Leader-follower and Leaderless Bipartite Formation-consensus over Undirected Coopetition Networks and under Proximity and Collision-avoidance Constraints

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ABSTRACT
We propose a method to control networks involving both cooperative and competitive interactions simultaneously under proximity and collision-avoidance constraints. The control design is of gradient type, using a barrier-Lyapunov function. Then, under the assumption that the network is undirected and structurally balanced, we establish asymptotic stability of the leaderless and leader-follower bipartite formation-consensus manifold. We assume that the agents are modeled by second-order integrators, but we also demonstrate the utility of our theoretical findings via numerical simulations on a problem of formation-consensus control of nonholonomic vehicles.

KEYWORDS
Formation consensus; coopetition networks; barrier-Lyapunov functions

1. Introduction

Most of the works done in the literature on consensus and synchronization of multi-agent systems apply to the case of networks having only cooperative interactions [15]. In these protocols, all agents reach a global agreement regarding a certain quantity of interest through collaboration [11]. This excludes several scenarios where some agents cooperate, and others compete, as *e.g.*, in the context of herding control [3, 7] or in social networks [1]. Networks in which competitive interactions are represented by negative weights on the edges and cooperative ones by positive weights, are called *coopetition networks* [8]. For these networks, agreement on a common value is generally not attainable. Still, an achievable goal is the so-called *multi-partite consensus*, in which case the system has more than one equilibrium point. More precisely, considering a structurally balanced and connected undirected signed network, *bipartite consensus* can be achieved and all the agents converge to the same state in modulus, but with opposite sign [1]. There are multiple studies on the bipartite consensus-control problem for single-integrators [1], double-integrators [17], and linear high-order dynamics [14].

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For these kinds of problems considered in the literature, the main objective is to attain the goal destination or to follow the reference trajectory. However, when application scenarios are considered, for instance, in formation of autonomous vehicles, the latter must also satisfy secondary objectives, such as guaranteeing collision avoidance between two agents or maintaining information exchange between the ones cooperating with each other. Those objectives are encoded as inter-agent constraints and are usually handled by barrier functions—see *e.g.*, [4], [2] and [10]. The latter uses Lyapunov-like barrier functions to encode collision avoidance and connectivity maintenance control goals and assures the convergence to desired destinations for a network of multiple nonholonomic vehicles.

In [3], the multi-swarm herding problem is solved under connectivity constraints using a mixed integer quadratically constrained program; in [7], a control strategy is proposed for the non-cooperative herding problem described by first-order dynamics and control barrier functions are used to prevent some agents from escaping from a protected zone. Yet, in [3] and [7], the system is modeled by a network in which all the agents cooperate and the control laws are optimization-based. Moreover, in [7] only a two-agent scenario is considered. In [6], the bipartite flocking-control problem is studied, and artificial potential functions are used to guarantee collision avoidance and connectivity maintenance. The statements rely on LaSalle's invariance principle. However, collisions are avoided in [6], but a minimal safety distance between agents is not guaranteed. Thus, only a few works in the literature focus on inter-agent constrained control problems for coopetition networks. In [13], a controller is proposed to solve the bipartite formation-control problem with inter-agent collision avoidance and connectivity maintenance for structurally balanced undirected networks. However, the proposed controller requires the global knowledge of the network topology. This paper proposes a fully distributed control law that reduces high communication burdens on the system.

We study the leaderless and leader-follower bipartite formation-consensus problems for structurally balanced undirected signed networks described by second-order systems under inter-agent distance constraints. We propose a distributed bipartite formation control law that prevents inter-vehicle collisions and keeps vehicles in their sensors' range. We base our control law and analysis on the edge representation for signed networks [5]. This also allows to recast the bipartite formation-consensus problem into the space of the error coordinates. In contrast to [3], [7], and [6], we encode the inter-agent distance constraints using barrier-Lyapunov functions, and our controller is based on the gradient of a barrier-Lyapunov function. With respect to [13], our control law is fully distributed, and we demonstrate our theoretical findings on more realistic scenarios with nonholonomic mobile robots following a leader.

Thus, relative to the existing literature, we contribute with a fully distributed control law that solves leader-follower and leaderless bipartite formation-consensus-control problems for structurally balanced undirected signed networks described by second-order systems. Our controller guarantees inter-agent collision avoidance by ensuring all agents keep a minimum safety distance between one another and ensures maintenance of information exchange for all agents having cooperative interactions. Furthermore, we establish asymptotic stability of the bipartite formation-consensus manifold by applying Lyapunov's direct method and employing edge-based formulation for signed networks.

2. Model and Problem Formulation

Consider a network of N dynamical systems having the following dynamics.

$$\dot{\bar{x}}_i = v_i, \tag{1a}$$
$$\dot{v}_i = u_i, \qquad \bar{x}_i \in \mathbb{R}^n,\ i \leq N, \tag{1b}$$

where $\bar{x}_i = x_i - d_i$, x_i is the agent's position, and d_i is a relative displacement with respect to a consensus value. For instance, for vehicles moving on the plane, $x_i \in \mathbb{R}^2$ may represent a pair of Cartesian coordinates and $d_i \in \mathbb{R}^2$ an offset relative to the center of a desired formation—see Section 4.

Let such systems be interconnected over an undirected network so that the ith agent exchanges state measurements with a set of neighbors with state $\bar{x}_j$, but under the assumption that some agents are cooperative and others are competitive. Such networks are referred to as *coopetitive* [8]. In general, the inputs of a coopetitive network are defined as—cf. [17],

$$u_i = -\sum_{j=1}^{N} |a_{ij}| \Big[k_1[\bar{x}_i - \operatorname{sgn}(a_{ij})\bar{x}_j] + k_2[v_i - \operatorname{sgn}(a_{ij})v_j] \Big], \qquad a_{ij} \in \{-1, 0, 1\}, \tag{2}$$

where $|a_{ij}| = 1$ if there is an interconnection between the ith and jth agents and $a_{ij} = 0$ if there is not. The expression (2) is reminiscent of classical consensus control laws for second-order systems [11], but with the important difference that $a_{ij} \in \{-1, 0, 1\}$; $a_{ij} = 1$ if the agents j and i are cooperative and $a_{ij} = -1$ if they are competitive. Because of the presence of competitive agents, in general, consensus cannot be reached, but following the framework of [1], which focuses on first-order systems, we may define the synchronization errors

$$\bar{e}_{x_k} := \bar{x}_i - \operatorname{sgn}(a_{ij})\bar{x}_j, \quad \bar{e}_{v_k} := v_i - \operatorname{sgn}(a_{ij})v_j, \quad k \leq M, \tag{3}$$

where k denotes the index of the interconnection between the ith and jth agents. Then, we are interested in the conditions under which the following property is attained.

Definition 1 (Leaderless bipartite consensus)**.** *The systems* (1)-(2) *are said to achieve bipartite position and velocity consensus if*

$$\lim_{t\to\infty} \bar{e}_{x_k}(t) = 0, \quad \lim_{t\to\infty} \bar{e}_{v_k}(t) = 0, \qquad k \leq M. \tag{4}$$

That is, for agents i and j cooperative with each other, bipartite consensus implies that $(x_i, v_i) \to (x_j, v_j)$ while for agents competitive with each other we have $(x_i, v_i) \to (-x_j, -v_j)$. In [17], it is established that the distributed control law in (2) guarantees bipartite consensus provided that the network enjoys the following property, which was introduced earlier, in [1].

Definition 2 (Structural balance)**.** *Let $\mathcal{V}$, of cardinality N, denote a set of vertices (nodes) ν_k, let $\mathcal{E} \subseteq \mathcal{V}^2$, of cardinality M, denote the set of edges (interconnections) $\varepsilon_k := (i, j)$, and let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ denote the corresponding graph. Then, a signed graph is structurally balanced if it may be split into two disjoint sets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}, \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ such that for every $i, j \in \mathcal{V}_p, p \in \{1, 2\}$, if $a_{ij} \geq 0$,*

while for every $i \in \mathcal{V}_p, j \in \mathcal{V}_q$, *with* $p, q \in \{1, 2\}, p \neq q$, *if* $a_{ij} \leq 0$. *It is structurally unbalanced, otherwise.*

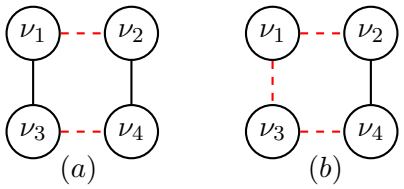

Figure 1. Example of two signed networks, where the cooperative interactions are represented by solid black lines and the competitive interactions by red dashed lines: (a) the nodes can be separated into two disjoint subsets such as $\mathcal{V}_1 = \{\nu_1,\ \nu_3\}$ and $\mathcal{V}_2 = \{\nu_2,\ \nu_4\}$ so the graph is structurally balanced, (b) the nodes cannot be separated into 2 disjoint subsets so the graph is structurally unbalanced.

