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Burning or collapsing the medial axis is unstable

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Abstract

The medial axis of a set consists of the points in the ambient space without a unique closest point in the original set. Since its introduction, the medial axis has been used extensively in many applications as a method of computing a skeleton topologically equivalent to the original set. Unfortunately, one limiting factor in the use of the medial axis of a smooth manifold is that it is not necessarily topologically stable under small perturbations of the manifold. To counter these instabilities, various prunings of the medial axis have been proposed in the computational geometry community. Here, we examine one type of pruning, called burning. Because of the good experimental results it was hoped that the burning method of simplifying the medial axis would be stable. In this work we show a simple example that dashes such hopes. Based on Bing's house with two rooms, we demonstrate an isotopy of a shape where the medial axis goes from collapsible to non-collapsible. More precisely, we consider the standard deformation retract from the closed ball to Bing's house with two rooms, but stop just short of the point where Bing's house becomes two dimensional. This way we obtain an isotopy from the 3-ball to a thickened version of Bing's

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house. Under this isotopy, the medial axis goes from collapsible to non-collapsible. We stress that this isotopy can be made generic, in the sense of singularity theory, as developed by Arnol'd and Thom.

Keywords: Medial axis, Burning, Stability theory, Singularity theory

MSC Classification: 68U05 , 65D18 , 58K40 , 58K60

1 Introduction

The (interior) medial axis is a skeleton that captures many geometric and topological features. Formally, the medial axis of a closed set $\mathcal{S}$ consists of all points in the ambient space that do not have a unique closed point on $\mathcal{S}$ (which lies on the boundary $\partial\mathcal{S}$).

As we will discuss, significant effort has gone into the study of the stability of the medial axis and its various prunings. In this paper we investigate the stability of two similar prunings of the medial axis: collapses and burnings (defined below) of the medial axis. We show that these prunings do not yield a geometrically or topologically stable core.

The introductions of the medial axis

The term *medial axis* was introduced by Blum [1], who suggested its use in the context of shape recognition. However, the concept predates Blum's work; to our best knowledge the concept was first probed by Erdős [2, 3]. In [4], Federer introduced the reach of a (closed) set $\mathcal{S} \subset \mathbb{R}^d$ as the minimum of the distance from $\mathcal{S}$ to the medial axis, $\text{ax}(\mathcal{S})$, and studied some properties of the (complement of the) medial axis. More recently, it has been reintroduced as the central set [5].

Spines and cut-loci. We note that the medial axis is closely related to both the cut-locus and spine. The cut-locus is the generalization of the medial axis to Riemannian manifolds [6]. A spine is a subset (subpolyhedra) to which the (triangulated) manifold with boundary can collapse; these fulfill a similar role in 3-manifold topology [7, 8].

Stability of the medial axis

Unfortunately, the medial axis is limited in its use by its (topological) instability under small perturbations [9]. Here small is understood to be small with respect to the Hausdorff distance, $d_H(\mathcal{S}, \mathcal{S}')$. See Figure 1 for a standard example of such an instability.

If we restrict ourselves to a smaller class of spaces and perturbations, there are stability results available: Chazal and Soufflet [10] proved that the medial axis is stable with respect to the Hausdorff distance under ambient diffeomorphisms, under strong conditions. Namely, Chazal and Soufflet assumed that

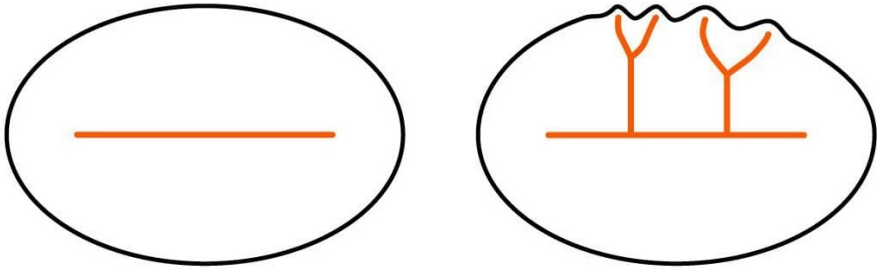


Fig. 1: Small perturbations (with respect to the Hausdorff distance) can lead to large perturbations of the medial axis.

the set of positive reach, $\mathcal{S}$, is a C^2 manifold and the distortion is a C^2 diffeomorphism of $\mathbb{R}^b$. Federer [4] proved that the reach is stable under $C^{1,1}$ diffeomorphisms of the ambient space, that is, a diffeomorphism of the ambient space whose derivative and the derivative of the inverse are Lipschitz. The medial axis does capture the homotopy type of $\mathcal{S}$ [11, 12]. With some extra information and in the smooth setting the medial axis contains much more information than just the homotopy type: if we know the medial axis of a shape and the distance to a smooth manifold (of codimension one) for each point on the cut locus (the radius function), we can recover the manifold, as noted by Wolter [12]. Damon [13–16] investigated how to recover the differential geometry of the manifold from the geometry of the medial axis and the radius function.

