

REDUCIBILITY OF QUANTUM HARMONIC OSCILLATOR ON \mathbb{R}^d PERTURBED BY A QUASI - PERIODIC POTENTIAL WITH LOGARITHMIC DECAY

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ABSTRACT. We prove the reducibility of quantum harmonic oscillators in \mathbb{R}^d perturbed by a quasi-periodic in time potential $V(x, \omega t)$ with *logarithmic decay*. By a new estimate built for solving the homological equation we improve the reducibility result by Grébert-Paturel (Annales de la Faculté des sciences de Toulouse : Mathématiques. **28**, 2019).

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1. INTRODUCTION AND MAIN RESULTS

1.1. Statement of the results. In this paper we consider the linear equation

$$i\partial_t u = (-\Delta + |x|^2)u + \epsilon V(x, \omega t)u, \quad u = u(t, x), \quad t \in \mathbb{R}, x \in \mathbb{R}^d, \quad (1.1)$$

where $\epsilon \geq 0$ is a small parameter and frequency vector ω of the forced oscillations is regarded as a parameter in $\mathcal{D}_0 := [0, 2\pi)^n$. We assume that the potential $V : \mathbb{R}^d \times \mathbb{T}^n \ni (x, \varphi) \mapsto V(x, \varphi) \in \mathbb{R}$ is continuous in all its variables and analytic in φ , where $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ denotes the n -dimensional torus. For $\sigma > 0$, the function $V(x, \varphi)$ analytically in φ extends to the strip $\mathbb{T}_\sigma^n = \{(a + bi) \in \mathbb{C}^n / 2\pi\mathbb{Z}^n : |b| < \sigma\}$ and for all $(x, \varphi) \in \mathbb{R}^d \times \mathbb{T}_\sigma^n$ verifies

$$|V(x, \varphi)| \leq C(1 + \ln(1 + |x|^2))^{-2\iota}, \quad (1.2)$$

where $\iota \geq 0$ and $C > 0$.

Before state the main results, we need some notations. For readers' convenience we will follow the notations in [28]. Let $H_0 = -\Delta + |x|^2 = -\Delta + x_1^2 + x_2^2 + \cdots + x_d^2$ be the d -dimensional quantum harmonic oscillator. Its spectrum is the sum of d copies of the odd integers set, i.e. the spectrum of H_0 equals $\widehat{\mathcal{E}} := \{d, d + 2, d + 4, \cdots\}$. Denote by E_j for $j \in \widehat{\mathcal{E}}$ the associated eigenspace whose dimension equals

$$\#\{(i_1, i_2, \cdots, i_d) \in (2\mathbb{N} + 1)^d : i_1 + i_2 + \cdots + i_d = j\} := d_j \leq j^{d-1}, \quad \text{where } \mathbb{N} = \{0, 1, 2, \cdots\}.$$

Denote by $\{\Phi_{j,l}, l = 1, 2, \cdots, d_j\}$ the basis of E_j obtained by d -tensor product of Hermite functions: $\Phi_{j,l} = \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_d}$ for some choice of $i_1 + i_2 + \cdots + i_d = j$. Then setting

$$\mathcal{E} := \{(j, l) \in \widehat{\mathcal{E}} \times \mathbb{N} : l = 1, 2, \cdots, d_j\} \text{ and } w_{j,l} = j \text{ for } (j, l) \in \mathcal{E}$$

$(\Phi_a)_{a \in \mathcal{E}}$ forms the Hermite basis of $L^2(\mathbb{R}^d)$, verifying $H_0 \Phi_a = w_a \Phi_a$, $a \in \mathcal{E}$. We define on \mathcal{E} an equivalent relation: $a \sim b \iff w_a = w_b$ and denote by $[a]$ the equivalence class

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corresponding with $a \in \mathcal{E}$. We notice for later use that $\#[a] \leq w_a^{d-1}$ and that abbreviate the eigenspace E_{w_a} as $E_{[a]}$.

For $s \geq 0$ denote by \mathcal{H}^s the form domain of H_0^s and the domain of $H_0^{s/2}$ endowed by the graph norm. For negative s , the space \mathcal{H}^s is the dual of \mathcal{H}^{-s} . In particular, for $s \geq 0$ an integer we have

$$\mathcal{H}^s = \{f \in L^2(\mathbb{R}^d) : x^\alpha \partial^\beta f \in L^2(\mathbb{R}^d), \forall \alpha, \beta \in \mathbb{N}^d, |\alpha| + |\beta| \leq s\}.$$

For Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , we will denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear operators from \mathcal{H}_1 to \mathcal{H}_2 and write $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$ as $\mathcal{B}(\mathcal{H}_1)$ for simplicity.

To a function $u \in \mathcal{H}^s$ we associate the sequence ξ of its Hermite coefficients by the formula $u(x) = \sum_{a \in \mathcal{E}} \xi_a \Phi_a(x)$ and define $\ell_s^2 := \{(\xi_a)_{a \in \mathcal{E}} : \sum_{a \in \mathcal{E}} w_a^s |\xi_a|^2 < \infty\}$. Next we will identify \mathcal{H}^s with ℓ_s^2 by endowing both spaces with the norm

$$\|u\|_{\mathcal{H}^s} = \|\xi\|_s = \left(\sum_{a \in \mathcal{E}} w_a^s |\xi_a|^2 \right)^{\frac{1}{2}}.$$

Our main theorem is the following.

Theorem 1.1. Assume that V satisfies (1.2) with $\iota \geq \frac{n+d+1}{2}$. There exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ there exists $\mathcal{D}_\epsilon \subset \mathcal{D}_0 := [0, 2\pi)^n$ such that for all $\omega \in \mathcal{D}_\epsilon$, the linear Schrödinger equation (1.1) reduces to a linear equation with constant coefficients in \mathcal{H}^1 .

More precisely, for any $\omega \in \mathcal{D}_\epsilon$, there exists a linear isomorphism $\Psi(\varphi) = \Psi_{\omega, \epsilon}(\varphi) \in \mathcal{B}(\mathcal{H}^p)$, for $p \in [0, 1]$, unitary on $L^2(\mathbb{R}^d)$, which analytically depends on $\varphi \in \mathbb{T}_{\sigma/2}^n$ and a bounded Hermitian operator $\mathcal{W} = \mathcal{W}_{\omega, \epsilon} \in \mathcal{B}(\mathcal{H}^s)$, for $s \geq 0$, such that $t \mapsto u(t, \cdot) \in \mathcal{H}^1$ satisfies

$$i\partial_t u = (-\Delta + |x|^2)u + \epsilon V(x, \omega t)u \quad (1.3)$$

if and only if $t \mapsto v(t, \cdot) = \Psi(\omega t)^{-1}u(t, \cdot)$ satisfies the autonomous equation

$$i\partial_t v = (-\Delta + |x|^2)v + \mathcal{W}(v). \quad (1.4)$$

Furthermore, there exists $C > 0$ such that

$$\begin{aligned} \text{Meas}(\mathcal{D}_0 \setminus \mathcal{D}_\epsilon) &\leq C\epsilon^{\frac{1}{6}}, \\ \|\mathcal{W}\|_{\mathcal{B}(\mathcal{H}^s)} &\leq C\epsilon, & \text{for all } s \geq 0, \\ \|\Psi(\varphi)^{\pm 1} - \text{Id}\|_{\mathcal{B}(\mathcal{H}^p)} &\leq C\epsilon^{\frac{5}{12}}, \quad \forall \varphi \in \mathbb{T}_{\sigma/2}^n, & \text{for all } p \in [0, 1]. \end{aligned} \quad (1.5)$$

Remark 1.2. The infinite matrix $(W_a^b)_{a, b \in \mathcal{E}}$ of the operator \mathcal{W} written in the Hermite basis $(W_a^b = \int_{\mathbb{R}^d} \Phi_a \mathcal{W}(\Phi_b) dx)$ is block diagonal and satisfies for some $C > 0$:

$$\|W_{[a]}^{[a]}\| \leq \frac{C\epsilon}{(1 + \ln w_a)^{2\iota}}, \quad \forall a \in \mathcal{E}.$$

As a consequence, we obtain the following corollaries concerning the Sobolev norm estimations on the solution of (1.1) and the spectra of the corresponding Floquet operator defined by

$$K = -i \sum_{j=1}^n \omega_j \frac{\partial}{\partial \varphi_j} - \Delta + |x|^2 + \epsilon V(x, \varphi).$$

Corollary 1.3. Assume that V satisfies (1.2) with $\iota \geq \frac{n+d+1}{2}$. There exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ and $\omega \in \mathcal{D}_\epsilon$, there exists a unique solution $u \in \mathcal{C}(\mathbb{R}, \mathcal{H}^1)$ such that $u(0) = u_0$. Moreover, u is almost-periodic in time and satisfies

$$(1 - C\epsilon)\|u_0\|_{\mathcal{H}^1} \leq \|u(t)\|_{\mathcal{H}^1} \leq (1 + C\epsilon)\|u_0\|_{\mathcal{H}^1},$$

for some $C > 0$.

Corollary 1.4. Assume that V satisfies (1.2) with $\iota \geq \frac{n+d+1}{2}$. There exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$ and $\omega \in \mathcal{D}_\epsilon$ the spectrum of the Floquet operator K is a pure point.

1.2. An introduction of the proof and related results. In this paper we will follow the strategy in [28] or [19]. As [28], if we endow the phase space $\mathcal{H}^s \times \mathcal{H}^s$ with the symplectic structure $\text{id}u \wedge d\bar{u}$, the equation (1.3) can be rewritten as the Hamiltonian system associated with the Hamiltonian

$$H(u, \bar{u}) = h(u, \bar{u}) + \epsilon q(\omega t, u, \bar{u}), \quad (1.6)$$

where $h(u, \bar{u}) = \int_{\mathbb{R}^d} |\nabla u|^2 + |x|^2 u \bar{u} dx$ and $q(\omega t, u, \bar{u}) = \int_{\mathbb{R}^d} V(x, \omega t) u \bar{u} dx$. Expanding u and \bar{u} on the Hermite basis of real valued functions $u = \sum_{a \in \mathcal{E}} \xi_a \Phi_a$, $\bar{u} = \sum_{a \in \mathcal{E}} \eta_a \Phi_a$, the Hamiltonian reads as $h = \sum_{a \in \mathcal{E}} w_a \xi_a \eta_a$, $q = \langle \xi, Q(\omega t) \eta \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the natural scalar product (no complex conjugation) and Q is the infinite matrix whose entries are

$$Q_a^b(\omega t) = \int_{\mathbb{R}^d} V(x, \omega t) \Phi_a(x) \Phi_b(x) dx, \quad (1.7)$$

and $(\xi, \eta) \in Y_s$, in which $Y_s = \{\zeta = (\zeta_a \in \mathbb{C}^2, a \in \mathcal{E}) : \|\zeta\|_s < \infty\}$. Therefore, Theorem 1.1 is equivalent to the reducibility problem for the non-autonomous Hamiltonian system associated with the Hamiltonian

$$\sum_{a \in \mathcal{E}} w_a \xi_a \eta_a + \epsilon \langle \xi, Q(\omega t) \eta \rangle. \quad (1.8)$$

As [28] we will construct a canonical change of variables and conjugate the Hamiltonian system with the Hamiltonian (1.8) to the Hamiltonian system associated with an autonomous Hamiltonian $\sum_{a \in \mathcal{E}} w_a \xi_a \eta_a + \langle \xi, W \eta \rangle$, where W is block diagonal and will be clear in the following sections.

Here we would like to compare our approaches with those of Grébert and Patrel [28]. By and large, because of the so-called small-divisor problem, in both [28] and the present paper the KAM technique is used to eliminate the dependence on φ of the perturbation $\langle \xi, Q(\varphi) \eta \rangle$. There is, however, an apparent difference. In [28], the perturbation matrix $Q(\varphi)$ belongs to $\overline{\mathcal{M}}_{s,\beta}$. In other words, for any $a, b \in \mathcal{E}$,

$$w_a^\beta w_b^\beta \|Q_{[a]}^{[b]}\| \left(\frac{\sqrt{\min\{w_a, w_b\}} + |w_a - w_b|}{\sqrt{\min\{w_a, w_b\}}} \right)^{\frac{s}{2}} \leq C. \quad (1.9)$$

The estimate comes from Cor. 3.2 from [30], the assumption of the perturbation and a delicate analysis. In this paper, we have a weaker assumption on $V(x, \varphi)$. Similarly, we also need an estimate on the perturbation matrix element $Q_{[a]}^{[b]}$ for every $a, b \in \mathcal{E}$. The following is the corresponding estimate:

Lemma 1.5. Assume that V satisfies (1.2) with $\iota \geq 0$, then we have

$$\left| \int_{\mathbb{R}^d} V(x, \varphi) \Psi_a(x) \Psi_b(x) dx \right| \leq \frac{C}{(1 + \ln w_a)^\iota (1 + \ln w_b)^\iota} \quad (1.10)$$

for all $\Psi_a \in E_{[a]}$, $\Psi_b \in E_{[b]}$ and $\|\Psi_a\| = \|\Psi_b\| = 1$, where the constant $C \equiv C(d, \iota) > 0$ and $\|\cdot\|$ denotes the $L^2(\mathbb{R}^d)$ norm.

