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# Simplicial subdivision of simplices of arbitrary dimension in a space of constant non-zero curvature with bounded quality

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## Abstract

In 1942 Freudenthal showed that a simplex in Euclidean space can be subdivided such that the quality (well-shapedness of the simplex, quantified in terms of e.g. fatness) of the simplices in the subdivision is lower bounded. This answered a question of Brouwer. Recently, Brunck discussed the same problem for simplices in two dimensional spaces of constant curvature and provided a closely related construction. In this note we exhibit the close relation between the two constructions and observe that, using radial projection, Freudenthal’s construction and result immediately generalize to simplices in spaces of constant non-zero curvature of arbitrary dimension. This note focuses on displaying classical results that may be unfamiliar to people in the field that do not speak German, rather than building new theory.

**Keywords:** Spaces of constant curvature, simplicial subdivision

**MSC Classification:** 65D18

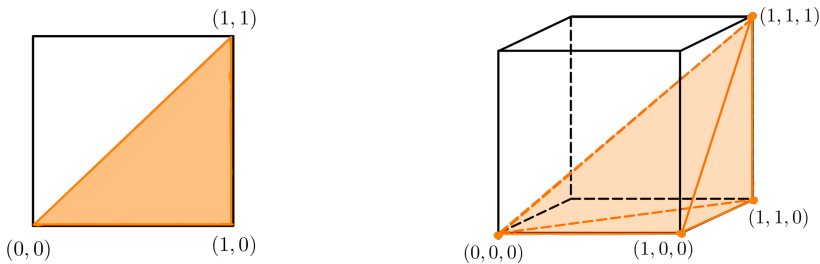
# 1 The Coxeter-Freudenthal-Kuhn triangulation

Because of historical reasons, Freudenthal's triangulation [1] of the Euclidean space is referred to as the *Coxeter-Freudenthal-Kuhn triangulation*. One of the equivalent definitions is as a hyperplane arrangement<sup>1</sup>. More precisely:

**Definition 1.1** *The Coxeter-Freudenthal-Kuhn triangulation is the hyperplane arrangement  $\mathcal{H}_{E_{FK}} = \{x \in \mathbb{R}^d \mid \langle x, u \rangle = \ell, u \in E_{FK}, \ell \in \mathbb{Z}\}$  associated to the set of vectors  $E_{FK} = \{e_1, \dots, e_d\} \cup \{u_{i,j} = e_j - e_i \mid 1 \leq i < j \leq d\}$ .*

As implied by the name, the cells of this hyperplane arrangement are simplices.

One can also define the triangulation by subdividing  $\mathbb{R}^d$  into unit cubes, and in turn subdivide each of these by looking at the coordinates of each point inside it modulo 1 (the parts of the coordinates after the decimal point) and sorting them. This yields a permutation of  $\{1, \dots, d\}$ ; points with the same permutation are assigned to the same simplex (as illustrated in Figure 1).



**Fig. 1:** Simplices in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  that correspond to the identity permutation — the coordinates are sorted in descending order.

One can rescale the Coxeter-Freudenthal-Kuhn triangulation by an integer factor  $k$ , which subdivides each simplex in the triangulation. We call this subdivision by a factor  $k$ . This subdivision was also used in [5].

The Coxeter-Freudenthal-Kuhn triangulation can be used to induce a subdivision of any simplex, since any such simplex can be mapped to a simplex in the Coxeter-Freudenthal-Kuhn triangulation using barycentric coordinates. The fatness<sup>2</sup> of simplices in the subdivision is independent of  $k$ . We refer to for example [6, 7] for a more general discussion on the quality of simplices in such arrangements.

<sup>1</sup>This result has a long history, see the bibliography and some historical remarks in [2]. We further note that Freudenthal's construction is closely related to Coxeter triangulations [2–4].

<sup>2</sup>The fatness is the volume divided by the longest edge to the power  $d$ . It is a common quality measure.