Generic stability and singularities. For generic points on the medial axis of a manifold embedded in low dimensions, the geometry is somewhat stable: generically a point on the medial axis in three dimensions has two closest points, in which case the geometry of a neighbourhood (or more precisely the second fundamental form) can be described in terms of the geometry near the two closest points, see Van Manen [17], see also [18–20]. This means that (again generically) the geometry of the medial axis is locally stable. The topology of the singularities of the medial axis can also be classified in a generic setting in low dimensions. This classification is the same as for the cut-locus, which is, roughly speaking, the medial axis for a single point on a manifold, see Figure 2.

This classification was done by Mather [21], Buchner [22] and Yomdin [23] based on work by Arnol’d [24], Thom [6] and Looijenga [25]; we refer to Van Manen [26] for an overview and history, see also [27, 28]. The classification of singularities is only explicitly possible in sufficiently low dimensions.

The generic transitions between singularities of the medial axis or perestroikas in a one parameter family isotopy between (generic) surfaces in $\mathbb{R}^3$ have been studied by Giblin and Kimia [29], based on earlier work by Bogaevsky [30–33]. We represent the transitions given by Giblin and Kimia,

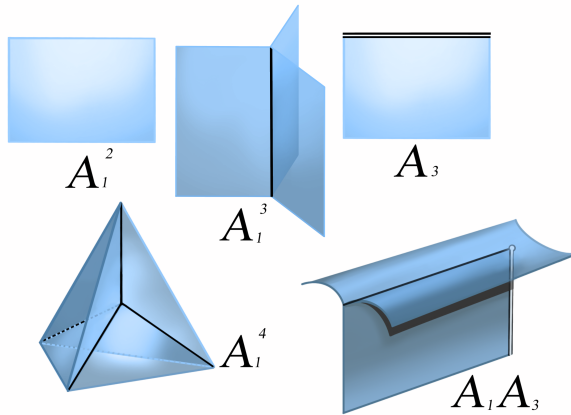


Fig. 2: The singularities of the medial axis of a generic surface embedded in $\mathbb{R}^3$. The double lines indicate open ends. The A_i^j notation for the types of singularities was introduced by Arnol'd.

and Bogaevsky in Figure 3. In this work the A_1^4 transition plays a significant role.

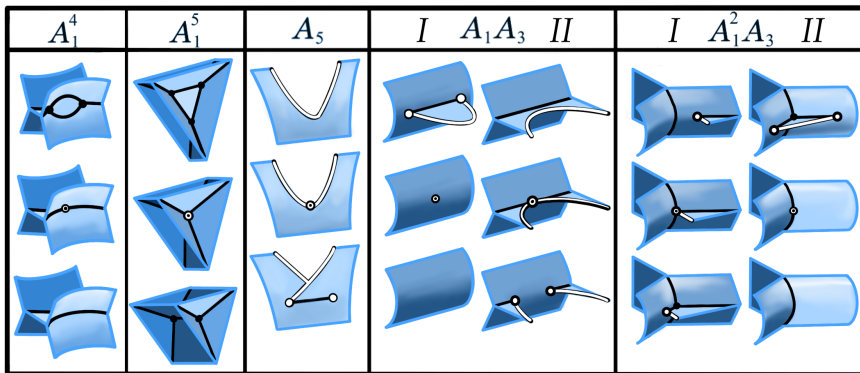


Fig. 3: The generic transitions of singularities (perestroikas) of the medial axis in $\mathbb{R}^3$. Adapted from figures by Bogaevsky [31]. We use the same notation as [29], which in turn follows Arnol'd. As we'll see in Figure 8, see also Figure 10, it is the transition of type A_1^4 that causes the (interior) medial axis to become non-collapsible (during the deformation of the cube into a thickened Bing's house).

Prunings

Significant effort has gone into the simplification (pruning) of the medial axis. This was motivated by applications in graphics (where the medial axis is used as a skeleton, see the surveys [34, 35]), data reduction, shape recognition, and learning (see for example [36–43]). Various prunings of the medial axis have been proposed in many different settings [40, 42, 44–50]. See Figure 4 for an illustration of some commonly used prunings and their pitfalls: the object angle, which is of historic importance in the community but can disconnect the medial axis, the λ -medial axis, which is used to compute a close approximation of the medial axis but which can truncate “thin” regions undesirably, and the burning method, which we consider in this work.

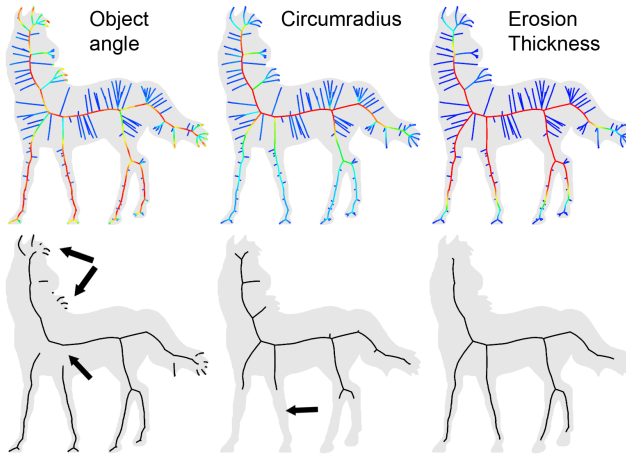


Fig. 4: Various pruning methods, from left to right: Object angles [45, 49], radius of the set of closest points [47] (the λ -medial axis, also used in our computation), and a burning method proposed in [43], with various undesirable features indicated. The value of the object angle, radius of the set of closest points, and burning time is indicated in colour on top. Reproduced from [43].