Remark 1.6. The above estimate was firstly proved by Z. Wang and one of the authors in [43] when $d = 1$.

Remark 1.7. From the above lemma, we obtain

$$(1 + \ln w_a)^\beta (1 + \ln w_b)^\beta \|Q_{[a]}^{[b]}\| \leq C.$$

Thus we need to introduce a new space \mathcal{M}_β which can be compared with the corresponding one $\overline{\mathcal{M}}_{s,\beta}$ in [28]. See section 2 for the definition of \mathcal{M}_β .

In [28] Grébert and Paturel used Lemma 4.3 called the key lemma to solve the homological equation, in which they assume $|\mu_a - \lambda_a| \leq \frac{C_0}{w_a^\delta}$ for all $a \in \mathcal{E}$. We remark this lemma was first built up in Proposition 2.2.4 [17] and also applied in [23] and [27]. But in our case we only have a weaker assumption $|\mu_a - \lambda_a| \leq \frac{C_0}{(1 + \ln w_a)^\delta}$ for all $a \in \mathcal{E}$ which comes from $Q \in \mathcal{M}_\beta$. On the first try one can follow the proof of Lemma 4.3 in [28], the estimate should be

$$\|B(k)_{[a]}^{[b]}\| \leq \frac{C_d K^{d-1} \exp\{C_{\delta,d} \kappa^{-\frac{1}{\delta}}\}}{\kappa(1 + |w_a - w_b|)} \|A_{[a]}^{[b]}\|. \quad (1.11)$$

If we choose $\kappa \sim \epsilon^\alpha$, the term $\exp\{C_{\delta,d} \kappa^{-\frac{1}{\delta}}\}$ is too big for KAM iteration whatever $\alpha > 0$ is chosen. If one chooses $\kappa \sim \frac{1}{|\ln \epsilon|^\delta}$, it follows $\exp\{C_{\delta,d} \kappa^{-\frac{1}{\delta}}\} \sim \epsilon^{-\frac{1}{100}}$ which seems to be enough for KAM. But from a further investigation we find that one will face big troubles relative with measure estimates. Now it is clear that we need to develop some new estimate for $\|B(k)_{[a]}^{[b]}\|$. In the following we denote by \mathcal{M} the set of infinite matrices $A : \mathcal{E} \times \mathcal{E} \mapsto \mathbb{C}$ which satisfies $\sup_{a,b \in \mathcal{E}} \|A_{[a]}^{[b]}\| < \infty$, where $A_{[a]}^{[b]}$ denotes the restriction of A on the block $[a] \times [b]$ and $\|\cdot\|$ denotes the operator norm. Here is the new estimate.

Lemma 1.8. Let $A \in \mathcal{M}$ and $B(k)$ defined for $k \in \mathbb{Z}^n$ with $|k| \leq K$ by

$$B(k)_j^l = \frac{A_j^l}{k \cdot \omega - \mu_j + \mu_l}, \quad j \in [a], l \in [b], \quad (1.12)$$

where $\omega \in [0, 2\pi)^n$ and $(\mu_a)_{a \in \mathcal{E}}$ is a sequence of positive real numbers satisfying

$$|\mu_a - \lambda_a| \leq \frac{C_0}{(1 + \ln w_a)^\delta}, \quad \text{for all } a \in \mathcal{E} \quad (1.13)$$

such that for all $a, b \in \mathcal{E}$ and all $|k| \leq K$,

$$|k \cdot \omega - \lambda_a + \lambda_b| \geq \gamma(1 + |w_a - w_b|), \quad (1.14)$$

$$|k \cdot \omega - \mu_j + \mu_l| \geq \kappa(1 + |w_a - w_b|), \quad j \in [a], l \in [b] \quad (1.15)$$

where $\gamma, \kappa, \delta, C_0 > 0$. Then $B(k) \in \mathcal{M}$ and there exists a positive constant $C_{\delta,d}$ such that

$$\|B(k)_{[a]}^{[b]}\| \leq \frac{\exp\{C_{\delta,d} \gamma^{-1/\delta}\}}{\kappa(1 + |w_a - w_b|)} \|A_{[a]}^{[b]}\|,$$

where $\|\cdot\|$ denotes the operator norm and $C_{\delta,d} = \frac{d-1}{2}(2C_0)^{\frac{1}{\delta}}$.

Remark 1.9. The term $\exp\{C_{\delta,d}\gamma^{-1/\delta}\}$ is clearly better than the one $\exp\{C_{\delta,d}\kappa^{-\frac{1}{\delta}}\}$ in (1.11). We explain it in a heuristic way. For example, if choose $\gamma \sim K^{-\alpha}$, it follows that $\exp\{C_{\delta,d}\gamma^{-1/\delta}\} \sim \exp\{K^{\frac{\alpha}{\delta}}\}$ and $\exp\{K^{\frac{\alpha}{\delta}}\} \leq C\epsilon^{-\frac{1}{4}}$ when $\alpha < \delta$. The KAM iteration can be easily set up as [28].

In the end let us review the previous works on the reducibility of some important PDEs and the behaviors of solutions in Sobolev spaces. We begin with the quantum harmonic oscillators (for short ‘‘QHO’’). See [15, 29, 43, 44] for the reducibility results on 1-d QHO with bounded perturbations. Bambusi [3, 4] firstly built the reducibility results for 1-d QHO with unbounded perturbations. His proof was based on the pseudodifferential calculus, in which he generalized the ideas from [1, 40] (see also [11, 25, 38]). It seems that the pseudodifferential calculus method hasn’t been applied for the unbounded perturbation terms such as $\langle x \rangle^\mu \cos(x - \omega t)$ ($\mu > 0$) (Remark 2.7 in [3]), which have been solved recently by Luo and one of the authors ([31]) when $0 \leq \mu < \frac{1}{3}$. For a general 1-d Schrödinger equation we recall the classical results [7] in which the potential grows at infinity like $|x|^{2\ell}$ with $\ell > 1$ and the perturbation is bounded by $|x|^\beta$ with $\beta < \ell - 1$. The limit case was solved by Liu-Yuan ([35]). Recently the upper bound for β was improved further by Bambusi ([3, 4]). When the perturbation is limited to a class of terms such as $\langle x \rangle^\mu \cos(x - \omega t)$ we ([33]) can improve the original upper boundedness from $\ell - 1$ to at least $\ell - \frac{3}{4}$.

Reducibility for PDEs in high dimension was initiated by Eliasson-Kuksin [19]. We can refer to [28] and [32] for higher-dimensional QHO with bounded potential. The reducibility result for n-d QHO was first built in [9] by Bambusi-Grébert-Maspero-Robert. Towards other PDEs Montalto [39] obtained the first reducibility result for linear wave equations with unbounded perturbations on \mathbb{T}^d , which can be applied to the linearized Kirchhoff equation in higher dimension. Bambusi, Langella and Montalto [5] obtained the reducibility results for transportation equations with unbounded perturbations ([22]). See also [23, 24] for a linear Schrödinger equation on Zoll manifold with unbounded potential. We remark that by implementing the above techniques the KAM-type results of quasi-linear PDEs such as incompressible Euler flows in 3D [2] and forced Kirchhoff equation on \mathbb{T}^d [14] have been built recently.

The reducibility results usually imply the boundedness of Sobolev norms. For the growth rate of the solutions with time in \mathcal{H}^s -norm, see [9] and [26] for a t^s -polynomial growth for 1-d QHO with time periodic perturbations. Delort [16] constructed a $t^{s/2}$ -polynomial growth for 1-d QHO with certain time periodic order zero perturbation (see [36] for a short proof). Combining with the ideas in [9] and [18], Zhao, Zhou and one of the authors [34] obtained the precise dynamics for one class of 1-d QHO with quasi-periodic in time quadratic perturbations, in which they also presented an exponential growth in time for 1-d QHO. Recently, for 2-d QHO with perturbation which is decaying in t , Faou-Raphaël [21] constructed a solution whose \mathcal{H}^1 -norm presents logarithmic growth with t . For 2-d QHO with perturbation being the projection onto Bargmann-Fock space, Thomann [42] constructed explicitly a traveling wave whose Sobolev norm presents polynomial growth with t , based on the study in [41] for linear Lowest Landau equations (LLL) with a time-dependent potential. There are also many literatures, e.g. [6, 8, 10, 12, 13, 20, 37, 45], which are closely relative to the upper growth bound of the solution in Sobolev space.

The rest paper will be organized as follows. In Section 2, a new reducibility theorem is presented. In Section 3, we check all the hypothesis of the reducibility theorem are

satisfied which follows the main theorem. In Section 4, after present the proof of key lemma we prove the reducibility theorem. Finally, the appendix contains some technical Lemmas.

2. REDUCIBILITY THEOREM

In this section we state an abstract reducibility theorem for quadratic quasiperiodic in time Hamiltonians of a general form $\sum_{a \in \mathcal{E}} \lambda_a \xi_a \eta_a + \epsilon \langle \xi, Q(\omega t) \eta \rangle$.

2.1. Setting. Following [28], we will introduce some spaces and relative algebraic properties for later use.

Linear space. Let $s \in \mathbb{R}$, we consider the complex weighted ℓ^2 space

$$\ell_s^2 = \{\xi = (\xi_a \in \mathbb{C}, a \in \mathcal{E}) : \|\xi\|_s < \infty\}, \text{ where } \|\xi\|_s^2 = \sum_{a \in \mathcal{E}} w_a^s |\xi_a|^2.$$

Then we define

$$Y_s = \ell_s^2 \times \ell_s^2 = \{\zeta = (\zeta_a = (\xi_a, \eta_a) \in \mathbb{C}^2, a \in \mathcal{E}) : \|\zeta\|_s < \infty\},$$

where

$$\|\zeta\|_s^2 = \sum_{a \in \mathcal{E}} w_a^s |\zeta_a|^2 = \sum_{a \in \mathcal{E}} w_a^s (|\xi_a|^2 + |\eta_a|^2).$$

We equip the space Y_s , $s \geq 0$, with the symplectic structure $\text{id}\xi \wedge d\eta$. Let f be a \mathcal{C}^1 smooth function, defined on a domain $\mathcal{O} \subset Y_s$, then we have the associated Hamiltonian system

$$\begin{cases} \dot{\xi} = -i\nabla_{\eta} f(\xi, \eta), \\ \dot{\eta} = i\nabla_{\xi} f(\xi, \eta), \end{cases}$$

where $\nabla f = (\nabla_{\xi} f, \nabla_{\eta} f)^T$ is the gradient with respect to the scalar product in Y_0 . Naturally, define the Poisson bracket for any \mathcal{C}^1 smooth functions f and g , defined on a domain $\mathcal{O} \subset Y_s$

$$\{f, g\} = -i \left(\sum_{a \in \mathcal{E}} \frac{\partial f}{\partial \xi_a} \cdot \frac{\partial g}{\partial \eta_a} - \frac{\partial g}{\partial \xi_a} \cdot \frac{\partial f}{\partial \eta_a} \right).$$

Infinite matrices. Let $\beta \geq 0$, we denote by \mathcal{M}_{β} the set of infinite matrices $A : \mathcal{E} \times \mathcal{E} \mapsto \mathbb{C}$ that satisfy

$$|A|_{\beta} := \sup_{a, b \in \mathcal{E}} (1 + \ln w_a)^{\beta} (1 + \ln w_b)^{\beta} \|A_{[a]}^{[b]}\| < \infty,$$

where $A_{[a]}^{[b]}$ denotes the restriction of A on the block $[a] \times [b]$ and $\|\cdot\|$ denotes the operator norm. Further we denote $\mathcal{M} = \mathcal{M}_0$. We will also need the more regular space $\mathcal{M}_{\beta}^{+} \subset \mathcal{M}_{\beta}$: an infinite matrix $A \in \mathcal{M}$ belongs to \mathcal{M}_{β}^{+} if

$$|A|_{\beta+} := \sup_{a, b \in \mathcal{E}} (1 + |w_a - w_b|) (1 + \ln w_a)^{\beta} (1 + \ln w_b)^{\beta} \|A_{[a]}^{[b]}\| < \infty.$$

The following structural lemma is proved in Appendix 5.1:

Lemma 2.1. Let $\beta > \frac{1}{2}$, there exists an absolute constant $C \equiv C(\beta) > 0$ such that

(i). Let $A \in \mathcal{M}_{\beta}$ and $B \in \mathcal{M}_{\beta}^{+}$. Then AB and BA belong to \mathcal{M}_{β} and

$$|AB|_{\beta}, |BA|_{\beta} \leq C |A|_{\beta} |B|_{\beta+}.$$

(ii). Let $A, B \in \mathcal{M}_{\beta}^{+}$. Then AB belongs to \mathcal{M}_{β}^{+} and $|AB|_{\beta+} \leq C |A|_{\beta+} |B|_{\beta+}$.