That is, for structurally balanced networks –see Figure 1, bipartite consensus means that the elements of the two sets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ converge to the same state in the module but with opposite signs. In [1], the final states of the agents are defined by their initial conditions. However, there are some cases in which an external factor decides the final states of the agents. Thus, a virtual leader ν_0 can be introduced to coordinate with at least one agent. Assume that the dynamics of the virtual leader are described by

$$\dot{x}_0 = 0, \quad x_0 \in \mathbb{R}^n. \tag{5}$$

Let $\phi_i = \{1,\ -1\}$ describe the interaction relationship between the virtual leader and the agents $\nu_i, i \leq N$. Then, we define

$$\hat{e}_i := \bar{x}_i - \phi_i x_0, \quad i \leq N, \tag{6}$$

and the leader-follower bipartite consensus is introduced next.

Definition 3 (Leader-follower bipartite consensus)**.** *Let* $x_0 \in \mathbb{R}^n$ *be the position of the virtual leader* ν_0, *whose dynamics are given in* (5), *provided to at least one follower. Then, the systems* (1)-(2) *are said to achieve leader-follower bipartite position and velocity consensus if* (4) *and*

$$\lim_{t\to\infty} \hat{e}_i(t) = 0, \qquad i \leq N. \tag{7}$$

That is, for agents i cooperative with the virtual leader, bipartite consensus implies that $(x_i, v_i) \to (x_0, 0)$ while for agents competitive with the virtual leader we have $(x_i, v_i) \to (-x_0, 0)$. This paper addresses these two problems in a scenario that imposes proximity and collision avoidance constraints between all agents $x_i,\ i \leq N$.

Consider a group of systems (1) interconnected through inputs u_i such that they form a communication network with a signed, undirected, structurally balanced graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$.

For each pair of communicating nodes ν_i and $\nu_j \in \mathcal{V}$, labeled $k \leq M$, let $\delta_k := x_i - x_j$, let $R_k > 0$ and $\Delta_k > 0$ be given. Also, consider the sets $\mathcal{V}_l$, with $l \in \{1, 2\}$, as defined in Definition 2. Then, define the set of proximity constraints $\mathcal{I}_r$ and the set of collision-

avoidance constraints $\mathcal{I}_c$ as

$$\mathcal{I}_r := \{\delta_k \in \mathbb{R}^n : |\delta_k| < R_k, \quad \forall i, j \in \mathcal{V}_l,\ l = \{1, 2\}\} \tag{8}$$
$$\mathcal{I}_c := \{\delta_k \in \mathbb{R}^n : |\delta_k| > \Delta_k \quad \forall k \leq M\}. \tag{9}$$

Under these conditions, it is required to design a distributed bipartite consensus control law u_i^* such that the synchronization errors $(\bar{e}_{x_k}, \bar{e}_{v_k})$ in (3) satisfy (4) and the agents' trajectories satisfy the proximity and collision-avoidance constraints, that is, $\delta_i(t) \in \mathcal{I}$ for all $t \geq 0$, with $\mathcal{I} := \mathcal{I}_r \cap \mathcal{I}_c$.

Remark 1. *Proximity constraints are not imposed on competitive interactions since it is assumed that the agents competitive with each other have different objectives, so they do not stay close to each other.* •

3. Bipartite consensus under constraints

We solve the bipartite formation-consensus problem over undirected signed networks using a barrier-Lyapunov function [10] and a gradient-based consensus control law. Moreover, we recast the problem stated above into the space of the synchronization errors using the edge-based framework [9]. Thus, we first recall some facts on signed networks [1] and on the edge-based formulation for signed networks [5].

Remark 2. *For notational simplicity, in this section, we assume that $x_i \in \mathbb{R}$, but with obvious modifications (involving the Kronecker product $\otimes$) all the statements may be reformulated for systems with $x_i \in \mathbb{R}^n$. In Section 4, e.g., we consider the problem of bipartite-formation consensus for vehicles moving on the plane, hence, with $n = 2$.* •

3.1. The edge-based formulation for signed networks

The elements of the Laplacian matrix of a signed graph, $L_s \in \mathbb{R}^{N \times N}$, are

$$\ell_{s_{ij}} = \begin{cases} \sum\limits_{h \leq N} |a_{ih}| & i = j \\ -a_{ij} & i \neq j. \end{cases} \tag{10}$$

In the following definition, we introduce the elements of the incidence matrix of a structurally balanced signed graph.

Definition 4. *Consider a structurally balanced undirected signed network $\mathcal{G}$ that contains N nodes and M edges. The incidence matrix $E_s \in \mathbb{R}^{N \times M}$ of $\mathcal{G}$ is defined as*

$$[E_s]_{ik} := \begin{cases} +1 & \text{if } v_i \text{ is the initial node of the edge } e_k; \\ -1 & \text{if } v_i, v_j \in \mathcal{V}_l \text{ and } v_i \text{ is the terminal node of the edge } e_k; \\ +1 & \text{if } v_i \in \mathcal{V}_p, v_j \in \mathcal{V}_q \text{ and } v_i \text{ is the terminal node of the edge } e_k; \\ 0 & \text{otherwise,} \end{cases}$$

where $e_k = v_i v_j,\ k \leq M,\ i, j \leq N$ are arbitrarily oriented edges, $l, p, q \in \{1, 2\}$ with $p \neq q$ and $\mathcal{V}_1$ and $\mathcal{V}_2$ are the two disjoint sets of vertices.

Using Definition 4, we express the synchronization errors in (3), in the vector forms

$$\bar{e}_x = E_s^\top \bar{x}, \quad \bar{e}_v = E_s^\top v. \tag{11}$$

We may also express the node and edge Laplacians using the definition of the incidence matrix. The Laplacian matrix L_s and the edge Laplacian matrix L_{e_s} of a structurally balanced graph $\mathcal{G}$ satisfy—see [13, Claim 1],

$$L_s = E_s E_s^\top, \quad L_{e_s} = E_s^\top E_s. \tag{12}$$

Furthermore, for a structurally balanced signed graph containing a spanning tree, we can distinguish the state variables corresponding to the spanning tree from the states corresponding to the cycles (the rest of the edges). The incidence matrix satisfies

$$E_s = [E_{t_s} \quad E_{c_s}], \tag{13}$$

where $E_{t_s} \in \mathbb{R}^{N\times N-1}$ is the incidence matrix representing the edges of the spanning tree corresponding to a spanning tree graph $\mathcal{G}_t$ and $E_{c_s} \in \mathbb{R}^{N\times M-(N-1)}$ is the incidence matrix representing the remaining edges corresponding to $\mathcal{G}_c := \mathcal{G}\backslash\mathcal{G}_t$. After (11) and (13) we partition the synchronization errors as

$$\bar{e}_x = [(E_{t_s}^\top \bar{x})^\top \ (E_{c_s}^\top \bar{x})^\top]^\top := [\bar{e}_{x_t}^\top \quad \bar{e}_{x_c}^\top]^\top, \quad \bar{e}_v = [(E_{t_s}^\top v)^\top \ (E_{c_s}^\top v)^\top]^\top =: [\bar{e}_{v_t}^\top \quad \bar{e}_{v_c}^\top]^\top.$$

The indices $_t$ and $_c$ correspond to states of the graphs $\mathcal{G}_t$ and $\mathcal{G}_c$, respectively. Then, we establish a relation between the errors of the spanning tree and the synchronization errors. For a structurally balanced signed graph, there exists a matrix R_s such that

$$E_s = E_{t_s} R_s, \tag{14}$$

where $R_s := [I_{N-1} \quad T_s]$ and $T_s := (E_{t_s}^\top E_{t_s})^{-1} E_{t_s}^\top E_{c_s}$—[13, Proposition 1]. This relation between the incidence matrices E_s and E_{t_s} also indicates a link between all synchronization errors $\bar{e}_x$ and the ones corresponding to the spanning-tree $\bar{e}_{x_t}$. We note that after (14) we have

$$\bar{e}_x = (E_{t_s} R_s)^\top \bar{x} = R_s^\top \bar{e}_{x_t}, \quad \bar{e}_v = (E_{t_s} R_s)^\top v = R_s^\top \bar{e}_{v_t}. \tag{15}$$

Then, the studied problem may be addressed as that of stabilizing the origin of a reduced-order system of the spanning-tree errors, and it follows that the objective (4) is attained if $\bar{e}_{x_t} \to 0$ and $\bar{e}_{v_t} \to 0$.

