Companion Paper: Moderate Exponential-time Quantum Dynamic Programming Across the Subsets for Scheduling Problems

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Camille Grange[1,2], Michael Poss[1], Eric Bourreau[1], Vincent T'kindt[3], and Olivier Ploton[3]

[1]University of Montpellier, LIRMM, CNRS, France

[2]SNCF, Technology, Innovation and Group Projects Department, France

[3]University of Tours, LIFAT, France

Structure of the companion paper. This companion paper describes the three algorithms regarding Q-DDPAS and its decision-based version (**Algorithm**, **Algorithm 2**, and **Algorithm 4**) of the main paper with a low-level description. We call low-level description the description of the quantum part of the above-mentioned hybrid algorithms with unitary circuits representing complete algorithms or single operations. This description aims to provide the necessary details to prove the correctness of these algorithms, and consequently enlighten possible implementations, but avoiding too many details as with a quantum gate description that seems unnecessary at this stage. Henceforth, we suppose the reader to be familiar with the basics of quantum circuits. Readers may refer to Nannicini (2020) for these notions. To clarify the references, we write in **bold** the one referring to the main paper.

1 Preliminaries

Let us introduce the building blocks required for the description of the algorithms in the next sections.

1.1 Building block quantum circuits

We specify the quantum circuit associated with two algorithms that we use in a black box way and which constitute fundamental subroutines in Q-DDPAS or Q-Dec-DDPAS algorithms. The first one is the Quantum Minimum Finding algorithm of Durr and Hoyer (1996) presented in **Definition 2.6**, needed for Q-DDPAS. The second one is the Grover Search Extension of Boyer et al. (1998) presented in **Definition 5.6**, needed for Q-Dec-DDPAS. But first, we make the following observation.

Observation 1.1 (Classical algorithm into quantum circuit (Bennett, 1973))**.** *Any classical algorithm $\mathcal{A}$ can be executed as a quantum circuit $U_{\mathcal{A}}$ by preserving the number of gates but increasing the size memory. This additional cost comes from the fact that $U_{\mathcal{A}}$ must be reversible. Specifically, if $\mathcal{A}$ uses T gates and S bits of memory, $U_{\mathcal{A}}$ uses $\mathcal{O}(T)$ gates and $\mathcal{O}(T+S)$ bits of memory.*

The circuit associated with the Quantum Minimum Finding (**Definition 2.6**) is the following.

Definition 1.2 (Circuit U_{QMF})**.** *Let $f : [n] \to \mathbb{Z}$ be a function and let U_f be its corresponding quantum circuit, specifically,*

$$U_f \left|i\right\rangle \left|0\right\rangle = \left|i\right\rangle \left|f(i)\right\rangle, \quad \forall i \in [n] .$$

We note $U_{\text{QMF}}[U_f]$ the quantum circuit corresponding to the Quantum Minimum Finding algorithm of Durr and Hoyer (1996) that computes with high probability the minimum value of f and the corresponding minimizer:

$$U_{\text{QMF}}[U_f] \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \left|i\right\rangle \left|0\right\rangle \left|0\right\rangle = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \left|i\right\rangle \left|\arg\min_{i\in[n]}\{f(i)\}\right\rangle \left|\min_{i\in[n]}\{f(i)\}\right\rangle .$$

Next, we present the circuit associated with the extension of Grover Search (**Definition 5.6**).

Definition 1.3 (Circuit U_G)**.** *Let $f : [n] \to \{0, 1\}$ be a function and let U_f be its corresponding quantum circuit, specifically,*

$$U_f \left|i\right\rangle \left|0\right\rangle = \left|i\right\rangle \left|f(i)\right\rangle, \quad \forall i \in [n] .$$

We note $U_G[U_f]$ the quantum circuit corresponding to the algorithm of Boyer et al. (1998) that computes with high probability the logical OR of all the f values. If it appends to be True, $U_G[U_f]$ also gives the corresponding set $I_f = \{i : f(i) = 1\}$. Specifically,

$$U_G[U_f] \sum_{i=1}^{N} \frac{1}{\sqrt{N}} \left|i\right\rangle \left|0\right\rangle \left|0\right\rangle = \sum_{i=1}^{N} \frac{1}{\sqrt{N}} \left|i\right\rangle \left|I_f\right\rangle \left|\bigvee_{i\in[N]} f(i)\right\rangle ,$$

Henceforth, we only look at the gate complexity of our algorithm. Thus, we deliberately ignore extra qubits required in Quantum Minimum Finding, Grover Search Extension, and classical computation as quantum circuits (see Observation 1.1).

1.2 Quantum circuits indexing

Before going into the description of the algorithms, we introduce some notations about indexing quantum circuits to be able to describe them rigorously. Let $reg = \left|q_1\right\rangle \ldots \left|q_n\right\rangle$ be a register of

n qubits and U be an operator acting on k qubits, with $k < n$. Let I be a k-tuple of distinct indices in $[n]$, $I = (i_1, \dots, i_k)$. We denote by U^I the operator acting on the full register *reg*, that applies U on $|q_{i_1}\rangle \dots |q_{i_k}\rangle$, and applies Id on the remaining qubits. For instance, if I is the tuple of contiguous indices $(3, \dots, k+3)$ with $k < n-3$, then

$$U^I := Id^{\otimes 2} \otimes U \otimes Id^{\otimes n-k-3} .$$

For $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$ two distinct tuples in $[n]$ (k-tuple and l-tuple where $i \neq j, \forall (i,j) \in I \times J$) , we note $I \oplus J$ the concatenation of I and J, namely $I \oplus J = (i_1, \dots, i_k, j_1, \dots, j_l)$. Regarding the Quantum Minimum Finding operator, let us denote the indexes related to the quantum circuit U_f of a function f as

$$U_f \underbrace{|i\rangle}_{I} \underbrace{|0\rangle}_{J} = \underbrace{|i\rangle}_{I} \underbrace{|f(i)\rangle}_{J} .$$

To clarify the computations detailed next, we index the corresponding Quantum Minimum Finding operator as $U_{\text{QMF}}[U_f^I]$. We omit the index J because this is an *auxiliary* register that does not appear in the output of $U_{\text{QMF}}[U_f]$. Similarly, we index the corresponding Grover Search Extension operator as $U_G[U_f^I]$ omitting the index J.

2 Hybrid algorithm Q-DDPAS

In this section, we describe in the gate-based quantum computing model our algorithm Q-DDPAS that applies to any problem that satisfies (Add-DPAS) and (Add-D-DPAS) or (Comp-DPAS) and (Comp-D-DPAS). We introduce in the two following subsections the sets and quantum circuits that constitute the building blocks of our algorithm Q-DDPAS, and we provide for each of them their complexity. Depending on the tackled problem $\mathcal{P}$ solved by the hybrid algorithm, these sets, respectively quantum circuits, slightly differ whether the related problem P satisfies (Add-DPAS) and (Add-D-DPAS), or the related auxiliary problem P' satisfies (Comp-DPAS) and (Comp-D-DPAS).

2.1 Additive DPAS sets and quantum circuits

Let us begin with the sets and related quantum circuits useful to the description of our algorithm for solving problems whose related problem P satisfies recurrences (Add-DPAS) and (Add-D-DPAS).