- (iii). Let $A \in \mathcal{M}_\beta^+$. Then $e^A - \text{Id}$ belongs to \mathcal{M}_β^+ and $|e^A - \text{Id}|_{\beta+} \leq e^{C|A|_{\beta+}}|A|_{\beta+}$.
- (iv). Let $A \in \mathcal{M}_\beta$. Then for any $s \geq 1$, $A \in \mathcal{B}(\ell_s^2, \ell_{-s}^2)$ and $\|A\|_{\mathcal{B}(\ell_s^2, \ell_{-s}^2)} \leq C|A|_\beta$.
- (v). Let $A \in \mathcal{M}_\beta^+$. Then for any $s \in [-1, 1]$, $A \in \mathcal{B}(\ell_s^2)$ and $\|A\|_{\mathcal{B}(\ell_s^2)} \leq C|A|_{\beta+}$.

Normal form.

Definition 2.2. A matrix $Q : \mathcal{E} \times \mathcal{E} \mapsto \mathbb{C}$ is in normal form, denoted by $Q \in \mathcal{NF}$, if

- (i). Q is block diagonal, i.e. $Q_a^b = 0$ for all $w_a \neq w_b$.
- (ii). Q is Hermitian, i.e. $Q_b^a = \overline{Q_a^b}$.

Notice that a block diagonal matrix with bounded blocks in operator norm defines a bounded operator on ℓ_s^2 , $\forall s \in \mathbb{R}$, which implies $\mathcal{M}_\beta \cap \mathcal{NF} \subset \mathcal{B}(\ell_s^2)$. To a matrix $Q = (Q_a^b)_{a,b \in \mathcal{E}} \in \mathcal{B}(\ell_s^2, \ell_{-s}^2)$ for $s \geq 0$, define in a unique way a quadratic form on $Y_s \ni (\zeta_a)_{a \in \mathcal{E}} = (\xi_a, \eta_a)_{a \in \mathcal{E}}$ by the formula $q(\xi, \eta) = \langle \xi, Q\eta \rangle = \sum_{a,b \in \mathcal{E}} Q_a^b \xi_a \eta_b$. A straightforward computation leads to

$$\{q_1, q_2\}(\xi, \eta) = i\langle \xi, [Q_1, Q_2]\eta \rangle, \quad (2.1)$$

where $[Q_1, Q_2]$ is the commutator of two matrices Q_1 and Q_2 .

In the following, by abuse of language, we will call both of q and Q the Hamiltonian.

Parameter. In all the paper ω will play the role of a parameter belonging to $\mathcal{D}_0 = [0, 2\pi)^n$. All the constructed maps will be dependent on ω with \mathcal{C}^1 regularity. When a map is only defined on a Cantor subset of \mathcal{D}_0 the regularity has to be understood in the Whitney sense.

A class of quadratic Hamiltonian. Let $\beta > \frac{1}{2}$, $\sigma > 0$ and $\mathcal{D} \subset \mathcal{D}_0$ and denote by $\mathcal{M}_\beta(\mathcal{D}, \sigma)$ the set of \mathcal{C}^1 mappings $\mathcal{D} \times \mathbb{T}_\sigma^n \ni (\omega, \varphi) \mapsto Q(\omega, \varphi) \in \mathcal{M}_\beta$, which is real analytic in $\varphi \in \mathbb{T}_\sigma^n$. Endow the space with the norm

$$[Q]_\beta^{\mathcal{D}, \sigma} := \sup_{\substack{\omega \in \mathcal{D}, j=0,1 \\ |\Im \varphi| < \sigma}} |\partial_\omega^j Q(\omega, \varphi)|_\beta.$$

In view of Lemma 2.1 (iv), to a matrix $Q \in \mathcal{M}_\beta(\mathcal{D}, \sigma)$ define the quadratic form in Y_1 $q(\xi, \eta; \omega, \varphi) = \langle \xi, Q(\omega, \varphi)\eta \rangle$ and a straightforward computation leads to

$$|q(\xi, \eta; \omega, \varphi)| \leq C[Q]_\beta^{\mathcal{D}, \sigma} \|(\xi, \eta)\|_1^2, \quad \text{for } (\xi, \eta) \in Y_1, (\omega, \varphi) \in \mathcal{D} \times \mathbb{T}_\sigma^n.$$

Denote by $\mathcal{M}_\beta^+(\mathcal{D}, \sigma)$ the subspace of $\mathcal{M}_\beta(\mathcal{D}, \sigma)$ formed by Hamiltonians S , verifying $S(\omega, \varphi) \in \mathcal{M}_\beta^+$ and endow it with the norm $[S]_{\beta+}^{\mathcal{D}, \sigma} := \sup_{\substack{\omega \in \mathcal{D}, j=0,1 \\ |\Im \varphi| < \sigma}} |\partial_\omega^j S(\omega, \varphi)|_{\beta+}$. The

space of Hamiltonians $N \in \mathcal{M}_\beta(\mathcal{D}, \sigma)$ that are independent on φ will be denoted by $\mathcal{M}_\beta(\mathcal{D})$ and equipped with the norm $[N]_\beta^{\mathcal{D}} := \sup_{\omega \in \mathcal{D}, j=0,1} |\partial_\omega^j N(\omega)|_\beta$.

Hamiltonian flow. To any $R \in \mathcal{M}_\beta^+$ with $\beta > \frac{1}{2}$, define in a unique way the symplectic linear change of variables on Y_s : $(\xi, \eta) \mapsto (e^{-iR^T \xi}, e^{iR\eta})$. It is well-defined and invertible in $\mathcal{B}(\ell_s^2)$ for all $s \in [-1, 1]$ by the assertion (v) of Lemma 2.1. Concretely, the change of variables can be regarded as the time-one flow generated by the quadratic Hamiltonian $\chi(\xi, \eta) = \langle \xi, R\eta \rangle$ and it preserves the symmetry $\eta = \bar{\xi}$ for any initial datum considered in the paper if and only if R is a Hermitian matrix, i.e.

$$R^T = \bar{R}. \quad (2.2)$$

When R also depends smoothly on φ , $\mathbb{T}_\sigma^n \ni \varphi \mapsto R(\varphi) \in \mathcal{M}_\beta^+$ we associate to R the symplectic linear change of variables on the extended phase space

$$\phi_\chi^1(y, \varphi, \xi, \eta) = (\tilde{y}, \varphi, e^{-iR^T \xi}, e^{iR\eta}), \quad (2.3)$$

where the Hamiltonian $\chi(y, \varphi, \xi, \eta) = \langle \xi, R(\varphi)\eta \rangle$ and $\tilde{y} = y - \int_0^1 \langle e^{-itR^T} \xi, \nabla_\varphi R e^{itR} \eta \rangle dt$.

2.2. Hypothesis on the spectrum. Now we present our hypothesis on λ_a , $a \in \mathcal{E}$:

Hypothesis H1 (asymptotics). Assume that there exists an absolute constant $c_0 > 0$ such that

$$\lambda_a \geq c_0 \text{ and } |\lambda_a - \lambda_b| \geq c_0 |w_a - w_b|, \text{ for all } a, b \in \mathcal{E}. \quad (2.4)$$

Hypothesis H2 (second Melnikov condition in measure). There exist absolute positive constants τ_1, τ_2 and C such that the following holds: for each $\gamma > 0$ and $K \geq 1$ there exists a closed subset $\mathcal{D}' = \mathcal{D}'(\gamma, K) \subset \mathcal{D}_0$ satisfying

$$\text{Meas}(\mathcal{D}_0 \setminus \mathcal{D}') \leq CK^{\tau_1} \gamma^{\tau_2}$$

such that for all $\omega \in \mathcal{D}'$, all $k \in \mathbb{Z}^n$ with $0 < |k| \leq K$ and all $a, b \in \mathcal{E}$ we have

$$|k \cdot \omega - \lambda_a + \lambda_b| \geq \gamma(1 + |w_a - w_b|).$$

2.3. The reducibility theorem. Consider the Hamiltonian

$$H_\omega(t, \xi, \eta) = \sum_{a \in \mathcal{E}} \lambda_a \xi_a \eta_a + \epsilon \langle \xi, Q(\omega t) \eta \rangle \quad (2.5)$$

and the associated non-autonomous Hamiltonian system on Y_s :

$$\begin{cases} \dot{\xi} = -iN_0\xi - i\epsilon Q^T(\omega t)\xi, \\ \dot{\eta} = iN_0\eta + i\epsilon Q(\omega t)\eta, \end{cases} \quad (2.6)$$

where $N_0 = \text{diag}(\lambda_a : a \in \mathcal{E})$.

Theorem 2.3. Let $\alpha = \frac{\alpha_1}{\alpha_2} + 1$ with $\alpha_1 = \max\{\tau_1, n+d\}$ and $\alpha_2 = \max\{\tau_2, 1\}$. Fix $\beta \geq \frac{\alpha}{2}$ and $\sigma > 0$. Assume that $(\lambda_a)_{a \in \mathcal{E}}$ satisfies Hypothesis H1, H2 and that $Q \in \mathcal{M}_\beta(\mathcal{D}_0, \sigma)$, then there exists $\epsilon_* > 0$ such that for all $0 \leq \epsilon < \epsilon_*$, there exists $C > 0$ and

(i). a Cantor set $\mathcal{D}_\epsilon \subset \mathcal{D}_0$ of asymptotically full measure:

$$\text{Meas}(\mathcal{D}_0 \setminus \mathcal{D}_\epsilon) \leq C\epsilon^{\alpha_2/6}; \quad (2.7)$$

(ii). a \mathcal{C}^1 family (in $\omega \in \mathcal{D}_\epsilon$) of real analytic (in $\varphi \in \mathbb{T}_{\sigma/2}^n$) linear, unitary and symplectic coordinate transformations on Y_0 :

$$\Phi_\omega(\varphi) : (\xi, \eta) \mapsto (\overline{M_\omega(\varphi)}\xi, M_\omega(\varphi)\eta), \quad (\omega, \varphi) \in \mathcal{D}_\epsilon \times \mathbb{T}_{\sigma/2}^n;$$

(iii). a \mathcal{C}^1 family of quadratic autonomous Hamiltonians in normal form $\mathcal{H}_\omega = \langle \xi, N_\omega \eta \rangle$, where $N_\omega \in \mathcal{NF}$ is close to N_0 :

$$\|N_\omega - N_0\|_\beta \leq C\epsilon, \quad \omega \in \mathcal{D}_\epsilon, \quad (2.8)$$

such that $t \mapsto (\xi(t), \eta(t)) \in Y_1$ is a solution of autonomous Hamiltonian system associated with \mathcal{H}_ω :

$$\begin{cases} \dot{\xi} = -iN_\omega^T \xi \\ \dot{\eta} = iN_\omega \eta \end{cases}$$

if and only if $t \mapsto \Phi_\omega(\omega t)(\xi(t), \eta(t)) \in Y_1$ is a solution of the original Hamiltonian system (2.6). Furthermore, $\Phi_\omega(\varphi)$ is a bounded operator from Y_p into itself for all $p \in [0, 1]$ and close to identity:

$$\|M_\omega(\varphi) - \text{Id}\|_{\mathcal{B}(\ell_p^2)} \leq C\epsilon^{\frac{5}{12}}. \quad (2.9)$$

3. APPLICATIONS TO THE QUANTUM HARMONIC OSCILLATOR ON \mathbb{R}^d

In this section we will prove Theorem 2.3 which concludes Theorem 1.1.

3.1. Verification of the hypothesis. We first verify the Hypothesis of Theorem 2.3 for the quantum harmonic oscillator equation (1.3).

Lemma 3.1 (see [27]). When $\lambda_a = w_a$, $a \in \mathcal{E}$. Hypothesis H1 and H2 hold true with $c_0 = 1$ and $\mathcal{D}_0 = [0, 2\pi)^n$ and $\tau_1 = n + 1$, $\tau_2 = 1$.

Proof of Lemma 1.5: Let $\Psi_a \in E_{[a]}$, $\Psi_b \in E_{[b]}$ and $\|\Psi_a\| = \|\Psi_b\| = 1$, then

$$\left| \int_{\mathbb{R}^d} V(x, \varphi) \Psi_a(x) \Psi_b(x) dx \right| \leq \|V^{\frac{1}{2}}(x, \varphi) \Psi_a(x)\| \|V^{\frac{1}{2}}(x, \varphi) \Psi_b(x)\|. \quad (3.1)$$

By the symmetry we only need to estimate $\|V^{\frac{1}{2}}(x, \varphi) \Psi_a(x)\|$.