3.2. Barrier-Lyapunov Function based controller

We first define a *barrier-Lyapunov function* (BLF)—cf. [9], [10], [12].

Definition 5. *Consider the system $\dot{x} = f(x)$ and let $\mathcal{I}$ be an open set containing the origin. A BLF is a positive definite function $W : \mathcal{I} \to \mathbb{R}_{\geq 0}$, $x \mapsto W(x)$, that is $\mathcal{C}^1$, satisfies $\nabla W(x) f(x) \leq 0$, where $\nabla W(x) := \partial V/\partial x$, and has the property that $W(x) \to \infty$, and $\nabla W(x) \to \infty$ as $x \to \partial\mathcal{I}$.*

We proceed to define a barrier-Lyapunov function of the synchronization errors $\bar{e}_{x_k}$. To that end, we remark that for a pair of agents cooperative with each other, the synchronization errors in position, defined in (3), are

$$\bar{e}_{x_k} = \bar{x}_i - \bar{x}_j = \delta_k - \bar{d}_k, \quad i,j \in \mathcal{V}_l,\ l \in \{1,2\}, \tag{16}$$

while for a pair of agents competitive with each other, the synchronization errors are

$$\bar{e}_{x_k} = \bar{x}_i + \bar{x}_j = \delta_k - \bar{d}_k + 2x_j, \quad i \in \mathcal{V}_p,\ j \in \mathcal{V}_q,\ p,q \in \{1,2\}\ p \neq q, \tag{17}$$

with $\bar{d}_k = d_i - \text{sgn}(a_{ij})d_j$. Then, from (16) and (17), the constraints sets in (8) and (9) may be rewritten as

$$\mathcal{I}_r = \{\bar{e}_{x_k} \in \mathbb{R}^n : |\bar{e}_{x_k} + \alpha_k| < R_k,\ \forall k \in \mathcal{E}_m\}, \tag{18a}$$
$$\mathcal{I}_c = \{\bar{e}_{x_k} \in \mathbb{R}^n : \Delta_k < |\bar{e}_{x_k} + \alpha_k|,\ \forall k \leq M\}. \tag{18b}$$

$\mathcal{E}_m$ consists in the indexes of m edges corresponding to cooperative interactions between agents, such that $0 \leq m < M$, and α_k is defined as

$$\alpha_k := \begin{cases} \bar{d}_k & \text{if} \quad i,j \in \mathcal{V}_l, \qquad l \in \{1,2\} \\ \delta_k - \bar{e}_{x_k} & \text{if} \quad i \in \mathcal{V}_p, j \in \mathcal{V}_q, \quad p,q \in \{1,2\},\ p \neq q. \end{cases} \tag{19}$$

Then, for each edge $e_k, k \leq M$, we define the BLF $W_k : \mathbb{R} \to \mathbb{R}_{\geq 0}$,

$$W_k(\bar{e}_{x_k}) = \frac{1}{2}[|\bar{e}_{x_k}|^2 + B_k(\bar{e}_{x_k})], \tag{20}$$

where $B_k(\bar{e}_{x_k})$ encodes the constraints in (18) and is given by

$$B_k(\bar{e}_{x_k}) = B_{c_k}(\bar{e}_{x_k}) + \frac{1}{2}(1+\sigma_k)B_{r_k}(\bar{e}_{x_k}). \tag{21}$$

$B_k(\bar{e}_{x_k})$ consists in two functions satisfying Definition 5. $B_{c_k}(\bar{e}_{x_k})$ encodes the inter-agent collision avoidance constraints for each edge, whether it is a cooperative or a competitive one. $B_{r_k}(\bar{e}_{x_k})$ encodes the inter-agent connectivity maintenance constraints for cooperative edges, such that $\sigma_k = 1$ if $k \in \mathcal{E}_m$, and $\sigma_k = -1$ otherwise. Furthermore, $B_k(\bar{e}_{x_k})$[1] is non-negative and satisfies $B_k(0) = 0$ and $B_k(\bar{e}_{x_k}) \to \infty$ as $|\bar{e}_{x_k}| \to \Delta_k$ for all $k \leq M$ and as $|\bar{e}_{x_k}| \to R_k$ for $k \in \mathcal{E}_m$. However, for the system's solution to lie within the constraint sets defined in (18) and for the agents to converge to the desired equilibrium point, the barrier function must be modified. To this end, we use the concept of a *gradient recentered barrier function* [16]. Let $\widehat{W}_k : \mathcal{I}_r \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ for competitive and $\widehat{W}_k : \mathcal{I}_r \cap \mathcal{I}_c \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ for cooperative edges be defined as

$$\widehat{W}_k(\alpha_k, \bar{e}_{x_k}) := W_k(\bar{e}_{x_k} + \alpha_k) - W_k(\alpha_k) - \frac{\partial W_k}{\partial s}(\alpha_k)\bar{e}_{x_k}. \tag{22}$$

The gradient recentered barrier function in (22) satisfies $\widehat{W}_k(\alpha_k, 0) = 0$, $\nabla_{\bar{e}_{x_k}}\widehat{W}_k(\alpha_k, 0) = 0$, where $\nabla_{\bar{e}_{x_k}}\widehat{W}_k = \frac{\partial \widehat{W}_k}{\partial \bar{e}_{x_k}}$, and $\widehat{W}_k(\alpha_k, \bar{e}_{x_k}) \to \infty$ as $|\delta_k| \to \Delta_k$

[1] A particular choice for $B_{r_k}(\bar{e}_{x_k}), B_{c_k}(\bar{e}_{x_k})$ is given in Section 4.

for $k \leq M$, and as $|\delta_k| \to R_k$ for all $\sigma_k = 1$. Furthermore, $\widehat{W}_k(\alpha_k, \bar{e}_{x_k})$ satisfies $\frac{\kappa_1}{2}\bar{e}_{x_k}^2 \leq \widehat{W}_k(\alpha_k, \bar{e}_{x_k}) \leq \kappa_2[\nabla_{\bar{e}_{x_k}}\widehat{W}_k]^2$.

Let k_1, $k_2 > 0$ and $k_3 \geq 0$. Then, we introduce the BLF-gradient-based bipartite formation-consensus control law,

$$\begin{aligned} u_i := &- k_1 \sum_{k=1}^{M} [E_s]_{ik} \nabla_{\bar{e}_{x_k}} \widehat{W}_k - k_1 \sum_{k=1}^{M} [\mathbb{E}]_{ik} \nabla_{\alpha_k} \widehat{W}_k - k_2 \sum_{k=1}^{M} [E_s]_{ik} \bar{e}_{v_k} \\ &- k_3 v_i - k_1 \sum_{i=1}^{N} a_{i_l} \hat{e}_i, \end{aligned} \tag{23}$$

where $\mathbb{E} := E - E_s$, with E corresponding to the incidence matrix of the unsigned version of the signed network.[2] Then, $\mathbb{E}$ is a matrix representing only the competitive edges in the network. We define

$$\bar{W}(\alpha, \bar{e}_x, \hat{e}) = \sum_{k=1}^{M} \widehat{W}_k(\alpha_k, \bar{e}_{x_k}) + \frac{1}{2} \sum_{i=1}^{N} a_{l_i} |\hat{e}_i|^2, \tag{24}$$

where $a_{l_i} = 1$ if there is an information exchange between the virtual leader and the agent i and $a_{l_i} = 0$ otherwise. Thus, the control law in (23) is given in the vector form as

$$u = -k_1 E_s \nabla_{\bar{e}_x} \bar{W} - k_1 \mathbb{E} \nabla_{\alpha} \bar{W} - k_2 E_s \bar{e}_v - k_3 v - k_1 \nabla_{\hat{e}} \bar{W}. \tag{25}$$

The first two terms in (25) ensure bipartite position consensus while the third guarantees velocity bipartite consensus, *i.e.*, that (4) holds. The second term is needed specifically to cope with competitive agents; technically, it stems from a Lyapunov-control redesign—see the proof of Proposition 1. In the fourth term, if $k_3 > 0$, the velocities $v \to 0$. Therefore, this gain may be set to a positive value or zero, depending on the control goal. For instance, for a swarm of autonomous vehicles, it may be desired that the vehicles continue to advance ($v \not\to 0$) or that they converge to two rendezvous points in formation ($v \to 0$), one for cooperative and one for competitive agents—see Section 4 for an example. The last term is equal to zero if there is no virtual leader, and all agents converge to a final state which is computed by their initial conditions, depending on k_3. If there is a virtual leader, it ensures that all agents cooperative with the virtual leader converge to the virtual leader's position and k_3 should be greater than zero.