2 Burning Bing’s house

The simplification which we focus on for this work is the burning of the medial axis [43], which generalizes Blum’s original “grassfire” analogy for the medial axis.

Burning and Bing’s house. The burning of the medial axis removes the extremities of the medial axis by ‘starting a fire’ at the boundary of the medial axis which stops if the fire hits an obstacle, as illustrated in Figure 5.

Because of the good experimental results, it was conjectured that the burning method of simplification of the medial axis would be stable [40], i.e. no

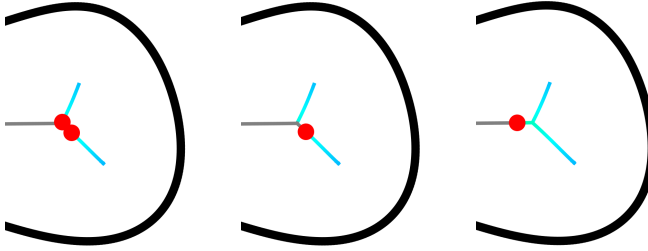
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Fig. 5: The fire front progression on a the medial axis (grey) of a curve (black). As the fire front (indicated by the red dot) hits an unburned junction, it stops. If the junction is already burned (with the colour indicating the burn time) the fire continues.

discontinuous jumps. In this work we show that this is not the case. The counter example is based on the standard deformation retract from the closed ball to Bing's house with two rooms [51], which is a contractible but not collapsible two dimensional simplicial complex, see Figure 7. Bing's house is not collapsible, as there is no boundary from which a collapsing sequence could begin.

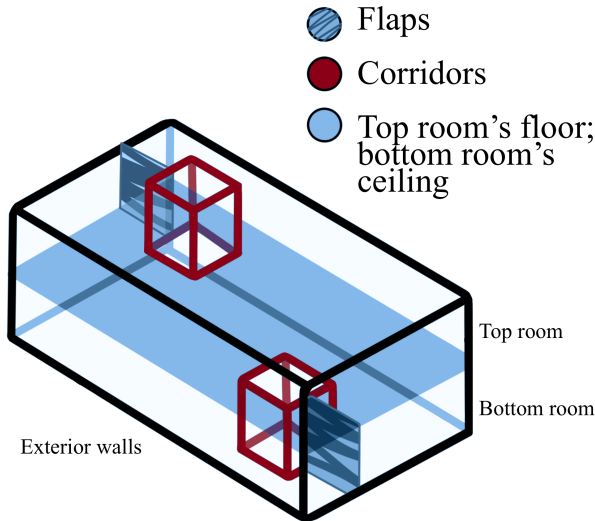


Fig. 6: The various parts of Bing's house indicated.

Deforming a cube into Bing's house. Bing constructed his house as a deformation retract from a solid cube, see Figure 7. The deformation retract goes as follows. During the entire construction, with the exception of the final step,

the set will remain three dimensional. As mentioned we start with a solid cube. We then start by digging two corridors. Once we have reached the part of the solid cube that will become a room, we start hollowing out the room by protruding a rectangular cuboid into the room. This can be continued until we come near the corridor that goes to the opposite room. Of course we cannot interfere with the corridor, instead we engulf the corridor from both sides. Once we are past the corridor, we extend the hollowed space until it almost touches itself near the flap. We finally flatten the three dimensional set to two dimensions, creating Bing's house.

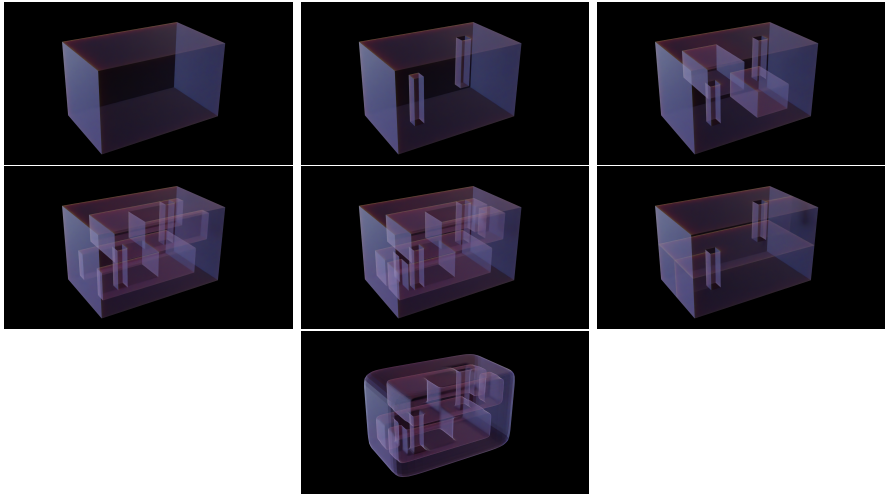


Fig. 7: The deformation retract of a solid cube (topological ball) to Bing's house. In the final frame we show the smoothed version of a thickened Bing's house used in the computation.

