FRANK-OSEEN TYPE SOLUTIONS WITH ORBITS AND VORTICES

Axel Osses, Nicolas Barnafi

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Abstract. We establish some exact and approximate analytical solutions of the Frank-Oseen equations under the one-constant hypothesis of the nematic theory of liquid crystals in two and three dimensions. In 2-d, we find a family of homogeneous solutions (or weakly harmonic map solutions) with infinite oscillations of type $\sin(\log(r/R))$ where r is the distance from the origin in a centered disk D of radius R. These solutions do not belong to the standard energy space $H^1(D)$ but to a weaker Sobolev space $W^{1,p}(D)$ with $p < 2$. In 3-d, we build analytic approximate solutions in a cylinder and in a sphere, in the sense that they satisfy the Frank-Oseen equations up to a source term that decays as $1/r^\gamma$ at some rate $\gamma > 0$, where r is the cylindrical or spherical radius respectively. While the 2-d solutions have singularities of type $\log(r)$, the 3-d solutions exhibit singularities of type $\log(r)$ or $1/r$ in the cylindrical of spherical coordinate examples respectively. All the 3-d solutions present vortices in the vertical axis. The fact that all the solutions both in 2-d and 3-d have infinite energy, motivated us to introduce p-nematic Frank-Oseen equations for $p \leq 2$. Many of the solutions encountered in this paper were motivated from the geometry of protoplanetary disks orbits and vortices observed in astrophysical data observations and models, even if the rigorous connection is not yet completely clear.

1. Introduction

The nematic liquid crystal theory studies the structure of the principal directions of the molecules of a liquid crystal, which is an intermediate state between a solid and a liquid [12]. Although this theory is typically used for studying electronic devices [4], other unconventional applications of ths theory have been established, such as proto-planet formation [21] and more recently the orientation of cardiac fibers, both experimentally [2] and mathematically [7]. Out work was motivated by the discovery of higher energy solutions of the Frank-Oseen equations on rings, computed in [7, Sec. 2.2.3].

The simplest case of the Frank-Oseen equations of nematic liquid crystals [17] consists in finding a spatial distribution of unit vectors that are critical points of a certain global energy. The following is the simplest case of the Frank-Oseen theory under the one-constant hypothesis of lyquid crystal theory [19].

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Definition 1. *Given a regular domain Ω in $\mathbb{R}^N$, $N = 2, 3$, we say that d is a 2-nematic field if it is a critical point of the functional*

$$\min_{s.t.|d|=1} \frac{1}{2} \int_D |\nabla d|^2 - \int_D f \cdot d, \tag{1}$$

where $f \in L^2(\Omega)$ is a given force and the solution is subject to the unit modulus constraint. That is, d is a unitary field solution of the following non linear system

$$\begin{aligned} \Delta d + 2\lambda_2(d)d &= -f, \quad d \cdot d = 1, \quad \text{in } D, \\ \lambda_2(d) &:= \frac{1}{2}\left(|\nabla d|^2 - f \cdot d\right), \end{aligned} \tag{2}$$

where λ_2 the point-wise energy potential. If $f = 0$ we say that the 2-nematic field is homogeneous, non-homogeneous otherwise and conservative if $f \cdot d = 0$ with total energy $E_2(\Omega) = \int_\Omega \lambda_2$ in the latter case. If the solution of (2) is in $H^1(\Omega)$ we say this is a finite energy solution and an infinite energy solution otherwise.

The subscript 2 makes emphasis on the presence of the L^2-norm for the gradient in (1), and will be instrumental for the definition of p-nematic liquid crystals in Section 5. In the classical theory [9], we can consider $f \in L^2(\Omega)^N$ and seek for solution of (1) or (2) with finite energy $d \in H^1(D)^N$, but we will build solutions that do not belongs to $H^1(D)^N$ in a neighborhood of the origin.

There are other more complete theories for nematic crystals such as the Ericksen [15], the Landau–De Gennes theories [13, 6, 5], and the Ginzburg-Landau theory of superconductors [18, 8].

Some of the solutions encountered in this paper were motivated from the geometry of the formation of planets and the distribution of their orbits, not only in our solar system but also in other planetary systems, geometries that exhibit orbits and vortices [3, 14]. This is part of the branch of Astronomy and Astrophysics that studies the dynamics of protoplanetary disks. We can think of a flat disk containing a fluid that rotates with an angular velocity and that under the action of gravity and magnetic fields is dragging matter until it accumulates in certain orbits. Even if the rigorous connection with minimization of energies of type (1) or equations of type (2) is not yet completely clear, there are some similarities with solutions obtained by perturbation analysis of a non-linear fluid accretion model in [21].

The shape and structure of these disks where planets are forming can be observed and studied with radio telescopes. Numerical simulations can also be performed from the equations of magneto-hydrodynamics [3] where after multiple spins we arrive to structures similar to those observed in radio-astronomy. The problem of fit of planet orbits, auto-similar and quantified solutions appears also in the context of the scale relativity theory [20].

There is also a recent connection between nematic theory and the distribution of the orientation of muscular cardiac fibers, both from the theoretical, numerical [7] and the

experimental point of view [2], where Frank-Oseen models naturally appears in the presence of external forces in the case of a single ventricle [7]. Again, the rigorous link with minimization of energies of type (1) is not yet completely clear.

Motivated from geometries from these applied fields, we build novel solutions of (2) that have infinite energy [10]. The solutions are exact in two dimensions (i.e. with $f = 0$) and approximate in the three dimensional case (i.e. with $f \neq 0$ but with a certain radial power decay from the origin). We focus only on the cases where the domain D is either a disk, a cylinder, and a sphere.

The rest of the paper is organized as follows. In Section 2 we compute analytic solutions on a disk. In Section 3 we extend the disk solutions to a cylinder. This formulation gives rise to a non-homogeneous problem, so we show three types of solutions where we try to maximize the power decay of the external force. In Section 4 we perform a similar study but for the case of a spherical domain. In Section 5 we define p-nematic liquid crystals for $p \leq 2$ and establish connections with the standard $p = 2$ model. Finally, we discuss our results in Section 6.

2. Exact solutions in a two-dimensional disk

Let us consider a disk of radius R centered at the origin of coordinates which we will represent in polar coordinates (r, θ) given by $D = \{|r| \leq R\}$. Problem (2) can be rewritten as the solution of the following nonlinear Frank-Oseen equation: find a unit vector field d such that:

$$\Delta d + \lambda d = d_{rr} + \frac{1}{r} d_r + \frac{1}{r^2} d_{\theta\theta} + 2\lambda_2 d = 0 \quad \text{in } D, \tag{3}$$

where

$$\lambda_2 = \frac{1}{2}|\nabla d|^2 = \frac{1}{2}\left(|d_r|^2 + \frac{1}{r^2}|d_\theta|^2\right). \tag{4}$$

We will look for solutions of this equation with boundary conditions on the boundary ∂D of D, yet to be defined. As we show later, we will not worry about possible singularities at $r = 0$—that is, we can obtain infinite energy solutions—because we can always remove a small neighborhood of the origin.

Henceforth it will be useful for us to use the rotation matrix at an angle α and two related matrices:

$$Q_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \quad , \quad I = Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad I^\perp = Q_{\pi/2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{5}$$

which satisfy the following properties: $\frac{dQ}{d\alpha} = I^\perp Q$, and $I^\perp I^\perp = -I$. These properties will be used extensively throughout this work.

In two dimensions, we will look for a solution with a rotating boundary condition

$$\widehat{e}(n\theta) = \begin{bmatrix} -\sin n\theta \\ \cos n\theta \end{bmatrix} \quad \text{if } r = R, \tag{6}$$

so we propose the following solution *ansatz*:

$$d(r,\theta) = Q_{\alpha(r)}\widehat{e}(n\theta), \tag{7}$$

where $\alpha(r)$ is a function to be determined. Leveraging the properties of the rotation matrix, we readily see that

$$\begin{aligned} d_r &= \alpha' I^{\perp} d, \quad d_{rr} = (-|\alpha'|^2 I + \alpha'' I^{\perp})d, \\ d_\theta &= -nI^{\perp}d, \quad d_{\theta\theta} = -n^2 d, \end{aligned} \tag{8}$$

which in (3) yields

$$-|\alpha'|^2 I + \alpha'' I^{\perp} + \frac{\alpha'}{r} I^{\perp} - \frac{n^2}{r^2} I + 2\lambda I = 0, \qquad \lambda_2 = \frac{1}{2}\left(|\alpha'|^2 + \frac{n^2}{r^2}\right). \tag{9}$$

We highlight that the components of I cancel out. By canceling also the components of $I^{\perp}$ and using the boundary condition we obtain

$$\begin{cases} \alpha'' + \dfrac{\alpha'}{r} = 0 & r \in (0,R), \\ \alpha(R) = 0, \end{cases}$$

from where we obtain the general solution for α:

$$\alpha = c_1 \log(r/R), \qquad \lambda_2 = \frac{c_1^2 + n^2}{2r^2}, \tag{10}$$

where c_1 is an arbitrary constant. Finally, we established the following result.

