . Necessarily  $w \in \text{out}(C)$ , otherwise  $P_x \cup C$  would be an  $x$ -tadpole, contradicting Proposition 4.1(f). Take  $P_{xw} := P_x$ .  $\square$

Of all pairs  $(C, P_{xw})$  as in Claim 8, we choose one where  $P_{xw}$  is longest. This choice ensures that:

Claim 9. *For any  $z$ -cycle  $C'$  in  $D$ , we have  $V(C') \cap V(P_{xw}) \neq \emptyset$ .*

*Proof of Claim 9.* Suppose for a contradiction that there exists a  $z$ -cycle  $C'$  such that  $V(C') \cap V(P_{xw}) = \emptyset$ . Since  $z \in V(C') \cap (V(C) \setminus \text{out}(C))$ , the projection  $P := \mathbf{P}_{V(C')}(w, C)$  is well defined, and it is of positive length because  $w \notin V(C')$ . Therefore, the path  $P'_x := [\overrightarrow{xP_{xw}w} \oplus \overrightarrow{wP}]$  is strictly longer than  $P_{xw}$ . For the same reason as  $P_x$  in the proof of Claim 8 above,  $P'_x$  satisfies  $\text{end}(\overrightarrow{xP'_x}) \cap V(C') = \{w'\}$  for some  $w' \in \text{out}(C')$ , and  $P'_x$  is an  $xw'$ -path. The pair  $(C', P'_x)$  thus contradicts the maximality of the length of  $P_{xw}$ .  $\square$

We will show that  $x \notin V(C)$  at the end of the proof. For now, let  $e$  be the only edge of  $C$  containing  $w$ , and let us show the existence of  $X \in \mathcal{O}_{x,z}(D)$  such that  $V(X) \cap V(P_{xw}) \neq \emptyset$  and  $e \setminus \{w\} \not\subseteq V(X)$ .

Let us first address the case  $z \in e$ . Since  $z \in \text{inn}(C)$  and  $w \in \text{out}(C)$ , we have  $z \neq w$ . Let  $v$  be the third vertex of  $e$ , so that  $e = \{w, z, v\}$ . By Proposition 4.1(b), there exists  $X^{\bar{v}} \in \mathcal{O}_{x,z}(D)$  such that  $v \notin V(X^{\bar{v}})$ , which implies  $e \setminus \{w\} \not\subseteq V(X^{\bar{v}})$ . Suppose for a contradiction that  $V(X^{\bar{v}}) \cap V(P_{xw}) = \emptyset$ . In particular  $X^{\bar{v}}$  is not a  $zx$ -path. We also know  $X^{\bar{v}}$  is not a  $z$ -snake by Proposition 4.5, so  $X^{\bar{v}} =: T$  is a  $z$ -tadpole. Since  $V(T) \cap (V(P_{xw}) \cup e) = \{z\}$ , the sequence  $\overrightarrow{xP_{xw}w} \oplus (w, e, z) \oplus \overrightarrow{zT}$  represents an  $x$ -tadpole in  $D$ , contradicting Proposition 4.1(f). In conclusion, we have  $V(X^{\bar{v}}) \cap V(P_{xw}) \neq \emptyset$ .

We can now assume that  $z \notin e$ . Write  $\overrightarrow{zC} = (z, e_1, \dots, e_L, z)$ : we have  $e = e_i$  for some  $1 \leq i \leq L$ . Actually, since  $z \notin e$ , we have  $L \geq 3$  and  $2 \leq i \leq L - 1$ . We can thus define  $w_1$  (resp.  $w_2$ ) as the only vertex in  $e_{i-1} \cap e_i$  (resp. in  $e_i \cap e_{i+1}$ ), and we have  $e = \{w, w_1, w_2\}$ . Therefore,  $P_1 := [(z, e_1, \dots, e_{i-1}, w_1)]$  is a  $zw_1$ -path and  $P_2 := [(z, e_L, e_{L-1}, \dots, e_{i+1}, w_2)]$  is a  $zw_2$ -path. These notations are summed up in Figure 26.

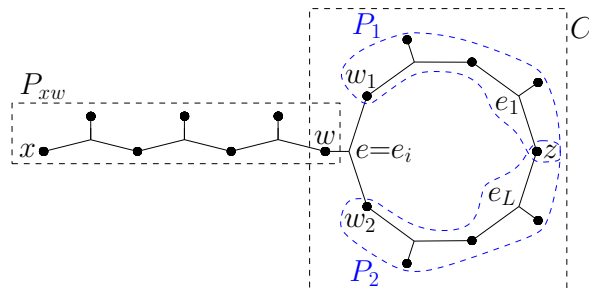


Figure 26: Summary of the notations in place.



Since  $z \notin e$ , we have  $w_1, w_2 \neq z$ . By Proposition 4.1(b), for all  $j \in \{1, 2\}$ , there exists  $X^{\overline{w_j}} \in \mathcal{O}_{x,z}(D)$  such that  $w_j \notin V(X^{\overline{w_j}})$ , which implies  $e \setminus \{w\} \not\subseteq V(X^{\overline{w_j}})$ . We choose  $X^{\overline{w_1}} = X^{\overline{w_2}}$  if possible i.e. if there exists an element of  $\mathcal{O}_{x,z}(D)$  containing neither  $w_1$  nor  $w_2$ . Suppose for a contradiction that  $V(X^{\overline{w_1}}) \cap V(P_{xw}) = \emptyset$  and  $V(X^{\overline{w_2}}) \cap V(P_{xw}) = \emptyset$ .

In particular,  $X^{\overline{w_1}}$  and  $X^{\overline{w_2}}$  are not  $zx$ -paths, moreover they are not  $z$ -snakes by Proposition 4.5 and they are not  $z$ -cycles by Claim 9. Therefore,  $X^{\overline{w_1}} =: T^{\overline{w_1}}$  and  $X^{\overline{w_2}} =: T^{\overline{w_2}}$  are  $z$ -tadpoles that are not cycles. We distinguish between two cases, obtaining a contradiction for both.

- First case:  $e_L \notin E(T^{\overline{w_1}})$  or  $e_1 \notin E(T^{\overline{w_2}})$ . By symmetry, assume that  $e_L \notin E(T^{\overline{w_1}})$ . It is impossible that  $V(T^{\overline{w_1}}) \cap V(P_2) = \{z\}$ , otherwise we would have  $V(T^{\overline{w_1}}) \cap (V(P_{xw}) \cup e \cup V(P_2)) = \{z\}$  so the sequence  $\overrightarrow{xP_{xw}w} \oplus (w, e, w_2) \oplus \overrightarrow{w_2P_2z} \oplus \overrightarrow{zT^{\overline{w_1}}}$  would represent an  $x$ -tadpole in  $D$ , contradicting Proposition 4.1(f). Therefore, the projection  $P^{\overline{w_1}} := \mathbf{P}_{V(P_2) \setminus \{z\}}(z, T^{\overline{w_1}})$  is well defined. Since  $P^{\overline{w_1}} \subseteq T^{\overline{w_1}}$ , we have  $V(P^{\overline{w_1}}) \cap (V(P_{xw}) \cup \{w_1\}) = \emptyset$  and  $e_L \notin E(P^{\overline{w_1}})$ . In particular  $\text{start}(\overrightarrow{zP^{\overline{w_1}}}) \neq e_L = \text{start}(\overrightarrow{zP_2w_2})$ , so we can apply Lemma 2.33. Since  $P_2 \cup P^{\overline{w_1}}$  cannot contain a  $z$ -cycle by Claim 9, it contains a  $w_2$ -tadpole  $T$ . We have  $V(T) \subseteq V(P_2) \cup V(P^{\overline{w_1}})$  hence  $V(T) \cap (V(P_{xw}) \cup e) = \{w_2\}$ , so the sequence  $\overrightarrow{xP_{xw}w} \oplus (w, e, w_2) \oplus \overrightarrow{w_2T}$  represents an  $x$ -tadpole in  $D$ , contradicting Proposition 4.1(f).
- Second case:  $e_L \in E(T^{\overline{w_1}})$  and  $e_1 \in E(T^{\overline{w_2}})$ . Since  $T^{\overline{w_1}}$  and  $T^{\overline{w_2}}$  are not cycles,  $z$  is of degree 1 in both of them, hence  $e_1 \notin E(T^{\overline{w_1}})$  and  $e_L \notin E(T^{\overline{w_2}})$ . Since  $e_1 \in E(T^{\overline{w_2}})$  and  $e_1 \notin E(T^{\overline{w_1}})$ , we have  $T^{\overline{w_1}} \neq T^{\overline{w_2}}$ , so our initial choice of  $T^{\overline{w_1}}$  and  $T^{\overline{w_2}}$  ensures that  $w_2 \in V(T^{\overline{w_1}})$  and  $w_1 \in V(T^{\overline{w_2}})$ .

- Firstly, suppose that  $w_2 \notin \text{out}(C_{T^{\overline{w_1}}})$  or  $w_1 \notin \text{out}(C_{T^{\overline{w_2}}})$ . By symmetry, assume that  $w_2 \notin \text{out}(C_{T^{\overline{w_1}}})$ . By Proposition 2.25,  $T^{\overline{w_1}}$  contains a  $w_2$ -tadpole  $T$ . Since  $V(T) \cap (V(P_{xw}) \cup e) = \{w_2\}$ , the sequence  $\overrightarrow{xP_{xw}w} \oplus (w, e, w_2) \oplus \overrightarrow{w_2T}$  represents an  $x$ -tadpole in  $D$ , which contradicts Proposition 4.1(f).
- Finally, suppose that  $w_2 \in \text{out}(C_{T^{\overline{w_1}}})$  and  $w_1 \in \text{out}(C_{T^{\overline{w_2}}})$ . Since  $J_1(\mathcal{D}_1, D)$  holds, there exists  $s \in I_{D+z}(z\mathcal{D}_1(D))$ . In particular,  $s \in V(T^{\overline{w_1}}) \cap V(C)$ . Since  $w \notin V(T^{\overline{w_1}})$ , we have  $s \neq w$ , hence  $s \in V(P_1)$  or  $s \in V(P_2)$ . By symmetry, assume  $s \in V(P_1)$ . In particular,  $s \neq w_2$ : by Proposition 2.23, the fact that  $w_2 \in \text{out}(C_{T^{\overline{w_1}}})$  thus ensures the existence of a  $zs$ -path  $P_{zs}^{\overline{w_2}}$  in  $T^{\overline{w_1}}$  that does not contain  $w_2$ . Since  $e_1 \notin E(T^{\overline{w_1}})$ , we have  $e_1 \notin E(P_{zs}^{\overline{w_2}})$ . Therefore  $\text{start}(\overrightarrow{zP_{zs}^{\overline{w_2}}}) \neq e_1 = \text{start}(\overrightarrow{zP_1w_1})$ , moreover  $V(P_{zs}^{\overline{w_2}}) \cap (V(P_1) \setminus \{z\}) \supseteq \{s\} \neq \emptyset$  so we can apply Lemma 2.33: since  $P_1 \cup P_{zs}^{\overline{w_2}}$  cannot contain a  $z$ -cycle by Claim 9, it contains a  $w_1$ -tadpole  $T$ . We have  $V(T) \subseteq V(P_1) \cup V(P_{zs}^{\overline{w_2}})$  hence  $V(T) \cap (V(P_{xw}) \cup e) = \{w_1\}$ , so the sequence  $\overrightarrow{xP_{xw}w} \oplus (w, e, w_1) \oplus \overrightarrow{w_1T}$  represents an  $x$ -tadpole in  $D$ , contradicting Proposition 4.1(f).