Deforming Bing's house into a collapsible complex. Before we go into the main statement, we consider a deformation of Bing's house which makes it collapsible. This deformation will be mirrored in the medial axis in our construction and is depicted in Figure 8; see Figure 6 for the nomenclature. In this construction we cut a flap open so that the room no longer completely runs around the corridor. This cutting exposes an edge of one of the walls of the corridor and path that goes from the edge to the bottom room. We can use this edge to collapse along the path into the bottom room, then the bottom room itself, and from this the rest of Bing's house.

Main result

We are now going to use the construction of Bing's house from a solid cube to Bing's house, but stop just before the set becomes two dimensional. This gives an isotopy of the boundary of the cube, namely $\mathbb{S}^2$, such that the (interior)

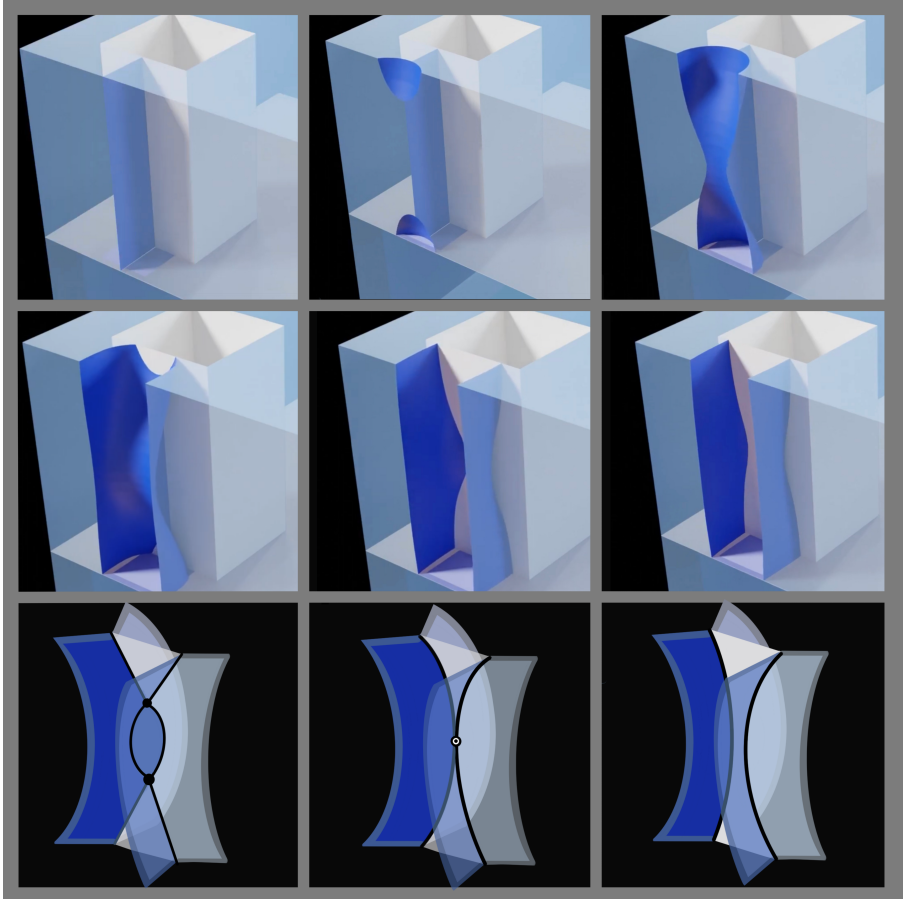


Fig. 8: This deformation (top six panels), which cuts a flap open, makes Bing's house collapsible. The topology of this transition (in the neighbourhood of the point where the flap disappears) is precisely that of the type A_1^4 . The transition of type A_1^4 is depicted in the bottom three panels; we have coloured the strata to indicate the correspondence with the construction directly above.

medial axis transitions from being collapsible and burnable to a point into Bing's house itself.

The precise result is the following:

Theorem 2.1 *There exists a smooth ambient isotopy $H_t : [0, 1] \times \mathbb{S}^2 \rightarrow \mathbb{R}^3$ such that:*

- *The (interior) medial axis $\text{ax}(H_0(\mathbb{S}^2))$ is collapsible/burnable to a single point.*
- *The (interior) medial axis $\text{ax}(H_1(\mathbb{S}^2))$ is Bing's house and is therefore non-collapsible/cannot burn (is fireproof).*

- The burning of $\text{ax}(H_t(\mathbb{S}^2))$ is not continuous in t with respect to the Hausdorff distance.
- The topology of the burned (interior) medial axis changes from a point to Bing's house with two rooms at a single $t_0 \in [0, 1]$.
- The isotopy H_t can be chosen to be generic in the sense of singularity theory as developed by Arnol'd and Thom [24], see in particular [29].

Proof Bing constructed his house as a deformation retract from a solid cube, see Figure 7. The isotopy of the sphere we consider is the boundary of this deformation. However instead of reducing to a two dimensional object we forego the final step so that at any point in the deformation the set remains a topological (solid) ball and its boundary a sphere. The end point of this deformation is a thickened version of Bing's house.