Theorem 2. *For each $n \geq 1$, there exist a family of solutions of the Frank-Oseen equation:*

$$\begin{cases} \Delta d + |\nabla d|^2 d = 0 & \text{in } D, \\ d \cdot d = 1 & \text{in } D, \\ d = \widehat{e}(n\theta) & \text{on } \partial D. \end{cases} \tag{11}$$

in the disk $D = \{(r,\theta), |r| \leq R\}$ where $\widehat{e}$ was defined in (6)*, given by the unitary field:*

$$d(r,\theta) = Q_{c_1 \log(r/R)}\, \widehat{e}(n\theta) \tag{12}$$

and where $c_1 \in \mathbb{R}$ is an arbitrary constant and Q_α is a rotation matrix defined in (5)*.*

See Figures 1 and 2 for a graphical representation of the family of solutions (12) of Theorem 2. We note that the solutions exhibits for $n = 1$ alternate attractive and repulsive orbits in rings with exponentially growing separation, for $n \geq 2$ the attractors are spirals.

Remark 3. *It is easy to verify that the solutions* (12) *are self-similar with respect to the origin. Indeed, if $d(r,\theta)$ is the solution given by* (12) *then $d_k(r',\theta') = d(r,\theta)$ for $r' = r/a_k$, $\theta' = \theta$ and $a_k = R\exp\frac{2k\pi}{c_1}$ are also solutions of the same equations* (2) *with the same boundary conditions for $r' = 1$.*

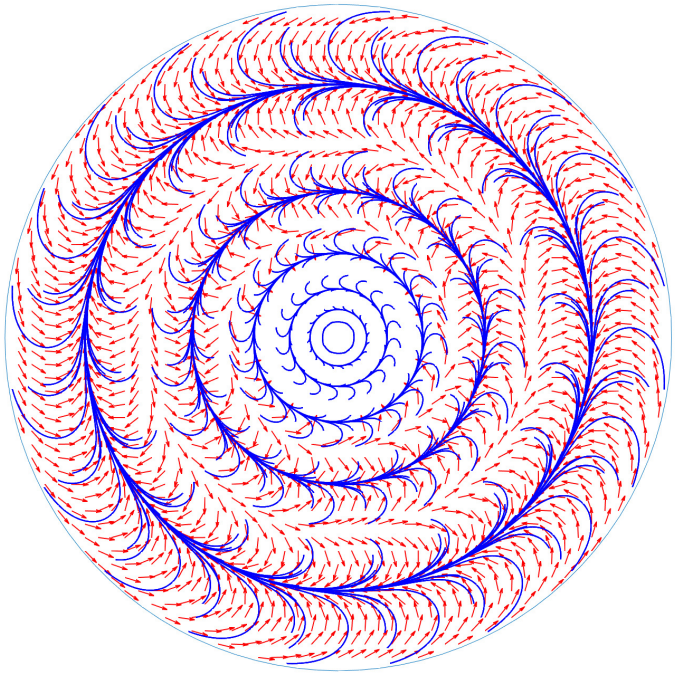

FIGURE 1. Vector field and trajectories of the 2D disk solutions of Theorem 2 for $n = 1$ with $R = 1$ and $c_1 = 3.84\pi$.

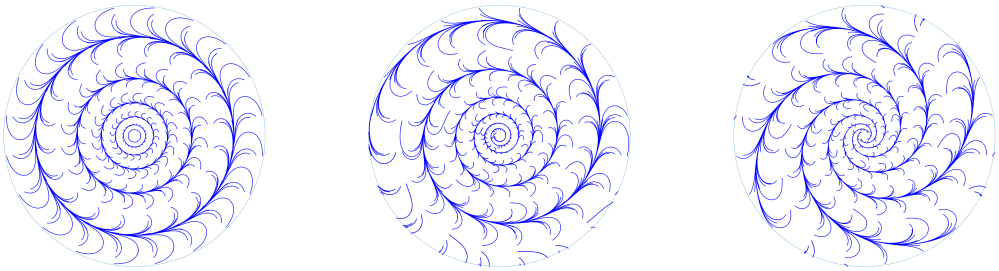

FIGURE 2. Trajectories of the 2D disk solutions of Theorem 2 for $n = 1$, $n = 2$ and $n = 5$ with $R = 1$ and $c_1 = 3.84\pi$.

Remark 4. *The family of solutions* (12) *present a singularity of type* $\log(r)$ *at the origin. This means that the solution is in* $L^2(D)$ *but not in* $H^1(D)$*. We note that this singularity can be removed simply by considering the equation in a ring* $D\backslash D_\epsilon$ *with* $D_\epsilon = \{\epsilon \leq |r| \leq R\}$*.*

In this ring, it is not difficult to check that the energy of the solution using (10) *is given by*

$$E_2(D \setminus D_\varepsilon) = \int_0^{2\pi} \int_\varepsilon^R \lambda_2(r,\theta) r \, dr d\theta = \pi(c_1^2 + n^2) \log(R/\varepsilon),$$

so we see that the H^1*-norm of the solution tends to infinity as* $\varepsilon \to 0$*. We note that if we fix* $n = 1$ *and* $c_1 = 0$*, we recover a minimal energy solution. For other values of* c_1 *and* n*, the solutions are critical points of the energy functional* (1) *(with* $f = 0$*) with higher energy levels. See Section 5 for a more detailed discussion about the energy of the solutions.*

3. Approximate solutions in a cylinder

In this section, we extend the solutions obtained in a disk to the case of a cylinder. For this we will consider the cylinder in cylindrical coordinates:

$$C = \{(r,\theta,z) \,|\, 0 \le r \le R,\, 0 \le \theta < 2\pi,\, 0 \le z \le H\}.$$

The Frank-Oseen equation in polar coordinates is written as searching for a unitary field d such that:

$$d_{rr} + \frac{1}{r} d_r + \frac{1}{r^2} d_{\theta\theta} + d_{zz} + 2\lambda_2 d = -f \quad \text{in } C, \tag{13}$$

with some right hand side f such that $f \cdot d = 0$, where

$$\lambda_2 = \frac{1}{2}\left(|d_r|^2 + \frac{1}{r^2}|d_\theta|^2 + |d_z|^2 \right). \tag{14}$$

We define

$$\widehat{e}(a,b) = \begin{bmatrix} -\sin a \cos b \\ \cos a \cos b \\ -\sin b \end{bmatrix}, \quad \widehat{e}^{\perp}(a,b) = \begin{bmatrix} \sin a \sin b \\ -\cos a \sin b \\ -\cos b \end{bmatrix}. \tag{15}$$

Notice that $\widehat{e} \cdot \widehat{e}^{\perp} = 0$ and $|\widehat{e}| = 1$, $|\widehat{e}^{\perp}| = 1$. We also introduce the rotation matrices in the xy plane of the form

(16)

$$Q_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, I_{12}^{\perp} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and I the usual identity matrix in $\mathbb{R}^3$. As before, we will make use of the following properties: $Q_\alpha Q_\alpha^t = I$, $I_{12} I_{12}^{\perp} = 0$, $I_{12}^{\perp} I_{12}^{\perp} = -I_{12}$. Additionally, Q_α and $I_{12}^{\perp}$ conmute. For the cylinder case, we consider a solution *ansatz* of the form

$$d(r,\theta,z) = Q_{\alpha(r)} \hat{e}(n\theta, \beta(\xi)),$$

where we will further consider three types of solution. One with a height rotation $\xi = z$ and two with an elevation rotation $\xi = z/r$. The difference between the last two will reside in the treatment of the forcing term.

3.1. **Cylindrical case 1.** In this subsection, we look for solutions of the form

$$d(r,\theta,z) = Q_{\alpha(r)}\widehat{e}\,(n\theta,\beta(z))\,,$$

where $\alpha(r)$ and $\beta(z)$ are functions to determine, with $\alpha(R)=0$ and $\beta(0)=0$ as suggested by the boundary conditions. The partial derivatives are given as follows:

$$\begin{array}{rcl} d_r &=& \alpha' I_{12}^{\perp} d, \quad d_{rr} = (-|\alpha'|^2 I_{12} + \alpha'' I_{12}^{\perp})d, \\ d_\theta &=& -n I_{12}^{\perp} d, \quad d_{\theta\theta} = -n^2 I_{12} d, \\ d_z &=& \beta' Q_\alpha \widehat{e}^{\perp}, \quad d_{zz} = -|\beta'|^2 d + \beta'' Q_\alpha \widehat{e}^{\perp}. \end{array}$$

By replacing them in (13) we obtain

$$\begin{aligned} &-\left(|\alpha'|^2 + \frac{n^2}{r^2}\right) I_{12} d - |\beta'|^2 d + \left(\alpha'' + \frac{\alpha'}{r}\right) I_{12}^{\perp} d + \beta'' Q_\alpha \widehat{e}^{\perp} + 2\lambda_2 d = -f_1, \\ &\lambda_2 = \frac{1}{2}\left(\cos^2\beta\left(|\alpha'|^2 + \frac{n^2}{r^2}\right) + |\beta'|^2\right), \\ &f_1 = g_1 Q_\alpha \widehat{e}^{\perp}, \quad g_1(r,z) = \left(|\alpha'|^2 + \frac{n^2}{r^2}\right)\sin\beta\cos\beta, \end{aligned} \tag{17}$$

where f_1 comes from the terms that do not vanish on the left hand side and can be rewritten as:

$$f_1 = \left(|\alpha'|^2 + \frac{n^2}{r^2}\right)(\sin^2\beta I_{12} - \cos^2\beta I_3)d = \sin\beta\cos\beta\left(|\alpha'|^2 + \frac{n^2}{r^2}\right) Q_\alpha \widehat{e}^{\perp}.$$