Let $\mathbb{D}_a = \{x \in \mathbb{R}^d : |x| \leq w_a^{1/(4d)}\}$ and denote by \mathbb{D}_a^c its complement, then one has

$$\|V^{\frac{1}{2}}(x, \varphi) \Psi_a(x)\| \leq \|V^{\frac{1}{2}}(x, \varphi) \Psi_a(x)\|_{L^2(\mathbb{D}_a)} + \|V^{\frac{1}{2}}(x, \varphi) \Psi_a(x)\|_{L^2(\mathbb{D}_a^c)}. \quad (3.2)$$

Clearly, since $|V(x, \varphi)| \leq C$ then one has

$$\begin{aligned} \|V^{\frac{1}{2}}(x, \varphi) \Psi_a(x)\|_{L^2(\mathbb{D}_a)} &\leq \|\Psi_a(x)\|_{L^p(\mathbb{D}_a)} \|V^{\frac{1}{2}}(x, \varphi)\|_{L^q(\mathbb{D}_a)} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \\ &\leq C \|\Psi_a(x)\|_{L^p(\mathbb{R}^d)} (\text{Meas}(\mathbb{D}_a))^{\frac{1}{q}} \\ &\leq C w_a^{-\frac{1}{2}(\frac{1}{2} - \frac{1}{p})} w_a^{1/(4d)} \quad \text{by Cor. 3.2 in [30] with } p = \frac{2(d+2)}{d+1} \\ &\leq C w_a^{-\frac{1}{4q}} = C w_a^{-\frac{1}{8(d+2)}}. \end{aligned} \quad (3.3)$$

Besides, for $x \in \mathbb{D}_a^c$ we have $|x| \geq w_a^{1/(4d)}$. Then $1 + \ln(1 + |x|^2) \geq 1 + \ln w_a^{1/(2d)} \geq \frac{1 + \ln w_a}{2d}$. By (1.2) one has $|V(x, \varphi)| \leq \frac{C}{(1 + \ln(1 + |x|^2))^{2\iota}} \leq \frac{C_{d,\iota}}{(1 + \ln w_a)^{2\iota}}$ for $x \in \mathbb{D}_a^c$. Hence, we obtain

$$\|V^{\frac{1}{2}}(x, \varphi) \Psi_a(x)\|_{L^2(\mathbb{D}_a^c)} \leq \frac{C_{d,\iota}}{(1 + \ln w_a)^\iota} \|\Psi_a(x)\|_{L^2(\mathbb{R}^d)} \leq \frac{C_{d,\iota}}{(1 + \ln w_a)^\iota}.$$

Combining (3.2), (3.3) with the last estimation we have

$$\|V^{\frac{1}{2}}(x, \varphi) \Psi_a(x)\| \leq \frac{C_{d,\iota}}{(1 + \ln w_a)^\iota}. \quad (3.4)$$

Collecting (3.1) and (3.4) concludes the result (1.10). \square

Remark 3.2. Using Lemma 1.5 and a similar method as [27] we have Q , defined by $Q_a^b(\varphi) = \int_{\mathbb{R}^d} V(x, \varphi) \Phi_a(x) \Phi_b(x) dx$, belongs to $\mathcal{M}_\iota(\mathcal{D}_0, \sigma)$.

Remark 3.3. For quantum harmonic oscillator equation (1.3), we have $\alpha = n + d + 1$ since $\alpha = \frac{\alpha_1}{\alpha_2} + 1$ with $\alpha_1 = \max\{\tau_1, n + d\}$ and $\alpha_2 = \max\{\tau_2, 1\}$, where $\tau_1 = n + 1$ and $\tau_2 = 1$.

3.2. Proof of Theorem 1.1. The Schrödinger equation (1.3) is a Hamiltonian system on $\mathcal{H}^s \times \mathcal{H}^s$ ($s \geq 1$) associated with the Hamiltonian function (1.6). Written on the orthonormal basis $(\Phi_a)_{a \in \mathcal{E}}$, it is equivalent to the Hamiltonian system on Y_s associated with (1.8) which reads as (2.5) with $\lambda_a = w_a$ and Q given by (1.7). By Lemma 3.1 and Remark 3.2, if V satisfies (1.2) with $\iota \geq \frac{n+d+1}{2}$, we can apply Theorem 2.3 to the Hamiltonian(1.8) which concludes Theorem 1.1. More precisely, in the new coordinates under a unitary transformation given by Theorem 2.3, $(\xi, \eta) = (\overline{M_\omega(\omega t)}\xi', M_\omega(\omega t)\eta')$, the original system

$$\begin{cases} \dot{\xi}_a = -iw_a\xi_a - i\epsilon(Q^T(\omega t)\xi)_a, \\ \dot{\eta}_a = iw_a\eta_a + i\epsilon(Q(\omega t)\eta)_a, \end{cases} \quad a \in \mathcal{E}$$

conjugates to an autonomous system as follows:

$$\begin{cases} \dot{\xi}'_a = -i(N_\omega^T \xi')_a, \\ \dot{\eta}'_a = i(N_\omega \eta')_a, \end{cases} \quad a \in \mathcal{E},$$

where $N_\omega \in \mathcal{NF}$ and $\overline{M_\omega(\omega t)}M_\omega^T(\omega t) = \text{Id}$. Furthermore, corresponding to the initial datum $u_0(x) = \sum_{a \in \mathcal{E}} \xi(0)_a \Phi_a(x) \in \mathcal{H}^1$ the solution $u(t, x)$ of (1.3) reads

$$u(t, x) = \sum_{a \in \mathcal{E}} \xi(t)_a \Phi_a(x) \text{ with } \xi(t) = \overline{M_\omega(\omega t)} e^{-itN_\omega^T} M_\omega^T(0) \xi(0).$$

Concretely, define the transformation $\Psi(\varphi) \in \mathcal{B}(\mathcal{H}^s)$ by

$$\Psi(\varphi) \left(\sum_{a \in \mathcal{E}} \xi'_a \Phi_a(x) \right) = \sum_{a \in \mathcal{E}} \left(\overline{M_\omega(\varphi)} \xi' \right)_a \Phi_a(x).$$

Then $v(t, x)$ satisfies (1.4) if and only if $u(t, x) = \Psi(\omega t)v(t, x)$ satisfies the original equation (1.3), where \mathcal{W} is defined by

$$\mathcal{W} \left(\sum_{a \in \mathcal{E}} \xi'_a \Phi_a(x) \right) = \sum_{a \in \mathcal{E}} (W \xi')_a \Phi_a(x) \text{ with } W = N_\omega - N_0.$$

By construction collecting (2.7)-(2.9) leads to (1.5) in Theorem 1.1. \square

For the proofs of Corollary 1.3 and 1.4 refer to [27].

4. PROOF OF REDUCIBILITY THEOREM

In this section we will prove the reducibility theorem presented in Sec.2 by the KAM methods. As we mentioned before, Lemma 1.8 is very important for solving homological equations, whose proof we present below.

4.1. Proof of Lemma 1.8.

Proof. Denote by $D_{[a]}$ the diagonal (square) matrix with entries μ_j for $j \in [a]$. The equation (1.12) reads

$$k \cdot \omega B_{[a]}^{[b]} - D_{[a]} B_{[a]}^{[b]} + B_{[a]}^{[b]} D_{[b]} = A_{[a]}^{[b]}. \quad (4.1)$$

From (1.14) for all $a, b \in \mathcal{E}$ and all $|k| \leq K$, $k \cdot \omega - \lambda_a + \lambda_b \neq 0$. We distinguish two cases. **Case 1:** suppose that $k \cdot \omega - \lambda_a + \lambda_b > 0$. Clearly, in this case by (1.14) we have

$k \cdot \omega - \lambda_a + \lambda_b \geq \gamma(1 + |w_a - w_b|) \geq \gamma$. By (1.13) for $j \in [a]$ and $l \in [b]$, if $\min\{w_a, w_b\} > \exp\left\{\left(\frac{2C_0}{\gamma}\right)^{1/\delta}\right\}$, we obtain

$$\begin{aligned} k \cdot \omega - \mu_j + \mu_l &= k \cdot \omega - \lambda_a + \lambda_b - \mu_j + \lambda_a - \lambda_b + \mu_l \\ &\geq \gamma - \frac{C_0}{(1 + \ln w_a)^\delta} - \frac{C_0}{(1 + \ln w_b)^\delta} \\ &\geq \gamma - \frac{2C_0}{(1 + \ln \min\{w_a, w_b\})^\delta} > 0. \end{aligned}$$

It follows that

$$k \cdot \omega + \min_{l \in [b]} \{\mu_l\} > \max_{j \in [a]} \{\mu_j\} > 0. \quad (4.2)$$

Subcase 1: $\min\{w_a, w_b\} > \exp\left\{\left(\frac{2C_0}{\gamma}\right)^{1/\delta}\right\}$. (4.2) proves that $k \cdot \omega I_{[b]} + D_{[b]}$ is an invertible operator and $\|(k \cdot \omega I_{[b]} + D_{[b]})^{-1}\| = \frac{1}{k \cdot \omega + \min_{l \in [b]} \{\mu_l\}}$. Thus (4.1) is equivalent to

$$B_{[a]}^{[b]} - D_{[a]} B_{[a]}^{[b]} (k \cdot \omega I_{[b]} + D_{[b]})^{-1} = A_{[a]}^{[b]} (k \cdot \omega I_{[b]} + D_{[b]})^{-1}.$$

Denote by $\mathcal{L}_{[a] \times [b]}$ the operator acting on matrices of size $[a] \times [b]$ such that

$$\mathcal{L}_{[a] \times [b]}(B_{[a]}^{[b]}) = D_{[a]} B_{[a]}^{[b]} (k \cdot \omega I_{[b]} + D_{[b]})^{-1}.$$

Then we have $\|\mathcal{L}_{[a] \times [b]}(B_{[a]}^{[b]})\| \leq \frac{\max_{j \in [a]} \{\mu_j\}}{k \cdot \omega + \min_{l \in [b]} \{\mu_l\}} \|B_{[a]}^{[b]}\|$ and $\|\mathcal{L}_{[a] \times [b]}\| < 1$ by (4.2).

The operator $\text{Id} - \mathcal{L}_{[a] \times [b]}$ is invertible and thus

$$\begin{aligned} \|B(k)_{[a]}^{[b]}\| &= \|(\text{Id} - \mathcal{L}_{[a] \times [b]})^{-1} A_{[a]}^{[b]} (k \cdot \omega I_{[b]} + D_{[b]})^{-1}\| \\ &\leq \frac{1}{1 - \frac{\max_{j \in [a]} \{\mu_j\}}{k \cdot \omega + \min_{l \in [b]} \{\mu_l\}}} \cdot \frac{\|A_{[a]}^{[b]}\|}{k \cdot \omega + \min_{l \in [b]} \{\mu_l\}} \\ &= \frac{\|A_{[a]}^{[b]}\|}{k \cdot \omega + \min_{l \in [b]} \{\mu_l\} - \max_{j \in [a]} \{\mu_j\}} \\ &\leq \frac{\|A_{[a]}^{[b]}\|}{\kappa(1 + |w_a - w_b|)} \quad \underline{\text{by (1.15) and (4.2)}}. \end{aligned} \quad (4.3)$$

Subcase 2: $\min\{w_a, w_b\} \leq \exp\left\{\left(\frac{2C_0}{\gamma}\right)^{1/\delta}\right\}$. In this situation we have $|B(k)_j^l| \leq \frac{|A_j^l|}{\kappa(1 + |w_a - w_b|)}$.

Then for any $\xi \in \ell_0^2$

$$\|B(k)_{[a]}^{[b]} \xi_{[b]}\|^2 = \sum_{j \in [a]} \left| \sum_{l \in [b]} B(k)_j^l \xi_l \right|^2 \leq \frac{1}{\kappa^2(1 + |w_a - w_b|)^2} \sum_{j \in [a]} \left(\sum_{l \in [b]} |A_j^l| |\xi_l| \right)^2.$$

On the other hand,

$$\begin{aligned} \sum_{j \in [a]} \left(\sum_{l \in [b]} |A_j^l| |\xi_l| \right)^2 &\leq \sum_{j \in [a]} \left(\sum_{l \in [b]} |A_j^l|^2 \right) \left(\sum_{l \in [b]} |\xi_l|^2 \right) \leq \|\xi_{[b]}\|^2 \sum_{j \in [a]} \sum_{l \in [b]} |A_j^l|^2 \\ &\leq \|\xi_{[b]}\|^2 \sum_{j \in [a]} \|A_{[a]}^{[b]}\|^2 \leq w_a^{d-1} \|A_{[a]}^{[b]}\|^2 \|\xi_{[b]}\|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{j \in [a]} \left(\sum_{l \in [b]} |A_j^l| |\xi_l| \right)^2 &\leq \|\xi_{[b]}\|^2 \sum_{j \in [a]} \sum_{l \in [b]} |A_j^l|^2 \leq \|\xi_{[b]}\|^2 \sum_{l \in [b]} \sum_{j \in [a]} |A_j^l|^2 \\ &\leq \|\xi_{[b]}\|^2 \sum_{l \in [b]} \|A_{[a]}^{[b]}\|^2 \leq w_b^{d-1} \|A_{[a]}^{[b]}\|^2 \|\xi_{[b]}\|^2. \end{aligned}$$

It follows that

$$\sum_{j \in [a]} \left(\sum_{l \in [b]} |A_j^l| |\xi_l| \right)^2 \leq (\min\{w_a, w_b\})^{d-1} \|A_{[a]}^{[b]}\|^2 \|\xi_{[b]}\|^2.$$

Therefore,

$$\|B(k)_{[a]}^{[b]}\| \leq \frac{\exp\{C_{\delta,d} \gamma^{-1/\delta}\}}{\kappa(1 + |w_a - w_b|)} \|A_{[a]}^{[b]}\|,$$

where $C_{\delta,d} = \frac{(d-1)}{2} (2C_0)^{1/\delta}$.