We will only consider the medial axis in the interior of the sphere and not the exterior. The medial axis of a thickened version of Bing's house is Bing's house itself. The deformation is depicted in Figure 9. The essential topological change only happens near the end of the deformation when the rooms wrap around their respective corridors, see Figure 10. At this moment the bisector between the corridor and the wall disappears and is replaced by the bisector between the two parts of the room that are wrapping around the corridor and the medial axis becomes non-collapsible.

The transition as depicted in Figure 9 is not generic, it is in fact quite symmetrical and at the critical transition, see Figure 10, there is an entire edge in the medial axis for which there are 4 nearest points on the boundary of the thickened Bing's house. For each point on the edge, the closest points are a point on the exterior wall, a point on the corridor and two points on either side that lie on the expanding top room. By perturbing the configuration slightly you can ensure that there are only isolated points of the medial axis that have 4 closest points. By doing so, the transition will look as depicted in Figure 8, which corresponds to a transition between singularities of type A_1^4 . $\square$

We immediately have the following corollary:

Corollary 2.2 *Collapsing or pruning the medial axis of a domain such that it becomes one-dimensional, as proposed in e.g. [46], is not always possible, even if the boundary of the domain is a smooth sphere.*

2.1 Computational method

The frames of the deformation illustrated in Figures 7, 9, and 10 are taken from our Symposium on Computational Geometry Media Exposition submission [52] (available at <https://youtu.be/CFmFP6CHVEk>). These animations were made using the λ -medial axis (see https://github.com/cdfillmore/lambda-medial_axis) and the open source software Blender [53]. Here λ is chosen very small to ensure that the λ -medial axis is a good approximation of the medial axis.

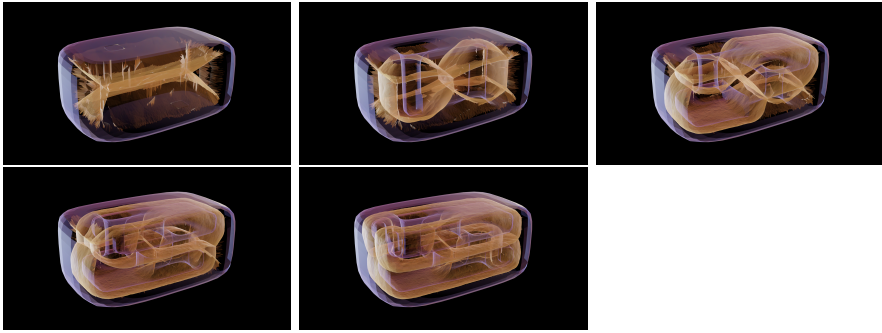


Fig. 9: The evolution of the interior medial axis (yellow) as the smoothed solid cube is deformed into a thickened version of Bing's house (blue/purple). The smoothed solid cubes in these five frames coincide with the unsmoothed solid cubes in the first five frames of Figure 7. The exact moment at which the medial axis becomes non-collapsible lies between frames four and five and is depicted explicitly in Figure 10.

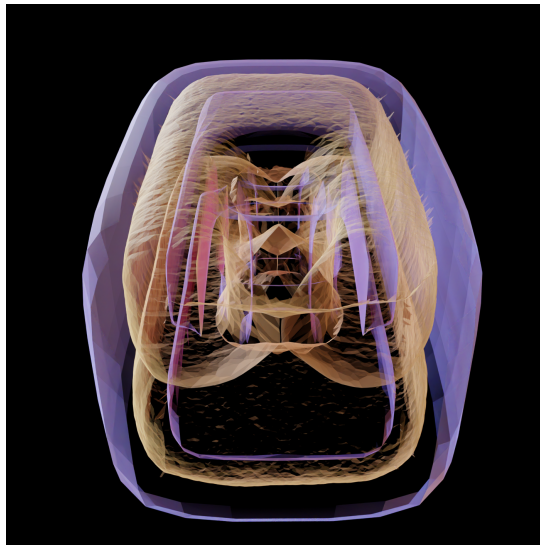


Fig. 10: The critical transition of the medial axis. There are points on the medial axis equidistant to the two parts of the room that wrap around the corridor, the corridor itself and the exterior wall, which can be avoided by a small perturbation. This transition occurs between frames 4 and 5 of Figure 7.

3 Conclusion and open question

In this paper we constructed an isotopy of the sphere such that the interior medial axis transitions from collapsible to non-collapsible. The exterior medial

axis throughout this isotopy remains collapsible (by which we mean after compactifying the ambient space to $\mathbb{S}^3$). This automatically leads some interesting questions:

Question *Does there exist an embedding of $\mathbb{S}^2$ in $\mathbb{S}^3$ where it is not possible to burn/partially collapse either the interior or exterior medial axis of $\mathbb{S}^2$ at all?*

We suspect that this question can be answered in the negative. We can also imagine that we can use the burning of the exterior medial axis to guide an isotopy of the sphere. See Figure 11 for an example of an isotopy guided by burning. For the thickened version of Bing's house, an isotopy guided by the burning of the medial axis would yield the inverse of the standard construction of Bing's house (which we described above and is depicted in Figure 7), see Figure 12 for a depiction of how the burning would guide the isotopy.