We define two sets Λ_{add} and Ω_{add} indexed by (J, t) for $J \subseteq [n]$ and $t \in T$. Essentially, the set $\Lambda_{\text{add}}(J, t)$ contains all the possible balanced bi-partitions of J and the associated parameter value of t_{shift}. The second set $\Omega_{\text{add}}(J, t)$ contains the optimal solutions for each bi-partition in

$\Lambda_{\text{add}}(J,t)$.

Definition 2.1 (Sets Λ_{add} and Ω_{add}). *For $J \subseteq [n]$ such that $|J|$ is even and for $t \in T$, we define the set*

$$\Lambda_{add}(J,t) = \left\{ (X, t, J \setminus X, t_{shift}(J,X,t)) : X \subseteq J, |X| = \frac{|J|}{2} \right\} ,$$

and the set

$$\Omega_{add}(J,t) = \left\{ (X, \mathrm{OPT}[X,t], J \setminus X, \mathrm{OPT}[J \setminus X, t_{shift}(J,X,t)], t) : X \subseteq J, |X| = \frac{|J|}{2} \right\} .$$

The two following quantum circuits $U_{\Lambda_{\text{add}}}$ and $U_{\Omega_{\text{add}}}$ amount, respectively, to put into uniform superposition the elements of Λ_{add} and Ω_{add}.

Definition 2.2 (Circuit $U_{\Lambda_{\text{add}}}$). *For $J \subseteq [n]$ such that $|J|$ is even, and for $t \in T$, we define $U_{\Lambda_{add}}$ as follows:*

$$U_{\Lambda_{add}} \left|J\right\rangle \left|t\right\rangle \left|0\right\rangle^{\otimes 6} = \left|J\right\rangle \left|t\right\rangle \sum_{(\lambda_1^s, \lambda_1^t, \lambda_2^s, \lambda_2^t) \in \Lambda_{add}(J,t)} \frac{1}{\sqrt{|\Lambda_{add}(J,t)|}} \left|\lambda_1^s\right\rangle \left|\lambda_1^t\right\rangle \left|0\right\rangle \left|\lambda_2^s\right\rangle \left|\lambda_2^t\right\rangle \left|0\right\rangle .$$

Observe that we index the objects that represent sets by s, and the objects that represent scalars by t, because these are equal to the values in T.

Proposition 2.3 (Complexity of $U_{\Lambda_{\text{add}}}$). *The complexity of $U_{\Lambda_{add}}$ is polynomial in the size of the input.*

Proof. First, let us prove that, for a given $J \subseteq [n]$ of size m for m even, the construction of the quantum superposition of subsets of J of size $m/2$ (i.e. superposition of balanced bi-partitions) is polynomial. We recall that we define $[\![a,b]\!] := \{a, a+1, \ldots, b\}$ for integers $a \leq b$.

Let $J \subseteq [n]$ be of size m, for m even. We note $\sigma_{\text{enum}} : J \mapsto [\![1,m]\!]$ the bijection that enumerates the elements of J. We note $\sigma_{\text{bipart}} : [\![1, \binom{m}{m/2}]\!] \mapsto \left\{(A, [\![1,m]\!] \setminus A) : |A| = \frac{m}{2}\right\}$ the bijection that enumerates the balanced bi-partitions of $[\![1,m]\!]$. Let $U_{\sigma_{\text{bipart}}}$ be the quantum circuit corresponding to the function σ_{bipart}. Specifically, for $i \in [\![1, \binom{m}{m/2}]\!]$,

$$U_{\sigma_{\text{bipart}}} \left|i\right\rangle \left|0\right\rangle \left|0\right\rangle = \left|i\right\rangle \underbrace{\left|A_i\right\rangle \left|[\![1,m]\!] \setminus A_i\right\rangle}_{\sigma_{\text{bipart}}(i)} .$$

Let $U_{\sigma_{\text{enum}}^{-1}}$ be the quantum circuit corresponding to the inverse of the function σ_{enum}. Thus,

$$U_{\sigma_{\text{enum}}^{-1}} \left|i\right\rangle \left|A_i\right\rangle \left|[\![1,m]\!] \setminus A_i\right\rangle = \left|i\right\rangle \left|X_i\right\rangle \left|J \setminus X_i\right\rangle ,$$

for $X_i = \sigma_{\text{enum}}^{-1}(A_i) \subseteq J$. We denote by $\sigma_{\text{enum}}^{-1}(S)$, for S a set, the operation of applying $\sigma_{\text{enum}}^{-1}$ to each element of S.

Consequently, we get a quantum superposition of all balanced bi-partitions of J by applying first $U_{\sigma_{\text{bipart}}}$ then $U_{\sigma_{\text{enum}}^{-1}}$ to a quantum register that represents the superposition of all elements in $[\![1, \binom{m}{m/2}]\!]$. For that, we require $n_q := \lceil \log_2(\binom{m}{m/2}) \rceil = \mathcal{O}(m)$ qubits, each one initially in state $|0\rangle$ on which we apply the Hadamard gate. Specifically,

$$\begin{aligned}
U_{\sigma_{\text{enum}}^{-1}} U_{\sigma_{\text{bipart}}} H^{\otimes n_q} |0\rangle^{\otimes n_q} |0\rangle |0\rangle &= U_{\sigma_{\text{enum}}^{-1}} U_{\sigma_{\text{bipart}}} \sum_{i=1}^{\binom{m}{m/2}} |i\rangle |0\rangle |0\rangle \\
&= U_{\sigma_{\text{enum}}^{-1}} \sum_{i=1}^{\binom{m}{m/2}} |i\rangle |A_i\rangle |[\![1, m]\!] \setminus A_i\rangle \\
&= \sum_{i=1}^{\binom{m}{m/2}} |i\rangle |X_i\rangle |J \setminus X_i\rangle .
\end{aligned}$$

Let us compute the complexity of $U_{\sigma_{\text{enum}}^{-1}} U_{\sigma_{\text{bipart}}} H^{\otimes n_q}$. For a given i, computing $\sigma_{\text{bipart}}(i)$, respectively $\sigma_{\text{enum}}^{-1}(i)$, is polynomial in m. According to Observation 1.1, the complexity of $U_{\sigma_{\text{bipart}}}$, respectively $U_{\sigma_{\text{enum}}^{-1}}$, is polynomial in m. Thus, the construction of the superposition of balanced bi-partitions of J is polynomial.

Eventually, the computation of the function t_{shift} is polynomial. Thus, the complexity of $U_{\Lambda_{\text{add}}}$ is polynomial. □

Definition 2.4 (Circuit $U_{\Omega_{\text{add}}}$). *For $J \subseteq [n]$ such that $|J|$ is even, and for $t \in T$, we define $U_{\Omega_{add}}$ as follows:*

$$U_{\Omega_{add}} |J\rangle |t\rangle |0\rangle = |J\rangle |t\rangle \sum_{\omega \in \Omega_{add}(J,t)} \frac{1}{\sqrt{|\Omega_{add}(J,t)|}} |\omega\rangle \ .$$

Proposition 2.5 (Complexity of $U_{\Omega_{\text{add}}}$). *Let J be the input set. If we suppose to have stored in the QRAM the values* $\mathrm{OPT}[X, t]$ *for all $X \subseteq J$ such that $|X| = |J|/2$ and for all $t \in T$, the complexity of $U_{\Omega_{add}}$ is polynomial in the size of the input.*

Proof. The proof follows essentially the same lines as the proof of Property 2.3. The quantum superposition of subsets is done in polynomial time, and instead of computing t_{shift}, we get values in the QRAM in constant time. □

We end this subsection with the definition of the quantum circuit of the addition required for recurrence (Add-D-DPAS).