Remark 3. *In [13], the authors propose a bipartite formation control law for multi-agent systems over undirected signed networks and under inter-agent distance constraints. In [13], the control law is given by*

$$u_i := - k_1 \sum_{k=1}^{M} [E_s]_{ik} \nabla_{\bar{e}_{x_k}} \widehat{W}_k - k_1 \sum_{k=1}^{M} [\mathbb{B}]_{ik} \nabla_{\alpha_k} \widehat{W}_k - k_2 \sum_{k=1}^{M} [E_s]_{ik} \bar{e}_{v_k} - k_3 v_i$$

where $\mathbb{B} = (1+\beta)(\beta I + E_{t_s} L_{e_{ts}}^{-1} E_{t_s}^{\top})^{-1}(E - E_s)$, $\beta > 0$, *is a term needing global knowledge*

[2] A structurally balanced signed network can be transformed into an unsigned one using the gauge transformation—see [1, 5].

of the network, making the controller not fully distributed. The control law proposed in this paper, in (23)*, is fully distributed and solves leaderless and leader-follower bipartite formation control problems.* •

3.3. Main statement

Using the definitions of Section 3.1 on the edge-based formulation, we analyze the stability of the system (1) in closed loop with the bipartite formation control law (25), with and without a virtual leader. We express the control law in terms of the spanning tree errors $\bar{e}_t$ by introducing a function $\tilde{W}$ as $\tilde{W}(\alpha, \bar{e}_{x_t}, \hat{e}) = \bar{W}(\alpha, R_s^\top \bar{e}_{x_t}, \hat{e})$, with $\bar{W}$ defined in (25). After (15), its gradient with respect to $\bar{e}_{x_t}$ gives

$$\nabla_{\bar{e}_{x_t}} \tilde{W} = \frac{\partial \tilde{W}(\alpha, \bar{e}_{x_t}, \hat{e})}{\partial \bar{e}_x}^\top \frac{\partial \bar{e}_x}{\partial \bar{e}_{x_t}} = \nabla_{\bar{e}_x} \bar{W}^\top R_s^\top. \tag{26}$$

Using (26), we first consider the leaderless and then the leader-follower bipartite formation-consensus-control problems.

3.3.1. Leaderless bipartite formation-consensus

After (14) and (26), the control law in (25) becomes

$$u = -k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} - k_3 v - k_1 \mathbb{E} \nabla_\alpha \tilde{W}. \tag{27}$$

Differentiating (11) on both sides and using $\dot{v} = u$ with u defined in (27), we obtain the following in terms of the errors corresponding to the spanning-tree

$$\dot{\bar{e}}_{x_t} = \bar{e}_{v_t} \tag{28a}$$

$$\dot{\bar{e}}_{v_t} = -k_1 L_{e_{ts}} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 L_{e_{ts}} R_s R_s^\top \bar{e}_{v_t} - k_3 \bar{e}_{v_t} - k_1 E_{t_s}^\top \mathbb{E} \nabla_\alpha \tilde{W} \tag{28b}$$

Remark 4. *$E_{t_s}^\top E_{t_s} = L_{e_{ts}}$ corresponds to the edge Laplacian of a spanning tree and has $N-1$ edges—see [13, Remark 3].* •

Proposition 1. *Consider the system* (1) *in closed loop with the distributed control law given by* (27)*. Assume that the resulting network is structurally balanced and contains a directed spanning tree. Then, the origin of the closed-loop system depicted in* (28) *is asymptotically stable under any initial conditions respecting the constraints in* (18)*, $|\alpha_k(0)| > \Delta_k$ for all $k \leq M$, and these constraints are satisfied for all t.* □

Proof. First, employing (15), we express the inter-agent distance constraints in (18) in terms of the errors corresponding to the spanning tree. Let $\mathcal{I}_t = \mathcal{I}_{r_t} \cap \mathcal{I}_{c_t}$, with

$$\mathcal{I}_{r_t} := \{\bar{e}_{x_{t_k}} \in \mathbb{R}^n : |r_{s_k}^\top \bar{e}_{x_{t_k}} + \alpha_k| < R_k,\ \forall k \in \mathcal{E}_m\}, \tag{29}$$

$$\mathcal{I}_{c_t} := \{\bar{e}_{x_{t_k}} \in \mathbb{R}^n : \Delta_k < |r_{s_k}^\top \bar{e}_{x_{t_k}} + \alpha_k|,\ \forall k \leq M\}. \tag{30}$$

where r_{s_k} is the kth column of R_s. Then, we consider the Lyapunov function candidate

$$V(\alpha, \bar{e}_{x_t}, v) = k_1 \tilde{W}(\alpha, \bar{e}_{x_t}) + \frac{1}{2}|v|^2. \tag{31}$$

Considering the definition of $\tilde{W}(\alpha, \bar{e}_{x_t}, \hat{e})$ in equation (24), and the positive nature of k_1, we note that V in (31) is positive definite with respect to the state variables $\bar{e}_{x_t}$ and $\bar{e}_{v_t}$ as long as $\bar{e}_t \neq 0$ and respects the constraints in (29) and (30). This positivity property holds uniformly in α. It is essential to note that in this part, where a virtual leader is not considered, $a_{il} = 0$ for all $i \leq N$, and $\tilde{W}$ only depends on α and $\bar{e}_{x_t}$ and V depends on α, $\bar{e}_{x_t}$ and v. In addition, by its construction, $V(\alpha, \bar{e}_{x_t}, v) \to \infty$ as $\bar{e}_{x_t}$ approaches the boundary $\partial\mathcal{I}_t$ of the set $\mathcal{I}_t$, for a given $\bar{e}_{v_t}$ and v.

Next, we calculate the total derivative of V. From the definition of α in (19), we have that $\alpha = E^\top x - E_s^\top \bar{x}$. Its derivative gives $\dot{\alpha} = (E - E_s)^\top v = \mathbb{E}^\top v$. For (1b) and (27), we have

$$\dot{v} = -k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} - k_3 v - k_1 \mathbb{E} \nabla_\alpha \tilde{W}. \tag{32}$$

Then,

$$\begin{aligned}\dot{V}(\alpha, \bar{e}_{x_t}, v) = k_1 \nabla_{\bar{e}_{x_t}} \tilde{W}^\top \dot{\bar{e}}_{x_t} + k_1 \nabla_\alpha \tilde{W}^\top \mathbb{E}^\top v + \dot{v}^\top \Big[& -k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} \\ & -k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} - k_3 v - k_1 \mathbb{E} \nabla_\alpha \tilde{W} \Big]\end{aligned}$$

and using (14) and (15), we obtain

$$\dot{V}(\alpha, \bar{e}_{x_t}, v) = -\, k_2 |\bar{e}_v|^2 - k_3 |v|^2,$$

which is negative semidefinite.

Next, Barbashin-Krasovskĭi's theorem is applied. For the set $\{(\bar{e},\, v) : \dot{V} = 0\}$ it is established that $\bar{e}_v = 0$ and $v = 0$. Consequently, $\dot{\alpha} = \mathbb{E}^\top v = 0$ leading to α being constant. Furthermore, following (32), we have

$$-k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} = k_1 \mathbb{E} \nabla_\alpha \tilde{W}. \tag{33}$$

After (22), it is also derived that

$$\nabla_\alpha \tilde{W} = \nabla_{\bar{e}_{x_t}} \tilde{W} - \frac{\partial}{\partial \alpha} \left\{ \frac{\partial W}{\partial \alpha}(\alpha) \right\} \bar{e}_{x_t}. \tag{34}$$

Since α remains constant on $\{\dot{V} = 0\}$, the last term on the right-hand-side of (34) equals zero. Thus, (33) holds if and only if $-k_1 [E_{t_s} + E_t - E_{t_s}] \nabla_{\bar{e}_{x_t}} \tilde{W} = 0$. Here, E_t corresponds to the incidence matrix of a spanning tree and is full rank. Then, we have $\nabla_{\bar{e}_{x_t}} \tilde{W}$ only vanishes at zero. It can be deduced that the origin is the only solution of (28) that remains in $\{(\bar{e},\, v) : \dot{V} = 0\}$ for all t. Thus, asymptotic stability in the large, within the domain of the definition of V, is established.