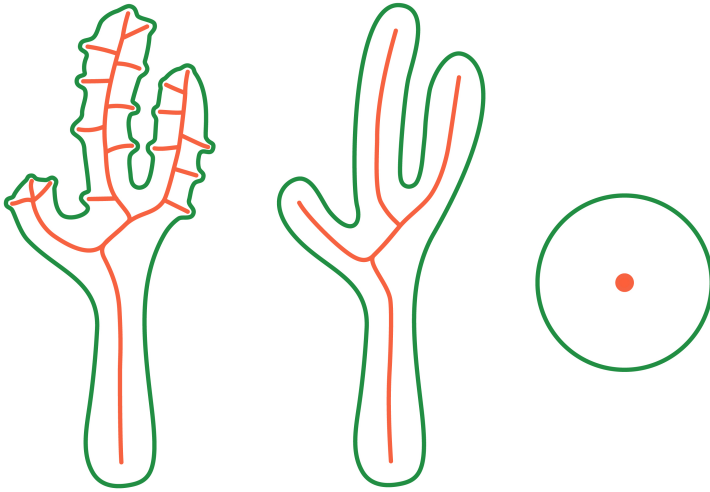


Fig. 11: The medial cactus: a two dimensional example of an isotopy guided by the burning of the medial axis. In this case the isotopy goes from a cactus shape into the (round) circle.

This leads us to the following stronger conjecture:

Conjecture 3.1 *Given any embedding of $\mathbb{S}^2$ in $\mathbb{S}^3$, there exists a series of isotopies of $\mathbb{S}^2$ based on successive burnings of both the interior and exterior medial axes, which yields the standard symmetrical embedding of the 2-sphere in $\mathbb{S}^3$.*

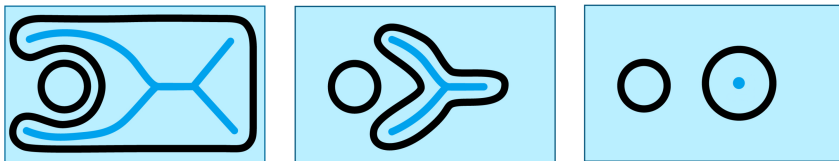


Fig. 12: In this figure we indicate an isotopy guided by the burning of the medial axis in the top room of a thickened version of Bing’s house. We see how the room deforms from enveloping the corridor (to the bottom room) to a circular remnant. After this deformation the shape is a smoothed version of the second frame of Figure 7. The corridor in this frame can again be isotoped away, using the burning of the medial axis as a guide.

We point out that some work on the correspondence between the singularities of the interior and exterior medial axes has already been done [28].

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Conflicts of interest/Competing interests

Financial interests. The authors declare they have no financial interests.

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References

- [1] Blum, H.: A Transformation for Extracting New Descriptors of Shape. In: Wathen-Dunn, W. (ed.) *Models for the Perception of Speech and Visual Form*, pp. 362–380. MIT Press, Cambridge (1967)
- [2] Erdős, P.: Some remarks on the measurability of certain sets. *Bulletin of the American Mathematical Society* **51**(10), 728–731 (1945)
- [3] Erdős, P.: On the Hausdorff dimension of some sets in Euclidean space. *Bulletin of the American Mathematical Society* **52**(2), 107–109 (1946). <https://doi.org/bams/1183507696>
- [4] Federer, H.: Curvature measures. *Transactions of the American Mathematical Society* **93**, 418–491 (1959). <https://doi.org/10.1090/S0002-9947-1959-0110078-1>
- [5] Milman, D., Waksman, Z.: On topological properties of the central set of a bounded domain in $\mathbb{R}^n$. *Journal of Geometry* **15**(1), 1–7 (1980)
- [6] Thom, R.: Sur le cut-locus d’une variété plongée. *Journal of Differential Geometry* **6**(4), 577–586 (1972). <https://doi.org/10.4310/jdg/1214430644>
- [7] Matveev, S.: Complexity theory of three-dimensional manifolds. *Acta Applicandae Mathematica* **19**, 101–130 (1990)
- [8] Matveev, S.: *Algorithmic Topology and Classification of 3-Manifolds. Algorithms and Computation in Mathematics*. Springer, Berlin, Heidelberg (2013). <https://books.google.at/books?id=DajvCAAQBAJ>
- [9] Attali, D., Boissonnat, J.-D., Edelsbrunner, H.: In: Möller, T., Hamann, B., Russell, R.D. (eds.) *Stability and Computation of Medial Axes - a State-of-the-Art Report*, pp. 109–125. Springer, Berlin, Heidelberg (2009). https://doi.org/10.1007/b106657_6. https://doi.org/10.1007/b106657_6
- [10] Chazal, F., Soufflet, R.: Stability and finiteness properties of medial axis and skeleton. *Journal of Dynamical and Control Systems* **10**(2), 149–170 (2004). <https://doi.org/10.1023/B:JODS.0000024119.38784.ff>
- [11] Lieutier, A.: Any open bounded subset of $\mathbb{R}^n$ has the same homotopy type as its medial axis. *Computer-Aided Design* **36**(11), 1029–1046 (2004). <https://doi.org/10.1016/j.cad.2004.01.011>. *Solid Modeling Theory and Applications*
- [12] Wolter, F.-E.: Cut locus and medial axis in global shape interrogation and representation. In: *MIT Design Laboratory Memorandum 92-2 and MIT Sea Grant Report* (1992)