Definition 2.6 (Circuit U_a). *We define the antecedent set $S_a = 2^{[n]} \times (\mathbb{Z} \cup \{+\infty\}) \times 2^{[n]} \times (\mathbb{Z} \cup \{+\infty\}) \times T$. Let $a : S_a \to \mathbb{Z} \cup \{+\infty\}$ be the function:*

$$a(\omega_1^s, \omega_1^v, \omega_2^s, \omega_2^v, \omega^t) = \omega_1^v + \omega_2^v + h(\omega_1^s \cup \omega_2^s, \omega_1^s, \omega^t) .$$

We note U_a the quantum circuit corresponding to a, namely:

$$\forall (\omega_1^s, \omega_1^v, \omega_2^s, \omega_2^v, \omega^t) \in S_a, \quad U_a |\omega\rangle |0\rangle = |\omega\rangle |a(\omega)\rangle \ ,$$

where $|\omega\rangle = |\omega_1^s\rangle |\omega_1^v\rangle |\omega_2^s\rangle |\omega_2^v\rangle |\omega^t\rangle$ *is encoded in five registers. Notice that we index the objects that represent numerical values by* v.

Notice that according to recurrence (Add-D-DPAS), the function a applies on objects of $\Omega_{\text{add}}(J,t)$ for $J \subseteq [n]$ and $t \in T$, explaining the choice of the antecedent set.

Proposition 2.7 (Complexity of U_a). *The complexity of* U_a *is polynomial in the size of the input.*

Proof. By assumption, the computation of h is polynomial. It implies that the computation of a is polynomial, and thus U_a has a polynomial complexity (see Observation 1.1). □

Remark 2.8. *Notice that for* $J \subseteq [n]$ *and* $t \in T$,

$$\mathrm{OPT}[J,t] = \min_{\omega \in \Omega_{add}(J,t)} a(\omega)\,.$$

2.2 Composed DPAS sets and quantum circuits

In this subsection, we define the sets and their associated quantum circuits used for the description of the hybrid algorithm that solves problems whose related auxiliary problem satisfies recurrences (Comp-DPAS) and (Comp-D-DPAS). Similarly to the previous subsection, we define two sets Λ_{comp} and Ω_{comp} indexed by (J,t,ϵ) for $J \subseteq [n]$, $t \in T$ and $\epsilon \in E$. In this case, the set $\Lambda_{\text{comp}}(J,t,\epsilon)$ contains all the possible balanced bi-partitions of J and the possible parameter values of T and E. The second set $\Omega_{\text{comp}}(J,t,\epsilon)$ contains the optimal solutions for each bi-partition and parameter values in $\Lambda_{\text{comp}}(J,t,\epsilon)$.

Definition 2.9 (Sets Λ_{comp} and Ω_{comp}). *For* $J \subseteq [n]$ *such that* $|J|$ *is even, for* $t \in T$ *and for* $\epsilon \in E$, *we define the set*

$$\Lambda_{comp}(J,t,\epsilon) = \left\{(X, t_i, \epsilon_i, J \setminus X, t, \epsilon - \epsilon_i) : X \subseteq J, |X| = \frac{|J|}{2}, \epsilon_i \in E, t_i \in T\right\},$$

and the set

$$\Omega_{comp}(J,t,\epsilon) = \left\{\begin{matrix}(X, \mathrm{OPT}_{\epsilon_i}[X,t_i], t_i, \epsilon_i, J \setminus X, \mathrm{OPT}_{\epsilon-\epsilon_i}[J \setminus X, t], t, \epsilon - \epsilon_i) : \\ \hfill X \subseteq J, |X| = \frac{|J|}{2}, \epsilon_i \in E, t_i \in T\end{matrix}\right\}.$$

Definition 2.10 (Circuit $U_{\Lambda_{\text{comp}}}$). *For* $J \subseteq [n]$ *such that* $|J|$ *is even, for* $t \in T$ *and for* $\epsilon \in E$, *we define* $U_{\Lambda_{comp}}$ *as follows:*

$$U_{\Lambda_{comp}} |J\rangle |t\rangle |0\rangle^{\otimes 8} = |J\rangle |t\rangle \sum_{\substack{(\lambda_1^s, \lambda_1^t, \lambda_1^e, \\ \lambda_2^s, \lambda_2^t, \lambda_2^e) \in \Lambda_{comp}(J,t,\epsilon)}} \frac{1}{\sqrt{|\Lambda_{comp}(J,t)|}} |\lambda_1^s\rangle |\lambda_1^t\rangle |\lambda_1^e\rangle |0\rangle |\lambda_2^s\rangle |\lambda_2^t\rangle |\lambda_2^e\rangle |0\rangle\,.$$

Observe that we index the objects that represent sets by s, the objects that represent scalars in T by t, and the objects that represent parameter values in E by e.

Proposition 2.11 (Complexity of $U_{\Lambda_{\text{comp}}}$)**.** *The complexity of $U_{\Lambda_{comp}}$ is polynomial in the size of the input.*

Definition 2.12 (Circuit $U_{\Omega_{\text{comp}}}$)**.** *For $J \subseteq [n]$ such that $|J|$ is even, for $t \in T$ and $\epsilon \in E$, we define $U_{\Omega_{comp}}$ as follows:*

$$U_{\Omega_{comp}} |J\rangle |t\rangle |\epsilon\rangle |0\rangle = |J\rangle |t\rangle |\epsilon\rangle \sum_{\omega \in \Omega_{comp}(J,t,\epsilon)} \frac{1}{\sqrt{|\Omega_{comp}(J,t,\epsilon)|}} |\omega\rangle \ .$$

Proposition 2.13 (Complexity of $U_{\Omega_{\text{comp}}}$)**.** *Let J be the input set. If we suppose to have stored in the QRAM the values $\text{OPT}_\epsilon[X,t]$ for all $X \subseteq J$ such that $|X| = |J|/2$, for all $t \in T$ and for all $\epsilon \in E$, the complexity of $U_{\Omega_{comp}}$ is polynomial in the size of the input.*

The proof of Proposition 2.11 (respectively Proposition 2.13) is similar to the proof of Proposition 2.3 (respectively Proposition 2.5).

The composition is the counterpart for (Comp-D-DPAS) of the addition for (Add-D-DPAS) (function a).

Definition 2.14 (Circuit U_c)**.** *We note the antecedent set $S_c = 2^{[n]} \times (\mathbb{Z} \cup \{+\infty\}) \times T \times E \times 2^{[n]} \times (\mathbb{Z} \cup \{+\infty\}) \times T \times E$. Let $c : S_c \to \mathbb{Z} \cup \{+\infty\}$ be the function:*

$$c(\omega_1^s, \omega_1^v, \omega_1^t, \omega_1^e, \omega_2^s, \omega_2^v, \omega_2^t, \omega_2^e) = \begin{cases} \omega_1^v & \textit{if } \omega_1^t = \omega_2^v \\ +\infty & \textit{else} \end{cases}$$

We note U_c the quantum circuit corresponding to c, namely:

$$\forall (\omega_1^s, \omega_1^v, \omega_1^t, \omega_1^e, \omega_2^s, \omega_2^v, \omega_2^t, \omega_2^e) \in S_c, \quad U_c |\omega\rangle |0\rangle = |\omega\rangle |c(\omega)\rangle \ ,$$

where $|\omega\rangle = |\omega_1^s\rangle |\omega_1^v\rangle \left|\omega_1^t\right\rangle |\omega_1^e\rangle |\omega_2^s\rangle |\omega_2^v\rangle \left|\omega_2^t\right\rangle |\omega_2^e\rangle$ is encoded in eight registers.