In conclusion, we have shown the existence of  $X \in \mathcal{O}_{x,z}(D)$  such that  $V(X) \cap V(P_{xw}) \neq \emptyset$  and  $e \setminus \{w\} \not\subseteq V(X)$ . To prove that  $D$  is of type (1), it only remains to show that  $x \notin V(C)$ . Suppose for a contradiction that  $x \in V(C)$  i.e.  $x = w$  i.e.  $V(P_{xw}) = \{x\}$ . Since  $V(X) \cap V(P_{xw}) \neq \emptyset$  by definition of  $X$ , we get  $x \in V(X)$ , so there exists an edge  $e'$  of  $X$  that is incident to  $x$ . Moreover,  $e$  is also incident to  $w = x$ . Since  $e \notin E(X)$  by definition of  $X$ , we have  $e' \neq e$ . Therefore,  $e$  and  $e'$  are two distinct edges of  $D$  that are incident to  $x$ , contradicting Proposition 4.6. This ends the proof.  $\blacksquare$

## 5 Proofs of the main results

### 5.1 Structural results

Theorems 3.15 and 3.16 can be deduced relatively easily from the following intermediate result, which we will prove in this section:

**Theorem 5.1.** *Let  $H$  be a 3-uniform marked hypergraph that is not a trivial Maker win, with  $|V(H) \setminus M(H)| \geq 2$ . Suppose that  $J_1(\mathcal{D}_1, H)$  holds. Then, for any  $x \in V(H) \setminus M(H)$  such that there exists an  $x$ -snake in  $H$ , we have  $I_{H+x}(x\mathcal{D}_2(H)) \neq \emptyset$ .*

In other words, under the assumptions of Theorem 5.1, if  $x$  sees a marked vertex then not only do its  $\mathcal{D}_1$ -dangers intersect (which is a given, since  $J_1(\mathcal{D}_1, H)$  holds) but actually even its  $\mathcal{D}_2$ -dangers do.

#### 5.1.1 Proof of Theorems 3.15 and 3.16 assuming Theorem 5.1

As announced in Section 3, we actually prove Theorems 3.19 and 3.20, which use the approximations  $\mathcal{D}_2$  and  $\mathcal{D}_1$  respectively. Theorems 3.15 and 3.16 then follow as corollaries.

*Proof of Theorem 3.19 assuming Theorem 5.1.* Item (i) is a direct consequence of Proposition 1.25. We now show item (ii) by induction on  $|V(H) \setminus M(H)|$ .

Let us start with the base case  $|V(H) \setminus M(H)| \in \{2, 3\}$ . Let  $x_1 \in V(H) \setminus M(H)$  and  $y_1 \in I_{H+x_1}(x_1\mathcal{D}_2(H))$ , which exists since  $J_1(\mathcal{D}_2, H)$  holds. The trivial danger of size 3 is in  $\mathcal{S} \subseteq \mathcal{D}_2$ , therefore all trivial dangers at  $x_1$  in  $H$  contain  $y_1$ , so the fact that  $H$  is not a trivial Maker win implies that  $H^{+x_1-y_1}$  is not a trivial Maker win either. Since  $|V(H^{+x_1-y_1}) \setminus M(H^{+x_1-y_1})| \leq 1$ , this means  $H^{+x_1-y_1}$  is a Breaker win, so  $H$  is a Breaker win.

For the induction step, assume  $|V(H) \setminus M(H)| \geq 4$  and the implication to be true for marked hypergraphs with less non-marked vertices than  $H$ . Let  $x_1 \in V(H) \setminus M(H)$  and  $y_1 \in I_{H+x_1}(x_1\mathcal{D}_2(H))$ , which exists since  $J_1(\mathcal{D}_2, H)$  holds: we must show that  $H^{+x_1-y_1}$  is a Breaker win. Let us first list a few important properties of  $H^{+x_1-y_1}$ :

- (a)  $|V(H^{+x_1-y_1}) \setminus M(H^{+x_1-y_1})| = |V(H) \setminus M(H)| - 2 \geq 2$ .
- (b)  $H^{+x_1-y_1}$  is not a trivial Maker win. Indeed, the trivial danger of size 3 is in  $\mathcal{S} \subseteq \mathcal{D}_2$ , therefore all trivial dangers at  $x_1$  in  $H$  contain  $y_1$ , so the fact that  $H$  is not a trivial Maker win implies that  $H^{+x_1-y_1}$  is not a trivial Maker win either.
- (c)  $J_1(\mathcal{D}_1, H^{+x_1-y_1})$  holds. Indeed,  $J_1(\mathcal{D}_1, H)$  holds because  $\mathcal{D}_1 \subseteq \mathcal{D}_2$  and  $J_1(\mathcal{D}_2, H)$  holds. Besides, since  $\mathcal{D}_2 = \mathcal{D}_1 \cup \mathcal{D}_1^{\mathfrak{O}, \text{rest}}$  by definition, we have  $I_{H+x_1}(x_1\mathcal{D}_2(H)) = I_{H+x_1}(x_1\mathcal{D}_1^*(H))$  by Proposition 1.42 hence  $y_1 \in I_{H+x_1}(x_1\mathcal{D}_1^*(H)) \subseteq I_{H+x_1}(x_1\mathcal{D}_1^{\mathfrak{O}}(H))$ . Therefore, Proposition 1.37 with  $\mathcal{F} = \mathcal{D}_1$  ensures that  $J_1(\mathcal{D}_1, H^{+x_1-y_1})$  holds.

Thanks to (a) and (b), checking that property  $J_1(\mathcal{D}_2, H^{+x_1-y_1})$  holds is sufficient to prove that  $H^{+x_1-y_1}$  is a Breaker win, according to the induction hypothesis. Let  $x \in V(H^{+x_1-y_1}) \setminus M(H^{+x_1-y_1})$ : we want to show that  $I_{(H^{+x_1-y_1})+x}(x\mathcal{D}_2(H^{+x_1-y_1})) \neq \emptyset$ . Assume that there exists some  $D_0 \in x\mathcal{D}_2(H^{+x_1-y_1})$ , otherwise  $I_{(H^{+x_1-y_1})+x}(x\mathcal{D}_2(H^{+x_1-y_1})) = I_{(H^{+x_1-y_1})+x}(\emptyset) = V((H^{+x_1-y_1})^{+x}) \setminus M((H^{+x_1-y_1})^{+x}) \neq \emptyset$  trivially since  $|V(H) \setminus M(H)| \geq 4$ .

- 1) First case: there is no  $xx_1$ -snake in  $H^{+x_1-y_1}$ .

What happens here is that any vertex that hit all the  $\mathcal{D}_2$ -dangers at  $x$  in  $H$  still works in  $H^{+x_1-y_1}$ , because the marking of  $x_1$  has not created any new  $\mathcal{D}_2$ -danger at  $x$ . Indeed, for all  $D \in x\mathcal{D}_2(H^{+x_1-y_1})$  (recall that  $\mathcal{D}_2 = \mathcal{S} \cup \mathcal{T} \cup \mathcal{D}_1^{\mathfrak{O}, \text{rest}}$  by definition):

- If  $(D, x) \in \mathcal{S}$ , then  $x_1 \notin V(D)$  since we are assuming that there is no  $xx_1$ -snake in  $H^{+x_1-y_1}$ .

- If  $(D, x) \in \mathcal{T}$ , then  $x_1 \notin V(D)$  since  $M(D) = \emptyset$  by definition of the family  $\mathcal{T}$ .
- If  $(D, x) \in \mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ , then  $x_1 \notin V(D)$  since  $M(D) = \emptyset$  by Proposition 4.5 (which (c) allows us to use).

Therefore, we have  $x\mathcal{D}_2(H^{+x_1-y_1}) \subseteq x\mathcal{D}_2(H^{-x_1-y_1}) \subseteq x\mathcal{D}_2(H)$ . Now, let  $y \in I_{H+x}(x\mathcal{D}_2(H))$ . To show that  $y \in I_{(H^{+x_1-y_1})+x}(x\mathcal{D}_2(H^{+x_1-y_1}))$ , since  $x\mathcal{D}_2(H^{+x_1-y_1}) \subseteq x\mathcal{D}_2(H)$ , it suffices to check that  $y \notin \{x_1, y_1\}$ . For this, we use  $D_0$ . On the one hand, we have  $D_0 \in x\mathcal{D}_2(H^{+x_1-y_1}) \subseteq x\mathcal{D}_2(H)$  hence  $y \in V(D_0)$ . On the other hand, we have  $D_0 \in x\mathcal{D}_2(H^{+x_1-y_1}) \subseteq x\mathcal{D}_2(H^{-x_1-y_1})$  hence  $x_1, y_1 \notin V(D_0)$ . In conclusion, we do have  $y \notin \{x_1, y_1\}$ , so  $y \in I_{(H^{+x_1-y_1})+x}(x\mathcal{D}_2(H^{+x_1-y_1}))$  hence  $I_{(H^{+x_1-y_1})+x}(x\mathcal{D}_2(H^{+x_1-y_1})) \neq \emptyset$ .

2) Second case: there is an  $xx_1$ -snake in  $H^{+x_1-y_1}$ .

Here, we have the  $x$ -snake that is necessary to apply Theorem 5.1 to  $H^{+x_1-y_1}$ . The other assumptions of this theorem are also verified thanks to (a), (b) and (c). In conclusion, Theorem 5.1 applies and yields  $I_{(H^{+x_1-y_1})+x}(x\mathcal{D}_2(H^{+x_1-y_1})) \neq \emptyset$  as desired. ■

*Proof of Theorem 3.20 assuming Theorem 5.1.* If  $H$  is a Breaker win then  $J_1(\mathcal{D}_1, H)$  holds by Proposition 1.30. Now assume that  $J_1(\mathcal{D}_1, H)$  holds. We claim that, actually,  $J_1(\mathcal{D}_2, H)$  holds: indeed, for all  $x \in V(H) \setminus M(H)$ , there exists an  $x$ -snake in  $H$  by assumption hence  $I_{H+x}(x\mathcal{D}_2(H)) \neq \emptyset$  by Theorem 5.1. Therefore,  $H$  is a Breaker win according to Theorem 3.19. ■

*Proof of Theorem 3.15 assuming Theorem 5.1.* Item (i) is a direct consequence of Proposition 1.25. Item (ii) follows from Theorem 3.19: indeed, since  $\mathcal{D}_2 \subseteq \mathcal{D}_0^{*2}$ ,  $J_1(\mathcal{D}_0^{*2}, H)$  implies  $J_1(\mathcal{D}_2, H)$  and  $I_{H+x_1}(x_1\mathcal{D}_2(H)) \supseteq I_{H+x_1}(x_1\mathcal{D}_0^{*2}(H))$ . As for the equivalence between  $J_1(\mathcal{D}_0^{*2}, H)$  and  $J_3(\mathcal{D}_0, H)$ , it is given by Proposition 1.40. Finally, the ultimate assertion of Theorem 3.15 is simply Proposition 3.14 with  $r = 3$ . ■

*Proof of Theorem 3.16 assuming Theorem 5.1.* The "only if" direction is given by Proposition 1.30. The "if" direction follows from Theorem 3.20 since  $J_1(\mathcal{D}_0^*, H)$  implies  $J_1(\mathcal{D}_1, H)$ . Finally, the ultimate assertion of Theorem 3.16 is simply Proposition 3.14 with  $r = 2$ . ■

### 5.1.2 Proof of Theorem 5.1

As we have just seen, all structural results will be proved once Theorem 5.1 is. Let  $H$  be a 3-uniform marked hypergraph that is not a trivial Maker win, and suppose that  $J_1(\mathcal{D}_1, H)$  holds. Let  $x \in V(H) \setminus M(H)$  and  $m \in M(H)$  such that there exists an  $xm$ -snake in  $H$ : we want to find some  $y \in I_{H+x}(x\mathcal{D}_2(H)) = I_{H+x}(x(\mathcal{D}_1 \cup \mathcal{D}_1^{\mathfrak{O}, \text{rest}})(H))$ . Since  $J_1(\mathcal{D}_1, H)$  holds, we already know that  $I_{H+x}(x\mathcal{D}_1(H)) \neq \emptyset$ , however picking an arbitrary  $y \in I_{H+x}(x\mathcal{D}_1(H))$  does not work in general, as shown in Figure 27. In this example, we can see that  $H$  satisfies the conditions of Theorem 5.1, and that the only  $\mathcal{D}_1$ -dangers at  $x$  in  $H$  are two  $xm$ -snakes whose intersection  $I_{H+x}(x\mathcal{D}_1(H))$  is represented by the vertices in square boxes. We can see that several of them are not in  $I_{H+x}(x\mathcal{D}_2(H))$ , because they miss  $D$  which is a  $\mathcal{D}_1^{\mathfrak{O}, \text{rest}}$ -danger at  $x$ : this is the case for the vertex  $y'$  for instance.