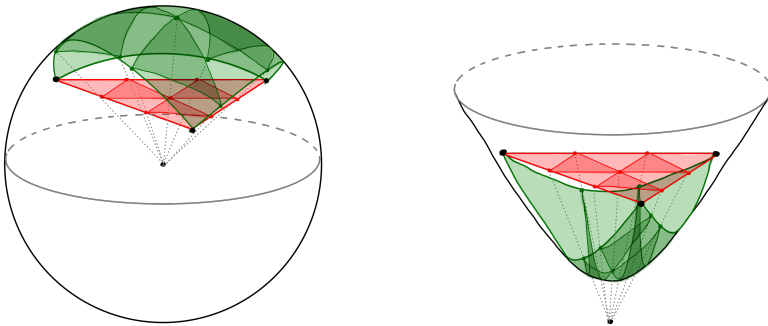
## 2 Models for simply connected spaces of constant curvature

We use the following two models for simply connected spaces of constant non-zero curvature: For spaces of positive constant curvature we use the symmetrically embedded sphere in  $\mathbb{R}^{d+1}$ , and for spaces of negative constant curvature we use the hyperboloid model in the Minkowski space  $\mathbb{R}^{d,1}$ . In both models geodesics are the intersections of two-dimensional planes through the origin in the ambient space and the model itself. This implies that the radial projection of an edge in  $\mathbb{R}^{d+1}$  (resp.  $\mathbb{R}^{d,1}$ ) yields a geodesic edge in the space.

Given a set  $S$  in a  $d$ -dimensional affine plane not containing the origin, the radial projection of the convex hull (in the affine space) of  $S$  is the convex hull (in the space of constant curvature) of the radial projection of  $S$ . Hence, the radial projection of a Euclidean simplex in the ambient space (whose affine hull does not contain the origin) yields a simplex on the space of constant curvature.

## 3 Subdividing simplices of constant curvature

We now consider a simplex  $\Delta$  of constant non-zero curvature, whose circumradius is either finite (if the curvature is negative) or is less than the circumradius of the model sphere (in the curvature is positive). The convex hull of the vertices in  $\mathbb{R}^{d+1}$  (resp.  $\mathbb{R}^{d,1}$ ) yields a Euclidean simplex, which can be subdivided by a factor  $k$  according to Freudenthal's scheme. Radial projection of this subdivision on the space of constant curvature yields a subdivision of  $\Delta$ . The scheme is illustrated in Figure 2.



**Fig. 2:** Radial projection of Freudenthal's scheme for  $d = 2$  and  $k = 3$ . The simplex  $\Delta$  is in green, the corresponding Euclidean simplex in red.

As long as there are lower and upper bounds on the distance between the plane containing  $\Delta$  and the origin, one can bound the quality of the simplices in the subdivision, as well as their dihedral angles.

## 4 Comparison of subdivision methods

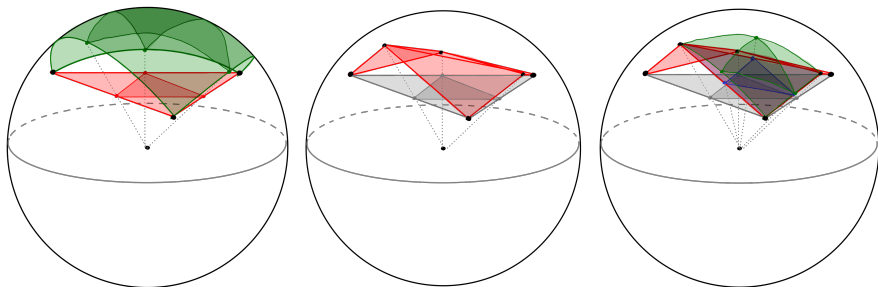
In [8], Brunck suggested the below construction in two dimensions. We reformulate his construction, elucidating that it generalizes to arbitrary dimensions. Instead of subdividing the Euclidean simplex using Freudenthal's scheme, Brunck iterates as follows (see also Figure 3):

**Step 1.** Consider the convex hull of the vertices in the ambient space, which yields a Euclidean simplex (as in the construction above).

**Step 2.** Subdivide the Euclidean simplex using Freudenthal's scheme of factor 2.

**Step 3.** Project the Euclidean subsimplices radially to obtain subsimplices in the space of constant curvature.

**Step 4.** Apply the steps of the construction on each of the subsimplices.



**Fig. 3:** Left: Steps 1-3 of Brunck's subdivision scheme. Centre: The Euclidean convex hulls of subsimplices, on which the scheme is to be iterated. Right: Steps 1-3 are performed on one of the subsimplices.

After  $m$  iterations, this scheme yields a subdivision which is close (for small simplices) to, but different<sup>3</sup> from, the subdivision one would obtain using Freudenthal's scheme of factor  $k = 2^m$ .

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<sup>3</sup>The difference stems from the following observation: If one subdivides a cord on a circle into  $k$  equal parts and radially projects the pieces on the circle then the resulting parts of the circle do not all have the same length.

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## Conflicts of interest/Competing interests

There are no conflicts of interest.

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