Notice that the function c is meant to be applied on objects of $\Omega_{\text{comp}}(J,t,\epsilon)$, for $J \subseteq [n]$, $t \in T$ and $\epsilon \in E$, according to recurrence (Comp-D-DPAS).

Proposition 2.15 (Complexity of U_c)**.** *The complexity of U_c is polynomial in the size of the input.*

The proof of the above proposition is the same as for Proposition 2.7.

Remark 2.16. *Notice that, for $J \subseteq [n]$, $t \in T$ and $\epsilon \in E$,*

$$\text{OPT}_\epsilon[J,t] = \min_{\omega \in \Omega_{comp}(J,t,\epsilon)} c(\omega) \, .$$

2.3 Algorithm for Additive DPAS (Algorithm 1)

Let us describe the hybrid algorithm Q-DDPAS in the gate-based quantum computing model. We begin with the description of **Algorithm 1** which is Q-DDPAS for problems $\mathcal{P}$ whose related problem P satisfies recurrences (Add-DPAS) and (Add-D-DPAS). **Algorithm 2**, which is Q-DDPAS for problems whose related auxiliary problem P' satisfies recurrences (Comp-DPAS) and (Comp-D-DPAS), derives directly as we explain later in Subsection 2.4.

We present the quantum circuits used in the quantum part, as well as the numbering of the different registers.

- Let $|\text{ini}\rangle$ be the initial state:

$$|\text{ini}\rangle := \underbrace{|[n]\rangle\,|0\rangle}_{I^1}\underbrace{|0\rangle^{\otimes 3}}_{I^2}\underbrace{|0\rangle^{\otimes 2}}_{I^3}\underbrace{|0\rangle^{\otimes 3}}_{I^4}\underbrace{|0\rangle^{\otimes 2}}_{I^5}\underbrace{|0\rangle^{\otimes 2}}_{I^6},$$

  where the tuples indexing the different registers are decomposed as follows:

$$\begin{aligned}
I^1 &= I^1_1 \oplus I^1_2\\
I^2 &= I^2_1 \oplus I^2_2 \oplus I^2_3\\
I^3 &= I^3_1 \oplus I^3_2\\
I^4 &= I^4_1 \oplus I^4_2 \oplus I^4_3\\
I^5 &= I^5_1 \oplus I^5_2\\
I^6 &= I^6_1 \oplus I^6_2
\end{aligned}$$

- Let

$$U_{\text{ini}} := (U^{I^2}_{\Omega_{\text{add}}} \otimes U^{I^4}_{\Omega_{\text{add}}}) \cdot U^{I^1\oplus I^2\oplus I^4}_{\Lambda_{\text{add}}} \tag{1}$$

  be the quantum circuit that, given initial quantum state $|\text{ini}\rangle$, superposes all the couples (X, X') such that $X, X' \subseteq [n]$, $|X| = |X'| = n/4$ and $X \cap X' = \emptyset$. For each couple, the optimal values and parameters associated are also superposed.

- The quantum circuit $U^{I^2_3\oplus I^3}_{\text{QMF}}[U^{I^2_3}_a] \otimes U^{I^4_3\oplus I^5}_{\text{QMF}}[U^{I^4_3}_a]$ applies two Quantum Minimum Finding *in parallel* (resulting from the tensor product of two quantum circuits) on the function a. Consequently, let

$$U_{\text{recur1}} := U_a^{I^2_1\oplus I^3_2\oplus I^4_1\oplus I^5_2\oplus I^1_2}\left(U^{I^2_3\oplus I^3}_{\text{QMF}}[U^{I^2_3}_a] \otimes U^{I^4_3\oplus I^5}_{\text{QMF}}[U^{I^4_3}_a]\right)$$

  be the quantum circuit that adds, with the help of of function a, the resulting values of the two registers.

- Eventually, let

$$U_{\text{recur}} := U_{\text{QMF}}^{I_1^2 \oplus I_2^3 \oplus I_1^4 \oplus I_2^5 \oplus I_2^1 \oplus I^6}[U_{\text{recur1}}] \tag{2}$$

  be the quantum circuit that applies Quantum Minimum Finding on the function represented by the circuit U_{recur1}.

We describe next the bounded-error hybrid algorithm Q-DDPAS (**Algorithm 1**) from a low-level point of view.

Algorithm 1: Q-DDPAS for Additive DPAS (low-level description)

Input: Problem P satisfying (Add-DPAS) and (Add-D-DPAS)

Output: $\text{OPT}[[n], 0]$ with high probability

begin classical part

    for $X \subseteq [n] : |X| = n/4$ *and* $t \in T$ **do**

        Compute the optimal value $\text{OPT}[X, t]$ and the corresponding permutation $\pi^*[X, t]$ by classical (Add-DPAS);

        Store the tuple $(X, t, \text{OPT}[X, t], \pi^*[X, t])$ in the QRAM;

begin quantum part

    Prepare quantum state $|\text{ini}\rangle$;

    Apply the quantum circuit $U_{\text{recur}} U_{\text{ini}}$ to $|\text{ini}\rangle$;

    Measure register of indexes I_2^6;

Return the outcome of the measurement

We recall **Theorem 2.8** which states Q-DDPAS complexity. Because the complexity proof has already been proven, we only provide the proof of the correctness, namely that the optimal value of $\mathcal{P}$ is stored in the register of indexes I_2^6 with high probability.

Theorem 2.8. *The bounded-error algorithm Q-DDPAS (Algorithm 1) solves $\mathcal{P}$ in $\mathcal{O}^*(|T| \cdot 1.754^n)$.*

Proof. Before entering the details of the computations, we give some intuition on the effect of the quantum circuit $U_{\text{recur}} U_{\text{ini}}$ and start by explaining the effect of U_{ini} defined in (1). First, the application of $U_{\Lambda_{\text{add}}}$ superposes all elements of $\Lambda_{\text{add}}([n], 0)$ in the registers of indexes I^2 (partition of J) and I^4 (partition of $[n] \setminus J$). This essentially amounts to superposing all the $\binom{n}{n/2}$ bi-partitions of $[n]$ where each partition is of size $n/2$ (parameters t included). Next, we apply $U_{\Omega_{\text{add}}}$ on register of index I^2, respectively I^4. This superposes all elements of $\Omega_{\text{add}}(J, t)$ (for a J of size $n/2$ and $t \in T$ previously described in registers of indexes I^2, respectively I^4).

This essentially amounts to superposing all the $\binom{n/2}{n/4}$ bi-partitions of $[n]$ where each partition is of size $n/2$, parameters t included, and the optimal value associated already stored in the QRAM.