Next, we establish inter-agent collision avoidance and ensure connectivity preservation. Referring to (29), we note that if $\bar{e}_{x_t} \in \mathcal{I}_t$ then $\bar{e}_x \in \mathcal{I}$, where $\mathcal{I} = \mathcal{I}_r \cap \mathcal{I}_c$. To prove the forward invariance of $\mathcal{I}_t$, we employ a proof by contradiction. Assume that there exist a $T > 0$ such that $\bar{e}_{x_t}(T) \notin \mathcal{I}_t$. This implies that either $|\bar{e}_{x_{t_k}} + \alpha_k| \to \Delta_k,\ k \leq M$ or $|\bar{e}_{x_{t_k}} + \alpha_k| \to R_k,\ k \in \mathcal{E}_m$ for at least one $k \leq M$. In either case, $\tilde{W}_k(\alpha_k, \bar{e}_{x_{t_k}}) \to \infty$, leading to $V(\alpha, \bar{e}_{x_t}, v) \to \infty$ as $t \to T$. However, this contradicts the fact that $\dot{V}(\alpha, \bar{e}_{x_t}, v) \leq 0$. Hence, inter-agent collision avoidance and connectivity maintenance are ensured.

We now demonstrate that the set $\mathcal{I}$ serves as the domain of attraction for the closed-loop system (28), by proving that all solutions originating from $\mathcal{I}_t$ converge to the origin. Consider any $\epsilon_1 \in (0, R_k)$ and $\epsilon_2 \in (0, \Delta_k)$. Define the subsets $\mathcal{I}_{\epsilon_{r_t}} \subset \mathcal{I}_{r_t}$ and $\mathcal{I}_{\epsilon_{c_t}} \subset \mathcal{I}_{c_t}$ as follows: $\mathcal{I}_{\epsilon_{r_t}} := \{\bar{e}_{x_k} \in \mathbb{R} : |\bar{e}_{x_k} + \alpha_k| < R_k - \epsilon_1,\ \forall k \in \mathcal{E}_m\}$ and $\mathcal{I}_{\epsilon_{c_t}} := \{\bar{e}_{x_k} \in \mathbb{R} : \Delta_k + \epsilon_2 < |\bar{e}_{x_k} + \alpha_k|,\ \forall k \leq M\}$ with $\mathcal{I}_{\epsilon_{r_t}} \cap \mathcal{I}_{\epsilon_{c_t}} = \mathcal{I}_{\epsilon_t}$. With the definition of $\tilde{W}(\alpha, \bar{e}_{x_t})$, it follows that $V(\alpha, \bar{e}_{x_t}, v)$ is positive definite for all $\bar{e}_{x_t} \in \mathcal{I}_{\epsilon_t}$ and $\bar{e}_{v_t} \in \mathbb{R}$ satisfying $a|\bar{e}_{x_t}|^2 + b|\bar{e}_{v_t}|^2 + c|v|^2 \leq V(\alpha, \bar{e}_{x_t}, v) \leq h(|\bar{e}_{x_t}|) + d|\bar{e}_{v_t}|^2 + f|v|^2$ with $a, b, c, d, f > 0$ and h is a strictly increasing everywhere in $\mathcal{I}_{\epsilon_t}$. Then, $V(\alpha, \bar{e}_{x_t}, v) \to 0$ as $\bar{e}_t \to 0$ and $v \to 0$. Hence, the origin is asymptotically stable for all trajectories for the closed-loop system starting in $\mathcal{I}_{\epsilon_t}$. Since ϵ_1 and ϵ_2 are arbitrarily small, letting $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ establishes the asymptotic stability of the origin of (28) for all trajectories originating from $\mathcal{I}_t$. Consequently, bipartite formation-consensus is achieved with inter-agent collision avoidance and connectivity. □

3.3.2. Leader-follower bipartite formation-consensus

In this Subsection, we introduce a virtual leader in the system, and we analyse the stability of (1) and (5), in closed loop with the bipartite formation control law (25). Now, after (15), (14) and (26), the control law in (25) becomes

$$u = -k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} - k_3 v - k_1 \mathbb{E} \nabla_\alpha \tilde{W} - k_1 \nabla_{\hat{e}} \tilde{W}. \tag{35}$$

Then, differentiating on both sides of (11) we obtain

$$\dot{\bar{e}}_{x_t} = \bar{e}_{v_t} \tag{36a}$$

$$\dot{\bar{e}}_{v_t} = -k_1 L_{e_{ts}} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 L_{e_{ts}} R_s R_s^\top \bar{e}_{v_t} - k_3 \bar{e}_{v_t} - k_1 E_{t_s}^\top \mathbb{E} \nabla_\alpha \tilde{W} - k_1 E_{t_s}^\top \nabla_{\hat{e}} \tilde{W} \tag{36b}$$

Proposition 2. *Consider the system* (1) *and* (5) *in closed loop with the distributed control law* (35) *and assume that the resulting network is structurally balanced and contains an underlying spanning tree. Then, the origin for the closed-loop system* (36) *is asymptotically stable for any initial conditions respecting the constraints in* (18)*, and* $|\alpha_k(0)| > \Delta_k$ *for all* $k \leq M$*. In addition, the constraints are satisfied for all* t*. Moreover,* $\bar{x} \to \Phi x_0$ *as* $t \to \infty$*.* □

Proof. We consider the Lyapunov function candidate in (31). In this context, where a virtual leader is taken into account within the network, the Lyapunov function also depends on $\hat{e}$ leading to $V(\alpha, \bar{e}_{x_t}, v, \hat{e}) = k_1 \tilde{W}(\alpha, \bar{e}_{x_t}, \hat{e}) + \frac{1}{2}|v|^2$. Next, we compute its total derivative. For (1b) and (35), we have

$$\dot{v} = -k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} - k_3 v - k_1 \mathbb{E} \nabla_\alpha \tilde{W} - k_1 \nabla_{\hat{e}} \tilde{W}. \tag{37}$$

Then,

$$\dot{V}(\alpha, \bar{e}_{x_t}, v, \hat{e}) = k_1 \nabla_{\bar{e}_{x_t}} \tilde{W}^\top \dot{\bar{e}}_{x_t} + k_1 \nabla_\alpha \tilde{W}^\top \mathbb{E}^\top v + k_1 \nabla_{\hat{e}} \bar{W}^\top \dot{\hat{e}} + v^\top \Big[-k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} - k_3 v - k_1 \mathbb{E} \nabla_\alpha \tilde{W} - k_1 \nabla_{\hat{e}} \tilde{W} \Big]$$

and using the fact that $\dot{\hat{e}} = v$ and, (14) and (15), we obtain

$$\dot{V}(\alpha, \bar{e}_{x_t}, v) = -\, k_2|\bar{e}_v|^2 - k_3|v|^2,$$

which is negative semidefinite.