This inspires us to choose  $y \in I_{H+x}(x\mathcal{D}_1(H))$  that maximizes  $\text{dist}_H(y, m)$ , as in Figure 27 for example. We now fix a such  $y$ , and we suppose for a contradiction that  $y \notin I_{H+x}(x\mathcal{D}_2(H))$ : there exists  $D \in x\mathcal{D}_1^{\mathfrak{O}, \text{rest}}(H)$  such that  $y \notin V(D)$ . Let  $z$  be a  $\mathcal{D}_1$ -dangerous vertex in  $(D, x)$ .

## I. Preliminary statements

Since  $H$  is not a trivial Maker win and  $J_1(\mathcal{D}_1, H)$  holds, all results from Section 4 apply to  $D$ . In particular:

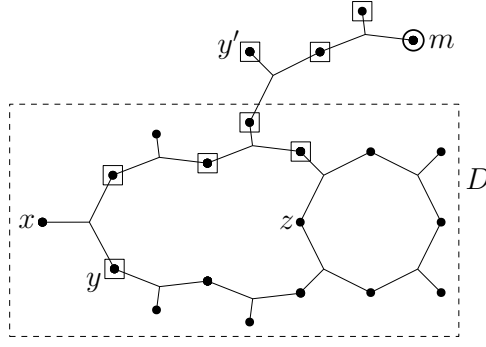


Figure 27: In this example, we have  $y \in I_{H+x}(x\mathcal{D}_2(H))$ , but  $y' \in I_{H+x}(x\mathcal{D}_1(H)) \setminus I_{H+x}(x\mathcal{D}_2(H))$  since  $y' \notin V(D)$ .

**Proposition 5.2.**  $D$  has the following properties:

- $M(D) = \emptyset$ . In particular,  $m \notin V(D)$ .
- $\mathcal{O}_{x,z}(D) = z\mathcal{T}(D^{-x}) \cup \mathcal{P}_{zx}(D)$ .
- There is exactly one edge of  $D$  that is incident to  $x$ : we call it  $e_x$ .

*Proof.* This is given by Propositions 4.5 and 4.6. ■

The next two properties can be summed up as follows:

- When following a path starting from  $m$ , we cannot enter  $D$  strictly before encountering  $y$ .
- When following a path starting from  $x$  by an edge other than  $e_x$ , we cannot re-enter  $D$  strictly before encountering  $y$ .

**Proposition 5.3.** Any  $m$ -path  $P_m$  in  $H$  such that  $V(P_m) \cap V(D) \neq \emptyset$  contains  $y$ .

*Proof.* Since  $J_1(\mathcal{D}_1, H)$  holds and  $m \notin V(D)$ , Lemma 4.3 with  $c = m$  ensures that  $D \cup P_m$  contains an  $m$ -tadpole or an  $mx$ -path (i.e. an  $xm$ -snake). There cannot be an  $m$ -tadpole in  $H$  according to Proposition 3.24, therefore  $D \cup P_m$  contains an  $xm$ -snake. Since  $y \in I_{H+x}(x\mathcal{D}_1(H))$ , that  $xm$ -snake must contain  $y$ , moreover  $y \notin V(D)$  by assumption so  $y \in V(P_m)$ . ■

**Proposition 5.4.** Any  $x$ -path  $P_x$  in  $H$  such that  $\text{start}(x\overrightarrow{P_x}) \neq e_x$  and  $V(P_x) \cap (V(D) \setminus \{x\}) \neq \emptyset$  contains  $y$ .

*Proof.* Since  $J_1(\mathcal{D}_1, H)$  holds, we have  $I_{H+z}(z\mathcal{D}_1(H)) \neq \emptyset$ , so Lemma 4.4 ensures that  $D \cup P_x$  contains an  $x$ -tadpole or an  $x$ -snake. In both cases, it contains  $y$ , moreover  $y \notin V(D)$  by assumption, so  $y \in V(P_x)$ . ■

We now state a useful preliminary lemma:

**Lemma 5.5.** Any  $v \in V(D) \setminus \{x\}$  satisfies  $\text{dist}_H(v, m) \geq \text{dist}_H(y, m)$ , moreover:

- If  $\text{dist}_H(v, m) > \text{dist}_H(y, m)$ , then there exists an  $xm$ -snake  $S_{xm}^{\overleftarrow{v}}$  in  $H$  that does not contain  $v$ .
- If  $\text{dist}_H(v, m) = \text{dist}_H(y, m)$ , then any shortest  $vm$ -snake  $S_{vm}$  in  $H$  satisfies  $V(S_{vm}) \cap V(D) = \{v\}$  and  $o(v, v\overrightarrow{S_{vm}}m) = y$ , moreover there is no  $v$ -tadpole in  $D$ .

*Proof.* The fact that  $\text{dist}_H(v, m) \geq \text{dist}_H(y, m)$  is a direct consequence of Proposition 5.3: since  $v \in V(D)$ , any  $vm$ -snake in  $H$  contains  $y$ .

- Suppose  $\text{dist}_H(v, m) > \text{dist}_H(y, m)$ .  
Let  $S_{ym}$  be a shortest  $ym$ -snake in  $H$ : note that there does exist one, since there exists

an  $xm$ -snake by assumption, which must contain  $y$  and therefore contains a  $ym$ -snake by Proposition 2.19. Since  $S_{ym}$  is shortest and  $\text{dist}_H(v, m) > \text{dist}_H(y, m)$ , we have  $v \notin V(S_{ym})$ .

We necessarily have  $v \notin I_{H+x}(x\mathcal{D}_1(H))$ , otherwise the fact that  $\text{dist}_H(v, m) > \text{dist}_H(y, m)$  would contradict our choice of  $y$ . Since  $v$  is non-marked (recall that  $M(D) = \emptyset$ ) and distinct from  $x$ , this means there exists some  $X \in x\mathcal{D}_1(H)$  such that  $v \notin V(X)$ . On the other hand, since  $y \in I_{H+x}(x\mathcal{D}_1(H))$ , we have  $y \in V(X)$  hence  $y \in V(X) \cap V(S_{ym})$ . This allows us to use the adequate union lemma in  $X \cup S_{ym}$  to find the desired  $xm$ -snake  $S_{xm}^{\bar{v}}$  (ensuring that  $v \notin V(S_{xm}^{\bar{v}})$  since  $v \notin V(X) \cup V(S_{ym})$ ):

- Suppose  $X =: S \in x\mathcal{S}(H)$ . If the marked vertex of  $S$  is  $m$ , then  $S_{xm}^{\bar{v}} := S$  is the desired  $xm$ -snake. Otherwise  $m \notin V(S)$ , so apply Lemma 2.37 with  $a = x$ ,  $S_{ab} = S$ ,  $c = m$  and  $P_c = S_{ym}$ . Since  $S \cup S_{ym} \subseteq H$  cannot contain an  $m$ -snake by Proposition 3.24, it contains an  $mx$ -path i.e. an  $xm$ -snake  $S_{xm}^{\bar{v}}$ .
  - Suppose  $X =: T \in x\mathcal{T}(H)$ . We have  $M(T) = \emptyset$  hence  $m \notin V(T)$ , so apply Lemma 2.34 with  $a = x$ ,  $c = m$  and  $P_c = S_{ym}$ . Since  $T \cup S_{ym} \subseteq H$  cannot contain an  $m$ -tadpole by Proposition 3.24, it contains an  $mx$ -path i.e. an  $xm$ -snake  $S_{xm}^{\bar{v}}$ .
- Suppose  $\text{dist}_H(v, m) = \text{dist}_H(y, m)$ .

Let  $S_{vm}$  be a shortest  $vm$ -snake in  $H$ . Since  $v \in V(D)$ , we have  $y \in V(S_{vm})$  by Proposition 5.3. If  $y \neq o(v, \overrightarrow{vS_{vm}m})$  (see Figure 28, top), then  $S_{vm}$  contains a  $ym$ -snake that is shorter than  $S_{vm}$ : this is impossible since  $\text{dist}_H(v, m) = \text{dist}_H(y, m)$  and  $S_{vm}$  has been chosen shortest. Therefore  $y = o(v, \overrightarrow{vS_{vm}m})$  (see Figure 28, bottom). The  $m$ -path  $S_{vm}^{-y-v}$  does not contain  $y$  and thus contains no vertex in  $D$  by Proposition 5.3, hence why  $V(S_{vm}) \cap V(D) = \{v\}$ . Finally, there cannot be a  $v$ -tadpole  $T$  in  $D$ , because  $S_{vm} \cup T$  would then an  $m$ -tadpole in  $H$  since  $V(S_{vm}) \cap V(T) = \{v\}$ , contradicting Proposition 3.24. ■

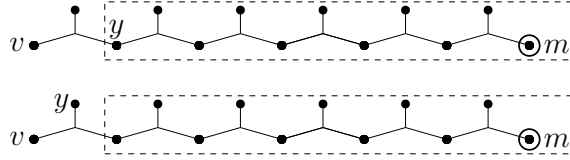


Figure 28: The snake  $S_{vm}$  if  $y \neq o(v, \overrightarrow{vS_{vm}m})$  (top, the contradictory  $ym$ -snake is highlighted) or if  $y = o(v, \overrightarrow{vS_{vm}m})$  (bottom, the snake  $S_{vm}^{-y-v}$  is highlighted).

As we have often done in Section 4, we fix a vertex  $s \in I_{H+z}(z\mathcal{D}_1(H))$ , given by Proposition 4.2 (which also tells us that  $s \in V(D)$ ). Before we engage in the core of the proof, let us summarize the objects involved and some of their basic properties that will be used thereafter, with Table 5:

## II. Roadmap of the proof

The idea of the proof is to eventually exhibit a vertex  $w \in V(D)$  such that  $I_{H+w}(w\mathcal{D}_1(H)) = \emptyset$ . Indeed, this is a contradiction since  $J_1(\mathcal{D}_1, H)$  holds. The roadmap to achieve this is given by the following result, which we will prove in this paragraph:

**Proposition 5.6.** *Let  $w \in V(D) \setminus \{x\}$ . Suppose that  $D$  contains the following three subhypergraphs:*

- (i) a  $z$ -cycle  $C$  containing  $w$ ;
- (ii) an  $xw$ -path  $P_{xw}$  such that  $V(P_{xw}) \cap \text{inn}(C) \subseteq \{w\}$ ;
- (iii) a  $w$ -tadpole that does not contain  $s$ .

$H$	<ul style="list-style-type: none"> <li>· not a trivial Maker win</li> <li>· <math>J_1(\mathcal{D}_1, H)</math> holds</li> </ul>	
$x$	<ul style="list-style-type: none"> <li>· <math>x \in V(H) \setminus M(H)</math></li> </ul>	
$D$	<ul style="list-style-type: none"> <li>· <math>\mathcal{D}_1^{\mathfrak{O}, \text{rest}}</math>-danger at <math>x</math> in <math>H</math></li> <li>· <math>\mathcal{O}_{x,z}(D) = z\mathcal{T}(D^{-x}) \cup \mathcal{P}_{zx}(D)</math></li> </ul>	
$z$	<ul style="list-style-type: none"> <li>· <math>\mathcal{D}_1</math>-dangerous vertex in <math>(D, x)</math></li> </ul>	
$m$	<ul style="list-style-type: none"> <li>· <math>m \in M(H)</math></li> <li>· <math>m \notin V(D)</math></li> </ul>	
$y$	<ul style="list-style-type: none"> <li>· <math>y \in I_{H+x}(x\mathcal{D}_1(H))</math></li> <li>· <math>y \notin V(D)</math></li> </ul>	
$s$	<ul style="list-style-type: none"> <li>· <math>s \in I_{H+z}(z\mathcal{D}_1(H))</math></li> <li>· <math>s \in V(D)</math></li> </ul>	
$e_x$	<ul style="list-style-type: none"> <li>· unique edge adjacent to <math>x</math> in <math>D</math></li> </ul>	

Table 5: The objects involved and some of their properties.