Let us explain the effect of U_{recur} defined in (2). The application of $U_{\text{QMF}}[U_a]$ on a register encoding $(J,\ t)$ and the superposition of elements of $\Omega_{\text{add}}(J,t)$ stores $\text{OPT}[J,t]$ (with high probability) in an output register, according to Equation (Add-D-DPAS). Thus, $U_{\text{QMF}}[U_a]$ on register of index I^2, respectively I^4, superposes all $\text{OPT}[J,t]$ in I^3, respectively I^5, according to Remark 2.8. In other words, the circuit $U_{\text{QMF}}^{I_3^2\oplus I^3}[U_a^{I_3^2\oplus I_1^3}]\otimes U_{\text{QMF}}^{I_3^4\oplus I^5}[U_a^{I_3^4\oplus I_1^5}]$ that appears in U_{recur1} superposes (with high probability) all optimal values of Equation (Add-D-DPAS) for J of size $n/2$. Now that the optimal values are known for sets of size $n/2$ (before, we only knew optimal values for sets of size $n/4$), we apply one more time $U_{\text{QMF}}[U_a]$ on these new registers: it outputs $\text{OPT}[[n],0]$ with high probability on the register of index I_2^6.

Next, we detail the computation of $U_{\text{recur}}U_{\text{ini}}\left|\text{ini}\right\rangle$ and show that $\text{OPT}[[n],0]$ is stored in register of indexes I_2^6 with high probability. We write the following computations as if the algorithm Quantum Minimum Finding was returning the optimal solution with probability 1. First, we compute $U_{\text{ini}}\left|\text{ini}\right\rangle$.

$$
\begin{aligned}
U_{\Lambda_{\text{add}}}^{I^1\oplus I^2\oplus I^4}\left|\text{ini}\right\rangle =& U_{\Lambda_{\text{add}}}^{I^1\oplus I^2\oplus I^4}\underbrace{\left|[n]\right\rangle\left|0\right\rangle}_{I^1}\underbrace{\left|0\right\rangle^{\otimes 3}}_{I^2}\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^3}\underbrace{\left|0\right\rangle^{\otimes 3}}_{I^4}\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^5}\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^6}\\
=&\underbrace{\left|[n]\right\rangle\left|0\right\rangle}_{I^1}\sum_{(\lambda_1^s,\lambda_1^t,\lambda_2^s,\lambda_2^t)\in\Lambda_{\text{add}}([n],0)}\frac{1}{\sqrt{|\Lambda_{\text{add}}([n],0)|}}\\
&\underbrace{\left|\lambda_1^s\right\rangle\left|\lambda_1^t\right\rangle\left|0\right\rangle}_{I^2}\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^3}\underbrace{\left|\lambda_2^s\right\rangle\left|\lambda_2^t\right\rangle\left|0\right\rangle}_{I^4}\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^5}\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^6}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
U_{\text{ini}}\left|\text{ini}\right\rangle =&(U_{\Omega_{\text{add}}}^{I^2}\otimes U_{\Omega_{\text{add}}}^{I^4})\cdot U_{\Lambda_{\text{add}}}^{I^1\oplus I^2\oplus I^4}\left|\text{ini}\right\rangle\\
=&(U_{\Omega_{\text{add}}}^{I^2}\otimes U_{\Omega_{\text{add}}}^{I^4})\underbrace{\left|[n]\right\rangle\left|0\right\rangle}_{I^1}\sum_{(\lambda_1^s,\lambda_1^t,\lambda_2^s,\lambda_2^t)\in\Lambda_{\text{add}}([n],0)}\frac{1}{\sqrt{|\Lambda_{\text{add}}([n],0)|}}\underbrace{\left|\lambda_1^s\right\rangle\left|\lambda_1^t\right\rangle\left|0\right\rangle}_{I^2}\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^3}\underbrace{\left|\lambda_2^s\right\rangle\left|\lambda_2^t\right\rangle\left|0\right\rangle}_{I^4}\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^5}\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^6}\\
=&\underbrace{\left|[n]\right\rangle\left|0\right\rangle}_{I^1}\sum_{(\lambda_1^s,\lambda_1^t,\lambda_2^s,\lambda_2^t)\in\Lambda_{\text{add}}([n],0)}\frac{1}{\sqrt{|\Lambda_{\text{add}}([n],0)|}}\underbrace{\left|\lambda_1^s\right\rangle}_{I_1^2}\underbrace{\left|\lambda_1^t\right\rangle}_{I_2^2}\left(\sum_{\omega\in\Omega_{\text{add}}(\lambda_1^s,\lambda_1^t)}\frac{1}{\sqrt{|\Omega_{\text{add}}(\lambda_1^s,\lambda_1^t)|}}\underbrace{\left|\omega\right\rangle}_{I_3^2}\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^3}\right)\\
&\underbrace{\left|\lambda_2^s\right\rangle}_{I_1^4}\underbrace{\left|\lambda_2^t\right\rangle}_{I_2^4}\left(\sum_{\omega\in\Omega_{\text{add}}(\lambda_2^s,\lambda_2^t)}\frac{1}{\sqrt{|\Omega_{\text{add}}(\lambda_2^s,\lambda_2^t)|}}\underbrace{\left|\omega\right\rangle}_{I_3^4}\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^5}\right)\underbrace{\left|0\right\rangle^{\otimes 2}}_{I^6}.
\end{aligned}
$$

Second, we apply the tensor product of the two first Quantum Minimum Finding to the previous state.

$$\left(U_{\mathrm{QMF}}^{I_3^2\oplus I^3}[U_a^{I_3^2}]\otimes U_{\mathrm{QMF}}^{I_3^4\oplus I^5}[U_a^{I_3^4}]\right)\underbrace{|[n]\rangle\,|0\rangle}_{I^1}\sum_{(\lambda_1^s,\lambda_1^t,\lambda_2^s,\lambda_2^t)\in\Lambda_{\mathrm{add}}([n],0)}\frac{1}{\sqrt{|\Lambda_{\mathrm{add}}([n],0)|}}$$

$$\underbrace{|\lambda_1^s\rangle}_{I_1^2}\underbrace{|\lambda_1^t\rangle}_{I_2^2}\left(\sum_{\omega\in\Omega_{\mathrm{add}}(\lambda_1^s,\lambda_1^t)}\frac{1}{\sqrt{|\Omega_{\mathrm{add}}(\lambda_1^s,\lambda_1^t)|}}\underbrace{|\omega\rangle}_{I_3^2}\underbrace{|0\rangle^{\otimes 2}}_{I^3}\right)$$

$$\underbrace{|\lambda_2^s\rangle}_{I_1^4}\underbrace{|\lambda_2^t\rangle}_{I_2^4}\left(\sum_{\omega\in\Omega_{\mathrm{add}}(\lambda_2^s,\lambda_2^t)}\frac{1}{\sqrt{|\Omega_{\mathrm{add}}(\lambda_2^s,\lambda_2^t)|}}\underbrace{|\omega\rangle}_{I_3^4}\underbrace{|0\rangle^{\otimes 2}}_{I^5}\right)\underbrace{|0\rangle^{\otimes 2}}_{I^6}$$

$$=\underbrace{|[n]\rangle\,|0\rangle}_{I^1}\sum_{(\lambda_1^s,\lambda_1^t,\lambda_2^s,\lambda_2^t)\in\Lambda_{\mathrm{add}}([n],0)}\frac{1}{\sqrt{|\Lambda_{\mathrm{add}}([n],0)|}}$$