Notice that d, $I_{12}^{\perp} d$ and $Q_\alpha \widehat{e}^{\perp}$ are mutually orthogonal since $I_{12}^{\perp} d \cdot d = 0$,

$$Q_\alpha \widehat{e}^{\perp} \cdot d = Q_\alpha \widehat{e}^{\perp} \cdot Q_\alpha \widehat{e} = \widehat{e}^{t} Q_\alpha^{t} Q_\alpha \widehat{e}^{\perp} = \widehat{e} \cdot \widehat{e}^{\perp} = 0,$$

and

$$I_{12}^{\perp} d \cdot Q_\alpha \widehat{e}^{\perp} = I_{12}^{\perp} Q_\alpha \widehat{e} \cdot Q_\alpha \widehat{e}^{\perp} = Q_\alpha^{t} I_{12}^{\perp} Q_\alpha \widehat{e} \cdot \widehat{e}^{\perp} = Q_\alpha^{t} Q_\alpha I_{12}^{\perp} \widehat{e} \cdot \widehat{e}^{\perp} = I_{12}^{\perp} \widehat{e} \cdot \widehat{e}^{\perp} = 0.$$

This implies in particular that $f_1 \cdot d = 0$. Equating the components of $I_{12}^{\perp}$ and $Q_\alpha e^{\perp}$ on the left hand side to zero, we obtain

$$\begin{cases} r\alpha'' + \alpha' = 0 & r \in (0,R), \\ \beta'' = 0 & z \in (0,Z), \\ \beta(0) = 0, & \\ \alpha(R) = 0, & \end{cases} \tag{18}$$

from which it follows that

$$\alpha = c_1 \log(r/R), \qquad \beta(z) = c_2 z, \tag{19}$$

for arbitrary real constants c_1 and c_2. We have thus obtained the following result.

Theorem 5. *For each $n \geq 1$, family of solutions of the non-homogeneous Frank-Oseen equations*

$$\begin{cases} \Delta d + |\nabla d|^2 d = -f_1 & \text{in } C, \\ d \cdot d = 1 & \text{in } C, \end{cases} \tag{20}$$

in the cylinder $C = \{(r,\theta,z) \,|\, 0 \leq r \leq R,\, 0 \leq \theta < 2\pi,\, 0 \leq z \leq H\}$ with right hand side f_1 such that $f_1 \cdot d = 0$ and f_1 of order $\mathcal{O}(1/r^2)$, is given by

$$d(r,\theta,z) = Q_{c_1 \log(r/R)} \widehat{e}\,(n\theta, c_2\, z) \tag{21}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants, Q_α is the rotation matrix defined in (16) *and $\widehat{e}$ was defined in* (15).

3.2. **Cylindrical case 2.** In the previous calculations, it would be interesting to absorb f_1 by solving $\beta'' + g_1 = 0$, but the solution of this equation is impossible since β only depends on z and g_1 depends on (r,z). This suggest to take $\beta = \beta(z,r)$. We will try the case $\beta = \beta(z/r)$, i.e. function of the elevation angle—but other choices could be possible—, that is, search now for a solution of type

$$d(r,\theta,z) = Q_{\alpha(r)} \widehat{e}\,(n\theta, \beta(z/r))\,, \tag{22}$$

where $\alpha(r)$ and $\beta(\xi)$, $\xi = z/r$ such that $\alpha(R) = 0$ and $\beta(0) = 0$ are functions to determine. The partial derivatives are now given by

$$\begin{aligned} d_r &= \alpha' I_{12}^{\perp} d - \frac{z}{r^2}\beta'\, Q_\alpha \widehat{e}^{\perp}, \\ d_{rr} &= (-|\alpha'|^2 I_{12} + \alpha'' I_{12}^{\perp}) d + \left(\frac{z^2}{r^4}\beta'' + 2\frac{z}{r^3}\beta'\right) Q_\alpha \widehat{e}^{\perp} \\ &\qquad - \frac{z^2}{r^4}|\beta'|^2 d - \frac{2z}{r^2}\alpha'\beta' I_{12}^{\perp} Q_\alpha \widehat{e}^{\perp}, \\ d_\theta &= -n I_{12}^{\perp} d, \quad d_{\theta\theta} = -n^2 I_{12} d, \\ d_z &= \frac{1}{r}\beta' Q_\alpha \widehat{e}^{\perp}, \quad d_{zz} = -\frac{1}{r^2}|\beta'|^2 d + \frac{1}{r^2}\beta'' Q_\alpha \widehat{e}^{\perp}. \end{aligned}$$

By replacing them in (13) we obtain

$$\begin{aligned} &- \left(|\alpha'|^2 + \frac{n^2}{r^2}\right) I_{12} d - \frac{1}{r^2}\left(1 + \frac{z^2}{r^2}\right)|\beta'|^2 d \\ &+ \left(\alpha'' + \frac{\alpha'}{r}\right) I_{12}^{\perp} d + \frac{1}{r^2}\left(\left(1 + \frac{z^2}{r^2}\right)\beta'' + \frac{z}{r}\beta'\right) Q_\alpha \widehat{e}^{\perp} + 2\lambda_2 d = -f_1 - f_2 \\ &\lambda_2 = \frac{1}{2}\left(\cos^2\beta \left(|\alpha'|^2 + \frac{n^2}{r^2}\right) + \frac{1}{r^2}\left(1 + \frac{z^2}{r^2}\right)|\beta'|^2\right) \\ &f_1 = g_1 Q_\alpha \widehat{e}^{\perp}, \quad g_1(r,z) = \left(|\alpha'|^2 + \frac{n^2}{r^2}\right)\sin\beta\cos\beta \\ &f_2 = -\frac{2z}{r^2}\alpha'\beta' I_{12}^{\perp} Q_\alpha \widehat{e}^{\perp} \end{aligned} \tag{23}$$

and the two terms f_1 and f_2 on the right hand satisfy $f_1 \cdot d = f_2 \cdot d = 0$. The last equality comes from the fact that

$$I_{12}^{\perp} Q_\alpha \widehat{e}^{\perp} \cdot d = Q_\alpha I_{12}^{\perp} \widehat{e}^{\perp} \cdot Q_\alpha \widehat{e} = Q_\alpha^t Q_\alpha I_{12}^{\perp} \widehat{e}^{\perp} \cdot Q_\alpha \widehat{e} = I_{12}^{\perp} \widehat{e}^{\perp} \cdot \widehat{e} = 0.$$

Equating the components of $I_{12}^{\perp}$ and of $Q_\alpha e^{\perp}$ on the left hand side to zero, we obtain

$$\begin{cases} r\alpha'' + \alpha' = 0 & r \in (0, R), \\ (1+\xi^2)\beta'' + \xi\beta' = 0 & \xi \in (0, \infty), \\ \alpha(R) = 0\,, \\ \beta(0) = 0\,. \end{cases} \tag{24}$$

Note that the domain of ξ was computed from $\xi = z/r$ as $z \to 0$ and as $r \to 0$. The solution of (24) is given by

$$\begin{aligned} \alpha &= c_1 \log(r/R), \\ \beta &= c_2 \sinh^{-1}(z/r), \end{aligned} \tag{25}$$

where $\sinh^{-1}(\xi) = \log\left(\xi + \sqrt{1+\xi^2}\right)$. Thus, we have obtained the following result.

Theorem 6. *For each $n \geq 1$, there is a family of solutions of the non-homogeneus Frank-Oseen equation*

$$\begin{cases} \Delta d + |\nabla d|^2 d = -f_1 - f_2 & \text{in } C, \\ d \cdot d = 1 & \text{in } C, \end{cases} \tag{26}$$

in the cylinder $C = \{(r, \theta, z) \,|\, 0 \leq r \leq R,\, 0 \leq \theta < 2\pi,\, 0 \leq z \leq H\}$, with right hand side such that $f_1 \cdot d = 0$, $f_2 \cdot d = 0$ and $f_1 = \mathcal{O}(1/r^2)$ and $f_2 = \mathcal{O}(z/r^3)$, is given by

$$d(r, \theta, z) = Q_{c_1 \log(r/R)} \widehat{e}\left(n\theta, c_2 \sinh^{-1}(z/r)\right), \tag{27}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants, Q_α is the rotation matrix defined in (16) *and $\widehat{e}$ was defined in* (15).