Then  $I_{H+w}(w\mathcal{D}_1(H)) = \emptyset$ .

Showing that  $I_{H+w}(w\mathcal{D}_1(H)) = \emptyset$  in the proof of Proposition 5.6 will require the ability, for every non-marked vertex  $v \neq w$ , to exhibit a  $\mathcal{D}_1$ -danger at  $w$  that does not contain  $v$ . The following lemma applied to  $d = w$  gives us that object under certain conditions.

**Lemma 5.7.** *Let  $d, v \in V(D) \setminus \{x\}$ . Suppose that  $\text{dist}_H(v, m) > \text{dist}_H(y, m)$  and that there exists a  $dx$ -path  $P_{dx}^v$  in  $D$  that does not contain  $v$ . Then there exists a  $dm$ -snake in  $H$  that does not contain  $v$ .*

*Proof.* Suppose for a contradiction that:

$$\text{All } dm\text{-snakes in } H \text{ contain } v. \quad (\text{C})$$

We are going to exhibit an  $m$ -tadpole in  $H$ , contradicting Proposition 3.24. This  $m$ -tadpole will be obtained inside the union of an  $xy$ -path and an  $xm$ -snake having specific properties, whose existence is given by the following two claims which we prove independently from each other. Define  $t := o(x, x\overrightarrow{P_{dx}^v}d)$ , and note that  $\text{start}(x\overrightarrow{P_{dx}^v}d) = e_x$  since  $e_x$  is the only edge adjacent to  $x$  in  $D$ .

Claim 10. *There exists an  $xy$ -path  $P_{xy}^v$  in  $H$  such that:*

- $V(P_{xy}^v) \subseteq V(P_{dx}^v) \cup \{y\}$ .
- $o(x, x\overrightarrow{P_{xy}^v}y) = t$ .

*Proof of Claim 10.* We have  $\text{dist}_H(v, m) > \text{dist}_H(y, m)$  by assumption, so by Lemma 5.5 there exists an  $xm$ -snake  $S_{xm}^v$  in  $H$  that does not contain  $v$ . Since  $P_{dx}^v \subseteq D$  and  $m \notin V(D)$ , the edge  $e^* := \text{end}(m\overrightarrow{\mathbf{P}_{V(P_{dx}^v)}}(m, S_{xm}^v))$  is well defined. According to (C), there is no  $dm$ -snake in  $P_{dx}^v \cup S_{xm}^v$ . Therefore, by Lemma 2.32 applied to  $a = d, b = x, c = m, P_{ab} = P_{dx}^v$  and  $P_c = S_{xm}^v$ :  $|e^* \cap V(P_{dx}^v)| = 2$ ,  $e^* \perp x\overrightarrow{P_{dx}^v}d$ , moreover there is an  $x$ -tadpole  $T$  in  $P_{dx}^v \cup e^*$ . Since  $|e^* \cap V(P_{dx}^v)| = 2$ , there is exactly one vertex of  $T$  that is not in  $P_{dx}^v$ . That vertex is necessarily  $y$ , as illustrated in Figure 29: indeed, we know  $y \in V(T)$  because  $y \in I_{H+x}(x\mathcal{D}_1(H))$ , and  $y \notin V(D) \supseteq V(P_{dx}^v)$ . By Proposition 2.24,  $T$  contains an  $xy$ -path  $P_{xy}^v$ , and we have  $V(P_{xy}^v) \subseteq V(T) \subseteq V(P_{dx}^v) \cup \{y\}$ . See Figure 29.

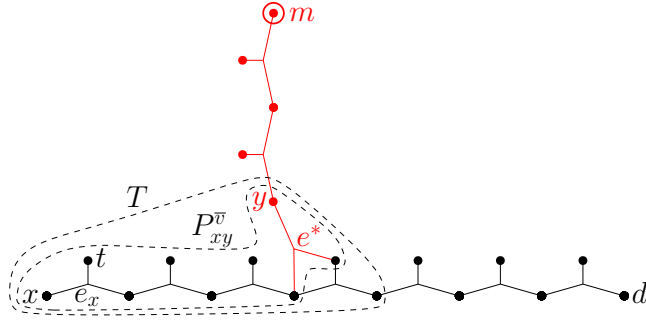


Figure 29: Definition of  $P_{xy}^{\bar{v}}$ . The represented paths are  $P_{dx}^{\bar{v}}$  (horizontal) and  $\mathbf{P}_{V(P_{dx}^{\bar{v}})}(m, S_{xm}^{\bar{v}})$  (vertical).

Finally, let us check that  $o(x, \overrightarrow{xP_{xy}^{\bar{v}}y}) = t$ . Since  $|e^* \cap V(P_{dx}^{\bar{v}})| = 2$  and  $e^* \perp \overleftarrow{xP_{dx}^{\bar{v}}d}$ , there are two possibilities:

- First possibility:  $\{x, t\} \subseteq e^*$ . Then  $P_{xy}^{\bar{v}}$  consists of the single edge  $e^* = \{x, t, y\}$ , so obviously  $o(x, \overrightarrow{xP_{xy}^{\bar{v}}y}) = t$ .
- Second possibility:  $\{x, t\} \cap e^* = \emptyset$ . Then  $\text{start}(\overrightarrow{xP_{xy}^{\bar{v}}y}) \neq e^*$ , so  $\text{start}(\overrightarrow{xP_{xy}^{\bar{v}}y}) = \text{start}(\overrightarrow{xP_{dx}^{\bar{v}}d}) = e_x \ni t$ . Moreover  $t$  is of degree 1 in  $P_{dx}^{\bar{v}} \cup e^* \supseteq P_{xy}^{\bar{v}}$ , so necessarily  $t \notin \text{inn}(P_{xy}^{\bar{v}})$  hence  $t = o(x, \overrightarrow{xP_{xy}^{\bar{v}}y})$ .  $\square$

Claim 11. *There exists an  $xm$ -snake  $S_{xm}^{\bar{t}}$  in  $H$  such that:*

- $t \notin V(S_{xm}^{\bar{t}})$ .
- $y \in \text{inn}(S_{xm}^{\bar{t}})$ . (We define  $P_{xy}^{\bar{t}}$ , resp.  $S_{ym}^{\bar{t}}$ , as the unique  $xy$ -path, resp.  $ym$ -snake, in  $S_{xm}^{\bar{t}}$ .)
- $(V(S_{xm}^{\bar{t}}) \cap V(P_{dx}^{\bar{v}})) \setminus \{x\} \subseteq \{u\}$  where  $u := o(y, \overleftarrow{xP_{xy}^{\bar{t}}y})$ .

*Proof of Claim 11.* It is impossible that  $\text{dist}_H(t, m) = \text{dist}_H(y, m)$ : indeed, a shortest  $tm$ -snake  $S_{tm}$  in  $H$  would then satisfy  $V(S_{tm}) \cap V(D) = \{t\}$  by Lemma 5.5, hence  $v \notin V(S_{tm})$ , so the sequence  $d\overrightarrow{P_{dx}^{\bar{v}}t} \oplus \overrightarrow{tS_{tm}m}$  would represent a  $dm$ -snake in  $H$  that does not contain  $v$ , contradicting (C). Therefore, Lemma 5.5 ensures that  $\text{dist}_H(t, m) > \text{dist}_H(y, m)$ , and that there exists an  $xm$ -snake  $S_{xm}^{\bar{t}}$  in  $H$  such that  $t \notin V(S_{xm}^{\bar{t}})$ . Since  $y \in I_{H+x}(x\mathcal{D}_1(H))$ , we obviously have  $y \in V(S_{xm}^{\bar{t}})$ .

Write  $x\overrightarrow{S_{xm}^{\bar{t}}m} = (x, e_1, \dots, e_L, m)$ . Recalling Notation 2.4, write  $x\overrightarrow{S_{xm}^{\bar{t}}m}|_{\{y\}} = (x, e_1, \dots, e_i)$  and  $\overleftarrow{xS_{xm}^{\bar{t}}m}|_{\{y\}} = (m, e_L, e_{L-1}, \dots, e_j)$ . Note that  $j = i + 1$  if  $y \in \text{inn}(S_{xm}^{\bar{t}})$  and  $j = i$  otherwise.

- By definition of the sequence  $x\overrightarrow{S_{xm}^{\bar{t}}m}|_{\{y\}}$ , the  $x$ -path  $[(x, e_1, \dots, e_{i-1})]$  does not contain  $y$ . Moreover  $e_1 \neq e_x$  because  $t \notin V(S_{xm}^{\bar{t}}) \supseteq e_1$ . Therefore,  $(e_1 \cup \dots \cup e_{i-1}) \cap (V(D) \setminus \{x\}) = \emptyset$  by Proposition 5.3.
- By definition of the sequence  $\overleftarrow{xS_{xm}^{\bar{t}}m}|_{\{y\}}$ , the  $m$ -path  $[(m, e_L, e_{L-1}, \dots, e_{j+1})]$  does not contain  $y$ . Therefore,  $(e_{j+1} \cup \dots \cup e_L) \cap V(D) = \emptyset$  by Proposition 5.4.

Suppose that  $y \notin \text{inn}(S_{xm}^{\bar{t}})$ : then  $i = j$  hence  $V(S_{xm}^{\bar{t}}) = e_1 \cup \dots \cup e_{i-1} \cup \{y\} \cup e_{j+1} \cup \dots \cup e_L$ . By the above, this yields  $V(S_{xm}^{\bar{t}}) \cap V(D) = \{x\}$  and in particular  $v \notin V(S_{xm}^{\bar{t}})$ , therefore the sequence  $d\overrightarrow{P_{dx}^{\bar{v}}x} \oplus \overrightarrow{xS_{xm}^{\bar{t}}m}$  represents a  $dm$ -snake that does contain  $v$ . This contradicts (C). Therefore, we have  $y \in \text{inn}(S_{xm}^{\bar{t}})$ . Let  $P_{xy}^{\bar{t}}$  (resp.  $S_{ym}^{\bar{t}}$ ) be the unique  $xy$ -path (resp.  $ym$ -snake) in  $S_{xm}^{\bar{t}}$ , and define  $u := o(y, \overleftarrow{xP_{xy}^{\bar{t}}y})$  and  $u' := o(y, \overleftarrow{yS_{ym}^{\bar{t}}m})$ , as in Figure 30. Since  $y \in \text{inn}(S_{xm}^{\bar{t}})$ , we have  $j = i + 1$  hence  $V(S_{xm}^{\bar{t}}) = e_1 \cup \dots \cup e_{i-1} \cup \{u, y, u'\} \cup e_{j+1} \cup \dots \cup e_L$ . By the above, this yields  $(V(S_{xm}^{\bar{t}}) \cap V(D)) \setminus \{x\} \subseteq \{u, u'\}$ , hence  $(V(S_{xm}^{\bar{t}}) \cap V(P_{dx}^{\bar{v}})) \setminus \{x\} \subseteq \{u, u'\}$ .

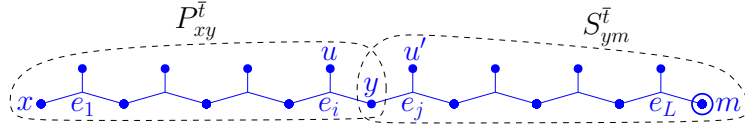


Figure 30: The  $xm$ -snake  $S_{xm}^{\bar{t}}$ .