$$\underbrace{|\lambda_1^s\rangle}_{I_1^2}\underbrace{|\lambda_1^t\rangle}_{I_2^2}\sum_{\omega\in\Omega_{\mathrm{add}}(\lambda_1^s,\lambda_1^t)}\frac{1}{\sqrt{|\Omega_{\mathrm{add}}(\lambda_1^s,\lambda_1^t)|}}\underbrace{|\omega\rangle}_{I_3^2}\underbrace{\left|\arg\min_{\omega} r(\omega)\right\rangle\left|\min_{\omega} r(\omega)\right\rangle}_{I_1^3\otimes I_2^3}$$

$$\underbrace{|\lambda_2^s\rangle}_{I_1^4}\underbrace{|\lambda_2^t\rangle}_{I_2^4}\sum_{\omega\in\Omega_{\mathrm{add}}(\lambda_2^s,\lambda_2^t)}\frac{1}{\sqrt{|\Omega_{\mathrm{add}}(\lambda_2^s,\lambda_2^t)|}}\underbrace{|\omega\rangle}_{I_3^4}\underbrace{\left|\arg\min_{\omega} r(\omega)\right\rangle\left|\min_{\omega} r(\omega)\right\rangle}_{I_1^5\otimes I_2^5}\underbrace{|0\rangle^{\otimes 2}}_{I^6}.$$

Thus, we apply the second circuit of Quantum Minimum Finding.

$$\begin{aligned}U_{\mathrm{recur}}U_{\mathrm{ini}}\,|\mathrm{ini}\rangle =&U_{\mathrm{QMF}}^{I_1^2\oplus I_2^3\oplus I_1^4\oplus I_2^5\oplus I_2^1\oplus I^6}[U_{\mathrm{recur1}}]U_{\mathrm{ini}}\,|\mathrm{ini}\rangle\\
=&U_{\mathrm{QMF}}^{I_1^2\oplus I_2^3\oplus I_1^4\oplus I_2^5\oplus I_2^1 I^6}[U_a^{I_1^2\oplus I_2^3\oplus I_1^4\oplus I_2^5\oplus I_2^1}]\underbrace{|[n]\rangle\,|0\rangle}_{I^1}\sum_{(\lambda_1^s,\lambda_1^t,\lambda_2^s,\lambda_2^t)\in\Lambda_{\mathrm{add}}([n],0)}\frac{1}{\sqrt{|\Lambda_{\mathrm{add}}([n],0)|}}\\
&\underbrace{|\lambda_1^s\rangle}_{I_1^2}\underbrace{|\lambda_1^t\rangle}_{I_2^2}\sum_{\omega\in\Omega_{\mathrm{add}}(\lambda_1^s,\lambda_1^t)}\frac{1}{\sqrt{|\Omega_{\mathrm{add}}(\lambda_1^s,\lambda_1^t)|}}\underbrace{|\omega\rangle}_{I_3^2}\underbrace{\left|\arg\min_{\omega} r(\omega)\right\rangle\left|\min_{\omega} r(\omega)\right\rangle}_{I_1^3\otimes I_2^3}\\
&\underbrace{|\lambda_2^s\rangle}_{I_1^4}\underbrace{|\lambda_2^t\rangle}_{I_2^4}\sum_{\omega\in\Omega_{\mathrm{add}}(\lambda_2^s,\lambda_2^t)}\frac{1}{\sqrt{|\Omega_{\mathrm{add}}(\lambda_2^s,\lambda_2^t)|}}\underbrace{|\omega\rangle}_{I_3^4}\underbrace{\left|\arg\min_{\omega} r(\omega)\right\rangle\left|\min_{\omega} r(\omega)\right\rangle}_{I_1^5\otimes I_2^5}\underbrace{|0\rangle^{\otimes 2}}_{I^6}\\
&=\underbrace{|[n]\rangle\,|0\rangle}_{I^1}\sum_{(\lambda_1^s,\lambda_1^t,\lambda_2^s,\lambda_2^t)\in\Lambda_{\mathrm{add}}([n],0)}\frac{1}{\sqrt{|\Lambda_{\mathrm{add}}([n],0)|}}\\
&\underbrace{|\lambda_1^s\rangle}_{I_1^2}\underbrace{|\lambda_1^t\rangle}_{I_2^2}\sum_{\omega\in\Omega_{\mathrm{add}}(\lambda_1^s,\lambda_1^t)}\frac{1}{\sqrt{|\Omega_{\mathrm{add}}(\lambda_1^s,\lambda_1^t)|}}\underbrace{|\omega\rangle}_{I_3^2}\underbrace{\left|\arg\min_{\omega} r(\omega)\right\rangle\left|\min_{\omega} r(\omega)\right\rangle}_{I_1^3\otimes I_2^3}\\
&\underbrace{|\lambda_2^s\rangle}_{I_1^4}\underbrace{|\lambda_2^t\rangle}_{I_2^4}\sum_{\omega\in\Omega_{\mathrm{add}}(\lambda_2^s,\lambda_2^t)}\frac{1}{\sqrt{|\Omega_{\mathrm{add}}(\lambda_2^s,\lambda_2^t)|}}\underbrace{|\omega\rangle}_{I_3^4}\\
&\underbrace{\left|\arg\min_{\omega} r(\omega)\right\rangle\left|\min_{\omega} r(\omega)\right\rangle}_{I_1^5\otimes I_2^5}\underbrace{\left|\arg\min_{\lambda\in\Lambda_{\mathrm{add}}([n],0)} r(\lambda_1^s,\min_{\omega\in\Omega_{\mathrm{add}}(\lambda_1^s,\lambda_1^t)} r(\omega),\lambda_2^s,\min_{\omega\in\Omega_{\mathrm{add}}(\lambda_2^s,\lambda_2^t)} r(\omega),0)\right\rangle}_{I_1^6}\end{aligned}$$

$$\underbrace{\left|\min_{\lambda\in\Lambda_{\mathrm{add}}([n],0)} r(\lambda_1^s, \min_{\omega\in\Omega_{\mathrm{add}}(\lambda_1^s,\lambda_1^t)} r(\omega), \lambda_2^s, \min_{\omega\in\Omega_{\mathrm{add}}(\lambda_2^s,\lambda_2^t)} r(\omega), 0)\right\rangle}_{I_2^6}.$$

According to definition of a and recurrence (Add-D-DPAS), the results stored in register of indexes I_2^6 is $\mathrm{OPT}[[n],0]$.

Notice that optimal permutation $\pi^*[[n],0]$ can be *rebuilt* with registers of indexes I_1^3, I_1^5 and I_1^6, and with the access to the results of the classical part in the QRAM. □

2.4 Adaptation for Composed DPAS (Algorithm 2)

In this subsection, we adapt Algorithm 1 for a problem $\mathcal{P}$ related to auxiliary problem P' satisfying recurrences (Comp-DPAS) and (Comp-D-DPAS). It essentially amounts to replacing Λ_{add} by Λ_{comp}, Ω_{add} by Ω_{comp} and function a by function c. Consequently, the quantum circuit $U_{\Lambda_{\mathrm{comp}}}$, respectively $U_{\Omega_{\mathrm{comp}}}$, apply on 8 registers, respectively 4 registers, that differ from Q-DDPAS for Additive DPAS. The resulting **Algorithm 2** is provided in a low-level description.

