# Simplicial subdivision of simplices of arbitrary dimension in a space of constant non-zero curvature with bounded quality 

Jean-Daniel Boissonnat, Florestan Brunck, Hana Dal Poz Kouřimská, Arijit Ghosh, Mathijs Wintraecken

## To cite this version:

Jean-Daniel Boissonnat, Florestan Brunck, Hana Dal Poz Kouřimská, Arijit Ghosh, Mathijs Wintraecken. Simplicial subdivision of simplices of arbitrary dimension in a space of constant non-zero curvature with bounded quality. 2023. hal-04297480

HAL Id: hal-04297480
https://hal.science/hal-04297480
Preprint submitted on 21 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Simplicial subdivision of simplices of arbitrary dimension in a space of constant non-zero curvature with bounded quality 

Jean-Daniel Boissonnat ${ }^{1}$, Florestan Brunck ${ }^{2}$, Hana Dal Poz Kouřimská ${ }^{2}$, Arijit Ghosh ${ }^{3}$ and Mathijs Wintraecken ${ }^{1,2}$<br>${ }^{1}$ Datashape, Université Côte d'Azur, Inria, 2004 Route des Lucioles, Sophia Antipolis, 06902, France.<br>${ }^{2}$ Edelsbrunner group, IST Austria, Am Campus 1, Klosterneuburg, 3400, Lower Austria, Austria.<br>${ }^{3}$ ACM Unit, Indian statistical institute, 203, B. T. Road, Kolkata, 700108, West Bengal, India.

Contributing authors: jean-daniel.boissonnat@inria.fr; florestan.brunck@mail.mcgill.ca; hana.kourimska@ist.ac.at; arijitiitkgpster@gmail.com; mathijs.wintraecken@inria.fr;


#### Abstract

In 1942 Freudenthal showed that a simplex in Euclidean space can be subdivided such that the quality (well-shapedness of the simplex, quantified in terms of e.g. fatness) of the simplices in the subdivision is lower bounded. This answered a question of Brouwer. Recently, Brunck discussed the same problem for simplices in two dimensional spaces of constant curvature and provided a closely related construction. In this note we exhibit the close relation between the two constructions and observe that, using radial projection, Freudenthal's construction and result immediately generalize to simplices in spaces of constant non-zero curvature of arbitrary dimension. This note focuses on displaying classical results that may be unfamiliar to people in the field that do not speak German, rather than building new theory.


Keywords: Spaces of constant curvature, simplicial subdivision

## 1 The Coxeter-Freudenthal-Kuhn triangulation

Because of historical reasons, Freudenthal's triangulation [1] of the Euclidean space is referred to as the Coxeter-Freudenthal-Kuhn triangulation. One of the equivalent definitions is as a hyperplane arrangement ${ }^{1}$. More precisely:

Definition 1.1 The Coxeter-Freudenthal-Kuhn triangulation is the hyperplane arrangement $\mathcal{H}_{E_{F K}}=\left\{x \in \mathbb{R}^{d} \mid\langle x, u\rangle=\ell, u \in E_{F K}, \ell \in \mathbb{Z}\right\}$ associated to the set of vectors $E_{F K}=\left\{e_{1}, \ldots, e_{d}\right\} \cup\left\{u_{i, j}=e_{j}-e_{i} \mid 1 \leq i<j \leq d\right\}$.

As implied by the name, the cells of this hyperplane arrangement are simplices.

One can also define the triangulation by subdividing $\mathbb{R}^{d}$ into unit cubes, and in turn subdivide each of these by looking at the coordinates of each point inside it modulo 1 (the parts of the coordinates after the decimal point) and sorting them. This yields a permutation of $\{1, \ldots, d\}$; points with the same permutation are assigned to the same simplex (as illustrated in Figure 1).


Fig. 1: Simplices in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$ that correspond to the identity permutation - the coordinates are sorted in descending order.

One can rescale the Coxeter-Freudenthal-Kuhn triangulation by an integer factor $k$, which subdivides each simplex in the triangulation. We call this subdivision by a factor $k$. This subdivision was also used in [5].

The Coxeter-Freudenthal-Kuhn triangulation can be used to induce a subdivision of any simplex, since any such simplex can be mapped to a simplex in the Coxeter-Freudenthal-Kuhn triangulation using barycentric coordinates. The fatness ${ }^{2}$ of simplices in the subdivision is independent of $k$. We refer to for example $[6,7]$ for a more general discussion on the quality of simplices in such arrangements.

[^0]
## 2 Models for simply connected spaces of constant curvature

We use the following two models for simply connected spaces of constant non-zero curvature: For spaces of positive constant curvature we use the symmetrically embedded sphere in $\mathbb{R}^{d+1}$, and for spaces of negative constant curvature we use the hyperboloid model in the Minkowski space $\mathbb{R}^{d, 1}$. In both models geodesics are the intersections of two-dimensional planes through the origin in the ambient space and the model itself. This implies that the radial projection of an edge in $\mathbb{R}^{d+1}$ (resp. $\mathbb{R}^{d, 1}$ ) yields a geodesic edge in the space.