3.3. **Cylindrical case 3.** We can go further and search for a solution that absorbs f_1 (but not f_2). Equating the components of $I_{12}^{\perp}$ on the left hand side to zero and the components of $Q_\alpha e^{\perp}$ on the left hand side to g_1, we obtain

$$\begin{cases} r\alpha'' + \alpha' = 0 & r \in (0, R), \\ (1+\xi^2)\beta'' + \xi\beta' + \left(r^2|\alpha'|^2 + n^2\right)\sin\beta\cos\beta = 0 & \xi \in (0, \infty) \\ \alpha(R) = 0\,, \\ \beta(0) = 0\,. \end{cases}$$

The solution of the first equation is again given by $\alpha(r) = c_1 \log(r/R)$. Therefore,

$$(1+\xi^2)\beta'' + \xi\beta' + A^2 \sin\beta\cos\beta = 0, \quad \beta(0) = 0,$$

where $A^2 = c_1^2 + n^2$. Now making the change of variables $\gamma(v) = \beta(\sinh v)$, we see that γ solves

$$\gamma'' + A^2 \sin\gamma\cos\gamma = 0, \quad \gamma(0) = 0,$$

whose solution is $\gamma(v) = \mathrm{am}(c_2 v; k^2)$, where am is the Jacobi elliptic amplitude function with parameter (modulus) $0 < k^2 < 1$ and $c_2 = k/A$ [1]. This is because the am function satisfies the second order equation $\mathrm{am}''(\varphi) + k^2 \,\mathrm{sn}(\varphi)\,\mathrm{cn}(\varphi) = 0$, $\sin(\mathrm{am}(\varphi)) = \mathrm{sn}(\varphi)$ and $\cos(\mathrm{am}(\varphi)) = \mathrm{cn}(\varphi)$. From this it follows that

$$\alpha = c_1 \log(r/R), \tag{28}$$

$$\beta = \mathrm{am}(\varphi), \quad \varphi = c_2 \sinh^{-1}(z/r), \quad c_2 = \frac{k}{\sqrt{c_1^2 + n^2}}. \tag{29}$$

If we use that $\mathrm{am}'(\varphi) = \mathrm{dn}(\varphi)$ then

$$f_2 = -\frac{2c_1 z}{r^3}\,\mathrm{dn}(\varphi) Q_\alpha I_{12}^{\perp} \widehat{e}^{\perp}(n\theta, \mathrm{am}(\varphi))$$

that satisfies $f_2 \cdot d = 0$ and because cn, sn and dn are bounded, we have that f_2 is of order $\mathcal{O}(z/r^3)$. Thus we have obtained the following result.

Theorem 7. *For each $n \geq 1$, there is a family of solutions of the non-homogeneus Frank-Oseen equation*

$$\begin{cases} \Delta d + |\nabla d|^2 d = -f_2 \quad \text{in } C, \\ d \cdot d = 1 \quad \text{in } C, \end{cases} \tag{30}$$

in the cylinder $C = \{(r,\theta,z)\,|\,0 \leq r \leq R,\, 0 \leq \theta < 2\pi,\, 0 \leq z \leq H\}$, with right hand side f_2 such that $f_2 \cdot d = 0$ and $f_2 = \mathcal{O}(z/r^3)$ given by

$$d(r,\theta,z) = Q_{c_1 \log(r/R)} \widehat{e}\left(n\theta, \mathrm{am}(c_2 \sinh^{-1}(z/r); k^2)\right), \quad c_2 = \frac{k}{\sqrt{c_1^2 + n^2}}, \tag{31}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants, Q_α is the rotation matrix defined in (16)*, $\mathrm{am}(\cdot; k^2)$ is the Jacobi amplitude of modulus $k^2 \in (0,1)$ and $\widehat{e}$ was defined in* (15)*.*

Remark 8. *The solutions in* (31) *can be easily computed using the Jacobi elliptic functions* sn *and* cn *because $\widehat{e}$ defined in* (15) *satisfies*

$$\widehat{e}\,(a, \mathrm{am}(b)) = \begin{bmatrix} -\sin a\,\mathrm{cn}\, b \\ \cos a\,\mathrm{cn}\, b \\ -\,\mathrm{sn}\, b \end{bmatrix}. \tag{32}$$

Remark 9. *It is natural to ask why we have not provided the computations for the fully harmonic case in which both load vectors f_1 and f_2 are absorbed through the scalar functions as in the previous cases. In fact, this is not possible, because the differential equations obtained in such case are given by*

$$\begin{gathered} r^2\alpha'' + r\alpha' + 2\xi r \alpha'\beta' \tan\beta = 0, \quad \alpha(R) = 0 \\ (1+\xi^2)\beta'' + \xi\beta' + \left(r^2|\alpha'|^2 + n^2\right)\sin\beta\cos\beta = 0, \quad \beta(0) = 0, \quad \text{with } \xi = z/r. \end{gathered} \tag{33}$$

Performing the changes of variables

$$\delta(u) = \alpha(e^u), \quad u = \log(r), \quad \gamma(v) = \beta(\sinh v), \quad v = \sinh^{-1}(\xi)$$

we obtain the following system of second order non linear ordinary differential equations

$$\begin{aligned} &\delta'' + 2\delta'\gamma' \tan\gamma = 0, \quad \delta(\log R) = 0, \\ &\gamma'' + \left(|\delta'|^2 + n^2\right) \sin\gamma\cos\gamma = 0, \quad \gamma(0) = 0. \end{aligned} \tag{34}$$

We note that this implies that α *the variables depend on the argument of the other variable, which contradicts the form of the* ansatz *considered. Indeed, previous computations were successful because they allowed for a decoupling of the scalar functions* α *and* β*.*

Remark 10. *Notice that the singularity at the origin in all the cases of this section is of type* $\log(r)$ *as for the two dimensional case, but around all the vertical axis* $z = 0$*, so it is a vortex* [22]*. We could also remove this singularity by considering the equations in a perforated cylinder* $C \setminus C_\varepsilon$ *with* $C_\epsilon = \{(r, \theta, z) \,|\, |r| < \varepsilon, 0 \le \theta < 2\pi,\ |z| \le H\}$*. Otherwise, we will see in Section 5 that these solutions are not in* $H^1(C)$ *but in* $W^{1,p}(C)$ *for* $p < 2$*.*

We have numerically computed the solutions of equations (20), (26), and (30), considering a symmetric cylinder with $z \in (-H, H)$. The 3D plot of the streamlines obtained from the solution is shown in Figure 3, where we highlight the presence of the orbits ($n = 1$) or spirals ($n \ge 2$) in the xy-plane and a vortex in the z axis for all the solutions. Instead, in Figure 5 we have plotted the first component of the solution in three different settings: the mid-xy plane, its sum on the same plane and then the solution on the mid-xz plane. Interestingly, self-similar structures are observable aloso along the xz plane (see the third column of Figure 5). In Figure 4 we provide the observation of the structure of a proto-planetary disk to see the similarities that motivated us to build this type of solutions.

4. Approximate solutions in a ball

Let us now consider the ball in spherical coordinates

$$B = \left\{(r, \theta, \phi) \,|\, |r| \le R,\, 0 \le \theta < 2\pi,\, -\frac{\pi}{2} \le \phi < \frac{\pi}{2}\right\}$$

where θ is the azimuth angle and ϕ is the elevation angle, i.e., $\phi = 0$ corresponds to the mid horizontal plane. The non-homogeneous Frank-Oseen equation in spherical coordinates given by searching for a unitary field d such that:

$$d_{rr} + \frac{2}{r} d_r + \frac{1}{r^2 \cos^2\phi} d_{\theta\theta} + \frac{1}{r^2 \cos\phi} (\cos\phi\, d_\phi)_\phi + 2\lambda_2 d = -f \quad \text{in } B \tag{35}$$

where $f \cdot d = 0$ and

$$\lambda_2 = \frac{1}{2}\left(|d_r|^2 + \frac{1}{r^2\cos^2\phi}|d_\theta|^2 + \frac{1}{r^2}|d_\phi|^2\right). \tag{36}$$

As in the cylinder case, we separate our cases according to the treatment of the right hand side.

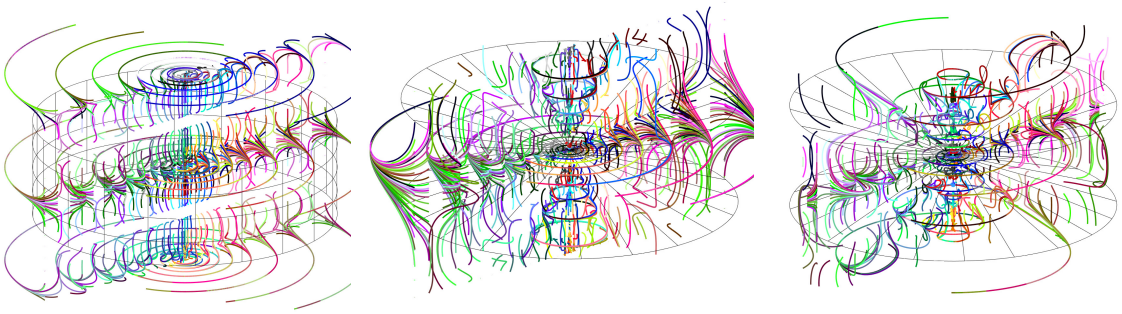

FIGURE 3. Trajectories of the 3D cylindrical solutions for Theorems 5, 6 and 7 for $R = 1$ and $H = \tan(\pi/10)$. The coefficients used were $n = 1$, $c_1 = 6\pi$, $c_2 = \pi$ for the first figure; $n = 1$, $c_1 = 6\pi$, $c_2 = \pi/2 - \arctan(H/R)$ for the second one; $n = 1$, $c_1 = 3\pi$, $A = \sqrt{c_1^2 + n^2}$, $k = 0.87$, $c_2 = A/k$ for the third one.

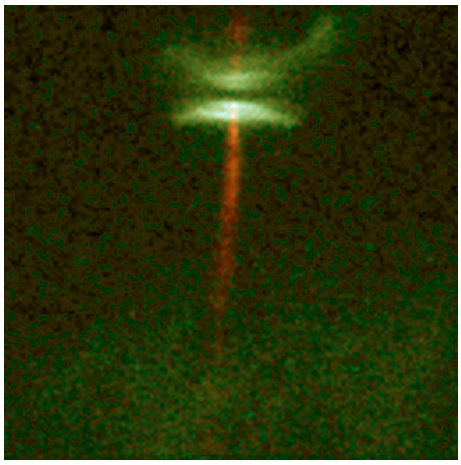

FIGURE 4. Protoplanetary disk HH-30 with jet, Taurus. Credit C. Burrows (STScI & ESA), the WFPC 2 Investigation Definition Team, and NASA [11].