Finally, it is impossible that  $u' \in V(P_{dx}^{\bar{v}})$ : indeed, this would imply that  $V(S_{ym}^{\bar{t}}) \cap V(P_{dx}^{\bar{v}}) = \{u'\}$  and that  $u' \neq v$  hence  $v \notin V(S_{ym}^{\bar{t}})$ , so the sequence  $\overrightarrow{dP_{dx}^{\bar{v}}x|_{\{u'\}}} \oplus \overrightarrow{u'S_{ym}^{\bar{t}}m}$  would represent a  $dm$ -snake not containing  $v$ , contradicting (C). Therefore  $(V(S_{xm}^{\bar{t}}) \cap V(P_{dx}^{\bar{v}})) \setminus \{x\} \subseteq \{u\}$ , which concludes the proof of the claim.  $\square$

We can now conclude the proof of this lemma by exhibiting an  $m$ -tadpole in  $H$ , which contradicts Proposition 3.24 since  $H$  is not a trivial Maker win and  $J_1(\mathcal{D}_1, H)$  holds.

Let  $P_{xy}^{\bar{v}}$  be as in Claim 10, and let  $S_{xm}^{\bar{t}}$ ,  $P_{xy}^{\bar{t}}$ ,  $S_{ym}^{\bar{t}}$ ,  $u$  be as in Claim 11. We have  $V(P_{xy}^{\bar{v}}) \subseteq V(P_{dx}^{\bar{v}}) \cup \{y\}$  by Claim 10, and  $(V(S_{xm}^{\bar{t}}) \cap V(P_{dx}^{\bar{v}})) \setminus \{x\} \subseteq \{u\}$  by Claim 11: therefore,  $V(S_{xm}^{\bar{t}}) \cap V(P_{xy}^{\bar{v}}) = \{x, y\}$  or  $V(S_{xm}^{\bar{t}}) \cap V(P_{xy}^{\bar{v}}) = \{x, y, u\}$ .

- Case 1:  $V(S_{xm}^{\bar{t}}) \cap V(P_{xy}^{\bar{v}}) = \{x, y\}$ . The sequence  $yS_{ym}^{\bar{t}}m \oplus yP_{xy}^{\bar{t}}x \oplus xP_{xy}^{\bar{v}}y$  clearly represents an  $m$ -tadpole (see Figure 31, top).

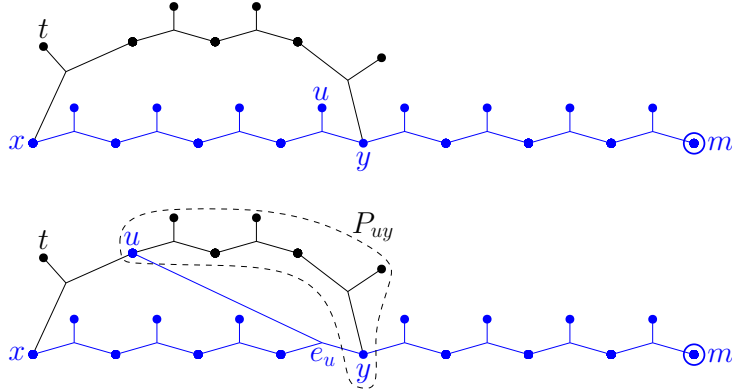


Figure 31: Conclusion of Lemma 5.7. The represented paths are  $S_{xm}^{\bar{t}}$  (horizontal) and  $P_{xy}^{\bar{v}}$ . Top: Case 1. Bottom: Case 2.

- Case 2:  $V(S_{xm}^{\bar{t}}) \cap V(P_{xy}^{\bar{v}}) = \{x, y, u\}$ . Let  $e_u$  be the edge of  $S_{xm}^{\bar{t}}$  containing  $u$ , and let  $P_{uy}$  be the unique  $uy$ -path in  $P_{xy}^{\bar{v}}$ . Since  $u \neq t$ , we have  $x \notin V(P_{uy})$ , therefore the sequence  $\overleftarrow{yS_{ym}^{\bar{t}}m} \oplus (y, e_u, u) \oplus \overrightarrow{uP_{uy}y}$  represents an  $m$ -tadpole (see Figure 31, bottom).  $\blacksquare$

**Corollary 5.8.** *We have  $\text{dist}_H(s, m) = \text{dist}_H(y, m)$ .*

*Proof.* Since  $s \in I_{H+z}(z\mathcal{D}_1(H))$ , there can be no  $zm$ -snake in  $H$  that does not contain  $s$ . Therefore, we can apply the contrapositive of Lemma 5.7 to  $d = z$  and  $v = s$ , which tells us that  $\text{dist}_H(s, m) \leq \text{dist}_H(y, m)$  or there is no  $zx$ -path in  $D$  that does not contain  $s$ . We know the latter is false: such a path  $P^s$  is given by Proposition 4.2. Therefore, the conclusion is that  $\text{dist}_H(s, m) \leq \text{dist}_H(y, m)$  hence  $\text{dist}_H(s, m) = \text{dist}_H(y, m)$  by Lemma 5.5.  $\blacksquare$

The previous corollary has a simple consequence which we will use extensively:

**Proposition 5.9.** *There is no  $s$ -tadpole in  $D$ . In particular, any  $z$ -tadpole  $T$  in  $D$  satisfies  $s \in \text{out}(C_T)$ .*



*Proof.* We have  $\text{dist}_H(s, m) = \text{dist}_H(y, m)$  by Corollary 5.8, so there is no  $s$ -tadpole in  $D$  according to Lemma 5.5. Let  $T$  be a  $z$ -tadpole in  $D$ : we know  $s \in V(T)$  by definition of  $s$ . If we had  $s \notin \text{out}(C_T)$ , then there would be an  $s$ -tadpole in  $T \subseteq D$  by Proposition 2.25, therefore  $s \in \text{out}(C_T)$ . ■

For example, one application of the previous proposition is the following:

**Proposition 5.10.** *There is no  $z$ -cycle of length 2 in  $D$ .*

*Proof.* Suppose for a contradiction that there exists a  $z$ -cycle  $C$  of length 2 in  $D$ . We have  $s \in \text{out}(C)$  by Proposition 5.9: write  $V(C) = \{z, s, u, v\}$  and  $E(C) = \{e_1, e_2\}$  where  $e_1 = \{z, u, s\}$  and  $e_2 = \{z, u, v\}$ . By Proposition 4.1(b), we know there exists some  $X^{\bar{u}} \in \mathcal{O}_{x,z}(D)$  such that  $u \notin V(X^{\bar{u}})$ .

- First suppose  $s \in V(X^{\bar{u}})$ . By Proposition 2.19 (if  $X^{\bar{u}}$  is a  $zx$ -path) or Proposition 2.24 (if  $X^{\bar{u}}$  is a  $z$ -tadpole), there exists a  $zs$ -path  $P_{zs}$  in  $X^{\bar{u}}$ . Since  $u \notin V(P_{zs})$ , we get an  $s$ -cycle  $P_{zs} \cup e_1$  in  $D$ , contradicting Proposition 5.9.
- Now suppose  $s \notin V(X^{\bar{u}})$ . Since  $s \in I_{H+z}(z\mathcal{D}_1(H))$ , this implies that  $X^{\bar{u}} =: P$  is a  $zx$ -path. If  $v \notin V(P)$ , then  $V(C) \cap V(P) = \{z\}$ , therefore  $P \cup C$  is an  $x$ -tadpole in  $D$ , contradicting Proposition 4.1(f). If  $v \in V(P)$ , then there exists a  $zv$ -path  $P_{zv}$  in  $P$  by Proposition 2.19, and we get a  $z$ -cycle  $P_{zv} \cup e_2$  in  $D$  that does not contain  $s$ , also a contradiction. ■

We can now prove Proposition 5.6.

*Proof of Proposition 5.6.* Let  $w \in V(D) \setminus \{x\}$  such that  $D$  contains the following three subgraphs:

- (i) a  $z$ -cycle  $C$  containing  $w$ ;
- (ii) an  $xw$ -path  $P_{xw}$  such that  $V(P_{xw}) \cap \text{inn}(C) \subseteq \{w\}$ ;
- (iii) a  $w$ -tadpole  $T$  that does not contain  $s$ .

We are going to consider  $C$ ,  $P_{xw}$  and  $T$  successively. Each of these three objects will imply the existence of some  $\mathcal{D}_1$ -dangers at  $w$ , which will improve our upper bound on  $I_{H+w}(w\mathcal{D}_1(H))$  until we get the desired conclusion that  $I_{H+w}(w\mathcal{D}_1(H)) = \emptyset$ .

- 1) Step 1: we show that  $I_{H+w}(w\mathcal{D}_1(H)) \subseteq \text{inn}(C) \cup \{s\} \cup (V(H) \setminus V(D))$ .

In this step, we use  $C$ . Recall that  $s \in \text{out}(C)$  by Proposition 5.9 and that  $C$  is of length at least 3 by Proposition 5.10.

Claim 12. *We have  $I_{H+w}(w\mathcal{D}_1(H)) \subseteq I_{C+w}(\mathcal{P}_{ws}(C)) \cup (V(H) \setminus V(D))$ .*

*Proof of Claim 12.* We know  $\text{dist}_H(s, m) = \text{dist}_H(y, m)$  by Corollary 5.8. Let  $S_{sm}$  be a shortest  $sm$ -snake in  $H$ : Lemma 5.5 thus ensures that  $V(S_{sm}) \cap V(D) = \{s\}$  hence  $V(S_{sm}) \cap V(C) = \{s\}$ . Therefore, any  $ws$ -path  $P_{ws}$  in  $C$  yields a  $wm$ -snake  $S_{wm} := P_{ws} \cup S_{sm}$  in  $H$  and:

$$\begin{aligned} I_{H+w}(w\mathcal{D}_1(H)) &\subseteq V(S_{wm}) \setminus \{w\} \\ &\subseteq (V(P_{ws}) \setminus \{w\}) \cup (V(S_{sm}) \setminus \{s\}) \\ &\subseteq I_{C+w}(\mathcal{P}_{ws}(C)) \cup (V(H) \setminus V(D)). \end{aligned} \quad \square$$

Using Claim 12, it suffices to show that  $I_{C+w}(\mathcal{P}_{ws}(C)) \subseteq \text{inn}(C) \cup \{s\}$ . This is a straightforward fact that actually holds in general for any two vertices  $w, s$  in a cycle  $C$ , but we do give a rigorous proof of it using sequences:

- Suppose  $w \in \text{inn}(C)$ , and write  $\overrightarrow{wC} = (w, e_1, \dots, e_L, w)$ . Let  $1 \leq i \leq L$  be the unique index such that  $s \in e_i$ . See Figure 32 (left).
  - If  $i \in \{1, L\}$ , then  $w \in e_i$ , so  $e_i$  is a  $ws$ -path of length 1 in  $C$  hence  $I_{C+w}(\mathcal{P}_{ws}(C)) \subseteq e_i \setminus \{w\} \subseteq \text{inn}(C) \cup \{s\}$ .
  - If  $i \notin \{1, L\}$ , then  $P_1 := [(w, e_1, \dots, e_i)]$  and  $P_2 := [(w, e_L, e_{L-1}, \dots, e_i)]$  are two  $ws$ -paths in  $C$ , so  $I_{C+w}(\mathcal{P}_{ws}(C)) \subseteq (V(P_1) \cap V(P_2)) \setminus \{w\} = e_i \subseteq \text{inn}(C) \cup \{s\}$ .
- Suppose  $w \in \text{out}(C)$ , and let  $e$  be the only edge of  $C$  containing  $w$ . Write  $e = \{w, w_1, w_2\}$  (we have  $w_1, w_2 \in \text{inn}(C)$ ) and  $\overleftarrow{w_1C} = (w_1, e_1, \dots, e_L, w_1)$ . We have  $e \in \{e_1, e_L\}$ : without loss of generality, assume  $e = e_1$ . Since  $L \geq 3$ , we have  $e_1 \cap e_2 = \{w_2\}$  and  $e_1 \cap e_L = \{w_1\}$ . Let  $1 \leq i \leq L$  be the unique index such that  $s \in e_i$ . See Figure 32 (right).
  - If  $i = 1$  then  $w = s$ , so  $[(w)]$  is a  $ws$ -path of length 0 in  $C$ , hence  $I_{C+w}(\mathcal{P}_{ws}(C)) = \emptyset \subseteq \text{inn}(C) \cup \{s\}$ .
  - If  $i \in \{2, L\}$ , then  $[(w, e_1, e_i, s)]$  is a  $ws$ -path of length 2 in  $C$  (because  $L \geq 3$ ), so  $I_{C+w}(\mathcal{P}_{ws}(C)) \subseteq (e_1 \cup e_i) \setminus \{w\} \subseteq \text{inn}(C) \cup \{s\}$ .
  - If  $3 \leq i \leq L - 1$ , then  $C$  contains two  $ws$ -paths  $P_1 := [(w, e_1, e_2, \dots, e_i)]$  and  $P_2 := [(w, e_1, e_L, e_{L-1}, \dots, e_i)]$ , so  $I_{C+w}(\mathcal{P}_{ws}(C)) \subseteq (V(P_1) \cap V(P_2)) \setminus \{w\} = (e_1 \cup e_i) \setminus \{w\} \subseteq \text{inn}(C) \cup \{s\}$ .