Case 2: suppose that $k \cdot \omega - \lambda_a + \lambda_b < 0$. Similarly, by (1.14) we have $\lambda_a - \lambda_b - k \cdot \omega \geq \gamma(1 + |w_a - w_b|) \geq \gamma$. Then (4.1) is equivalent to

$$-(D_{[a]} - k \cdot \omega I_{[a]}) B_{[a]}^{[b]} + B_{[a]}^{[b]} D_{[b]} = A_{[a]}^{[b]}.$$

The following proof is similar as case 1 in which we use $\mathcal{L}'_{[a] \times [b]}$ instead of $\mathcal{L}_{[a] \times [b]}$, where

$$\mathcal{L}'_{[a] \times [b]}(B_{[a]}^{[b]}) = (D_{[a]} - k \cdot \omega I_{[a]})^{-1} B_{[a]}^{[b]} D_{[b]}.$$

□

Remark 4.1. In (4.3), we use a cancellation.

4.2. Homological equation. In this section consider a homological equation of the form

$$-\omega \cdot \nabla_{\varphi} S + i[N, S] + Q = \text{remainder}$$

where $N \in \mathcal{NF}$ close to N_0 and $Q \in \mathcal{M}_{\beta}$. We will construct a solution $S \in \mathcal{M}_{\beta}^+$ in the following proposition.

Proposition 4.2. Denote $\mathcal{D} \subset \mathcal{D}_0$. Let $\mathcal{D} \ni \omega \mapsto N(\omega) \in \mathcal{NF}$ be a \mathcal{C}^1 mapping that verifies

$$[N - N_0]_{\beta}^{\mathcal{D}} \leq 2\epsilon_0 \quad (4.4)$$

and $Q \in \mathcal{M}_{\beta}(\mathcal{D}, \sigma)$. Assume that $K \geq 1$ and $0 < \kappa \leq \gamma \leq \frac{\epsilon_0}{4}$, verifying

$$\exp \left\{ 8d \left(\frac{\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\} \kappa \leq \gamma. \quad (4.5)$$

Then there exists a subset $\mathcal{D}' = \mathcal{D}'(\gamma, \kappa, K)$, satisfying

$$\text{Meas}(\mathcal{D} \setminus \mathcal{D}') \leq CK^{\alpha_1} \gamma^{\alpha_2} \quad (4.6)$$

and \mathcal{C}^1 mappings $\tilde{N} : \mathcal{D}' \mapsto \mathcal{M}_\beta \cap \mathcal{NF}$, $R : \mathcal{D}' \times \mathbb{T}_\sigma^n \mapsto \mathcal{M}_\beta$ and $S : \mathcal{D}' \times \mathbb{T}_\sigma^n \mapsto \mathcal{M}_\beta^+$, Hermitian and analytic in φ , such that

$$-\omega \cdot \nabla_\varphi S + i[N, S] = \tilde{N} - Q + R \quad (4.7)$$

and for any $0 < \sigma' < \sigma$

$$[\tilde{N}]_\beta^{\mathcal{D}'} \leq [Q]_\beta^{\mathcal{D}, \sigma}, \quad (4.8)$$

$$[R]_\beta^{\mathcal{D}', \sigma'} \leq \frac{C e^{-\frac{\kappa}{2}(\sigma - \sigma')}}{(\sigma - \sigma')^n} [Q]_\beta^{\mathcal{D}, \sigma}, \quad (4.9)$$

$$[S]_{\beta+}^{\mathcal{D}', \sigma'} \leq \frac{CK\gamma}{\kappa^3(\sigma - \sigma')^n} [Q]_\beta^{\mathcal{D}, \sigma}, \quad (4.10)$$

where the constant $C > 0$ depends on n, d, β and $\alpha_1 = \max\{\tau_1, n + d\}$, $\alpha_2 = \max\{\tau_2, 1\}$.

Proof. Written in Fourier variables (w.r.t. φ), the homological equation (4.7) reads

$$-ik \cdot \omega \widehat{S}(k) + i[N, \widehat{S}(k)] = \delta_{k,0} \tilde{N} - \widehat{Q}(k) + \widehat{R}(k),$$

where $\delta_{k,j}$ denotes the Kronecker symbol.

Decompose the equation on each product block $[a] \times [b]$:

$$L\widehat{S}_{[a]}^{[b]}(k) = i\delta_{k,0} \tilde{N}_{[a]}^{[b]} - i\widehat{Q}_{[a]}^{[b]}(k) + i\widehat{R}_{[a]}^{[b]}(k), \quad (4.11)$$

where $L := L(k, [a], [b], \omega)$ is the linear operator, acting on the space of complex $[a] \times [b]$ -matrices defined by

$$LM = (k \cdot \omega - N_{[a]}(\omega))M + MN_{[b]}(\omega) \text{ with } N_{[a]} = N_{[a]}^{[a]}. \quad (4.12)$$

First solve this equation when $|k| + |w_a - w_b| = 0$ (i.e. $k = 0$, $w_a = w_b$) by defining

$$\widehat{S}_{[a]}^{[a]}(0) = 0, \quad \widehat{R}_{[a]}^{[a]}(0) = 0 \text{ and } \tilde{N}_{[a]}^{[a]} = \widehat{Q}_{[a]}^{[a]}(0).$$

Then setting $\tilde{N}_{[a]}^{[b]} = 0$ for $w_a \neq w_b$ we obtain $\tilde{N} \in \mathcal{M}_\beta \cap \mathcal{NF}$ satisfies $|\tilde{N}|_\beta \leq |\widehat{Q}(0)|_\beta$. The estimates of the derivatives (w.r.t. ω) are obtained by differentiating the expressions of \tilde{N} . Taking all the estimates leads to (4.8).

Now turn to the other cases when $|k| + |w_a - w_b| > 0$. Diagonalize the (Hermitian) matrix $N_{[a]}$ in an orthonormal basis: $\overline{P}_{[a]}^T N_{[a]} P_{[a]} = D_{[a]}$ and denote $\widehat{S}'_{[a]}^{[b]} = \overline{P}_{[a]}^T \widehat{S}_{[a]}^{[b]} P_{[b]}$, $\widehat{Q}'_{[a]}^{[b]} = \overline{P}_{[a]}^T \widehat{Q}_{[a]}^{[b]} P_{[b]}$ and $\widehat{R}'_{[a]}^{[b]} = \overline{P}_{[a]}^T \widehat{R}_{[a]}^{[b]} P_{[b]}$. Here we note for later use that $\|\widehat{M}'_{[a]}^{[b]}\| = \|\widehat{M}_{[a]}^{[b]}\|$ for $M = S, Q, R$. In this new variables the homological equation (4.11) reads

$$(k \cdot \omega - D_{[a]}) \widehat{S}'_{[a]}^{[b]}(k) + \widehat{S}'_{[a]}^{[b]}(k) D_{[b]} = -i\widehat{Q}'_{[a]}^{[b]}(k) + i\widehat{R}'_{[a]}^{[b]}(k).$$

We solve it term by term: let $a, b \in \mathcal{E}$ and set

$$\begin{aligned} \widehat{R}'_{[a]}^{[b]}(k) &= 0, & \text{for } |k| \leq K, \\ \widehat{R}'_{jl}(k) &= \widehat{Q}'_{jl}(k), \quad j \in [a], l \in [b], & \text{for } |k| > K \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} \widehat{S}'_{[a]}^{[b]}(k) &= 0, & \text{for } |k| > K \text{ or } |k| + |w_a - w_b| = 0, \\ \left(\widehat{S}'_{[a]}^{[b]}(k)\right)_{jl} &= \frac{-i \left(\widehat{Q}'_{[a]}^{[b]}(k)\right)_{jl}}{k \cdot \omega - \alpha_j + \beta_l}, & \text{in the other cases,} \end{aligned} \quad (4.14)$$

where $\alpha_j(\omega)$ and $\beta_l(\omega)$ denote eigenvalues of $N_{[a]}(\omega)$ and $N_{[b]}(\omega)$, respectively. Before the estimations of such matrices, first remark the following assertion:

$$\overline{\left(\widehat{Q}'_{[a]}^{[b]}(k)\right)_{jl}} = \left(\widehat{Q}'_{[b]}^{[a]}(-k)\right)_{lj} \Rightarrow \overline{\left(\widehat{S}'_{[a]}^{[b]}(k)\right)_{jl}} = \left(\widehat{S}'_{[b]}^{[a]}(-k)\right)_{lj}.$$

Hence, if Q' verifies condition (2.2), so it is with S' which implies the flow generated by S preserves the symmetry $\eta = \bar{\xi}$.

Canonically, (4.13) leads to

$$|R(\varphi)|_\beta = |R'(\varphi)|_\beta \leq \frac{C e^{-\frac{\kappa}{2}(\sigma - \sigma')}}{(\sigma - \sigma')^n} \sup_{|\Im \varphi| < \sigma} |Q(\varphi)|_\beta, \text{ for } |\Re \varphi| < \sigma'. \quad (4.15)$$

Facing the small divisors

$$k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega), \quad j \in [a], l \in [b] \text{ and } [a], [b] \in \widehat{\mathcal{E}},$$

we distinguish two cases depending on whether $k = 0$ or not.

The case $k = 0$. In this case we know that $w_a \neq w_b$ which implies $|w_a - w_b| \geq 2$. Using (2.4) and (4.4) we get that, if $\kappa \leq \gamma \leq \frac{c_0}{4}$ and $\epsilon_0 \leq \frac{c_0}{4}$,

$$|-\lambda_a + \lambda_b| \geq c_0 |w_a - w_b| \geq \frac{c_0}{2} (1 + |w_a - w_b|) \geq 2\gamma (1 + |w_a - w_b|)$$

and

$$|-\alpha_j(\omega) + \beta_l(\omega)| \geq |-\lambda_a + \lambda_b| - 4\epsilon_0 \geq \frac{c_0}{2} |w_a - w_b| \geq \kappa (1 + |w_a - w_b|).$$

Collecting the last two estimates and condition (4.4) allows us to utilize Lemma 1.8 to conclude that

$$|\widehat{S}(0)|_{\beta_+} \leq \kappa^{-1} \exp \left\{ d \left(\frac{\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\} |\widehat{Q}(0)|_\beta. \quad (4.16)$$

The case $k \neq 0$. Concretely, in this case we only solve the main terms of Fourier series truncated at order K . Utilizing Hypothesis H2, for any $\gamma > 0$, there exists a subset $D_1 = \mathcal{D}(2\gamma, K)$, satisfying $\text{Meas}(\mathcal{D}_0 \setminus D_1) \leq CK^{\tau_1} \gamma^{\tau_2}$, such that for all $\omega \in D_1$ and $0 < |k| \leq K$, $|k \cdot \omega - \lambda_a + \lambda_b| \geq 2\gamma(1 + |w_a - w_b|)$. By (4.4) this implies

$$\begin{aligned} |k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega)| &\geq |k \cdot \omega - \lambda_a + \lambda_b| - |\alpha_j(\omega) - \lambda_a| - |\beta_l(\omega) - \lambda_b| \\ &\geq 2\gamma(1 + |w_a - w_b|) - \frac{2\epsilon_0}{(1 + \ln w_a)^{2\beta}} - \frac{2\epsilon_0}{(1 + \ln w_b)^{2\beta}} \\ &\geq \underline{\gamma(1 + |w_a - w_b|)}, \quad \text{if } w_b \geq w_a \geq \exp \left\{ \left(\frac{4\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\}. \end{aligned}$$

Now let $w_a \leq \exp \left\{ \left(\frac{4\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\}$. In fact if $|w_a - w_b| \geq CK$, we can prove that

$$|k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega)| \geq \kappa(1 + |w_a - w_b|).$$

We face the case $|w_a - w_b| \leq CK$ which follows $w_b \leq CK \exp \left\{ \left(\frac{4\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\}$. Since $|\partial_\omega(k \cdot \omega)(\frac{k}{|k|})| = |k| \geq 1$, condition (4.4) implies $\left| \partial_\omega(k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega)) \left(\frac{k}{|k|} \right) \right| \geq \frac{|k|}{2}$. The last estimate allows us to use Lemma 5.2 to conclude that

$$|k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega)| \geq \kappa(1 + |w_a - w_b|), \quad \forall j \in [a], l \in [b]$$

except a set $F_{[a],[b],k}$ whose measure is smaller than $Cw_a^{d-1}w_b^{d-1}\frac{\kappa(1+|w_a-w_b|)}{|k|}$. Denoting F be the union of $F_{[a],[b],k}$ for $[a], [b] \in \widehat{\mathcal{E}}$ and $0 < |k| \leq K$ such that $w_a \leq \exp \left\{ \left(\frac{4\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\}$ and $w_b \leq CK \exp \left\{ \left(\frac{4\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\}$ with $|w_a - w_b| \leq CK$, condition (4.5) leads to

$$\text{Meas}(F) \leq Cw_a^d w_b^d K^n \kappa \leq CK^{n+d} \exp \left\{ 8d \left(\frac{\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\} \kappa \leq CK^{n+d} \gamma.$$