Let us describe the slightly different quantum circuits adapting the number of registers and the registers on which they apply. Let $\epsilon_0 \in E$. The initial state is

$$|\mathrm{ini}\rangle = \underbrace{|[n]\rangle\,|0\rangle\,|\epsilon_0\rangle}_{I^1}\underbrace{|0\rangle^{\otimes 4}}_{I^2}\underbrace{|0\rangle^{\otimes 2}}_{I^3}\underbrace{|0\rangle^{\otimes 4}}_{I^4}\underbrace{|0\rangle^{\otimes 2}}_{I^5}\underbrace{|0\rangle^{\otimes 2}}_{I^6},$$

where the tuples indexing the different registers are decomposed as follows:

$$\begin{aligned}
I^1 &= I_1^1 \oplus I_2^1 \oplus I_3^1\\
I^2 &= I_1^2 \oplus I_2^2 \oplus I_3^2 \oplus I_4^2\\
I^3 &= I_1^3 \oplus I_2^3\\
I^4 &= I_1^4 \oplus I_2^4 \oplus I_3^4 \oplus I_4^4\\
I^5 &= I_1^5 \oplus I_2^5\\
I^6 &= I_1^6 \oplus I_2^6
\end{aligned}$$

The three quantum circuits that appear on the quantum part are:

$$U_{\mathrm{ini}} = (U_{\Omega_{\mathrm{comp}}}^{I^2} \otimes U_{\Omega_{\mathrm{comp}}}^{I^4}) \cdot U_{\Lambda_{\mathrm{comp}}}^{I^1\oplus I^2\oplus I^4},$$

$$U_{\mathrm{recur1}} = U_c^{I_1^2\oplus I_2^3\oplus I_2^2\oplus I_3^2\oplus I_1^4\oplus I_2^5\oplus I_2^4\oplus I_3^4}\left(U_{\mathrm{QMF}}^{I_4^2\oplus I^3}[U_c^{I_4^2}] \otimes U_{\mathrm{QMF}}^{I_4^4\oplus I^5}[U_c^{I_4^4}]\right),$$

$$U_{\mathrm{recur}} = U_{\mathrm{QMF}}^{I_1^2\oplus I_2^3\oplus I_2^2\oplus I_3^2\oplus I_1^4\oplus I_2^5\oplus I_2^4\oplus I_3^4\oplus I^6}[U_{\mathrm{recur1}}].$$

Next, we describe with a low level of details **Algorithm 2** which is the adaptation of Q-DDPAS to solve $P'([n], 0, \epsilon_0)$ for a given $\epsilon_0 \in E$.

Algorithm 2: Q-DDPAS for Composed DPAS (low-level description)

Input: $\epsilon_0 \in E$, auxiliary problem P' satisfying (Comp-DPAS) and (Comp-D-DPAS)

Output: $\mathrm{OPT}[[n], 0, \epsilon_0]$ with high probability

```
begin classical part
    for X ⊆ [n] : |X| = n/4 and t ∈ T do
        Compute the optimal value OPT[X, t, ϵ0] and the corresponding permutation
          π*[X, t, ϵ0] by classical (Comp-DPAS);
        Store the tuple (X, t, OPT[X, t, ϵ0], π*[X, t, ϵ0]) in the QRAM;
begin quantum part
    Prepare quantum state |ini⟩;
    Apply the quantum circuit U_recur U_ini to |ini⟩;
    Measure register of indexes I_2^6;
Return the outcome of the measurement
```

The proof of correctness of **Lemma 3.5** is the same as for **Theorem 2.8**. To lighten the reading, and because the approach is very similar, we do not detail the calculations here.

3 Decision-based hybrid algorithm Q-Dec-DPAS (Algorithm 4)

In what follows, we define the sets and their associated quantum circuits to describe the Q-Dec-DDPAS (**Algorithm 4**).

Definition 3.1 (Sets Λ_{dec} and Ω_{dec})**.** *For $J \subseteq [n]$ such that $|J|$ is even and for $\vec{\beta}, \vec{\epsilon} \in T^2$, we define the set*

$$\Lambda_{dec}(J, \vec{\beta}, \vec{\epsilon}) = \left\{ (X, \vec{\beta}, \vec{t}, J \setminus X, \vec{t}, \vec{\epsilon}) : X \subseteq J, |X| = \frac{|J|}{2}, \vec{t} \in [\vec{\beta}, \vec{\epsilon}] \right\} ,$$

and the set

$$\Omega_{dec}(J, \vec{\beta}, \vec{\epsilon}) = \left\{ (X, D[X, \vec{\beta}, \vec{t}], \vec{\beta}, \vec{t}, J \setminus X, D[J \setminus X, \vec{t}, \vec{\epsilon}], \vec{t}, \vec{\epsilon}) : X \subseteq J, |X| = \frac{|J|}{2}, \vec{t} \in [\vec{\beta}, \vec{\epsilon}] \right\} .$$

The quantum circuits associated with these two sets are the following.

Definition 3.2 (Circuit $U_{\Lambda_{\mathrm{dec}}}$)**.** *For $J \subseteq [n]$ such that $|J|$ is even, and for $\vec{\beta}, \vec{\epsilon} \in T^2$, we define*

$U_{\Lambda_{dec}}$ *as follows:*

$$U_{\Lambda_{dec}} |J\rangle \left|\vec{\beta}\right\rangle |\vec{\epsilon}\rangle |0\rangle^{\otimes 8} =$$
$$|J\rangle \left|\vec{\beta}\right\rangle |\vec{\epsilon}\rangle \sum_{\substack{(\lambda_1^s, \lambda_1^{tb}, \lambda_1^{te}, \\ \lambda_2^s, \lambda_2^{tb}, \lambda_2^{te}) \in \Lambda_{dec}(J, \vec{\beta}, \vec{\epsilon})}} \frac{1}{\sqrt{|\Lambda_{dec}(J, \vec{\beta}, \vec{\epsilon})|}} |\lambda_1^s\rangle \left|\lambda_1^{tb}\right\rangle |\lambda_1^{te}\rangle |0\rangle |\lambda_2^s\rangle \left|\lambda_2^{tb}\right\rangle |\lambda_2^{te}\rangle |0\rangle \ .$$

Notice that we index the objects that represent sets by s, and the objects that represent scalars in T^2 by tb if it represents a couple of beginning times, or by te if it represents a couple of ending times.