We have  $I_{C+w}(\mathcal{P}_{ws}(C)) \subseteq \text{inn}(C) \cup \{s\}$  in all cases, so this concludes Step 1.

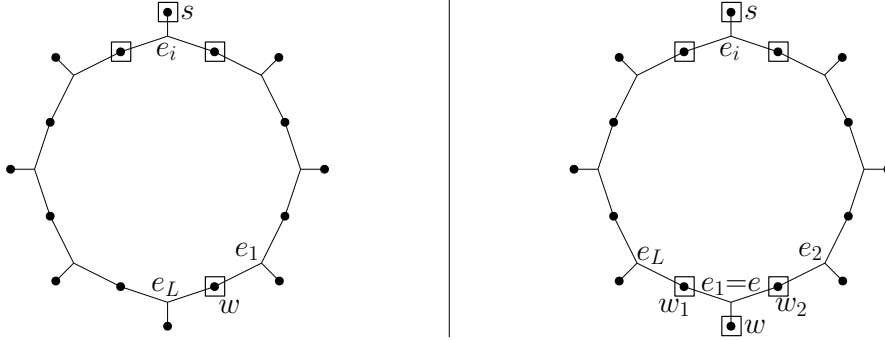


Figure 32: The cycle  $C$  if  $w \in \text{inn}(C)$  (left) or  $w \in \text{out}(C)$  (right). The vertices in square boxes represent the intersection of the  $ws$ -paths in  $C$ .

2) Step 2: we show that  $I_{H+w}(w\mathcal{D}_1(H)) \subseteq \{s\} \cup (V(H) \setminus V(D))$ .

In this step, we use  $P_{xw}$ . Comparing with Step 1, we need to show that  $I_{H+w}(w\mathcal{D}_1(H))$  is disjoint from  $\text{inn}(C)$ . Let  $v \in \text{inn}(C)$ . If  $v = w$ , then obviously  $v \notin I_{H+w}(w\mathcal{D}_1(H))$ , so assume  $v \neq w$ . By definition of  $P_{xw}$ , we then have  $v \notin V(P_{xw})$ , therefore Lemma 5.7 applies with:  $d = w$ , our vertex  $v$ , and  $P_{dx} = P_{xw}$ . We get a  $wm$ -snake in  $H$  that does not contain  $v$ , hence  $v \notin I_{H+w}(w\mathcal{D}_1(H))$ , which concludes Step 2.

3) Step 3: we show that  $I_{H+w}(w\mathcal{D}_1(H)) = \emptyset$ .

In this step, we use  $T$ . We already know that  $I_{H+w}(w\mathcal{D}_1(H)) \subseteq \{s\} \cup (V(H) \setminus V(D))$ . Moreover,  $I_{H+w}(w\mathcal{D}_1(H)) \subseteq V(T)$  because  $T$  is a  $w$ -tadpole, where  $V(T)$  is disjoint from  $\{s\} \cup (V(H) \setminus V(D))$  by definition. In conclusion,  $I_{H+w}(w\mathcal{D}_1(H)) = \emptyset$ . ■

Our goal is now to show that, for a suitable vertex  $w$ ,  $D$  contains all three subhypergraphs listed in Proposition 5.6. A lot of the work has already been done through Proposition 4.7: we now separate the case where  $D$  is of type (1) from the case where  $D$  is of type (2).

### III. Finishing the proof when $D$ is of type (2)

We first suppose that  $D$  is of type (2). By definition (recall Proposition 4.7), this means  $D$  contains a  $z$ -cycle  $C$  such that  $x \notin V(C)$  as well as an  $xw$ -path  $P_{xw}$ , for some  $w \in V(C)$ , such that  $V(P_{xw}) \cap V(C) = \{w, w'\}$  where  $w' := o(w, \overleftarrow{xP_{xw}w})$ .

Recall that  $C$  is of length at least 3 by Proposition 5.10, and that  $s \in \text{out}(C)$  by Proposition 5.9. Define  $e^* := \text{end}(\overrightarrow{xP_{xw}w})$ .

Note that  $s \notin \{w, w'\}$ : indeed, let  $P$  be a  $ww'$ -path in  $C$  (which exists by Proposition 2.22), if we had  $s \in \{w, w'\}$  then  $P \cup e^*$  would be an  $s$ -cycle in  $D$ , contradicting Proposition 5.9. Therefore, since  $s \in \text{out}(C)$ , Proposition 2.21 ensures that there exists a unique  $ww'$ -path  $P_{ww'}$  in  $C$  that does not contain  $s$ .

Define  $C' := P_{ww'} \cup e^*$ :  $C'$  is both a  $w$ -cycle and a  $w'$ -cycle in  $D$ , and it does not contain  $s$ . Moreover, we have  $z \notin V(P_{ww'})$ : indeed, we would otherwise have  $z \in \text{inn}(C) \cap V(P_{ww'}) = \{w, w'\} \cup \text{inn}(P_{ww'}) = \text{inn}(C')$ , so  $C'$  would be a  $z$ -cycle not containing  $s$ , contradicting Proposition 5.9. See Figure 33.

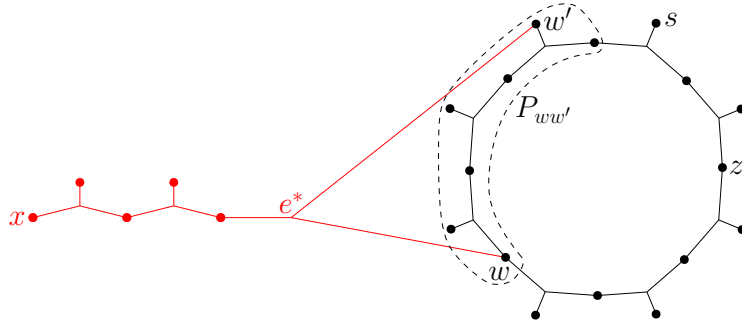


Figure 33: The cycle  $C$  (on the far right) and the path  $P_{xw}$ . In this drawing, we have  $w \in \text{inn}(C)$  and  $w' \in \text{out}(C)$ .

Claim 13.  $w \in \text{out}(C)$  or  $w' \in \text{out}(C)$ .

*Proof of Claim 13.* Suppose for a contradiction that  $w, w' \in \text{inn}(C)$ . Write  $\overrightarrow{zC} = (z, e_1, \dots, e_L, z)$ . Since  $L \geq 3$  and  $w, w' \in \text{inn}(C) \setminus \{z\}$ , there exist  $1 \leq i \neq i' \leq L - 1$  such that  $e_i \cap e_{i+1} = \{w\}$  and  $e_{i'} \cap e_{i'+1} = \{w'\}$ . Since  $w$  and  $w'$  have symmetrical roles, assume  $i < i'$ . Let  $1 \leq j \leq L$  be the unique index such that  $s \in e_j$ .

Since  $e_1$  and  $e_L$  are the only edges of  $C$  containing  $z$ , the  $ww'$ -path represented by the sequence  $(w, e_{i+1}, \dots, e_{i'}, w')$  does not contain  $z$ , so it is necessarily  $P_{ww'}$  according to the unicity statement of Proposition 2.21. Since  $s \notin V(P_{ww'})$ , this yields  $1 \leq j \leq i$  or  $i' + 1 \leq j \leq L$ : by symmetry, assume  $i' + 1 \leq j \leq L$ . Then  $(z, e_1, \dots, e_i, w) \oplus \overrightarrow{wP_{ww'}w'} \oplus (w', e^*, w)$  represents a  $z$ -tadpole not containing  $e_j$  i.e. not containing  $s$ , a contradiction which concludes the proof of the claim.  $\square$

Using Claim 13, assume  $w' \in \text{out}(C)$  by symmetry. This ensures that  $V(P_{xw}) \cap \text{inn}(C) \subseteq \{w\}$ . In conclusion, we can apply Proposition 5.6 to the vertex  $w$ , with: the  $z$ -cycle  $C$  containing  $w$ , the  $xw$ -path  $P_{xw}$  which satisfies  $V(P_{xw}) \cap \text{inn}(C) \subseteq \{w\}$ , and the  $w$ -cycle  $C'$  which does not contain  $s$ . We get  $I_{H+w}(w\mathcal{D}_1(H)) = \emptyset$ , contradicting property  $J_1(\mathcal{D}_1, H)$ . This ends the proof of Theorem 5.1 when  $D$  is of type (2).

### IV. Finishing the proof when $D$ is of type (1)

We now suppose that  $D$  is of type (1). By definition (see Proposition 4.7), this means  $D$  contains:

- a  $z$ -cycle  $C$  such that  $x \notin V(C)$ ;
- an  $xw$ -path  $P_{xw}$  for some  $w \in \text{out}(C)$  such that  $V(P_{xw}) \cap V(C) = \{w\}$ ;
- some  $X \in \mathcal{O}_{x,z}(D)$  such that  $V(X) \cap V(P_{xw}) \neq \emptyset$  and  $\{w_1, w_2\} \not\subseteq V(X)$  where  $e = \{w, w_1, w_2\}$  denotes the unique edge of  $C$  containing  $w$ .

Recall that  $C$  is of length at least 3 by Proposition 5.10 and that  $s \in \text{out}(C)$  by Proposition 5.9. Since  $C$  contains  $w$  and  $V(P_{xw}) \cap \text{inn}(C) = \emptyset \subseteq \{w\}$ , the only subhypergraph in  $D$  that we are missing to apply Proposition 5.6 is a  $w$ -tadpole that does not contain  $s$ . The rest of the proof consists in finding a  $w$ -cycle in  $D$  that does not contain  $s$ .

Claim 14. *There exists a  $w$ -path  $P_w$  in  $D$  such that:*

- (a) *The only edge of  $P_w$  that intersects  $V(C) \setminus \{w\}$  is  $e^* := \text{end}(\overrightarrow{wP_w})$ . In particular:  $|V(P_w) \cap (V(C) \setminus \{w\})| = |e^* \cap (V(C) \setminus \{w\})| \in \{1, 2\}$ .*
- (b)  *$\{w_1, w_2\} \not\subseteq V(P_w)$ .*
- (c)  *$s \notin V(P_w)$ .*

*Proof of Claim 14.* Since  $V(X) \cap V(P_{xw}) \neq \emptyset$ , the projection  $\mathbf{P}_{V(P_{xw})}(z, X)$  is well defined. There is no  $x$ -tadpole in  $D \supseteq P_{xw} \cup \mathbf{P}_{V(P_{xw})}(z, X)$  by Proposition 4.1(f), so Lemma 2.32 with  $a = w$ ,  $b = x$  and  $c = z$  ensures that  $P_{xw} \cup \mathbf{P}_{V(P_{xw})}(z, X)$  contains a  $wz$ -path  $P_{wz}$ . Now, since  $V(P_{wz}) \cap (V(C) \setminus \{w\}) \supseteq \{z\} \neq \emptyset$ , the projection  $P_w := \mathbf{P}_{V(C) \setminus \{w\}}(w, P_{wz})$  is well defined.