4.1. **Spherical case 1.** We will look for solutions in the form of

$$d(r, \theta, \phi) = Q_{\alpha(r)} \widehat{e}(n\theta, \beta(\phi)) \tag{37}$$

where $\alpha(r)$ and $\beta(\phi)$ are functions to be determined such that $\alpha(R) = 0$, $\beta(0) = 0$, that is, the vector field rotates horizontally in the central plane xy. The partial derivatives are

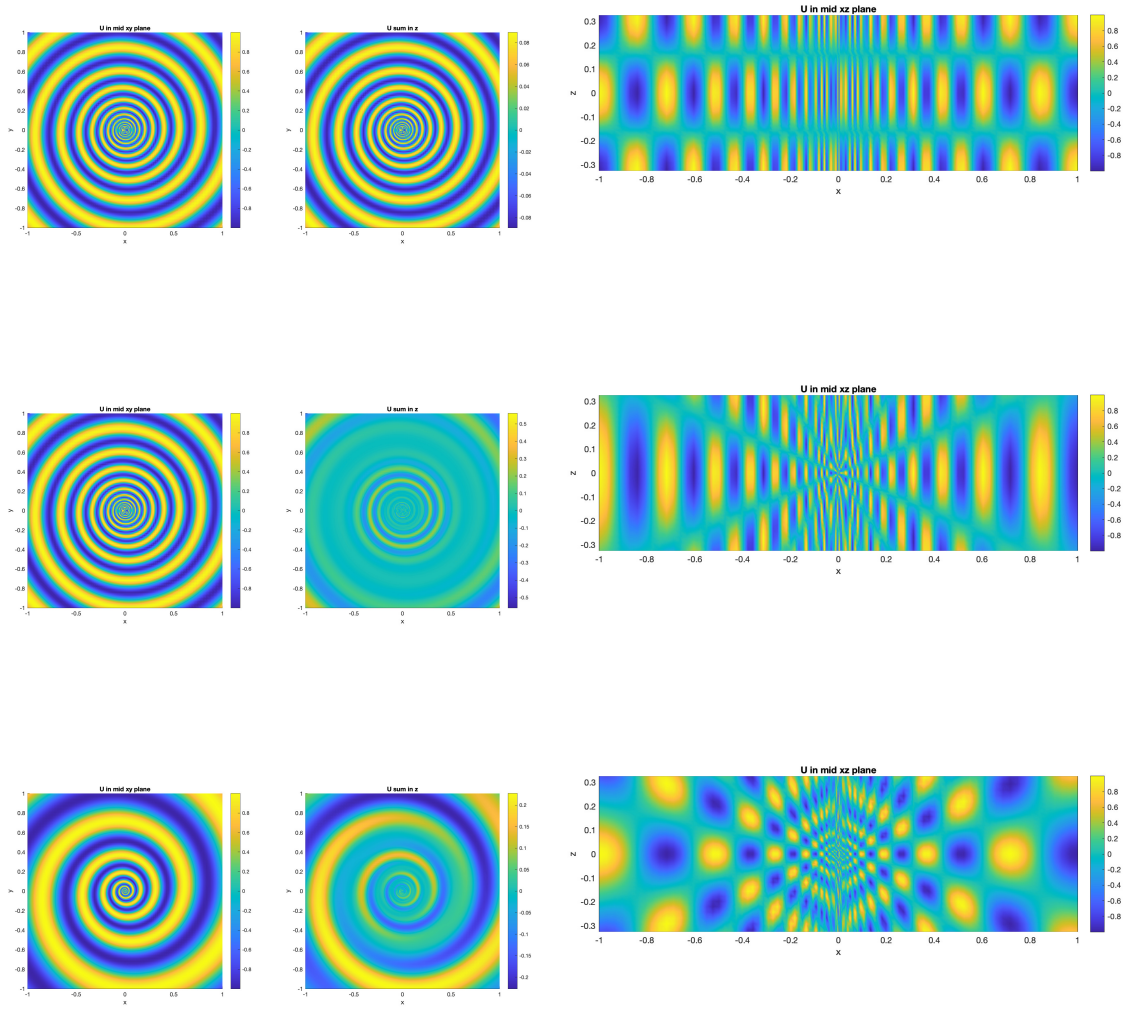

FIGURE 5. First component of the unitary field d from the cylinder case in the mid xy plane (left), sum in the z axis (center) and first component in the mid xz plane (right). The same three cases of Figure 3 are plotted in each row correspondingly.

given by

$$\begin{aligned} d_r &= \alpha' I_{12}^{\perp} d, \quad d_{rr} = (-|\alpha'|^2 I_{12} + \alpha'' I_{12}^{\perp}) d, \\ d_\theta &= -n I_{12}^{\perp} d, \quad d_{\theta\theta} = -n^2 I_{12} d, \end{aligned}$$

$$d_\phi \;=\; \beta' Q_\alpha \widehat{e}^{\perp}, \quad d_{\phi\phi} = -|\beta'|^2 d + \beta'' Q_\alpha \widehat{e}^{\perp}.$$

Replacing in (35) yields

$$\begin{aligned} -|\alpha'|^2 I_{12} d + \left(\alpha'' + \frac{2\alpha'}{r}\right) I_{12}^{\perp} d - \frac{n^2}{r^2\cos^2\phi} I_{12} d \\ -\frac{|\beta'|^2}{r^2} d + \frac{1}{r^2}(\beta'' - \tan(\phi)\beta') Q_\alpha \widehat{e}^{\perp} + 2\lambda_2 I = -f_3 - f_4, \\ \lambda_2 = \frac{1}{2}\left(\cos^2\beta\left(|\alpha'|^2 + \frac{n^2}{r^2\cos^2\phi}\right) + \frac{1}{r^2}|\beta'|^2\right), \end{aligned} \tag{38}$$

$$f_3 = g_3 Q_\alpha \widehat{e}^{\perp},\ g_3 = \sin\beta\cos\beta|\alpha'|^2,\ f_4 = g_4 Q_\alpha \widehat{e}^{\perp},\ g_4 = \sin\beta\cos\beta\frac{n^2}{r^2\cos^2\phi}.$$

Equating the components of $I_{12}^{\perp}$ and of $Q_\alpha \widehat{e}^{\perp}$ on the left hand side, we obtain

$$\begin{cases} r\alpha'' + 2\alpha' = 0 & r \in (0, R), \\ \beta'' - \tan(\phi)\beta' = 0 & \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ \alpha(R) = 0, \\ \beta(0) = 0, \end{cases}$$

from which we have that

$$\alpha \;=\; c_1\left(\frac{1}{R} - \frac{1}{r}\right), \tag{39}$$

$$\beta \;=\; c_2 \log\left(\frac{1+\sin\phi}{\cos\phi}\right), \tag{40}$$

We have thus obtained the following result.

Theorem 11. *For each $n \geq 1$ there is a family of solutions of the non-homogeneous Frank-Oseen equations*

$$\begin{cases} \Delta d + |\nabla d|^2 d = -f_3 - f_4 & \text{in } B, \\ d \cdot d = 1 & \text{in } B, \end{cases} \tag{41}$$

in the ball $B = \left\{(r,\theta,\phi)\,|\,|r| \leq R,\, 0 \leq \theta < 2\pi,\, -\frac{\pi}{2} \leq \phi < \frac{\pi}{2}\right\}$ with $f_3 \cdot d = 0$, $f_4 \cdot d = 0$, $f_3 = \mathcal{O}(1/r^2)$ and $f_4 = \mathcal{O}(1/r^4)$ of the form

$$d(r,\theta,\phi) = Q_{c_1\left(\frac{1}{R} - \frac{1}{r}\right)} \widehat{e}\left(n\theta, c_2 \log\left(\frac{1+\sin\phi}{\cos\phi}\right)\right), \tag{42}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants, Q_α is the rotation matrix defined in (16) *and $\widehat{e}$ was defined in* (15).

Remark 12. *For the solutions* (42) *of the spherical case in the previous theorem, they do not present orbits in the mid xy-plane, but only spirals.*

4.2. **Spherical case 2.** If we want to absorb f_3 from (35) we would need to solve

$$\begin{cases} r\alpha'' + 2\alpha' = 0 \quad r \in (0,R), \\ \cos^2\phi\beta'' - \sin\phi\cos\phi\beta' + n^2\sin\beta\cos\beta = 0 \quad \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ \alpha(R) = 0\,, \\ \beta(0) = 0\,. \end{cases} \tag{43}$$

The solution for α is the same as for the previous case. For β, if we define

$$v = h(\phi) = \log\left(\frac{1+\sin\phi}{\cos\phi}\right) \tag{44}$$

with $h(0) = 0$, now if we make the change of variables $\gamma(v) = \beta(h^{-1}(v))$ then $\gamma'' = \cos^2\phi\beta'' - \sin\phi\cos\phi\beta'$ and γ is a solution of

$$\gamma'' + n^2\sin\gamma\cos\gamma = 0, \quad \gamma(0) = 0. \tag{45}$$

Again, the solution is given by $\gamma(v) = \mathrm{am}(c_2 v; k^2)$ of modulus k, where $c_2 = n/k$. From the previous analysis we have that

$$\alpha = c_1\left(\frac{1}{R} - \frac{1}{r}\right), \tag{46}$$

$$\beta = \mathrm{am}\left(c_2\log\left(\frac{1+\sin\phi}{\cos\phi}\right); k^2\right),\ c_2 = \frac{k}{n}, \tag{47}$$

and we obtain the following result.