Next, Barbashin-Krasovskĭi's theorem is applied. For the set $\{(\bar{e},\, v,\, \hat{e}) \,:\, \dot{V} = 0\}$ it is established that $\bar{e}_v = 0$ and $v = 0$, which gives that α is constant. In turn, after (37), we have

$$-k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_1 \nabla_{\hat{e}} \tilde{W} = k_1 \mathbb{E} \nabla_{\alpha} \tilde{W}. \tag{38}$$

On the other hand, since we have $\alpha \equiv \text{const}$ on $\{\dot{V} = 0\}$, it follows that with (34)

$$-E_t \nabla_{\bar{e}_{x_t}} \tilde{W} - \nabla_{\hat{e}} \tilde{W} = 0 \Leftrightarrow - \begin{bmatrix} E_t & I_N \end{bmatrix} \begin{bmatrix} \nabla_{\bar{e}_{x_t}} \tilde{W} \\ \nabla_{\hat{e}} \tilde{W} \end{bmatrix} = 0.$$

E_t corresponds to the incidence matrix of a spanning-tree, so it is full rank, and I_N is also full rank. The matrix $\begin{bmatrix} E_t & I_N \end{bmatrix}$ has linearly independent columns, so is also full rank and has a pseudoinverse, such that $\begin{bmatrix} E_t & I_N \end{bmatrix}^+ \begin{bmatrix} E_t & I_N \end{bmatrix} = I$. Then, it is deduced that the only solution is $\nabla_{\bar{e}_{x_t}} \tilde{W} = 0$ and $\nabla_{\hat{e}} \tilde{W} = 0$ so $\bar{e}_{x_t} = 0$ and $\hat{e} = 0$, which gives $\bar{x}_i - \text{sgn}(a_{ij})\bar{x}_j = 0$ and $\bar{x}_i = \phi_i x_0$ for all $i, j \leq N$. It follows that the only solution of (36) that remains in $\{(\bar{e},\, v,\, \hat{e}) \,:\, \dot{V} = 0\}$ for all t, is the origin. Thus, asymptotic stability in the large, within the domain of the definition of V, is established.

Next, we establish inter-agent collision avoidance and ensure connectivity preservation, as in the proof of Proposition 1. Referring to (29), we note that if $\bar{e}_{x_t} \in \mathcal{I}_t$ then $\bar{e}_x \in \mathcal{I}$, where $\mathcal{I} = \mathcal{I}_r \cap \mathcal{I}_c$. To prove the forward invariance of $\mathcal{I}_t$, we employ a proof by contradiction. Assume that there exist a $T > 0$ such that $\bar{e}_{x_t}(T) \notin \mathcal{I}_t$. This implies that either $|\bar{e}_{x_{t_k}} + \alpha_k| \to \Delta_k,\ k \leq M$ or $|\bar{e}_{x_{t_k}} + \alpha_k| \to R_k,\ k \in \mathcal{E}_m$ for at least one $k \leq M$, which makes $\tilde{W}_k(\alpha_k, \bar{e}_{x_{t_k}}, \hat{e}) \to \infty$, leading to $V(\alpha, \bar{e}_{x_t}, v, \hat{e}) \to \infty$ as $t \to T$. However, the latter contradicts the fact that $\dot{V}(\alpha, \bar{e}_{x_t}, v, \hat{e}) \leq 0$. Hence, inter-agent collision avoidance and connectivity maintenance are ensured.

Then, we demonstrate that the set $\mathcal{I}$ serves as the domain of attraction for the closed-loop system (36), by proving that all solutions originating from $\mathcal{I}_t$ converge to the origin. For any $\epsilon_1 \in (0, R_k)$ and $\epsilon_2 \in (0, \Delta_k)$, define subsets $\mathcal{I}_{\epsilon_{r_t}} \subset \mathcal{I}_{r_t}$ and $\mathcal{I}_{\epsilon_{c_t}} \subset \mathcal{I}_{c_t}$ as in the proof of Proposition 1. From the definition of $\tilde{W}(\alpha, \bar{e}_{x_t}, \hat{e})$, it follows that $V(\alpha, \bar{e}_{x_t}, v, \hat{e})$ is positive definite for all $\bar{e}_{x_t} \in \mathcal{I}_{\epsilon_t}$ and $\bar{e}_{v_t} \in \mathbb{R}$ and satisfies $a_1|\bar{e}_{x_t}|^2 + b_1|\bar{e}_{v_t}|^2 + c_1|v|^2 + d_1|\hat{e}|^2 \leq V(\alpha, \bar{e}_{x_t}, v, \hat{e}) \leq h(|\bar{e}_{x_t}|) + b_2|\bar{e}_{v_t}|^2 + c_2|v|^2 + d_2|\hat{e}|^2$ with $a_1, b_1, c_1, d_1, b_2, c_2, d_2 > 0$ and h is a strictly increasing everywhere in $\mathcal{I}_{\epsilon_t}$. Then, $V(\alpha, \bar{e}_{x_t}, v, \hat{e}) \to 0$ as $\bar{e}_t \to 0,\ v \to 0$ and $\hat{e} \to 0$. Hence, the origin is asymptotically stable for all trajectories for the closed-loop system starting in $\mathcal{I}_{\epsilon_t}$. Since ϵ_1 and ϵ_2 are arbitrarily small, letting $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ establishes the asymptotic stability of the origin of (28) for all trajectories originating from $\mathcal{I}_t$. Consequently, bipartite formation-consensus is achieved with inter-agent collision avoidance and connectivity. □

4. Simulation Results

In this section, we present some numerical examples to show the performance of our control laws. We consider a structurally balanced signed network of nonholonomic

mobile robots. Let $r_i = [r_{x_i}\ r_{y_i}]^\top$ be the inertial position, θ_i the orientation, v_i the linear speed, ω_i the angular speed, m_i the mass, J_i the moment of inertia, η_i be the applied force and torque, and F_i be Lagrangian dynamics. Then, the robot's model is given by

$$\begin{bmatrix}\dot{r}_{x_i}\\ \dot{r}_{y_i}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}v_i\cos\theta_i\\ v_i\sin\theta_i\\ \omega_i\end{bmatrix}, \quad \begin{bmatrix}\dot{v}_i\\ \dot{\omega}\end{bmatrix} = \begin{bmatrix}\frac{1}{m_i} & 0\\ 0 & \frac{1}{J_i}\end{bmatrix}[\eta_i + F_i]. \tag{39}$$

Then, we define $s_{\theta_i} := \sin\theta_i$ and $c_{\theta_i} := \cos\theta_i$ and choose the reference point $p_i = r_i + \delta_i\begin{bmatrix}c_{\theta_i}\ s_{\theta_i}\end{bmatrix}^\top$ located at a distance $\delta_i = 0.1$m along the line that is perpendicular to the wheels' axis and we define

$$\zeta := \begin{bmatrix} r_{x_i} + \delta_i c_{\theta_i}\\ r_{y_i} + \delta_i s_{\theta_i}\\ v_i c_{\theta_i} - \delta_i\omega_i s_{\theta_i}\\ v_i s_{\theta_i} + \delta_i\omega_i c_{\theta_i}\\ \theta_i\end{bmatrix}. \tag{40}$$

In transformed coordinates, with $p_i = [\zeta_{1_i}^\top\ \zeta_{2_i}^\top]^\top$, we have

$$\begin{aligned}
\begin{bmatrix}\dot{\zeta}_{1_i}\\ \dot{\zeta}_{2_i}\end{bmatrix} &= \begin{bmatrix}\zeta_{3_i}\\ \zeta_{4_i}\end{bmatrix},\\
\begin{bmatrix}\dot{\zeta}_{3_i}\\ \dot{\zeta}_{4_i}\end{bmatrix} &= \begin{bmatrix}-v_i\omega_i s_{\theta_i} - \delta_i\omega_i^2 c_{\theta_i}\\ v_i\omega_i c_{\theta_i} - \delta_i\omega_i^2 s_{\theta_i}\end{bmatrix} + \begin{bmatrix}\frac{1}{m_i}c_{\theta_i} & -\frac{\delta_i}{J_i}s_{\theta_i}\\ \frac{1}{m_i}s_{\theta_i} & \frac{\delta_i}{J_i}c_{\theta_i}\end{bmatrix}[\eta_i + F_i],\\
\dot{\zeta}_{5_i} &= -\frac{1}{2\delta_i}\zeta_{3_i}\sin\zeta_{5_i} + \frac{1}{2\delta_i}\zeta_{4_I}\cos\zeta_{5_i}.
\end{aligned} \tag{41}$$

The feedback-linearizing control η_i is given by

$$\eta_i = \begin{bmatrix}\frac{1}{m_i}c_{\theta_i} & -\frac{\delta_i}{J_i}s_{\theta_i}\\ \frac{1}{m_i}s_{\theta_i} & \frac{\delta_i}{J_i}c_{\theta_i}\end{bmatrix}^{-1}\left[u_i - \begin{bmatrix}-v_i\omega_i s_{\theta_i} - \delta_i\omega_i^2 c_{\theta_i}\\ v_i\omega_i c_{\theta_i} - \delta_i\omega_i^2 s_{\theta_i}\end{bmatrix}\right] - F_i \tag{42}$$

which yields $[\dot{\zeta}_{3_i}\ \dot{\zeta}_{4_i}]^\top = u_i$.