Define  $e^* := \text{end}(\overrightarrow{wP_w})$ : by definition of a projection, the edge  $e^*$  is the only edge of  $P_w$  that intersects  $V(C) \setminus \{w\}$ , and  $|e^* \cap (V(C) \setminus \{w\})| \in \{1, 2\}$ , hence item (a).

Since  $P_w \subseteq P_{wz} \subseteq X \cup P_{xw}$ , where  $\{w_1, w_2\} \not\subseteq V(X)$  by definition of  $X$  and  $\{w_1, w_2\} \cap V(P_{xw}) = \emptyset$ , we have  $\{w_1, w_2\} \not\subseteq V(P_w)$  i.e. item (b).

Finally, suppose for a contradiction that  $s \in V(P_w)$ :

- First suppose  $s = w$ . By Proposition 2.20,  $C^{-s}$  is a  $w_1w_2$ -path. By Lemma 2.32 with  $a = w_1$ ,  $b = w_2$  and  $c = w = s$ ,  $C^{-s} \cup P_w$  contains an  $sw_1$ -path or an  $sw_2$ -path. Take a shortest  $sw_1$ -path or  $sw_2$ -path in  $C^{-s} \cup P_w$ : by symmetry, assume it is an  $sw_1$ -path  $\overleftarrow{P_{sw_1}}$ . The minimality of the length ensures that either  $w_2 \notin V(P_{sw_1})$  or  $w_2 = o(w_1, \overleftarrow{P_{sw_1}})$ .
  - If  $w_2 \notin V(P_{sw_1})$ , then  $P_{sw_1} \cup e$  is an  $s$ -cycle in  $D$ , contradicting Proposition 5.9.
  - If  $w_2 = o(w_1, \overleftarrow{P_{sw_1}})$ , then  $\text{end}(\overrightarrow{sP_{sw_1}w_1})$  is an edge of  $C^{-s} \cup P_w$  containing both  $w_1$  and  $w_2$ . However, there is no such edge in  $C^{-s}$  because  $C^{-s}$  is a  $w_1w_2$ -path of length at least 2 (indeed, recall that  $C$  is of length at least 3), and there is no such edge in  $P_w$  because  $\{w_1, w_2\} \not\subseteq V(P_w)$  by item (b). We have a contradiction.
- Now suppose  $s \neq w$ . By item (a),  $e^* := \text{end}(\overrightarrow{wP_w})$  is the only edge of  $P_w$  that intersects  $V(C) \setminus \{w\}$ , so  $P_w$  is a  $ws$ -path and either  $V(P_w) \cap V(C) = \{w, s\}$  or  $V(P_w) \cap V(C) = \{w, s, o(s, \overrightarrow{sP_w w})\}$ .
  - If  $V(P_w) \cap V(C) = \{w, s\}$ , then let  $P_{ws}$  be a  $ws$ -path in  $C$  (which exists by Proposition 2.22): since  $P_w$  and  $P_{ws}$  are both  $ws$ -paths and  $V(P_w) \cap V(P_{ws}) = \{w, s\}$ , we get an  $s$ -cycle  $P_w \cup P_{ws}$ , contradicting Proposition 5.9.
  - If  $V(P_w) \cap V(C) = \{w, s, t\}$  where  $t := o(s, \overrightarrow{sP_w w}) \in e^*$ , then let  $P_{st}$  be an  $st$ -path in  $C$  that does not contain  $w$  (which exists by Proposition 2.21 because  $w \in \text{out}(C)$ ): since  $V(P_{st}) \cap e^* = \{s, t\}$ , we get an  $s$ -cycle  $P_{st} \cup e^*$ , contradicting Proposition 5.9.

We have a contradiction in all cases, hence item (c).  $\square$

From now, the action takes place in  $C \cup P_w$  exclusively: we are going to exhibit a  $w$ -cycle in  $C \cup P_w$  that does not contain  $s$ . The idea is simply to get such a cycle by using  $P_w$  to go from  $w$  to  $C$  and then rejoining  $w$  by rotating along  $C$  in the correct direction so as to avoid  $s$

(for instance, see Figure 34, left and middle). This is always possible, unless this direction is blocked by a cycle of length 2, which cannot happen because there would then be an  $s$ -tadpole, contradicting Proposition 5.9 (for instance, see Figure 34, right). We now carry out the rigorous proof of this, distinguishing between two cases.

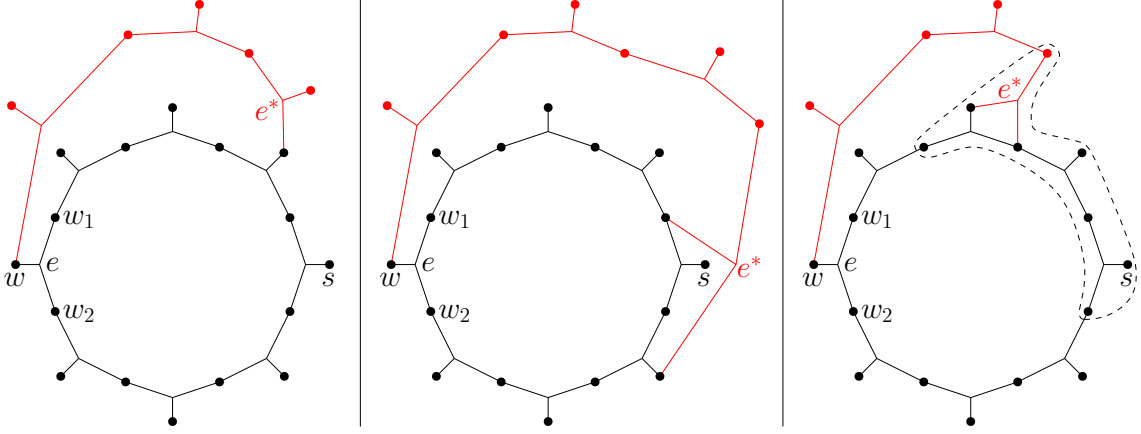


Figure 34: Represented here are  $C$  and  $P_w$ . In the left and middle examples, there is a  $w$ -cycle not containing  $s$ . In the right example, there is none but there is an  $s$ -tadpole (highlighted).

1) Case 1:  $w_1 \in V(P_w)$  or  $w_2 \in V(P_w)$ .

By symmetry, assume  $w_1 \in V(P_w)$ . By Claim 14(b), we have  $\{w_1, w_2\} \not\subseteq V(P_w)$  hence  $w_2 \notin V(P_w)$ . Therefore,  $P_w$  is a  $ww_1$ -path that does not contain  $w_2$ , so  $C' := P_w \cup e$  is a  $w$ -cycle. Moreover,  $s \notin V(C') = V(P_w) \cup \{w_2\}$ : indeed, we have  $s \notin V(P_w)$  by Claim 14(c), and  $s \in \text{out}(C)$  whereas  $w_2 \in \text{inn}(C)$ . Therefore,  $C'$  is the desired cycle.

2) Case 2:  $w_1, w_2 \notin V(P_w)$ .

By Proposition 2.20,  $C^{-w}$  is a  $w_1w_2$ -path. Write  $w_1C^{-w}w_2 = (w_1, e_1, \dots, e_L, w_2)$ . We have  $s \in \text{out}(C)$ , moreover  $s \neq w$  by Claim 14(c), so there exists a unique index  $1 \leq i \leq L$  such that  $s \in e_i$ . If  $i \neq 1$ , define  $s_1$  as the only vertex in  $e_{i-1} \cap e_i$  and  $P_1 := [(w_1, e_1, \dots, e_{i-1}, s_1)]$ , otherwise define  $s_1 = w_1$  and  $P_1 := [(w_1)]$ . Similarly, if  $i \neq L$ , define  $s_2$  as the only vertex in  $e_i \cap e_{i+1}$  and  $P_2 := [(w_2, e_L, e_{L-1}, \dots, e_{i+1}, s_2)]$ , otherwise define  $s_2 = w_2$  and  $P_2 := [(w_2)]$ . For all  $j \in \{1, 2\}$ ,  $P_j$  is a  $w_j s_j$ -path in  $C$ , and  $V(P_1) \cap V(P_2) = \emptyset$ . These notations are summed up in Figure 35.

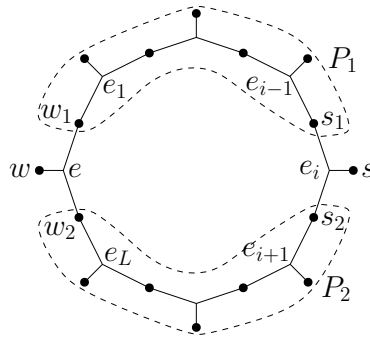


Figure 35: The cycle  $C$ .

By Claim 14(a), we have  $|V(P_w) \cap (V(C) \setminus \{w\})| \in \{1, 2\}$ . Note that  $V(C) \setminus \{w\} = V(P_1) \cup V(P_2) \cup \{s\}$ . Since  $s \notin V(P_w)$  by Claim 14(c), we obtain that  $|V(P_w) \cap V(P_1)| \in \{1, 2\}$  or  $|V(P_w) \cap V(P_2)| \in \{1, 2\}$ . By symmetry, assume that  $|V(P_w) \cap V(P_1)| \in \{1, 2\}$ .

- First suppose  $|V(P_w) \cap V(P_1)| = 1$ . Let  $u$  be the only vertex in  $V(P_w) \cap V(P_1)$ : in particular,  $P_w$  is a  $wu$ -path. Recall that  $w_2 \notin V(P_w)$  by assumption, moreover  $w_2 \notin V(P_1)$  by definition of  $P_1$ . Therefore, the sequence  $(w, e, w_1) \oplus \overrightarrow{w_1 P_1|_{\{u\}}} \oplus \overrightarrow{u P_w w}$  represents a  $w$ -cycle in  $C \cup P_w$ , which does not contain  $s$  since  $s \notin e \cup V(P_1) \cup V(P_w)$ . This is the desired cycle.
- Now suppose  $|V(P_w) \cap V(P_1)| = 2$ . Since  $V(P_1) \cap V(P_2) = \emptyset$ , this yields  $V(P_w) \cap V(P_2) = \emptyset$  by Claim 14(a). In particular  $s_2 \notin V(P_w)$ , so there cannot be an  $s_1$ -tadpole  $T$  in  $P_w \cup P_1$ : indeed, since  $s, s_2 \notin V(P_w) \cup V(P_1)$ , the sequence  $(s, e_i, s_1) \oplus \overrightarrow{s_1 T}$  would otherwise represent an  $s$ -tadpole in  $D$ , contradicting Proposition 5.9. Therefore, Lemma 2.32 with  $a = w_1$ ,  $b = s_1$  and  $c = w$  ensures that  $P_1 \cup P_w$  contains a  $ww_1$ -path  $P_{ww_1}$ . Since  $w_2 \notin V(P_1)$ , and  $w_2 \notin V(P_w)$  by assumption, the sequence  $\overrightarrow{w P_{ww_1} w_1} \oplus (w_1, e, w)$  represents a  $w$ -cycle, which does not contain  $s$  since  $s \notin e \cup V(P_{ww_1})$ . This is the desired cycle.