- [13] Damon, J.: Smoothness and geometry of boundaries associated to skeletal structures I: Sufficient conditions for smoothness. In: *Annales de L'institut Fourier*, vol. 53, pp. 1941–1985 (2003). <https://doi.org/10.5802/aif.1997>
- [14] Damon, J.: Smoothness and geometry of boundaries associated to skeletal structures, II: Geometry in the Blum case. *Compositio Mathematica* **140**(6), 1657–1674 (2004). <https://doi.org/10.1112/S0010437X04000570>
- [15] Damon, J.: Global geometry of regions and boundaries via skeletal and medial integrals. *Communications in Analysis and Geometry* **15**(2), 307–358 (2007). <https://doi.org/10.4310/CAG.2007.v15.n2.a5>
- [16] Damon, J.: Determining the geometry of boundaries of objects from medial data. *International Journal of Computer Vision* **63**(1), 45–64 (2005)
- [17] van Manen, M.: The geometry of conflict sets. PhD thesis, Universiteit Utrecht, Faculteit der Wiskunde en Informatica (2002)
- [18] Siersma, D.: Properties of conflict sets in the plane. *Banach Center Publications* **50**, 267–276 (1999). <https://doi.org/10.4064/-50-1-267-276>
- [19] Sotomayor, J., Siersma, D., Garcia, R.: Curvatures of conflict surfaces in Euclidean 3-space. *Banach Center Publications* **50**(1), 277–285 (1999)
- [20] van Manen, M.: Curvature and torsion formulas for conflict sets. *Banach Center Publications* **62**, 209–222 (2003)
- [21] Mather, J.N.: Distance from a submanifold in Euclidean-space. In: *Proceedings of Symposia in Pure Mathematics*, vol. 40, pp. 199–216 (1983). American Mathematical Society
- [22] Buchner, M.A.: The structure of the cut locus in dimension less than or equal to six. *Compositio Mathematica* **37**(1), 103–119 (1978)
- [23] Yomdin, Y.: On the local structure of a generic central set. *Compositio Mathematica* **43**(2), 225–238 (1981)
- [24] Arnol'd, V.: *Singularities of Caustics and Wave Fronts*. Mathematics and its Applications, vol. 62. Springer, Berlin, Heidelberg (2013)
- [25] Looijenga, E.: Structural stability of smooth families of C^∞ -functions. PhD thesis, Universiteit van Amsterdam (1974)
- [26] van Manen, M.: Maxwell strata and Caustics. *Singularities in Geometry and Topology*, pp. 787–824 (2007). https://doi.org/10.1142/9789812706812_0028

- [27] Damon, J.: The global medial structure of regions in $\mathbb{R}^3$. *Geometry & Topology* **10**(4), 2385–2429 (2006). <https://doi.org/10.2140/gt.2006.10.2385>
- [28] Damon, J., Gasparovic, E.: Medial/skeletal linking structures for multi-region configurations. *Memoirs of the American Mathematical Society* **250** (2014). <https://doi.org/10.1090/memo/1193>
- [29] Giblin, P.J., Kimia, B.B., Pollitt, A.J.: Transitions of the 3D medial axis under a one-parameter family of deformations. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **31**(5), 900–918 (2008). <https://doi.org/10.1109/TPAMI.2008.120>
- [30] Bogaevsky, I.A.: Metamorphoses of singularities of minimum functions, and bifurcations of shock waves of the burgers equation with vanishing viscosity. *Algebra i Analiz* **4**, 1–16 (1989). English translation in *St.Petersburg (Leningrad) Mathematical Journal*, 1 (1990), no.4, 807–823.
- [31] Bogaevsky, I.A.: Perestroikas of shocks and singularities of minimum functions. *Physica D: Nonlinear Phenomena* **173**(1-2), 1–28 (2002)
- [32] Bogaevsky, I.A.: Singularities of viscosity solutions of hamilton-jacobi equation (singularity theory and differential equations). *Suri kaiseiki kenkyujo kokyuroku* **1111**, 138–143 (1999)
- [33] Bogaevsky, I.A.: Perestroikas of fronts in evolutionary families. *Trudy Matematicheskogo Instituta imeni V.A. Steklova* **209**, 65–83 (1995). English translation in *Proceedings of the Steklov Institute of Mathematics* 209 (1995), 57–72.
- [34] Saha, P.K., Borgfors, G., di Baja, G.S.: A survey on skeletonization algorithms and their applications. *Pattern recognition letters* **76**, 3–12 (2016)
- [35] Tagliasacchi, A., Delame, T., Spagnuolo, M., Amenta, N., Telea, A.: 3D skeletons: A state-of-the-art report. In: *Computer Graphics Forum*, vol. 35, pp. 573–597 (2016). <https://doi.org/10.1111/cgf.12865>. Wiley Online Library
- [36] Bal, G., Diebold, J., Chambers, E.W., Gasparovic, E., Hu, R., Leonard, K., Shaker, M., Wenk, C.: Skeleton-based recognition of shapes in images via longest path matching. In: Leonard, K., Tari, S. (eds.) *Research in Shape Modeling*, pp. 81–99. Springer, Cham (2015)
- [37] Chang, M.-C., Kimia, B.B.: Measuring 3d shape similarity by graph-based