Theorem 13. *For each $n \geq 1$ there is a family of solutions of the non-homogeneous Frank-Oseen equations*

$$\begin{cases} \Delta d + |\nabla d|^2 d = -f_4 & \text{in } B, \\ d\cdot d = 1 & \text{in } B, \end{cases} \tag{48}$$

in the ball $B = \left\{(r,\theta,\phi)\,|\,|r| \leq R,\ 0 \leq \theta < 2\pi,\ -\frac{\pi}{2} \leq \phi < \frac{\pi}{2}\right\}$ with $f_4 \cdot d = 0$, $f_4 = \mathcal{O}(1/r^4)$ of the form

$$d(r,\theta,\phi) = Q_{c_1\left(\frac{1}{R}-\frac{1}{r}\right)}\widehat{e}\left(n\theta, \mathrm{am}\left(c_2\log\left(\frac{1+\sin\phi}{\cos\phi}\right); k^2\right)\right), \quad c_2 = \frac{k}{n}, \tag{49}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants, Q_α is the rotation matrix defined in (16)*, $\mathrm{am}(\cdot; k^2)$ is the Jacobi amplitude of modulus $k^2 \in (0,1)$ and $\widehat{e}$ was defined in* (15)*.*

Remark 14. *Again, all the solutions* (49) *of the spherical case in the previous theorem do not exhibit orbits in the mid xy-plane, but only spirals. See Figure 8 and compare with Figure 2.*

Remark 15. *Note that the singularity of the solution at the origin in the spherical cases are of type $1/r$. There is also a singularity on the vertical axis (elevation $\phi = \pm\pi/2$) of type $\log\left(\frac{\pi}{2} - |\phi|\right)$ which is a vortex* [22]*. This means the solutions are not in $H^1(B)$. We*

could remove these singularities on the origin and the vertical axis by considering the ball B minus $B_\varepsilon = \{(r,\theta,\phi)\,|\,|r|<\varepsilon, 0\le\theta<2\pi, |\phi|<\arctan(R/\varepsilon)\}$.

Remark 16. *All of the missing combinations of terms that can be absorbed into the left hand side yield the same contradiction detailed in Remark 9. We omit further details.*

We have numerically computed the solutions of (41) and (48), with its streamlines shown in Figure 6, where the presence of a radial spiral instead of the logarithmic orbits can be appreciated. In Figure 7 we plot the first component of both solutions in the xy and xz planes. Interestingly, the solution quickly fades to the origin, which prevents the formation of orbits. It is also possible to see the discontinuity along the $\pm\pi/2$ elevations. We conclude this by showing how the solution changes in the partially non-homogeneous case as the angle parameter n changes in Figure 8. We stretch the analogy of liquid crystals and protoplanets formation to show that, only anecdotically, the solutions resemble protoplanetary jet observations, as the one shown in Figure 4.

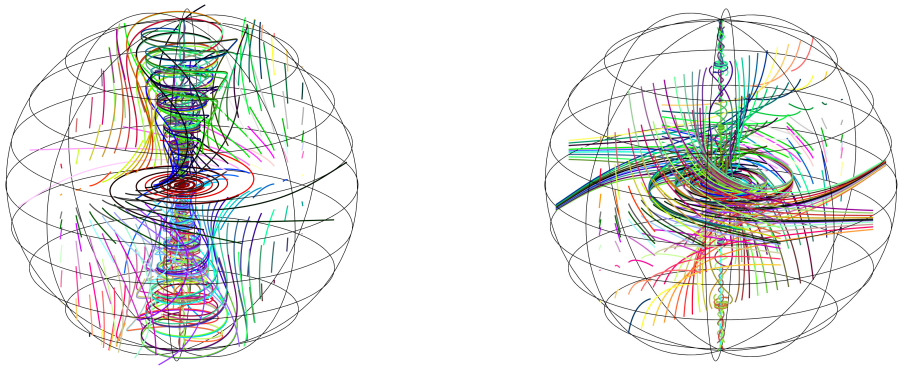

FIGURE 6. Streamlines for the spherical solutions of Theorems 11 and 13 in the unit sphere given by (42) and (49) respectively. Left: $n=1$, $c_1=3$ and $c_2=-\pi/2$. Right: $n=1$, $c_1=3$, $k=0.87$.

5. p-NEMATIC FIELDS

In the previously constructed solutions, we have obtained that they satisfy the Frank-Oseen equations (2), but they have infinite energy in $H^1(\Omega)$, where Ω is either a disk in two dimension or a cylinder or a sphere in three dimensions. It is natural to ask if there is a modified potential which is (i) finite and (ii) such that the solutions are critical points (or minima). Given that the functions have been observed to belong to $W^{1,p}(\Omega)$, we define a vector field to be p-nematic by the following definition.

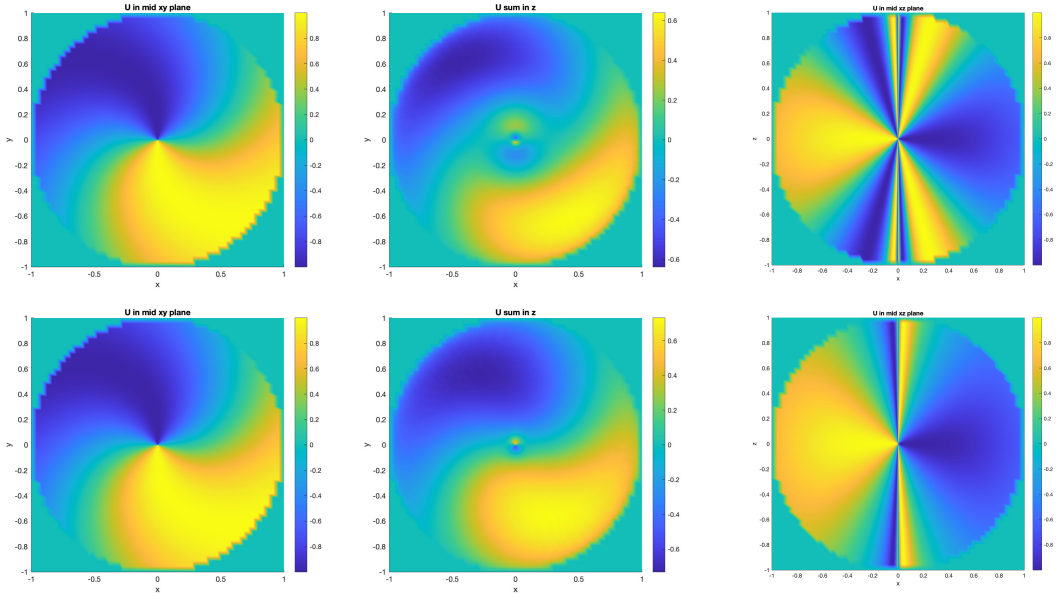

FIGURE 7. First component of the unitary field d of the same spherical cases shown in Figure 6, first component of the solution (left), sum over z (center) and mid xz plane (right).

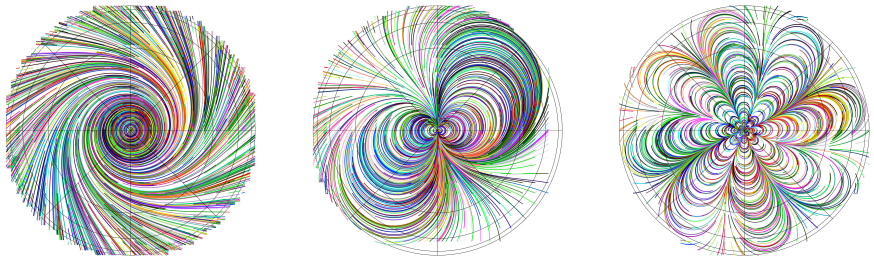

FIGURE 8. Streamlines for spherical solutions in the spherical case of Theorem 13 for varying values of the parameter n. From left to right $n = 1, 2, 5$.

Definition 17. *Given a regular domain $\Omega \in \mathbb{R}^N$, $N = 2, 3$, we say that d is a p-nematic liquid crystal in $W^{1,p}(\Omega)$, if given f in the dual space of $W^{1,p}(\Omega)$, it is a critical point of:*

$$\min_{s.t.|d|=1} \frac{1}{p} \int_\Omega |\nabla d|^p - \int_\Omega f \cdot d \tag{50}$$

for $p \geq 1$ in a certain domain Ω. That is, a unit vector field d that satisfies the generalized Frank-Oseen equations:

$$\Delta_p d + p\lambda_p(d)d \quad = \quad -f, \quad d \cdot d = 1 \quad \text{in } \Omega,$$

$$\lambda_p(d) := \frac{1}{p}\left(|\nabla d|^p - |\nabla d|^{p-2} f \cdot d\right), \tag{51}$$

where

$$\Delta_p d = \nabla \cdot (|\nabla d|^{p-2}\nabla d) \tag{52}$$

is the p−Laplacian [16]. *We say that the p-nematic field is homogeneous if $f = 0$, non-homogeneous otherwise and conservative if $f \cdot d = 0$ with total energy $E_p(\Omega) = \int_\Omega \lambda_p$ in the latter case.*

Notice that the case $p = 2$ corresponds to the 2-nematic solutions encountered in the previous sections and introduced in Definition 1, but we recall that they had finite energy only away from the the origin or the vertical axis. We have the following purely algebraic relationship between the $p = 2$ case and the general case that will be useful in the following.