4.1. Leaderless bipartite formation-consensus

We first provide a numerical example to show the performance of our control law (25) with $k_1 = 1$, $k_2 = 1.2$, $k_3 = 1$ and the barrier-Lyapunov function in (22), with

$$B_{r_k}(s) = \ln\left(\frac{R_k^2}{R_k^2 - |s|^2}\right), \quad B_{c_k}(s) = \ln\left(\frac{|s|^2}{|s|^2 - \Delta_k^2}\right), \tag{43}$$

We implement (42) with u_i as in (25), $x_i = [\zeta_{1_i}\ \zeta_{2_i}]^\top$, $v_i = [\zeta_{3_i}\ \zeta_{4_i}]^\top$, $m_i = 8$kg and $J_i = 0.12\text{kg/m}^2$, $\forall i \leq N$.

We consider an undirected signed network of 10 agents and 10 edges as the one depicted in Figure 2, subject to inter-agent collision avoidance and connectivity maintenance restrictions. We define the orientation of the edges as fol-

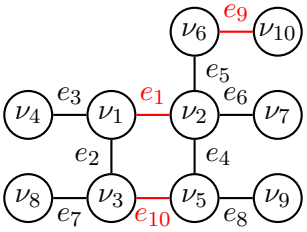

Figure 2. Network 1: A structurally balanced, leaderless, and undirected signed network of 10 mobile robots. The black lines represent cooperative edges, and the red lines represent the competitive ones.

lows: $e_1 = \nu_1 + \nu_2$, $e_2 = \nu_1 - \nu_3$, $e_3 = \nu_1 - \nu_4$, $e_4 = \nu_2 - \nu_5$, $e_5 = \nu_2 - \nu_6$, $e_6 = \nu_2 - \nu_7$, $e_7 = \nu_3 - \nu_8$, $e_8 = \nu_5 - \nu_9$, $e_9 = \nu_6 + \nu_{10}$ and $e_{10} = \nu_3 + \nu_5$. The incidence matrix corresponding to the graph is

$$E_s = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

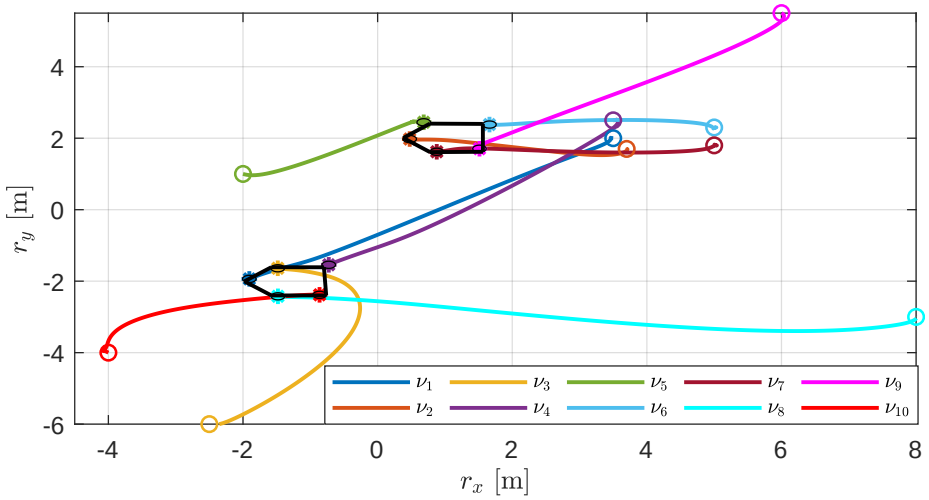

Figure 3. Bipartite formation of system (39) with control input (25) in the leader centered reference frame. The asterisks are the inertial positions of the robots. The reference points p_i of the mobile robots, on the black circles around the asterisks, of the two disjoint subgroups form a formation around two symmetrical consensus points.

The set of nodes may be split into two disjoint subsets, such as $\mathcal{V}_1 = \{\nu_1, \nu_3, \nu_4, \nu_8, \nu_{10}\}$ and $\mathcal{V}_2 = \{\nu_2, \nu_5, \nu_6, \nu_7, \nu_9\}$ so the network is structurally balanced. From the decomposition in (13), edges e_i, $i \leq 9$ correspond to $\mathcal{G}_t$ and the remaining edge e_{10} corresponds to $\mathcal{G}_c$. The eigenvalues of L_s and L_{e_s}

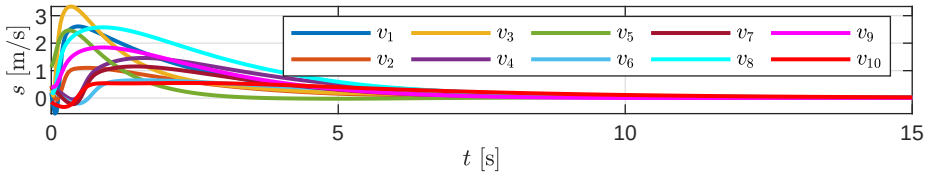

Figure 4. Bipartite formation of system (39) with control input (25) on velocity, where $k_3 > 0$. The velocities of all agents converge to zero.

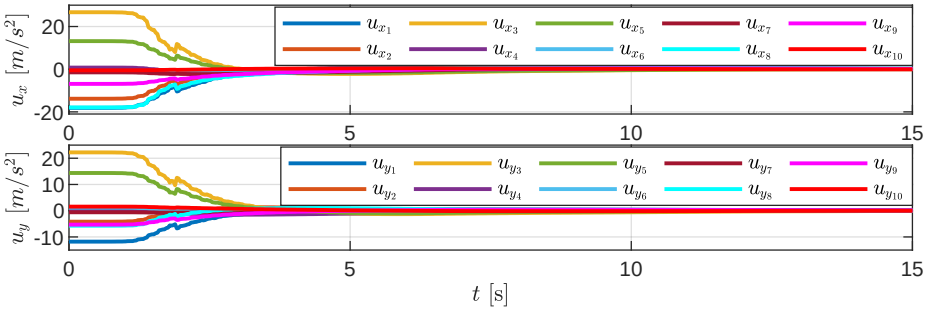

Figure 5. Control input (25) of all agents converge to zero.

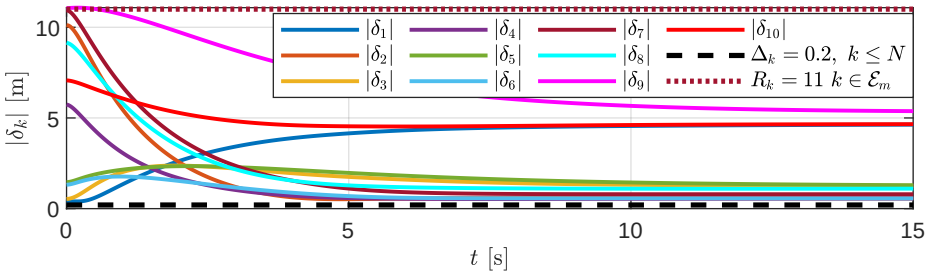

Figure 6. Trajectories of the norm of the inter-agent distances with control input (25). The dashed lines are the minimum, and the dotted lines are the maximum distance constraints for agents. All inter-agent safety proximity constraints are respected.

are $\lambda_{L_s} = \lambda_{L_{e_s}} = \{0,\ 0.33,\ 0.58,\ 0.67,\ 0.79,\ 2,\ 2.5,\ 3.41,\ 4,\ 5.7\}$ as the considered network has the same number of edges as the number of nodes. The respective agents' initial states are $r_x(0) = [3.5,\ 3.7,\ -2.5,\ 3.5,\ -2,\ 5,\ 5,\ 8,\ 6,\ -4]^\top$, $r_y(0) = [2,\ 1.7,\ -6,\ 2.5,\ 1,\ 2.3,\ 1.8,\ -3,\ 5.5,\ -4]^\top$, $v(0) = [0.2,\ 0.45,\ 0.32,\ 0.32,\ 1.12,\ 0.3,\ 0.4,\ 0.22,\ 0.42,\ -0.14]^\top$, $\theta(0) = [0,\ -0.46,\ -1.25,\ 0.32,\ -0.46,\ 0,\ 0,\ -1.11,\ -0.79,\ -0.79]^\top$ and the relative displacements are $d_x = [-0.8,\ -0.8,\ -0.4,\ 0.4,\ -0.4,\ 0.4,\ -0.4,\ -0.4,\ 0.4,\ 0.4]^\top$ and $d_y = [0,\ 0,\ 0.4,\ 0.4,\ 0.4,\ 0.4,\ -0.4,\ -0.4,\ -0.4,\ -0.4]^\top$. The constraint sets are $\Delta_k = 0.2$ for all $k \leq M$ and $R_k = 11$ for all $k \in \mathcal{E}_m$. The paths of each agent up to bipartite formation are depicted in Figure 3. The reference points of the mobile robots reach the desired formation around two symmetric consensus points. The velocities of mobile robots are depicted in Figure 4, and velocities converge to zero. Moreover, it is clear from Figure 6 that the inter-agent collision avoidance and connectivity maintenance constraints in (8)–(9) are always respected.