- matching of the medial scaffolds. *Computer Vision and Image Understanding* **115**(5), 707–720 (2011). <https://doi.org/10.1016/j.cviu.2010.10.013>
- [38] Demir, I., Hahn, C., Leonard, K., Morin, G., Rahbani, D., Panotopoulou, A., Fondevilla, A., Balashova, E., Durix, B., Kortylewski, A.: SkelNetOn 2019: Dataset and challenge on deep learning for geometric shape understanding. In: 2019 IEEE/CVF Conference on Computer Vision and Pattern Recognition Workshops (CVPRW), pp. 1143–1151 (2019). <https://doi.org/10.1109/CVPRW.2019.00149>
- [39] Ho, S.-B., Dyer, C.R.: Shape smoothing using medial axis properties. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **PAMI-8**(4), 512–520 (1986). <https://doi.org/10.1109/TPAMI.1986.4767815>
- [40] Liu, L., Chambers, E.W., Letscher, D., Ju, T.: Extended grassfire transform on medial axes of 2d shapes. *Computer-Aided Design* **43**(11), 1496–1505 (2011). <https://doi.org/10.1016/j.cad.2011.09.002>. Solid and Physical Modeling 2011
- [41] Sebastian, T.B., Klein, P.N., Kimia, B.B.: Recognition of shapes by editing their shock graphs. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **26**(5), 550–571 (2004). <https://doi.org/10.1109/TPAMI.2004.1273924>
- [42] Shaked, D., Bruckstein, A.M.: Pruning medial axes. *Computer Vision and Image Understanding* **69**(2), 156–169 (1998). <https://doi.org/10.1006/cviu.1997.0598>
- [43] Yan, Y., Sykes, K., Chambers, E., Letscher, D., Ju, T.: Erosion thickness on medial axes of 3d shapes. *ACM Transactions on Graphics* **35**(4), 38–13812 (2016). <https://doi.org/10.1145/2897824.2925938>
- [44] Amenta, N., Choi, S., Kolluri, R.K.: The power crust, unions of balls, and the medial axis transform. *Computational Geometry: Theory and Applications* **19**(2-3), 127–153 (2001). [https://doi.org/10.1016/S0925-7721\(01\)00017-7](https://doi.org/10.1016/S0925-7721(01)00017-7)
- [45] Attali, D., Montanvert, A.: Modeling noise for a better simplification of skeletons. In: *Proceedings of 3rd IEEE International Conference on Image Processing*, vol. 3, pp. 13–16 (1996). <https://doi.org/10.1109/ICIP.1996.560357>. IEEE
- [46] Blanc-Beyne, T., Morin, G., Leonard, K., Hahmann, S., Carlier, A.: A salience measure for 3D shape decomposition and sub-parts classification. *Graphical Models* **99**, 22–30 (2018). <https://doi.org/10.1016/j.gmod.2018.07.003>

- [47] Chazal, F., Lieutier, A.: The “ λ -medial axis”. *Graphical Models* **67**(4), 304–331 (2005). <https://doi.org/10.1016/j.gmod.2005.01.002>
- [48] Dey, T.K., Sun, J.: Defining and computing curve-skeletons with medial geodesic function. In: *Proceedings of the Fourth Eurographics Symposium on Geometry Processing*. SGP '06, pp. 143–152. Eurographics Association, Goslar, DEU (2006)
- [49] Dey, T.K., Zhao, W.: Approximating the medial axis from the Voronoi diagram with a convergence guarantee. *Algorithmica* **38**(1), 179–200 (2004). <https://doi.org/10.1007/s00453-003-1049-y>
- [50] Giesen, J., Miklos, B., Pauly, M., Wormser, C.: The scale axis transform. In: *Proceedings of the Twenty-Fifth Annual Symposium on Computational Geometry*, pp. 106–115. Association for Computing Machinery, New York, NY, USA (2009). <https://doi.org/10.1145/1542362.1542388>
- [51] Bing, R.: Some aspects of the topology of 3-manifolds related to the Poincaré conjecture. In: Saaty, T.L. (ed.) *Lectures on Modern Mathematics* vol. II, pp. 93–128. John Wiley and Sons, USA (1964)
- [52] Chambers, E., Fillmore, C., Stephenson, E., Wintraecken, M.: A Cautionary Tale: Burning the Medial Axis Is Unstable. In: Goaoc, X., Kerber, M. (eds.) *38th International Symposium on Computational Geometry (SoCG 2022)*. *Leibniz International Proceedings in Informatics (LIPIcs)*, vol. 224, pp. 66–1669. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany (2022). <https://doi.org/10.4230/LIPIcs.SoCG.2022.66>. <https://drops.dagstuhl.de/opus/volltexte/2022/16074>
- [53] Community, B.O.: Blender - a 3D Modelling and Rendering Package. Blender Foundation, Stichting Blender Foundation, Amsterdam (2018). Blender Foundation. <http://www.blender.org>