Lemma 18. *Let d be a vector field in $\mathbb{R}^N$, then*

$$\lambda_p = \frac{2}{p}|\nabla d|^{p-2}\lambda_2, \tag{53}$$

which in the conservative case $f \cdot d = 0$ reduces to

$$\lambda_p = \frac{2^{p/2}}{p}\lambda_2^{p/2}, \tag{54}$$

and the corresponding energy in the conservative case is given by

$$E_p(\Omega) = \int_\Omega \lambda_p = \frac{2^{p/2}}{p}\int_\Omega \lambda_2^{p/2}. \tag{55}$$

In the following, we provide estimates of the p-energy for most of the solutions obtained in the previous sections.

Theorem 19. *The energy of the solutions of the Frank-Oseen solutions of the previous sections are estimated by:*

- **Disk case***: The solutions of Theorem 2 in the disk D of radius $R > 0$ do not belong to $H^1(D)$ and they belong to $W^{1,p}(D)$ for all $1 \le p < 2$. More precisely, if $D_\varepsilon = \{(r,\theta),\ |r| < \varepsilon, 0 \le \theta < 2\pi\}$, then*

$$\begin{aligned} E_2(D) &= +\infty \\ E_2(D \setminus D_\varepsilon) &= \pi(c_1^2 + n^2)\log(R/\varepsilon), \\ E_p(D) &= \frac{2\pi}{p(2-p)}(c_1^2 + n^2)^{p/2}R^{2-p}. \end{aligned}$$

- **Cylindrical cases***: The solutions of Theorems 5, 6 and 7 in the cylinder C of radius $R > 0$ and height $2H > 0$ do not belong to $H^1(C)$ and they belong to $W^{1,p}(C)$ for all $1 \le p < 2$. More precisely, if $C_\varepsilon = \{(r,\theta,z),\ |r| < \varepsilon,\ 0 \le \theta < 2\pi,\ |z| \le H\}$, then*

$$\begin{aligned} E_2(C) &= +\infty \\ E_2(C \setminus C_\varepsilon) &= \mathcal{O}\left(\log(R/\varepsilon)\right) \ \text{as } \varepsilon \to 0, \end{aligned}$$

$$E_p(C) \;=\; \mathcal{O}\left(\frac{R^{2-p}}{2-p}\right) \text{ as } p \to 2.$$

- **Spherical cases**: *The solutions of Theorems 11 and 13 in the ball B of radius $R > 0$ do not belong to $H^1(B)$ and they belong to $W^{1,p}(B)$ for all $1 \le p < 3/2$. More precisely, if $B_{\varepsilon,\delta} = \{(r,\theta,\phi),\ |r| < \varepsilon,\ 0 \le \theta < 2\pi,\ |\phi| < \pi/2 - \delta\}$, then*

$$\begin{aligned} E_2(B) &= +\infty \\ E_2(B \setminus B_{\varepsilon,\delta}) &= \mathcal{O}\left(\left(\frac{1}{\varepsilon} - \frac{1}{R}\right)\log(1/\delta)\right) \text{ as } \varepsilon \to 0 \text{ and } \delta \to 0, \\ E_p(B) &= \mathcal{O}\left(\frac{R^{3-2p}}{3-2p}\right) \text{ as } p \to 3/2. \end{aligned}$$

Proof. In the two dimensional case of a disk, it is easy to verify from (9), (10) and that

$$E_2(D \setminus D_\varepsilon) = \int_{D\setminus D_\varepsilon} \lambda_2 = \pi \int_\varepsilon^R \left[\frac{c_1^2 + n^2}{r^2}\right]^{p/2} r\, dr = \pi(c_1^2 + n^2)\log(R/\varepsilon)$$

where we see that as $\varepsilon \to 0$ the integral diverges to $+\infty$ and using the previous proposition for $1 \le p < 2$

$$E_p(D) = \frac{2^{p/2}}{p}\int_D \lambda_2^{p/2} = \frac{2\pi}{p}\int_0^R \left[\frac{c_1^2 + n^2}{r^2}\right]^{p/2} r\, dr = \frac{2\pi}{p(2-p)}(c_1^2 + n^2)^{p/2} R^{2-p}$$

from which we obtain the statement of the theorem for this case.

In the cylindrical case 1, using (17) and (19), we see that the energy in the cylinder C is given by

$$\begin{aligned} E_p(C) &= \frac{2^{p/2}}{p}\int_0^{2\pi}\int_{-H}^{H}\int_0^R \lambda_2^{p/2} r\, dr dz d\theta \\ &= \frac{2\pi}{p}\int_{-H}^{H}\int_0^R |\cos(c_2 z)|^p \left[\frac{c_1^2 + n^2}{r^2} + c_2^2\right]^{p/2} r\, dr dz \\ &\le \frac{4\pi H}{p}\frac{(c_1^2 + n^2 + c_2^2 R^2)^{p/2}}{2-p} R^{p-2}. \end{aligned}$$

For $1 \le p < 2$ we have $x^p \ge x^2$ for $x \in [0,1]$ and $\int_{-H}^{H} cos^2(c_2 z)dz = H\,(1 + \mathrm{sinc}(2c_2 H))$, using also that $|x|^{p/2}$ is a non decreasing function we have that

$$E_p(C) \ge \frac{2\pi H(1 + \mathrm{sinc}(2c_2 H))}{p}\frac{(c_1^2 + n^2)^{p/2}}{2-p} R^{p-2}.$$

In the same way, it is easy to check that

$$E_2(C \setminus C_\varepsilon) \le 2\pi H(c_1^2 + n^2 + c_2 R^2)\log(R/\varepsilon)$$

and

$$E_2(C \setminus C_\varepsilon) \ge \pi H(1 + \mathrm{sinc}(2c_2 H))(c_1^2 + n^2)\log(R/\varepsilon).$$

In the cylindical case 2, from (23) and (25), the energy in the cylinder C for $1 \leq p < 2$ is given by

$$E_p(C) = \frac{2\pi}{p}\int_{-H}^{H}\int_0^R \left(\cos^2(\beta(z/r))\left[\frac{c_1^2+n^2}{r^2}\right] + |\beta'(z/r)|^2\left[\frac{1}{r^2}+\frac{z^2}{r^4}\right]\right)^{p/2} r dr dz$$
$$\leq \frac{4\pi H}{p}\int_0^R \left(\frac{c_1^2+n^2+2c_2^2}{r^2}\right)^{p/2} r dr \leq \frac{4\pi H}{p(2-p)}(c_1^2+n^2+2c_2^2)^{p/2}R^{2-p},$$

where $\beta(\xi) = c_2 \sinh^{-1}(\xi)$, and we used that $|\cos(\beta)| \leq 1$ and $\xi(\sinh^{-1})'(\xi) = \xi(1+\xi^2)^{-1/2} \leq 1$ for $\xi = z/r$. It is also easy to see that since $(1+\xi^2)^{-1/2} \geq (1+(H/R)^2)^{-1/2}$ then

$$E_p(C) \geq \frac{4\pi H}{p(2-p)}(1+(R/H)^2)^{-p/2}R^{2-p}.$$

In the same way, it is easy to check in this case that

$$E_2(C \setminus C_\varepsilon) \leq 2\pi H(c_1^2+n^2+2c_2)\log(R/\varepsilon)$$

and

$$E_2(C \setminus C_\varepsilon) \geq 2\pi H(1+(R/H)^2)^{-1}\log(R/\varepsilon).$$

For the cylindrical case 3, in which β is given this time by (28)), the analysis is analogous to the previous one by replacing $\beta(\xi) = \mathrm{am}(c_2\sinh^{-1}(\xi))$, because in this case $\beta'(\xi) = c_2\,\mathrm{dn}(c_2\sinh^{-1}(\xi))(1+\xi^2)^{-1/2}$ and the Jacobi function dn is strictly positive and more precisely has range in $[\sqrt{1-k^2},1]$. We obtain thus the same upper bounds of the previous case and the same lower bounds up to the factor $\sqrt{1-k^2}$.