Given a set $S$ in a $d$-dimensional affine plane not containing the origin, the radial projection of the convex hull (in the affine space) of $S$ is the convex hull (in the space of constant curvature) of the radial projection of $S$. Hence, the radial projection of a Euclidean simplex in the ambient space (whose affine hull does not contain the origin) yields a simplex on the space of constant curvature.

## 3 Subdividing simplices of constant curvature

We now consider a simplex $\Delta$ of constant non-zero curvature, whose circumradius is either finite (if the curvature is negative) or is less than the circumradius of the model sphere (in the curvature is positive). The convex hull of the vertices in $\mathbb{R}^{d+1}$ (resp. $\mathbb{R}^{d, 1}$ ) yields a Euclidean simplex, which can be subdivided by a factor $k$ according to Freudenthal's scheme. Radial projection of this subdivision on the space of constant curvature yields a subdivision of $\Delta$. The scheme is illustrated in Figure 2.


Fig. 2: Radial projection of Freudenthal's scheme for $d=2$ and $k=3$. The simplex $\Delta$ is in green, the corresponding Euclidean simplex in red.

As long as there are lower and upper bounds on the distance between the plane containing $\Delta$ and the origin, one can bound the quality of the simplices in the subdivision, as well as their dihedral angles.

## 4 Comparison of subdivision methods

In [8], Brunck suggested the below construction in two dimensions. We reformulate his construction, elucidating that it generalizes to arbitrary dimensions. Instead of subdividing the Euclidean simplex using Freudenthal's scheme, Brunck iterates as follows (see also Figure 3):

Step 1. Consider the convex hull of the vertices in the ambient space, which yields a Euclidean simplex (as in the construction above).
Step 2. Subdivide the Euclidean simplex using Freudenthal's scheme of factor 2.

Step 3. Project the Euclidean subsimplices radially to obtain subsimplices in the space of constant curvature.
Step 4. Apply the steps of the construction on each of the subsimplices.


Fig. 3: Left: Steps 1-3 of Brunck's subdivision scheme. Centre: The Euclidean convex hulls of subsimplices, on which the scheme is to be iterated. Right: Steps 1-3 are performed on one of the subsimplices.

After $m$ iterations, this scheme yields a subdivision which is close (for small simplices) to, but different ${ }^{3}$ from, the subdivision one would obtain using Freudenthal's scheme of factor $k=2^{m}$.

## Acknowledgements

We are greatly indebted to Ramsay Dyer for discussion.

## Funding

The research leading to these results has received funding from the European Research Council (ERC) under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement No. 339025 GUDHI (Algorithmic Foundations of Geometry Understanding in Higher Dimensions).

[^1]The first author is further supported by the French government, through the 3IA Côte d'Azur Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR-19-P3IA0002. The last author is supported by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 754411 and the Austrian science fund (FWF) M-3073.

## Conflicts of interest/Competing interests

There are no conflicts of interest.

## References

[1] Freudenthal, H.: Simplizialzerlegungen von beschrankter flachheit. Annals of Mathematics, 580-582 (1942)
[2] Boissonnat, J.-D., Kachanovich, S., Wintraecken, M.: Tracing Isomanifolds in $\mathbb{R}^{d}$ in Time Polynomial in d using Coxeter-Freudenthal-Kuhn Triangulations (2021). https://hal.inria.fr/hal-03006663
[3] Coxeter, H.S.M.: Discrete groups generated by reflections. Annals of Mathematics, 588-621 (1934)
[4] Dobkin, D.P., Wilks, A.R., Levy, S.V.F., Thurston, W.P.: Contour tracing by piecewise linear approximations. ACM Transactions on Graphics (TOG) 9(4), 389-423 (1990). https://doi.org/10.1145/88560.88575
[5] Edelsbrunner, H., Grayson, D.R.: Edgewise subdivision of a simplex. In: Proceedings of the Fifteenth Annual Symposium on Computational Geometry, pp. 24-30 (1999)
[6] Choudhary, A., Kachanovich, S., Wintraecken, M.: Coxeter triangulations have good quality. Mathematics in Computer Science 14, 141-176 (2020)
[7] Choudhary, A., Ghosh, A.: Delaunay simplices in diagonally distorted lattices. Computational Geometry 81, 33-44 (2019). https://doi.org/10. 1016/j.comgeo.2019.01.010
[8] Brunck, F.: Iterated medial triangle subdivision in surfaces of constant curvature. Discrete \& Computational Geometry, 1-31 (2023)


[^0]:    ${ }^{1}$ This result has a long history, see the bibliography and some historical remarks in [2]. We further note that Freudenthal's construction is closely related to Coxeter triangulations [2-4].
    ${ }^{2}$ The fatness is the volume divided by the longest edge to the power $d$. It is a common quality measure.

[^1]:    ${ }^{3}$ The difference stems from the following observation: If one subdivides a cord on a circle into $k$ equal parts and radially projects the pieces on the circle then the resulting parts of the circle do not all have the same length.