Let $\mathcal{D}' = D_1 \cup D_2$ with $D_2 = \mathcal{D} \setminus F$ and $\alpha_1 = \max\{\tau_1, n + d\}$ and $\alpha_2 = \max\{\tau_2, 1\}$, then

$$\text{Meas}(\mathcal{D} \setminus \mathcal{D}') \leq \text{Meas}(\mathcal{D}_0 \setminus D_1) + \text{Meas}(F) \leq CK^{\tau_1} \gamma^{\tau_2} + CK^{n+d} \gamma \leq CK^{\alpha_1} \gamma^{\alpha_2}.$$

Further, by construction, for all $\omega \in \mathcal{D}'$, $0 < |k| \leq K$, $a, b \in \mathcal{E}$ and $j \in [a], l \in [b]$ we have

$$|k \cdot \omega - \lambda_a + \lambda_b| \geq 2\gamma(1 + |w_a - w_b|) \text{ and } |k \cdot \omega - \alpha_j(\omega) + \beta_l(\omega)| \geq \kappa(1 + |w_a - w_b|).$$

Hence, in view of (4.14), utilizing Lemma 1.8 concludes that $\widehat{S}'(k) \in \mathcal{M}_\beta^+$ satisfies

$$|\widehat{S}(k)|_{\beta+} = |\widehat{S}'(k)|_{\beta+} \leq \kappa^{-1} \exp \left\{ d \left(\frac{\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\} |\widehat{Q}(k)|_\beta, \quad 0 < |k| \leq K.$$

Combining the last estimate with (4.16) we obtain a solution S satisfying for all $|\Im\varphi| < \sigma'$

$$|S(\varphi)|_{\beta+} \leq \frac{C}{\kappa(\sigma - \sigma')^n} \exp \left\{ d \left(\frac{\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\} \sup_{|\Im\varphi| < \sigma} |Q(\varphi)|_\beta. \quad (4.17)$$

To obtain the estimates for the derivative (w.r.t. ω) we differentiate (4.11):

$$L(\partial_\omega \widehat{S}_{[a]}^{[b]}(k, \omega)) = -(\partial_\omega L) \widehat{S}_{[a]}^{[b]}(k, \omega) - i\partial_\omega \widehat{Q}_{[a]}^{[b]}(k, \omega) + i\partial_\omega \widehat{R}_{[a]}^{[b]}(k, \omega)$$

which is an equation of the same type as (4.11) for $\partial_\omega \widehat{S}_{[a]}^{[b]}(k, \omega)$ and $\partial_\omega \widehat{R}_{[a]}^{[b]}(k, \omega)$ where $-i\widehat{Q}_{[a]}^{[b]}(k, \omega)$ is replaced by $B_{[a]}^{[b]}(k, \omega) = -(\partial_\omega L) \widehat{S}_{[a]}^{[b]}(k, \omega) - i\partial_\omega \widehat{Q}_{[a]}^{[b]}(k, \omega)$. Solve this equation by defining

$$\begin{aligned} \partial_\omega \widehat{S}_{[a]}^{[b]}(k, \omega) &= \chi_{|k| \leq K} L^{-1}(k, [a], [b], \omega) B_{[a]}^{[b]}(k, \omega), \\ \partial_\omega \widehat{R}_{[a]}^{[b]}(k, \omega) &= i\chi_{|k| > K} B_{[a]}^{[b]}(k, \omega) = \chi_{|k| > K} \partial_\omega \widehat{Q}_{[a]}^{[b]}(k, \omega). \end{aligned}$$

Collecting condition (4.4) and the definition (4.12) leads to $|(\partial_\omega L)\widehat{S}(k, \omega)|_\beta \leq CK|\widehat{S}(k, \omega)|_\beta$ which implies

$$|B(k, \omega)|_\beta \leq \frac{CK}{\kappa} \exp \left\{ d \left(\frac{\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\} \left(|\widehat{Q}(k)|_\beta + |\partial_\omega \widehat{Q}(k)|_\beta \right).$$

Following the same strategy as in the resolution of (4.11) we get for $|\Im\varphi| < \sigma'$

$$|\partial_\omega S(\varphi)|_{\beta+} \leq \frac{CK}{\kappa^2(\sigma - \sigma')^n} \exp \left\{ 2d \left(\frac{\epsilon_0}{\gamma} \right)^{\frac{1}{2\beta}} \right\} \left(\sup_{|\Im\varphi| < \sigma} |Q(\varphi)|_\beta + \sup_{|\Im\varphi| < \sigma} |\partial_\omega Q(\varphi)|_\beta \right), \quad (4.18)$$

$$|\partial_\omega R(\varphi)|_\beta \leq \frac{Ce^{-\frac{\kappa}{2}(\sigma - \sigma')}}{(\sigma - \sigma')^n} \sup_{|\Im\varphi| < \sigma} |\partial_\omega Q(\varphi)|_\beta.$$

Collecting (4.15), (4.17) and the last estimates (4.18) and taking into account (4.5) leads to (4.9) and (4.10). \square

4.3. The KAM iteration. In this section we will prove Theorem 2.3 by an iterative KAM procedure. Let us begin with the initial Hamiltonian $H_\omega = h_0 + q_0$ where

$$h_0(y, \varphi, \xi, \eta) = \omega \cdot y + \langle \xi, N_0 \eta \rangle,$$

$N_0 = \text{diag}(\lambda_a : a \in \mathcal{E}), \omega \in \mathcal{D}_0$ and the quadratic perturbation $q_0(\varphi, \xi, \eta) = \langle \xi, Q_0(\varphi)\eta \rangle$ with $Q_0 = \epsilon Q \in \mathcal{M}_\beta(\mathcal{D}_0, \sigma_0)$ and $\sigma_0 = \sigma$. Building iteratively the change of variables $\phi_{\chi_m}^1$, we obtain the normal form $h_m = \omega \cdot y + \langle \xi, N_m(\omega)\eta \rangle$ and the perturbation $q_m = \langle \xi, Q_m(\omega, \varphi)\eta \rangle$ with $Q_m \in \mathcal{M}_\beta(\mathcal{D}_m, \sigma_m)$ as follows: assume that the construction has been built up to step $m \geq 0$ then

- (i). we utilize Proposition 4.2 to construct $S_{m+1}(\omega, \varphi)$ solution of the homological equation verifying for $(\omega, \varphi) \in \mathcal{D}_{m+1} \times \mathbb{T}_{\sigma_{m+1}}^n$

$$-\omega \cdot \nabla_\varphi S_{m+1} + i[N_m, S_{m+1}] = \widetilde{N}_m - Q_m + R_m \quad (4.19)$$

where $\widetilde{N}_m(\omega), R_m(\omega, \varphi)$ defined for $(\omega, \varphi) \in \mathcal{D}_{m+1} \times \mathbb{T}_{\sigma_{m+1}}^n$ by

$$\widetilde{N}_m(\omega) = \left(\delta_{[j],[l]} \widehat{Q}_m(0)_{jl} \right)_{j,l \in \mathcal{E}}, \quad (4.20)$$

$$R_m(\omega, \varphi) = \sum_{|k| > K_{m+1}} \widehat{Q}_m(\omega, k) e^{ik \cdot \varphi}; \quad (4.21)$$

- (ii). we define N_{m+1}, Q_{m+1} for $(\omega, \varphi) \in \mathcal{D}_{m+1} \times \mathbb{T}_{\sigma_{m+1}}^n$ by

$$N_{m+1} = N_m + \widetilde{N}_m, \quad (4.22)$$

$$Q_{m+1} = R_m + i \int_0^1 e^{-itS_{m+1}} [(1-t)(\widetilde{N}_m + R_m) + tQ_m, S_{m+1}] e^{itS_{m+1}} dt. \quad (4.23)$$

By construction, if Q_m and N_m are Hermitian, so it is with all of \widetilde{N}_m, R_m and S_{m+1} , by resolution of the homological equation, and also N_{m+1} and Q_{m+1} . Let

$$\begin{aligned} h_{m+1}(y, \varphi, \xi, \eta; \omega) &= \omega \cdot y + \langle \xi, N_{m+1}(\omega)\eta \rangle, \\ \chi_{m+1}(y, \varphi, \xi, \eta; \omega) &= \langle \xi, S_{m+1}(\omega, \varphi)\eta \rangle, \\ q_{m+1}(y, \varphi, \xi, \eta; \omega) &= \langle \xi, Q_{m+1}(\omega, \varphi)\eta \rangle. \end{aligned} \quad (4.24)$$

Recall that ϕ_χ^t denotes the time t flow generated by S (see (2.3)), then

$$f \circ \phi_{\chi_{m+1}}^1 = f + \int_0^1 \{f, \chi_{m+1}\} \circ \phi_{\chi_{m+1}}^t dt$$

or

$$f \circ \phi_{\chi_{m+1}}^1 = f + \{f, \chi_{m+1}\} + \int_0^1 (1-t) \{\{f, \chi_{m+1}\}, \chi_{m+1}\} \circ \phi_{\chi_{m+1}}^t dt.$$

Therefore collecting (2.1) and (2.3) leads to for $\omega \in \mathcal{D}_{m+1}$

$$\begin{aligned} (h_m + q_m) \circ \phi_{\chi_{m+1}}^1 &= h_m \circ \phi_{\chi_{m+1}}^1 + q_m \circ \phi_{\chi_{m+1}}^1 \\ &= h_m + \{h_m, \chi_{m+1}\} + q_m + \int_0^1 \{(1-t)\{h_m, \chi_{m+1}\} + q_m, \chi_{m+1}\} \circ \phi_{\chi_{m+1}}^t dt \\ &= h_m + \langle \xi, (\tilde{N}_m + R_m)\eta \rangle + \int_0^1 \{\langle \xi, ((1-t)(\tilde{N}_m + R_m) + tQ_m)\eta \rangle, \chi_{m+1}\} \circ \phi_{\chi_{m+1}}^t dt \\ &:= h_{m+1} + q_{m+1}. \end{aligned}$$

4.4. Iterative lemma. Following the general iterative procedures (4.19)-(4.24) we have

$$(h_0 + q_0) \circ \phi_{\chi_1}^1 \circ \phi_{\chi_2}^1 \circ \cdots \circ \phi_{\chi_m}^1 = h_m + q_m$$

where $h_m = \omega \cdot y + \langle \xi, N_m \eta \rangle$ with $N_m \in \mathcal{NF}$ and $q_m = \langle \xi, Q_m \eta \rangle$ with $Q_m \in \mathcal{M}_\beta(\mathcal{D}_m, \sigma_m)$. At the step m the Fourier series are truncated at order K_m and the small divisors are controlled by γ_m and κ_m . Specifically, we choose all of the parameters for $m \geq 0$ in term of ϵ_m which will control $[Q_m]_\beta^{\mathcal{D}_m, \sigma_m}$ as follows: first define $\sigma_0 = \sigma$ and ϵ_0 verifying $[Q_0]_\beta^{\mathcal{D}_0, \sigma_0} \leq \epsilon_0$ and denote $\alpha = \frac{\alpha_1}{\alpha_2} + 1$, then for $m \geq 1$ let

$$\begin{aligned} \epsilon_m &= \epsilon_{m-1}^{5/4}, \kappa_m = \epsilon_{m-1}^{1/4}, \gamma_m = \epsilon_0^{1/6} (\ln \epsilon_{m-1}^{-1})^{-\alpha}, \\ \sigma_{m-1} - \sigma_m &= C_* \sigma_0 m^{-2}, K_m = 2(\sigma_{m-1} - \sigma_m)^{-1} \ln \epsilon_{m-1}^{-1}, \end{aligned} \quad (4.25)$$

where $(C_*)^{-1} = 2 \sum_{m \geq 1} m^{-2}$ and $\alpha_1 = \max\{\tau_1, n + d\}$, $\alpha_2 = \max\{\tau_2, 1\}$.

Remark 4.3. From (4.25), the assumption (4.5) in the KAM iteration is equivalent to

$$8d\epsilon_0^{\frac{5}{12\beta}} (\ln \epsilon_{m-1}^{-1})^{\frac{\alpha}{2\beta}} + \frac{1}{6} \ln \epsilon_0^{-1} + \alpha \ln (\ln \epsilon_{m-1}^{-1}) \leq \frac{1}{4} \ln \epsilon_{m-1}^{-1}, \quad \forall m \geq 1. \quad (4.26)$$

Clearly, if $\beta \geq \frac{\alpha}{2}$, (4.26) or (4.5) always holds true for $\epsilon_0 \ll 1$.