In conclusion, Proposition 5.6 applies and yields  $I_{H+w}(w\mathcal{D}_1(H)) = \emptyset$ , contradicting property  $J_1(\mathcal{D}_1, H)$ . This ends the proof of Theorem 5.1.

## 5.2 Algorithmic result

The algorithm derives from the reduction, given by Theorem 3.15, of the MAKERBREAKER problem on 3-uniform marked hypergraphs to the problem of the existence of a path between two given vertices in a 3-uniform hypergraph. This latter problem is independently known to be tractable:

**Definition 5.11.** Let  $H$  be a 3-uniform hypergraph and let  $a \in V(H)$ . The *linear connected component of  $a$  in  $H$*  is defined as the set  $LCC_H(a)$  of all vertices  $b$  of  $H$  such that there exists an  $ab$ -path in  $H$ .

**Theorem 5.12.** [8] *There exists an algorithm that, given a 3-uniform hypergraph  $H$  and a vertex  $x \in V(H)$ , computes  $LCC_H(x)$  in  $O(m^2)$  time where  $m = |E(H)|$ .*

*Proof of Theorem 3.17.* Consider a 3-uniform marked hypergraph  $H$ . Let  $n, m, \Delta$  be the number of vertices, the number of edges, and the maximum degree of  $H$  respectively. Since MAKERBREAKER is obviously in  $O(1)$  time on 3-uniform marked hypergraphs with less than 6 non-marked vertices, assume  $|V(H) \setminus M(H)| \geq 6$ . By Theorem 3.15,  $H$  is a Maker win if and only if:

$$\exists x_1 \in V(H) \setminus M(H), \forall y_1 \in \dots, \exists x_2 \in \dots, \forall y_2 \in \dots, \exists x_3 \in \dots, \forall y_3 \in \dots, \\ H^{+x_1-y_1+x_2-y_2+x_3-y_3} \text{ contains a fully marked edge, a nunchaku or a necklace.}$$

Suppose that we are given some  $x_1, y_1, x_2, y_2, x_3, y_3$ :

- Clearly,  $H^{+x_1-y_1+x_2-y_2+x_3-y_3}$  contains a fully marked edge or a nunchaku if and only if it contains a path between two marked vertices. This can be checked in  $O(n^2m^2)$  time: for all marked  $x$ , compute  $LCC_{H^{-y_1-y_2-y_3}}(x)$  using Theorem 5.12 and check whether it contains a marked vertex other than  $x$ .
- If  $H^{+x_1-y_1+x_2-y_2+x_3-y_3}$  contains no fully marked edge and no nunchaku, then it contains a necklace if and only if it contains some edge  $\{x, a, b\}$  with  $x$  marked such that there exists an  $xa$ -path that does not contain  $b$  (the union of that path and the edge  $\{x, a, b\}$  is the necklace). This can be checked in  $O(n\Delta m^2)$  time: for all marked  $x$  and all edge  $e = \{x, a, b\}$ , compute  $LCC_{H^{-y_1-y_2-y_3-b}}(x)$  using Theorem 5.12 and check whether it contains  $a$ , then repeat when exchanging the roles of  $a$  and  $b$ .

In conclusion, we get a  $O(n^6(n^2m^2 + n\Delta m^2)) = O(n^7m^2(n + \Delta))$  time algorithm for MAKER-BREAKER on 3-uniform marked hypergraphs. Note that this algorithm is not optimized. Actually, the time complexity can easily be improved to  $O(\max(n^5m^2, n^6\Delta))$ , with some preprocessing and removal of some redundant computations. ■

### 5.3 "Fast-winning Maker strategy" result

*Proof of Theorem 3.18.* By Theorem 3.15, Maker has a strategy ensuring that, after three rounds of play with successive picks  $x_1, y_1, x_2, y_2, x_3, y_3$ , there is a fully marked edge, a nunchaku or a necklace in  $H^{+x_1-y_1+x_2-y_2+x_3-y_3}$ . Proposition 1.21 thus ensures that, to conclude the proof, it suffices to show that any nunchaku or necklace  $N$  in  $H^{+x_1-y_1+x_2-y_2+x_3-y_3}$  satisfies  $\tau_M(N) \leq \lceil \log_2(|V(H) \setminus M(H)| - 5) \rceil$ .

A nunchaku  $N$  is of length  $\frac{|V(N)|-1}{2}$ , so it satisfies  $\tau_M(N) = 1 + \lceil \log_2\left(\frac{|V(N)|-1}{2}\right) \rceil = \lceil \log_2(|V(N)| - 1) \rceil$  according to Proposition 3.8. Moreover, a nunchaku  $N$  in  $H^{+x_1-y_1+x_2-y_2+x_3-y_3}$  has two marked vertices and all its other vertices are in  $V(H) \setminus (M(H) \cup \{x_1, y_1, x_2, y_2, x_3, y_3\})$ , so it satisfies  $|V(N)| \leq 2 + (|V(H) \setminus M(H)| - 6)$  hence  $\tau_M(N) \leq \lceil \log_2(|V(H) \setminus M(H)| - 5) \rceil$ .

A necklace  $N$  is of length  $\frac{|V(N)|}{2}$ , so it satisfies  $\tau_M(N) = \lceil \log_2(|V(N)|) \rceil$  according to Proposition 3.11. Moreover, a necklace  $N$  in  $H^{+x_1-y_1+x_2-y_2+x_3-y_3}$  has one marked vertex and all its other vertices are in  $V(H) \setminus (M(H) \cup \{x_1, y_1, x_2, y_2, x_3, y_3\})$ , so it satisfies  $|V(N)| \leq 1 + (|V(H) \setminus M(H)| - 6)$  hence  $\tau_M(N) \leq \lceil \log_2(|V(H) \setminus M(H)| - 5) \rceil$ . ■

## Conclusion and perspectives

In this paper, we have introduced a general notion of danger in a marked hypergraph, and then applied it in the 3-uniform case to get structural results about the Maker-Breaker game on 3-uniform marked hypergraphs (which correspond to hypergraphs of rank 3). We have shown that Breaker wins if and only if he can manage the  $\mathcal{D}_0$ -dangers during three rounds.

As a consequence, the winner of the Maker-Breaker game on a hypergraph of rank 3 can be decided in polynomial time. Since the 6-uniform case is PSPACE-complete [15], it remains to determine the complexity for hypergraphs of rank 4 and 5.

We have also obtained a (basically) optimal logarithmic bound for  $\tau_M$  on 3-uniform Maker wins. On the other hand, for fixed  $k \geq 4$ , it is not difficult to exhibit a  $k$ -uniform hypergraph  $H_n$  on  $n$  vertices such that  $\tau_M(H_n) \geq \lceil \frac{n}{2} \rceil - O(1)$ . Therefore, the question of finding the best general bound for  $\tau_M$  depending on the size of the edges is resolved.

All these questions remain open for the more general version of the game which is played on a 3-CNF formula instead of a hypergraph of rank 3. The concept of danger and the results from Section 1 should translate well to this version, except that there would be two types of dangers at  $x$  depending on which value False must assign to  $x$ . Property  $J_r(\cdot)$  should then be checked for both types of dangers. It is possible that our proofs generalize to this version, in which case this would show that, apart from some trivial cases, True wins if and only if he can break any manriki that appears during the first three rounds of play, thus validating Rahman and Watson's conjecture [14]. However, we have not looked into it. For now, we only know that this holds for positive 3-CNF formulas.

Positional games also exist in a *biased* version ( $p : q$ ), where the players get to pick  $p$  and  $q$  vertices respectively each round instead of only one vertex. Many instances of biased Maker-Breaker games have been studied in the literature, notably to study the *threshold bias* which is the smallest  $q$  such that Breaker wins with a  $(1 : q)$  bias [10]. The concept of danger

and the results from Section 1 naturally generalize to the biased version of the game, and might be useful in some cases. For example, one could look into biased games on hypergraphs of rank 3 with a danger-based approach, defining elementary dangers depending on the bias, and trying to get a characterization of Breaker wins similar to the one we obtained without bias.

## Acknowledgments

We thank Md Lutfar Rahman and Thomas Watson for useful exchanges. This work was supported by the ANR P-GASE project (grant ANR-21-CE48-0001).

## References

- [1] L. Bahack. Solving the general case of rank-3 Maker-Breaker games in polynomial time. Preprint (2022). <https://arxiv.org/abs/2209.11202>.
- [2] J. Beck. Remarks on positional games. *Acta Math. Hungar.*, **40**(1–2) (1982), pp. 65–71. <https://doi.org/10.1007/BF01897304>.
- [3] J. Beck. *Combinatorial games: tic-tac-toe theory*, Cambridge University Press (2008). <https://doi.org/10.1017/CB09780511735202>.
- [4] J. Beck, L. Csirmaz. Variations on a game. *J. Comb. Theory, Ser. A*, **33**(3) (1982), pp. 297–315. [https://doi.org/10.1016/0097-3165\(82\)90042-5](https://doi.org/10.1016/0097-3165(82)90042-5).
- [5] V. Chvátal, P. Erdős. Biased positional games. *Ann. Disc. Math.*, **2** (1978), pp. 221–229. [https://doi.org/10.1016/S0167-5060\(08\)70335-2](https://doi.org/10.1016/S0167-5060(08)70335-2).
- [6] P. Erdős, J. Selfridge. On a combinatorial game. *J. Comb. Theory, Ser. A*, **14**(3) (1973), pp. 298–301. <https://doi.org/10.1016/0097-3165%2873%2990005-8>.
- [7] M. Gardner. "Recreational topology", in *The second scientific american book of mathematical puzzles and diversions*, University of Chicago Press (1961).
- [8] F. Galliot, S. Gravier, I. Sivignon.  $(k - 2)$ -linear connected components in hypergraphs of rank  $k$ . Preprint, submitted (2022). <https://hal.archives-ouvertes.fr/hal-03412083>.
- [9] A. W. Hales, R. I. Jewett. Regularity and positional games. *Trans. Amer. Math. Soc.*, **106** (1963), pp. 222–229. [http://doi.org/10.1007/978-0-8176-4842-8\\_23](http://doi.org/10.1007/978-0-8176-4842-8_23).
- [10] D. Hefetz, M. Krivelevich, M. Stojaković, T. Szabó. *Positional Games*, Springer Basel (2014). <https://doi.org/10.1007/978-3-0348-0825-5>.
- [11] M. Kutz. "Weak positional games", in *The angel problem, positional games, and digraph roots*, PhD thesis, Freie Universität Berlin (2004).
- [12] M. Kutz. Weak positional games on hypergraphs of rank three. *Discrete Math. Theor. Comput. Sci.*, **AE**, DMTCS Proceedings of the 3rd European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05) (2005), pp. 31–36. <https://doi.org/10.46298/dmtcs.3422>.
- [13] A. Lehman. A solution of the Shannon switching game. *J. Soc. Indust. Appl. Math.*, **12**(4) (1964), pp. 687–725. <https://doi.org/10.1137/0112059>.



- [14] M. L. Rahman, T. Watson. Tractable unordered 3-CNF games. *Lect. Notes Comput. Sci.*, **12118**, Proceedings of the 14th Latin American Theoretical Informatics Symposium (LATIN 2020) (2020), pp. 360–372. [https://doi.org/10.1007/978-3-030-61792-9\\_29](https://doi.org/10.1007/978-3-030-61792-9_29).
- [15] M. L. Rahman, T. Watson. 6-uniform Maker-Breaker game is PSPACE-complete. *LIPICs*, **187**, Proceedings of the 38th International Symposium on Theoretical Aspects of Computer Science (STACS 2021) (2021), pp. 57:1–57:15. <https://doi.org/10.4230/LIPICs.STACS.2021.57>.
- [16] T. J. Schaefer. On the complexity of some two-person perfect-information games. *J. Comput. Syst. Sci.*, **16**(2) (1978), pp. 185–225. [https://doi.org/10.1016/0022-0000\(78\)90045-4](https://doi.org/10.1016/0022-0000(78)90045-4).