4.2. Leader-follower bipartite formation-consensus

We then provide a numerical example to show the performance of our control law (35), in the presence of a virtual leader ν_0, with $k_1 = 1$, $k_2 = 1.2$, $k_3 = 1$ and the

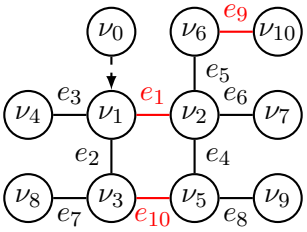

Figure 7. Network 2: A structurally balanced undirected signed network of 10 mobile robots with a virtual leader ν_0.

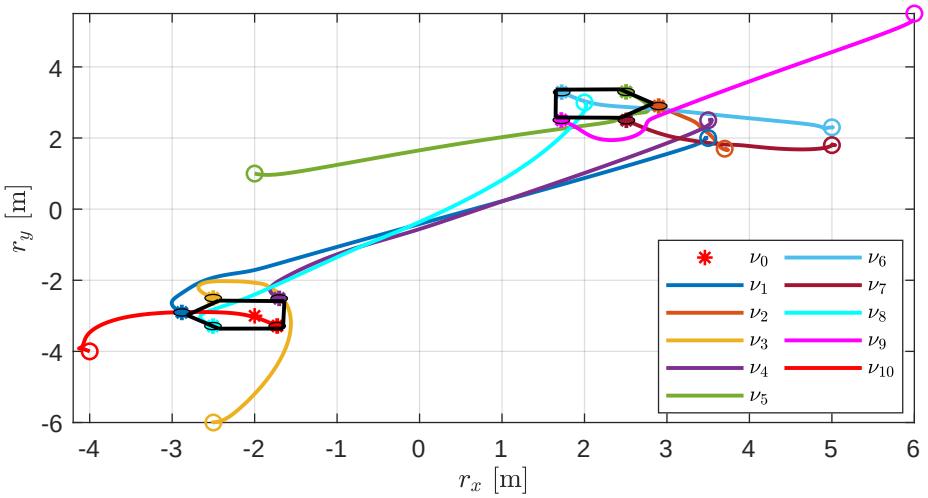

Figure 8. Bipartite formation of system (39) with control input (25). The asterisks are the inertial positions of the robots. The reference points p_i of the mobile robots, on the black circles around the asterisks, of the two disjoint subgroups form a formation around two symmetrical consensus points.

barrier-Lyapunov function in (22), with (43). We consider the same undirected signed network in Figure 2 with a virtual leader ν_0 giving information to agent ν_1, depicted in Figure 7. The virtual leader's position is $x_0 = [-2\ -3]^\top$, $A_l = \text{diag}(1, 0, \ldots, 0)$ and $\Phi = [1, -1, 1, 1, -1, -1, -1, 1, -1, 1]^\top$. Thus, we implement (42) with u_i as in (35), $x_i = [\zeta_{1_i}\ \zeta_{2_i}]^\top$, $v_i = [\zeta_{3_i}\ \zeta_{4_i}]^\top$, $m_i = 8\text{kg}$ and $J_i = 0.12\text{kg/m}^2$, $\forall i \leq N$. The respective agents' inertial positions, velocities, and orientations are the same as before, while the relative displacements are $d_x = [-0.8,\ 0.8,\ -0.4,\ 0.4,\ 0.4,\ -0.4,\ 0.4,\ -0.4,\ -0.4,\ 0.4]^\top$ and $d_y = [0,\ 0,\ 0.4,\ 0.4,\ 0.4,\ 0.4,\ -0.4,\ -0.4,\ -0.4,\ -0.4]^\top$. The constraints are also set as before. The paths of each agent up to leader-follower bipartite formation are depicted in Figure 8. Agents in $\mathcal{V}_1$ converge around the virtual leader's position, while agents in $\mathcal{V}_2$ reach formation around the virtual leader's symmetric position. The velocities of mobile robots are depicted in Figure 10, and velocities converge to zero. Moreover, it is clear from Figure 11 that the inter-agent collision avoidance and connectivity maintenance constraints in (8)–(9) are always respected.

5. Conclusions

We presented a BLF-based distributed control law to solve the bipartite formation-consensus control problem for double integrators over structurally balanced undirected signed networks. The proposed control law guarantees inter-agent collision avoidance

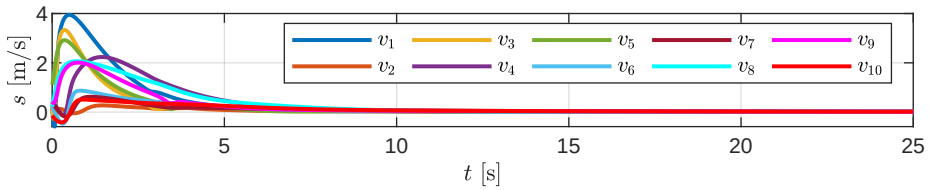

Figure 9. Bipartite formation of system (39) with control input (25) on velocity, where $k_3 > 0$. The velocities of all agents converge to zero.

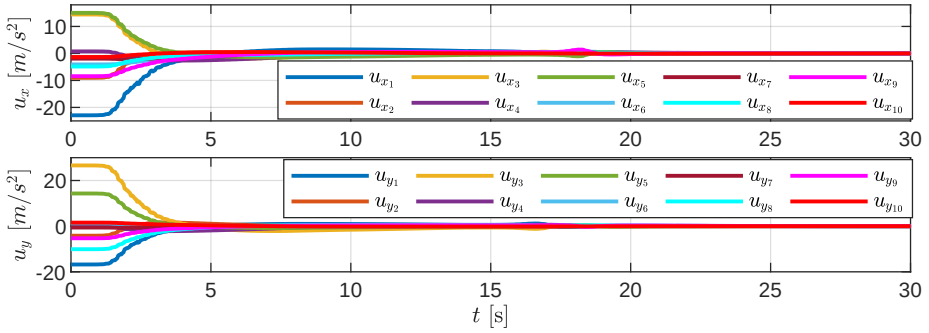

Figure 10. Control input (25) of all agents converge to zero.

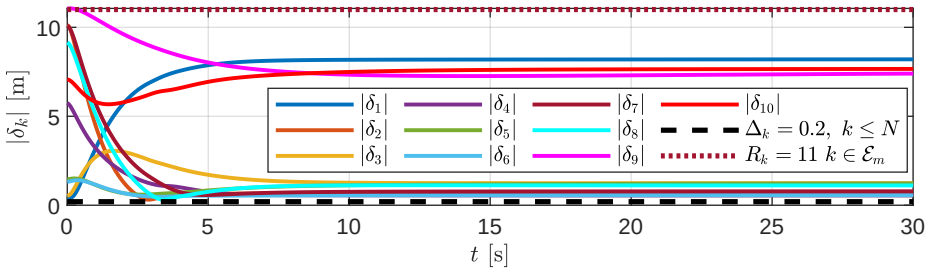

Figure 11. Trajectories of the norm of the inter-agent distances with control input (25). The dashed lines are the minimum, and the dotted lines are the maximum distance constraints for agents. All inter-agent safety proximity constraints are respected.

for every agent in the networks and connectivity maintenance for two cooperative agents. We analyzed the asymptotic stability of the system in terms of the errors and using Lyapunov's direct method. We also illustrated the practical application of our results through numerical simulations involving the formation-consensus control of nonholonomic vehicles. Further research is focused on extending these results to directed signed networks with one or more leaders.

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