Lemma 4.4. Let $\alpha = \frac{\alpha_1}{\alpha_2} + 1$ with $\alpha_1 = \max\{\tau_1, n + d\}$, $\alpha_2 = \max\{\tau_2, 1\}$ and $\beta \geq \frac{\alpha}{2}$. There exists ϵ_* depending on $\sigma, d, n, \beta, \tau_1, \tau_2$ and h_0 such that, for $0 \leq \epsilon_0 < \epsilon_*$ and $\epsilon_m = \epsilon_0^{(5/4)^m}$, $m \geq 0$, the followings hold for all $m \geq 1$: there exists $\mathcal{D}_m \subset \mathcal{D}_{m-1}$, $S_m \in \mathcal{M}_\beta^+(\mathcal{D}_m, \sigma_m)$, $h_m = \omega \cdot y + \langle \xi, N_m \eta \rangle$ with $N_m \in \mathcal{NF}$ and $Q_m \in \mathcal{M}_\beta(\mathcal{D}_m, \sigma_m)$ such that

(i). for $p \in [-1, 1]$ the transformation

$$\phi_m(\cdot, \omega, \varphi) := \phi_{\chi_m}^1 : Y_p \mapsto Y_p, \quad \forall (\omega, \varphi) \in \mathcal{D}_m \times \mathbb{T}_{\sigma_m}^n \quad (4.27)$$

is linear (unitary in Y_0) isomorphism conjugating the Hamiltonian at step $m-1$ to the Hamiltonian at step m , i.e.

$$(h_{m-1} + q_{m-1}) \circ \phi_m = h_m + q_m;$$

(ii). the following estimations hold:

$$\text{Meas}(\mathcal{D}_{m-1} \setminus \mathcal{D}_m) \leq \epsilon_0^{\alpha_2/6} (\ln \epsilon_{m-1}^{-1})^{-\frac{\alpha_2}{2}}, \quad (4.28)$$

$$[\tilde{N}_{m-1}]_{\beta}^{\mathcal{D}_m} \leq \epsilon_{m-1}, \quad (4.29)$$

$$[Q_m]_{\beta}^{\mathcal{D}_m, \sigma_m} \leq \epsilon_m, \quad (4.30)$$

$$[S_m]_{\beta+}^{\mathcal{D}_m, \sigma_m} \leq \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{2\alpha_2}} \epsilon_{m-1}^{1/4}$$

and for all $p \in [-1, 1]$ the transformation satisfies

$$\|\phi_m(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} \leq \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{3\alpha_2}} \epsilon_{m-1}^{1/4}, \quad \forall (\omega, \varphi) \in \mathcal{D}_m \times \mathbb{T}_{\sigma_m}^n. \quad (4.31)$$

Proof. At step 1, the initial $h_0 = \omega \cdot y + \langle \xi, N_0 \eta \rangle$ and thus condition (4.4) is trivially satisfied. Remark 4.3 allows us to apply Proposition 4.2 for constructing S_1, \tilde{N}_0, R_0 and \mathcal{D}_1, σ_1 such that for $(\omega, \varphi) \in \mathcal{D}_1 \times \mathbb{T}_{\sigma_1}^n$, $-\omega \cdot \nabla_{\varphi} S_1 + i[N_0, S_1] = \tilde{N}_0 - Q_0 + R_0$. Then utilizing (4.6), we obtain for $\epsilon_0 \ll 1$

$$\text{Meas}(\mathcal{D}_0 \setminus \mathcal{D}_1) \leq CK_1^{\alpha_1} \gamma_1^{\alpha_2} \leq C \epsilon_0^{\alpha_2/6} (\ln \epsilon_0^{-1})^{-\alpha_2} \leq \epsilon_0^{\alpha_2/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_2}{2}}.$$

Due to (4.10) we have for $\epsilon_0 \ll 1$

$$[S_1]_{\beta+}^{\mathcal{D}_1, \sigma_1} \leq \frac{CK_1 \gamma_1}{\kappa_1^3 (\sigma_0 - \sigma_1)^n} [Q_0]_{\beta}^{\mathcal{D}_0, \sigma_0} \leq C \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{\alpha_2}} \epsilon_0^{1/4} \leq \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{2\alpha_2}} \epsilon_0^{1/4}.$$

Thus, in view of (2.3), (4.27) and assertion (iii),(v) of Lemma 2.1 we get for all $p \in [-1, 1]$

$$\|\phi_1(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} \leq C e^{C[S_1]_{\beta+}^{\mathcal{D}_1, \sigma_1}} [S_1]_{\beta+}^{\mathcal{D}_1, \sigma_1} \leq \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{3\alpha_2}} \epsilon_0^{1/4}, \quad \text{if } \epsilon_0 \ll 1.$$

Collecting (4.8) and (4.9) leads to $[\tilde{N}_0]_{\beta}^{\mathcal{D}_1} \leq \epsilon_0$ and for $\epsilon_0 \ll 1$

$$[R_0]_{\beta}^{\mathcal{D}_1, \sigma_1} \leq \frac{C e^{-\frac{\kappa_1}{2}(\sigma_0 - \sigma_1)}}{(\sigma_0 - \sigma_1)^n} [Q_0]_{\beta}^{\mathcal{D}_0, \sigma_0} \leq C \epsilon_0^2 \leq \frac{1}{2} \epsilon_0^{5/4} = \frac{\epsilon_1}{2}.$$

Besides, (4.23) infers that for $\epsilon_0 \ll 1$

$$[Q_1]_{\beta}^{\mathcal{D}_1, \sigma_1} \leq [R_0]_{\beta}^{\mathcal{D}_1, \sigma_1} + C [Q_0]_{\beta}^{\mathcal{D}_0, \sigma_0} [S_1]_{\beta+}^{\mathcal{D}_1, \sigma_1} \leq \frac{\epsilon_1}{2} + C \epsilon_0^{5/4+1/6} \leq \epsilon_1.$$

Now assume that we have verified Lemma 4.4 up to step m , then we consider the step $m+1$. Since $h_m = \omega \cdot y + \langle \xi, N_m \eta \rangle$ and $[N_m - N_0]_{\beta}^{\mathcal{D}_m} \leq \sum_{l=0}^{m-1} [\tilde{N}_l]_{\beta}^{\mathcal{D}_{l+1}} \leq \sum_{l=0}^{m-1} \epsilon_l \leq 2\epsilon_0$, if $\epsilon_0 \ll 1$. Then condition (4.4) verifies and Remark 4.3 allows us to apply Proposition 4.2 for constructing $S_{m+1}, \tilde{N}_m, R_m$ and $\mathcal{D}_{m+1}, \sigma_{m+1}$ such that for $(\omega, \varphi) \in \mathcal{D}_{m+1} \times \mathbb{T}_{\sigma_{m+1}}^n$

$$-\omega \cdot \nabla_{\varphi} S_{m+1} + i[N_m, S_{m+1}] = \tilde{N}_m - Q_m + R_m.$$

Similarly, utilizing (4.6) we obtain for $\epsilon_0 \ll 1$

$$\text{Meas}(\mathcal{D}_m \setminus \mathcal{D}_{m+1}) \leq CK_{m+1}^{\alpha_1} \gamma_{m+1}^{\alpha_2} \leq C \epsilon_0^{\alpha_2/6} \frac{(m+1)^{2\alpha_1}}{(\ln \epsilon_m^{-1})^{\alpha_2}} \leq \epsilon_0^{\alpha_2/6} (\ln \epsilon_m^{-1})^{-\frac{\alpha_2}{2}}.$$

Due to (4.10) we have for $\epsilon_0 \ll 1$

$$[S_{m+1}]_{\beta+}^{\mathcal{D}_{m+1}, \sigma_{m+1}} \leq \frac{CK_{m+1} \gamma_{m+1}}{\kappa_{m+1}^3 (\sigma_m - \sigma_{m+1})^n} [Q_m]_{\beta}^{\mathcal{D}_m, \sigma_m} \leq \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{2\alpha_2}} \epsilon_m^{1/4}$$

Thus, in view of (2.3),(4.27) and assertion (iii),(v) of Lemma 2.1 we get for $p \in [-1, 1]$

$$\|\phi_{m+1}(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} \leq C [S_{m+1}]_{\beta+}^{\mathcal{D}_{m+1}, \sigma_{m+1}} \leq \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{3\alpha_2}} \epsilon_m^{1/4}, \quad \text{if } \epsilon_0 \ll 1.$$

Collecting (4.8) and (4.9) leads to $[\tilde{N}_m]_\beta^{\mathcal{D}^{m+1}} \leq \epsilon_m$ and

$$[R_m]_\beta^{\mathcal{D}^{m+1}, \sigma_{m+1}} \leq \frac{C e^{-\frac{K_{m+1}}{2}(\sigma_m - \sigma_{m+1})}}{(\sigma_m - \sigma_{m+1})^n} [Q_m]_\beta^{\mathcal{D}^m, \sigma_m} \leq \frac{\epsilon_{m+1}}{2}, \quad \text{if } \epsilon_0 \ll 1.$$

In addition, (4.23) implies that for $\epsilon_0 \ll 1$

$$\begin{aligned} [Q_{m+1}]_\beta^{\mathcal{D}^{m+1}, \sigma_{m+1}} &\leq [R_m]_\beta^{\mathcal{D}^{m+1}, \sigma_{m+1}} + C [Q_m]_\beta^{\mathcal{D}^m, \sigma_m} [S_{m+1}]_\beta^{\mathcal{D}^{m+1}, \sigma_{m+1}} \\ &\leq \frac{\epsilon_{m+1}}{2} + C \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{2\alpha_2}} \epsilon_m^{5/4} \leq \epsilon_{m+1}. \end{aligned}$$

□

4.5. Transition to the limit and proof of reducibility theorem. Let $\mathcal{D}_\epsilon = \bigcap_{m \geq 0} \mathcal{D}_m$. In view of (4.28), this is a Borel set satisfying for $\epsilon_0 \ll 1$

$$\text{Meas}(\mathcal{D}_0 \setminus \mathcal{D}_\epsilon) \leq \sum_{m \geq 0} \epsilon_0^{\alpha_2/6} (\ln \epsilon_m^{-1})^{-\frac{\alpha_2}{2}} = \epsilon_0^{\alpha_2/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_2}{2}} \sum_{m \geq 0} \left((4/5)^{\frac{\alpha_2}{2}} \right)^m \leq \epsilon_0^{\alpha_2/6}.$$

This leads to the assertion (i) of Theorem 2.3.

In the following, let $p \in [0, 1]$, $(\omega, \varphi) \in \mathcal{D}_\epsilon \times \mathbb{T}_{\sigma/2}^n$ and $\epsilon_0 \ll 1$. Collecting (4.30) and (4.29) we conclude the direct lemmas as follows:

Lemma 4.5. $\{Q_m(\omega, \varphi)\}_{m \geq 1}$ is a Cauchy sequence in \mathcal{M}_β and $Q_m(\omega, \varphi) \rightarrow \mathbf{0}$ when $m \rightarrow \infty$. Furthermore, (4.30) infers the uniformly convergence on (ω, φ) .

Lemma 4.6. $\{N_m(\omega) - N_0\}_{m \geq 1}$ is a Cauchy sequence in \mathcal{M}_β . Letting $W(\omega) \in \mathcal{M}_\beta$ be the limit mapping we have $N_m(\omega) - N_0 \rightarrow W(\omega)$ when $m \rightarrow \infty$. Moreover, (4.29) implies the uniformly convergence on ω , which leads to the \mathcal{C}^1 regularity.

To estimate the change of variables, we need the following two lemmas.