For the spherical case 1, given by (38) and (39), the energy in the ball B (since in this case $\beta'(\phi) = c_2/\cos\phi$) is given by

$$E_p(B) = \frac{2\pi}{p}\int_{-\pi/2}^{\pi/2}\int_0^R \left(\frac{c_1^2\cos^2\beta(\phi)}{r^4} + \frac{n^2\cos^2\beta(\phi)+c_2^2}{r^2\cos^2\phi}\right)^{p/2} r^2\cos\phi\, dr d\phi.$$

We can bound the energy from above by

$$\begin{aligned} E_p(B) &\leq \frac{2\pi}{p}\max\{c_1^p,(n^2+c_2^p)^p\}\int_{-\pi/2}^{\pi/2}\int_0^R\left(\frac{1+R^2}{r^4\cos^2\phi}\right)^{p/2} r^2\cos\phi\, dr d\phi, \\ &\leq \frac{2\pi}{p}\max\{c_1^p,(n^2+c_2^p)^p\}(1+R^2)^{p/2}\int_0^R r^{2-2p}\,dr\int_{-\pi/2}^{\pi/2}(\cos\phi)^{1-p}\,d\phi, \end{aligned}$$

which gives that the radial integral is finite for $p < 3/2$, and for the angular integral for the elevation angle ϕ we have

$$\int_{-\pi/2}^{\pi/2}(\cos\phi)^{1-p}\,d\phi = 2\int_0^{\pi/2}(\sin\psi)^{1-p}d\psi$$

and $\sin\psi \approx \psi$ for ψ small so the integral is finite (say a constant $C_3 > 0$) for $p < 2$. Finally

$$E_p(B) \leq C_3\frac{2\pi}{p}\max\{c_1^p,(n^2+c_2^p)^p\}(1+R^2)^{p/2}\frac{R^{3-2p}}{3-2p}.$$

For a lower bound, we have that

$$E_p(B) \geq \frac{2\pi c_1^p}{p} \int_0^R r^{2-2p} dr \int_{-\pi/2}^{\pi/2} \cos^2(\beta(\phi)) \cos\phi \, d\phi,$$

and the integral in ϕ is strictly positive, since its argument is non negative, continuous and thus positive at some point $\phi = \phi_0$ with $-\pi/2 < \phi_0 < \pi/2$, so we conclude by continuity that there exists a constant $C_4 > 0$ such that

$$E_p(B) \geq C_4 \frac{2\pi c_1^p}{p} \frac{R^{3-2p}}{3-2p}.$$

In the same way, using the same arguments as before, it is easy to check that there exist positive constants C_5 and C_6 such that

$$E_2(B \setminus B_{\varepsilon,\delta}) \leq C_5 \left(\frac{1}{\varepsilon} - \frac{1}{R}\right) \log(1/\delta)$$

and

$$E_2(B \setminus B_{\varepsilon,\delta}) \geq C_6 \left(\frac{1}{\varepsilon} - \frac{1}{R}\right) \log(1/\delta).$$

The factor $\log(1/\delta)$ comes from the fact that

$$\int_{-\pi/2+\delta}^{\pi/2-\delta} (\cos\phi)^{-1} \, d\phi = 2 \int_0^{\pi/2-\delta} (\sin\psi)^{-1} d\psi = \mathcal{O}\left(\log(1/\delta)\right).$$

The proof for spherical case 2 given as in the precedent case by (38) but only changing the function β by (46), is analogous to the previous one. □

One may naturally wonder if a 2-nematic solution behaves also as a p-nematic solution, under certain circumstances. We have the following.

Theorem 20. *If d is a non-homogeneous 2-nematic solution associated to f with $f \cdot d = 0$, then d is also a p-nematic non homogeneous solution associated to $\widetilde{f}$ defined by*

$$\widetilde{f} = |\nabla d|^{p-2} f - (p-2)|\nabla d|^{p-4} \nabla d \cdot [\nabla d : \nabla^2 d] \tag{56}$$

with the property that $\widetilde{f} \cdot d = 0$.

Proof. For this, we consider d a 2-nematic solution, and compute its p-Laplacian:

$$\begin{aligned}
\Delta_p d &= \left((d_{k,l} d_{k,l})^{\frac{p-2}{2}} d_{i,j}\right)_{,j} = (p-2)(d_{k,l} d_{k,l})^{\frac{p-4}{2}} d_{k,lj} d_{k,l} d_{i,j} + |\nabla d|^{p-2} d_{i,jj} \\
&= (p-2)\,|\nabla d|^{p-4} \nabla d \cdot [\nabla d : \nabla^2 d] + |\nabla d|^{p-2} \Delta d, \\
&= (p-2)|\nabla d|^{p-4} \nabla d \cdot [\nabla d : \nabla^2 d] + |\nabla d|^{p-2}(-|\nabla d|^2 + f \cdot d) d - |\nabla d|^{p-2} f \\
&= -(|\nabla d|^p + |\nabla d|^{p-2} f \cdot d) d - \widetilde{f}, \qquad (57)
\end{aligned}$$

where $\widetilde{f}$ was defined in (56). Also we have that

$$\widetilde{f} \cdot d = |\nabla d|^{p-2} f \cdot d. \tag{58}$$

Indeed, the second term in (56) vanishes when multiplying by d as can be seen by differentiating the unitary condition $d \cdot d = 1$. This yields $(\nabla d)^t d = 0$, that is $d_{i,j} d_i = 0$, and then also $d_{k,lj} d_{k,l} d_{i,j} d_i = 0$. In particular if $f \cdot d = 0$ then also $\widetilde{f} \cdot d = 0$. Notice that if $f \cdot d = 0$ then $\widetilde{f} \cdot d = 0$ so in this case $\lambda_p = \frac{1}{p}|\nabla d|^p$. □

Let us see, in the case of the 2-nematic two dimensional disk solution of Theorem 2, which is the p-nematic equation that this solution satisfies. We have the following result.

Theorem 21. *The 2-nematic solutions $d \in W^{1,p}(D)$ for $1 \leq p < 2$ of Theorem 2 defined in the disk $D = \{(r,\theta),\ 0 \leq r \leq R, 0 \leq \theta \leq 2\pi\}$ satisfy the non-homogeneous $p-$nematic equations of Definition* (17) *with right hand side given by*

$$\widetilde{f}(r,\theta) = \frac{c_1(p-2)2^{p/2-2}(c_1^2+n^2)^{p/2-1}}{r^{p/2}} I^{\perp} d(r,\theta),\ (r,\theta) \in D \tag{59}$$

with $\tilde{f} \cdot d = 0$, $\widetilde{f} = \mathcal{O}\left(\frac{1}{r^{p/2}}\right)$ and $\widetilde{f} \in L^q(D)$ for all $q < \frac{4}{p}$.

Proof. In this case, the solution is homogeneous ($f = 0$), so by Theorem 20 we have that

$$\tilde{f} = -(p-2)|\nabla d|^{p-4} \nabla d \cdot [\nabla d : \nabla^2 d].$$

Changing variables we obtain

$$\nabla_{x,y}\, d = \frac{1}{r} \nabla_{r,\theta}\, d \begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \frac{1}{2} I^{\perp} \begin{bmatrix} \frac{c_1}{r} d & -nd \end{bmatrix} \begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

and we notice the following simplification (in tensor notation):

$$\nabla d : \nabla^2 d = d_{k,l} d_{k,lj} = \frac{1}{2}(d_{k,l} d_{k,l})_j = \nabla\left(\frac{1}{2}|\nabla d|^2\right) = \nabla \lambda_2,$$

which allows us to compute

$$\nabla d : \nabla^2 d = \nabla \lambda_2(x,y) = \frac{-2}{r^2} \lambda_2 \begin{bmatrix} x \\ y \end{bmatrix}.$$

Using that $|\nabla d|^{p-4} = 2^{p/2-2} \lambda_2^{p/2-2}$, we can finally compute the desired term:

$$\begin{aligned} \tilde{f} &= -(p-2)2^{p/2-2}\lambda_2^{p/2-2} \cdot \left(\frac{-2\lambda_2}{r^2}\right) \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \frac{(p-2)2^{p/2-1}}{r^2} \lambda_2^{p/2-1} \nabla d \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \frac{(p-2)2^{p/2-1}(c_1^2+n^2)^{p/2-1}}{r^{p/2+1}} \nabla d \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \frac{(p-2)2^{p/2-1}(c_1^2+n^2)^{p/2-1}}{r^{p/2}} \nabla d \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \end{aligned}$$

and we note that

$$\nabla d \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \frac{1}{2} I^{\perp} \begin{bmatrix} \frac{c_1}{r} d & -n\, d \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} = \frac{c_1}{2} I^{\perp} d.$$

This results in

$$\tilde{f} = \frac{c_1 (p-2) 2^{p/2-2} (c_1^2 + n^2)^{p/2-1}}{r^{p/2}} I^{\perp} d,$$

which naturally yields $\tilde{f} \cdot d = 0$ and the required order. The regularity L^q-regularity of $\widetilde{f}$ is given by

$$\int_0^R \left(\frac{1}{r^{p/2}} \right)^q r dr = \frac{R^{2-pq/2}}{2 - pq/2}$$

where the integral is finite if $q < 4/p$, this complete the proof. □

6. Summary and discussion

In this work we have obtained analytical solutions of the Frank-Oseen equations that are homogeneous or exact on a disk, and non-homogenoeus or approximate in a cylinder and a ball. The solutions possess infinite energy in the natural H^1 energy space, but they have instead finite energy in $W^{1,p}$ spaces for some $p < 2$. We have shown that the exponent p is strictly smaller than 2 for the disk and cylinder cases, whereas for the sphere solution there are cases where the energy is finite only for $p < 3/2$. We have additionally defined a modification of the Frank-Oseen equations for allow infinite energy solutions as p-nematic liquid crystals. There is an interesting algebraic connection between 2-nematic and p-nematic liquid crystals, whose regularity properties will be addressed in future studies. One of the main difficulties lies in the non-homogeneous loading term that arises on such a formulation, which we have shown it depends on the second order derivatives of the original solution. The solutions exhibit qualitative behavior with self-similar orbits or spirals in the horizontal plane and vortices in the vertical axis. This type of singularities naturally appear in nematic fluids. Finally, notice that some of the ansatz of the solutions were inspired in hydrodynamics orbits and singularities observed in proto-planet formation models. The rigorous connection between nematic fluids and these type of hydrodynamic or magneto-hydrodynamic models has not been yet established, and it will be matter of further research.

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