Proposition 3.3 (Complexity of $U_{\Lambda_{\text{dec}}}$). *The complexity of $U_{\Lambda_{dec}}$ is polynomial in the size of the input.*

Definition 3.4 (Circuit $U_{\Omega_{\text{dec}}}$). *For $J \subseteq [n]$ such that $|J|$ is even, and for $\vec{\beta}, \vec{\epsilon} \in T^2$, we define $U_{\Omega_{dec}}$ as follows:*

$$U_{\Omega_{dec}} |J\rangle \left|\vec{\beta}\right\rangle |\vec{\epsilon}\rangle |0\rangle = |J\rangle \left|\vec{\beta}\right\rangle |\vec{\epsilon}\rangle \sum_{\omega \in \Omega_{dec}(J, \vec{\beta}, \vec{\epsilon})} \frac{1}{\sqrt{|\Omega_{add}(J, \vec{\beta}, \vec{\epsilon})|}} |\omega\rangle \ .$$

Proposition 3.5 (Complexity of $U_{\Omega_{\text{dec}}}$). *Let J be the input set. If we suppose to have stored in the QRAM the values $D[X, \vec{\beta}, \vec{\epsilon}]$ for all $X \subseteq J$ such that $|X| = |J|/2$ and for all $\vec{\beta}, \vec{\epsilon} \in T^2$, the complexity of $U_{\Omega_{dec}}$ is polynomial in the size of the input.*

The proof of Proposition 3.3, respectively Proposition 3.5, is similar to the proof of Proposition 2.3, respectively Proposition 2.5. Notice that $\vec{t} \in [\vec{\beta}, \vec{\epsilon}]$ can be replaced by $\vec{t} \in T^2$ in sets $U_{\Lambda_{\text{dec}}}$ and $U_{\Omega_{\text{dec}}}$ so that the circuits that superpose all elements of these sets are easier to conceive. Indeed, (Dec-DPAS) and (Dec-D-DPAS) are less accurate but still valid with this replacement.

The operation in recurrence (Dec-D-DPAS) is not the addition (represented by the function a for (Add-D-DPAS)) nor the composition (represented by the function c for (Comp-D-DPAS)) but the logical AND. We define below its corresponding quantum circuit.

Definition 3.6 (Circuit U_{and}). *We note the antecedent set $S_{and} = 2^{[n]} \times \{0,1\} \times T^2 \times T^2 \times 2^{[n]} \times \{0,1\} \times T^2 \times T^2$. Let $and : S_{and} \to \{0,1\}$ be the function:*

$$and(\omega_1^s, \omega_1^b, \omega_1^{tb}, \omega_1^{te}, \omega_2^s, \omega_2^b, \omega_2^{tb}, \omega_2^{te}) = \begin{cases} 1 & \textit{if } \omega_1^b = \omega_2^b \\ 0 & \textit{else} \end{cases}$$

We note U_{and} the quantum circuit associated to the function, specifically,

$$\forall \omega = (\omega_1^s, \omega_1^b, \omega_1^{tb}, \omega_1^{te}, \omega_2^s, \omega_2^b, \omega_2^{tb}, \omega_2^{te}) \in S_{and}, \quad U_{and} |\omega\rangle |0\rangle = |\omega\rangle |and(\omega)\rangle \ .$$

Notice that objects representing boolean values are indexed by b. Note that according to recurrence (Dec-D-DPAS), the function *and* applies on objects of sets $\Omega_{\mathrm{dec}}(J, \vec{\beta}, \vec{\epsilon})$ for $J \subseteq [n]$ and $\vec{\beta}, \vec{\epsilon} \in T^2$.

Proposition 3.7 (Complexity of U_{and})**.** *The complexity of U_{and} is polynomial in the size of the input.*

The proof of the above proposition is the same as the one of Proposition 2.7.

Remark 3.8. *Notice that for $J \subseteq [n]$ and $\vec{\beta}, \vec{\epsilon} \in T^2$,*

$$D[J, \vec{\beta}, \vec{\epsilon}] = \bigvee_{\omega \in \Omega_{dec}(J, \vec{\beta}, \vec{\epsilon})} and(\omega)\,.$$

Next, we describe the different quantum circuits for the quantum part of Q-Dec-DDPAS (**Algorithm 4**). Let $\vec{\beta}_0, \vec{\epsilon}_0 \in T^2$. The initial state is

$$|\mathrm{ini}\rangle = \underbrace{|[n]\rangle \left|\vec{\beta}_0\right\rangle |\vec{\epsilon}_0\rangle}_{I^1} \underbrace{|0\rangle^{\otimes 4}}_{I^2} \underbrace{|0\rangle^{\otimes 2}}_{I^3} \underbrace{|0\rangle^{\otimes 4}}_{I^4} \underbrace{|0\rangle^{\otimes 2}}_{I^5} \underbrace{|0\rangle^{\otimes 2}}_{I^6},$$

where the tuples indexing the different registers are decomposed as follows:

$$\begin{aligned}
I^1 &= I^1_1 \oplus I^1_2 \oplus I^1_3 \\
I^2 &= I^2_1 \oplus I^2_2 \oplus I^2_3 \oplus I^2_4 \\
I^3 &= I^3_1 \oplus I^3_2 \\
I^4 &= I^4_1 \oplus I^4_2 \oplus I^4_3 \oplus I^4_4 \\
I^5 &= I^5_1 \oplus I^5_2 \\
I^6 &= I^6_1 \oplus I^6_2
\end{aligned}$$

The three quantum circuits that appear on the quantum part are:

$$U_{\mathrm{ini}} = (U^{I^2}_{\Omega_{\mathrm{dec}}} \otimes U^{I^4}_{\Omega_{\mathrm{dec}}}) \cdot U^{I^1 \oplus I^2 \oplus I^4}_{\Lambda_{\mathrm{dec}}}\,,$$

$$U_{\mathrm{recur1}} = U^{I^2_1 \oplus I^3_2 \oplus I^2_2 \oplus I^2_3 \oplus I^4_1 \oplus I^5_2 \oplus I^4_2 \oplus I^4_3}_{and} \left(U^{I^2_4 \oplus I^3}_{G} [U^{I^2_4}_{and}] \otimes U^{I^4_4 \oplus I^5}_{G} [U^{I^4_4}_{and}] \right)\,,$$

$$U_{\mathrm{recur}} = U^{I^2_1 \oplus I^3_2 \oplus I^2_2 \oplus I^2_3 \oplus I^4_1 \oplus I^5_2 \oplus I^4_2 \oplus I^4_3 \oplus I_6}_{G} [U_{\mathrm{recur1}}]\,.$$

We provide below the description of **Algorithm 4** with the details about the quantum circuits.

Similarly to Q-DPAS for the Additive or the Composed version, the correctness of **Lemma 5.7** can be verified by the same type of computations.

Algorithm 4: Q-Dec-DDPAS for 3-machine flowshop (low-level description)

Input: $\vec{\beta}_0, \vec{\epsilon}_0 \in T^2$, decision problem D satisfying (Dec-DPAS) and (Dec-D-DPAS)

Output: $D[[n], \vec{\beta}_0, \vec{\epsilon}_0]$ with high probability

begin classical part

  for $X \subseteq [n] : |X| = n/4$ *and* $\vec{\beta}, \vec{\epsilon} \in T^2$ **do**

    Compute the optimal value $D[X, \vec{\beta}, \vec{\epsilon}]$ and the corresponding permutation $\pi^*[X, \vec{\beta}, \vec{\epsilon}]$ by classical (Dec-DPAS);

    Store the tuple $(X, \vec{\beta}, \vec{\epsilon}, D[X, \vec{\beta}, \vec{\epsilon}], \pi^*[X, \vec{\beta}, \vec{\epsilon}])$ in the QRAM;

begin quantum part

  Prepare quantum state $|\text{ini}\rangle$;

  Apply the quantum circuit $U_{\text{recur}} U_{\text{ini}}$ to $|\text{ini}\rangle$;

  Measure register of indexes I_2^6;

Return the outcome of the measurement

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