Lemma 4.7. Let $\Phi_m = \phi_1 \circ \phi_2 \circ \dots \circ \phi_m$ for $m \geq 1$, then we have

$$\|\Phi_m(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} \leq \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{4\alpha_2}} \sum_{l=0}^{m-1} \epsilon_l^{1/4}. \quad (4.32)$$

Proof. First (4.31) implies the above estimate (4.32) holds true for $m = 1$. Now assume that we have verified (4.32) up to $m > 1$, then we consider the case for $m + 1$. From the definition,

$$\Phi_{m+1} - \text{Id} = \Phi_m \circ \phi_{m+1} - \text{Id} = \Phi_m \circ (\phi_{m+1} - \text{Id}) + \Phi_m - \text{Id}.$$

Therefore, collecting the assumption and (4.31) leads to

$$\begin{aligned} \|\Phi_{m+1}(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} &\leq \|\Phi_m(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} + C \|\phi_{m+1}(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} \\ &\leq \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{4\alpha_2}} \sum_{l=0}^{m-1} \epsilon_l^{1/4} + C \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{3\alpha_2}} \epsilon_m^{1/4} \leq \epsilon_0^{1/6} (\ln \epsilon_0^{-1})^{-\frac{\alpha_1}{4\alpha_2}} \sum_{l=0}^m \epsilon_l^{1/4}. \end{aligned}$$

By induction, we complete the proof. □

Lemma 4.8. $\{\Phi_m(\cdot, \omega, \varphi)\}_{m \geq 1}$ is a Cauchy sequence in $\mathcal{B}(Y_p)$. Letting $\Phi_\infty(\cdot, \omega, \varphi) \in \mathcal{B}(Y_p)$ be the limit mapping we have $\Phi_m(\cdot, \omega, \varphi) \rightarrow \Phi_\infty(\cdot, \omega, \varphi)$ when $m \rightarrow \infty$. Furthermore, (4.32) implies the uniformly convergence on (ω, φ) , which leads to $\Phi_\infty(\cdot, \omega, \varphi)$ is analytic in φ and \mathcal{C}^1 in ω . Moreover, (4.32) implies

$$\|\Phi_\infty(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} \leq \epsilon_0^{5/12}. \quad (4.33)$$

Proof. From the definition we have

$$\Phi_{m+1} - \Phi_m = \Phi_m \circ \phi_{m+1} - \Phi_m = \Phi_m \circ (\phi_{m+1} - \text{Id}).$$

Collecting (4.31) and (4.32) leads to

$$\|\Phi_{m+1}(\cdot, \omega, \varphi) - \Phi_m(\cdot, \omega, \varphi)\|_{\mathcal{B}(Y_p)} \leq C \|\phi_{m+1}(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} \leq \epsilon_m^{1/4}.$$

Hence, given $m_2 \geq m_1 \geq 1$ we have

$$\|\Phi_{m_2}(\cdot, \omega, \varphi) - \Phi_{m_1}(\cdot, \omega, \varphi)\|_{\mathcal{B}(Y_p)} \leq \sum_{l=m_1}^{m_2-1} \epsilon_l^{1/4} \leq 2\epsilon_{m_1} \rightarrow 0 \text{ when } m_1 \rightarrow \infty.$$

The last estimate implies the Cauchy sequence, which concludes the results. \square

Define $\mathcal{B}(\ell_p^2) \ni M_m(\omega, \varphi) = e^{iS_1(\omega, \varphi)} \circ e^{iS_2(\omega, \varphi)} \circ \dots \circ e^{iS_m(\omega, \varphi)}$ for $m \geq 1$. In view of (2.3) we get that

$$\phi_m(\xi, \eta, \omega, \varphi) = (e^{-iS_m^T(\omega, \varphi)} \xi, e^{iS_m(\omega, \varphi)} \eta)$$

and

$$\Phi_m(\xi, \eta, \omega, \varphi) = (\overline{M_m(\omega, \varphi)} \xi, M_m(\omega, \varphi) \eta). \quad (4.34)$$

From a straightforward computation, given $m_2 \geq m_1 \geq 1$

$$\|M_{m_2}(\omega, \varphi) - M_{m_1}(\omega, \varphi)\|_{\mathcal{B}(\ell_p^2)} \leq \|\Phi_{m_2}(\cdot, \omega, \varphi) - \Phi_{m_1}(\cdot, \omega, \varphi)\|_{\mathcal{B}(Y_p)}.$$

This implies that $\{M_m(\omega, \varphi)\}_{m \geq 1}$ is a Cauchy sequence in $\mathcal{B}(\ell_p^2)$. Letting $M_\infty(\omega, \varphi)$ be the limit mapping, the uniform convergence leads to that $(\omega, \varphi) \mapsto M_\infty(\omega, \varphi)$ is analytic in φ and \mathcal{C}^1 in ω . Moreover, due to (4.34)

$$\Phi_\infty(\xi, \eta, \omega, \varphi) = (\overline{M_\infty(\omega, \varphi)} \xi, M_\infty(\omega, \varphi) \eta) \quad (4.35)$$

and taking into account (4.33) leads to

$$\|\overline{M_\infty(\omega, \varphi)} - \text{Id}\|_{\mathcal{B}(\ell_p^2)} = \|M_\infty(\omega, \varphi) - \text{Id}\|_{\mathcal{B}(\ell_p^2)} \leq \|\Phi_\infty(\cdot, \omega, \varphi) - \text{Id}\|_{\mathcal{B}(Y_p)} \leq \epsilon_0^{5/12}. \quad (4.36)$$

By construction the map $\Phi_m(\cdot, \omega, \omega t)$ conjugates the original Hamiltonian system associated with $H_\omega(t, \xi, \eta) = \langle \xi, N_0 \eta \rangle + \epsilon \langle \xi, Q(\omega t) \eta \rangle$ into the Hamiltonian system associated with $H_m(t, \xi, \eta) = \langle \xi, N_m(\omega) \eta \rangle + \langle \xi, Q_m(\omega, \omega t) \eta \rangle$. Collecting Lemma 4.5 and 4.6 one concludes $Q_m(\omega, \omega t) \rightarrow 0$ and $N_m(\omega) \rightarrow N_\omega$ when $m \rightarrow \infty$, where the operator $N_\omega \equiv N_\infty(\omega) = N_0 + W(\omega) \in \mathcal{NF}$ is \mathcal{C}^1 in ω with

$$|W|_\beta = |N_\omega - N_0|_\beta \leq 2\epsilon_0. \quad (4.37)$$

Let $\Phi_\omega(\varphi) := \Phi_\infty(\cdot, \omega, \varphi)$ and $M_\omega(\varphi) = M_\infty(\omega, \varphi)$, then (4.35) reads $\Phi_\omega(\varphi)(\xi, \eta) = (\overline{M_\omega(\varphi)} \xi, M_\omega(\varphi) \eta)$. Furthermore, denoting the limiting Hamiltonian $\mathcal{H}_\omega = \langle \xi, N_\omega \eta \rangle = \langle \xi, N_0 \eta \rangle + \langle \xi, W \eta \rangle$, the symplectic coordinate transformation $\Phi_\omega(\varphi)$ conjugates the original Hamiltonian system associated with H_ω into the autonomous Hamiltonian system associated with \mathcal{H}_ω . Collecting (4.36) and (4.37) leads to (2.8) and (2.9) in Theorem 2.3. \square

5. APPENDIX

5.1. **Proof of Lemma 2.1.** Recall that $\beta > \frac{1}{2}$.

(i). The proof results from Lemma 5.1 with $\beta > \frac{1}{2}$ and

$$\sum_{c \in \widehat{\mathcal{E}}} \frac{1}{(1 + \ln w_c)^{2\beta} (1 + |w_b - w_c|)} \leq C. \quad (5.1)$$

(ii). Similarly, collecting (5.1) and $\beta > \frac{1}{2}$ concludes the results.

(iii). Use assertion (ii) of Lemma 2.1.

(iv). Let $A \in \mathcal{M}_\beta$ and $s \geq 1$. Then for any $\xi \in \ell_s^2$, we have

$$\begin{aligned} \|A\xi\|_{-s}^2 &= \sum_{a \in \widehat{\mathcal{E}}} w_a^{-s} \left\| \sum_{b \in \widehat{\mathcal{E}}} A_{[a]}^{[b]} \xi_{[b]} \right\|^2 \leq \sum_{a \in \widehat{\mathcal{E}}} w_a^{-s} \left(\sum_{b \in \widehat{\mathcal{E}}} \|A_{[a]}^{[b]}\| \cdot \|\xi_{[b]}\| \right)^2 \\ &\leq |A|_\beta^2 \sum_{a \in \widehat{\mathcal{E}}} w_a^{-s} \left(\sum_{b \in \widehat{\mathcal{E}}} \frac{w_b^{s/2} \|\xi_{[b]}\|}{(1 + \ln w_a)^\beta (1 + \ln w_b)^\beta w_b^{s/2}} \right)^2 \\ &\leq |A|_\beta^2 \sum_{a \in \widehat{\mathcal{E}}} \frac{1}{(1 + \ln w_a)^{2\beta} w_a^s} \left(\sum_{b \in \widehat{\mathcal{E}}} \frac{1}{(1 + \ln w_b)^{2\beta} w_b^s} \right) \left(\sum_{b \in \widehat{\mathcal{E}}} w_b^s \|\xi_{[b]}\|^2 \right) \\ &\leq |A|_\beta^2 \sum_{a \in \widehat{\mathcal{E}}} \frac{1}{(1 + \ln w_a)^{2\beta} w_a} \left(\sum_{b \in \widehat{\mathcal{E}}} \frac{1}{(1 + \ln w_b)^{2\beta} w_b} \right) \|\xi\|_s^2 \\ &\leq C^2 |A|_\beta^2 \|\xi\|_s^2. \end{aligned}$$

(v). Case 1: $s \in [0, 1]$. In this case we first prove

$$(I) := \sum_{b \in \widehat{\mathcal{E}}} \frac{(w_a/w_b)^s}{(1 + \ln w_b)^{2\beta} (1 + |w_a - w_b|)} \leq C, \quad \forall s \in [0, 1] \text{ and } \beta > \frac{1}{2}. \quad (5.2)$$

We split the series above into two parts as follows:

$$(I) = \left(\sum_{w_b > \frac{w_a}{2}} + \sum_{w_b \leq \frac{w_a}{2}} \right) \frac{(w_a/w_b)^s}{(1 + \ln w_b)^{2\beta} (1 + |w_a - w_b|)} := (I_1) + (I_2).$$

For the former, $(w_a/w_b)^s \leq 2^s \leq 2$. Thus

$$(I_1) \leq \sum_{b \in \widehat{\mathcal{E}}} \frac{2}{(1 + \ln w_b)^{2\beta} (1 + |w_a - w_b|)} \leq C. \quad (5.3)$$

Then turn to the latter. Since $w_b \leq \frac{w_a}{2}$, then $1 + |w_a - w_b| \geq w_a - w_b \geq \frac{w_a}{2} \geq w_b$. Thus $1 + |w_a - w_b| = (1 + |w_a - w_b|)^s (1 + |w_a - w_b|)^{1-s} \geq (w_a/2)^s w_b^{1-s} \geq \frac{1}{2} w_a^s w_b^{1-s}$. Therefore, one obtains $(I_2) \leq \sum_{b \in \widehat{\mathcal{E}}} \frac{2}{(1 + \ln w_b)^{2\beta} w_b} \leq C$. Collecting the last estimate and (5.3) leads to the results (5.2).

Now, we prepare to prove the assertion of (v) when $s \in [0, 1]$. Since $A \in \mathcal{M}_\beta^+$,

then for any $\xi \in \ell_s^2$,

$$\begin{aligned}
\|A\xi\|_s^2 &\leq \sum_{a \in \widehat{\mathcal{E}}} w_a^s \left(\sum_{b \in \widehat{\mathcal{E}}} \|A_{[a]}^{[b]}\| \cdot \|\xi_{[b]}\| \right)^2 \\
&\leq \sum_{a \in \widehat{\mathcal{E}}} \frac{|A|_{\beta+}^2}{(1 + \ln w_a)^{2\beta}} \left(\sum_{b \in \widehat{\mathcal{E}}} \frac{(w_a/w_b)^{s/2}}{(1 + \ln w_b)^\beta (1 + |w_a - w_b|)^{1/2}} \frac{w_b^{s/2} \|\xi_{[b]}\|}{(1 + |w_a - w_b|)^{1/2}} \right)^2 \\
&\leq \sum_{a \in \widehat{\mathcal{E}}} \frac{|A|_{\beta+}^2}{(1 + \ln w_a)^{2\beta}} \left(\sum_{b \in \widehat{\mathcal{E}}} \frac{(w_a/w_b)^s}{(1 + \ln w_b)^{2\beta} (1 + |w_a - w_b|)} \right) \left(\sum_{b \in \widehat{\mathcal{E}}} \frac{w_b^s \|\xi_{[b]}\|^2}{1 + |w_a - w_b|} \right) \\
&\leq C |A|_{\beta+}^2 \sum_{b \in \widehat{\mathcal{E}}} w_b^s \|\xi_{[b]}\|^2 \sum_{a \in \widehat{\mathcal{E}}} \frac{1}{(1 + \ln w_a)^{2\beta} (1 + |w_a - w_b|)} \quad \text{by (5.2)} \\
&\leq C^2 |A|_{\beta+}^2 \|\xi\|_s^2. \quad \text{by Lemma 5.1}
\end{aligned}$$

Next turn to the other case: $s \in [-1, 0)$. Repeating similar procedures as the first case and noting that

$$\sum_{a \in \widehat{\mathcal{E}}} \frac{(w_a/w_b)^s}{(1 + \ln w_a)^{2\beta} (1 + |w_a - w_b|)} \leq C, \quad \forall s \in [-1, 0) \text{ and } \beta > \frac{1}{2},$$

we complete the proof. □

5.2. Some auxiliary lemmas.

Lemma 5.1 (see Lemma A1 in [43]). For $j \geq 1$ and $\delta > 1$, there exists a positive constant $C \equiv C(\delta)$ independent of j such that $\sum_{l \geq 1} \frac{1}{(1 + \ln l)^\delta (1 + |l - j|)} \leq C$.

The following lemma is classical.

Lemma 5.2. Let $f : [0, 1] \mapsto \mathbb{R}$ be a \mathcal{C}^1 map satisfying $|f'(x)| \geq \delta$ for all $x \in [0, 1]$ and let $\kappa > 0$ then $\text{Meas}(\{x \in [0, 1] : |f(x)| \leq \kappa\}) \leq \frac{2\kappa}{\delta}$.

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