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► **To cite this version:**

Véronique Maume-Deschamps. Estimation of multivariate generalized gamma convolutions, application to dependence structure modelling in insurance.. Conference on Industrial and Applied Mathematics - ECMI 2023, Jun 2023, Wrocław, Poland. hal-04294063

**HAL Id: hal-04294063**

**<https://hal.science/hal-04294063v1>**

Submitted on 19 Nov 2023

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# Estimation of multivariate generalized gamma convolutions, application to dependence structure modelling in insurance.

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Partially funded by SCOR

ECMI 2023



# Plan

- 1 Introduction / motivation
- 2 Multivariate generalized Gamma convolutions
- 3 Simulations

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# Introduction

A (re-)insurer observes many (positive) random variables that represent losses from several portfolios of risks.

A major topic is the [evaluation and modeling of the dependency](#) between those risks. Indeed, the properties of this dependency has a huge impact on the behavior of the total loss, and therefore on the capital management of the company.

Risks can be heavy-tailed, skewed, and are usually dependant with each other, including in the extremes.

The infinite divisibility may be a wanted property.

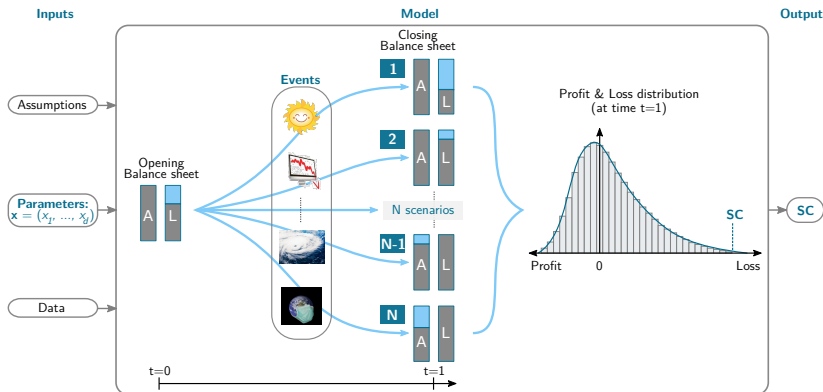
# Why infinite divisibility ?

## Definition (ID r.v. $X$ )

- 1  $X$  is  $n$ -divisible if and only if there exists  $X_1, \dots, X_n$  i.i.d. such that  $X = X_1 + \dots + X_n$
- 2  $X$  is ID iff this can be done for all  $n$ .
- 3  $X_i$  is called a *piece* of  $X$ .

- Dependencies are largely unobservables in reinsurance;
- Experts are providing information as divisibility parameters: "20% of this is driven by the same risk factors as 10% of that.." *this means comonotonic pieces accross marginals.*

# Company's Solvency



**SC (Solvency Capital):** regulatory capital calculated using a risk measure that the re-insurer must have to absorb potential losses. For example, in the Solvency II directive of the European Union, it is the  $VaR(99.5\%)$  of the P&L distribution.

# Risk factors modelisation

In the **internal modeling** point of view, the distributions of several losses  $X_1, \dots, X_d$  are supposed to be known, but the dependence structure between them must still be evaluated and taken into account to assess the variability and behavior of the total loss:

$$\sum_{i=1}^d X_i.$$

It might be achieved by a **risk factors decomposition**:  $X_i = \sum_{j=1}^k X_{i,j}$ ,

where  $X_{i,j}, X_{\ell,j}$  are either independent or comonotonic.

**Divisibility** allows to interpret the  $X_{i,j}$  as pieces of  $X_i$ .

⇒ multivariate generalized Gamma convolutions give an answer to that question.



# Plan

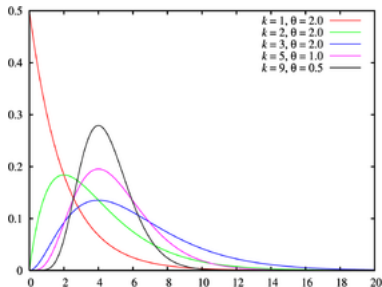
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# Recall Gamma distributions

Well known class of infinite divisible distributions.

$X \sim \mathcal{G}(k, \theta)$  if its density function is:

$$\frac{x^{k-1} e^{-x/\theta}}{\Gamma(k) \theta^k}.$$



# Multivariate Gamma convolutions classes $\mathcal{G}_{d,n}$ and $\mathcal{G}_d$

$\mathbf{x}, \mathbf{t} \in \mathbb{R}^d$ , **Recall:** Cumulant generating function:

$$K(\mathbf{t}) = \ln \left( \mathbb{E} \left( e^{\langle \mathbf{t}, \mathbf{X} \rangle} \right) \right).$$

Multivariate Thorin Classes <sup>1</sup>

$$\mathbf{X} \sim \mathcal{G}_{d,n}(\alpha, \mathbf{s}) \Leftrightarrow K(\mathbf{t}) = - \sum_{i=1}^n \alpha_i \ln (1 - \langle \mathbf{s}_i, \mathbf{t} \rangle)$$

$$\mathbf{X} \sim \mathcal{G}_d(\nu) \Leftrightarrow K(\mathbf{t}) = - \int \ln (1 - \langle \mathbf{s}, \mathbf{t} \rangle) \nu(d\mathbf{s}).$$

## Example

All gammas, log-Normals, Paretos,  $\alpha$ -stables, and sums and/or products of these are in  $\mathcal{G}_1$ .

No estimation procedures were available before our work.

<sup>1</sup> Lennart Bondesson (Nov. 2009). en. In: *Journal of Statistical Planning and Inference*

# Motivation: interpretability of $\mathcal{G}_{d,n}$ models

$\mathbf{X} \in \mathcal{G}_{d,n}$  has an additive risk-factor structure: there exists gamma random variables  $G_1, \dots, G_n$  such that:

$$\begin{pmatrix} X_1 \\ \dots \\ X_d \end{pmatrix} = \begin{pmatrix} s_{1,1} & \dots & \dots & s_{1,n} \\ \dots & \dots & \dots & \dots \\ s_{d,1} & \dots & \dots & s_{d,n} \end{pmatrix} \cdot \begin{pmatrix} G_1 \\ \dots \\ G_n \end{pmatrix}$$

- 1 For all  $i \in 1, \dots, n$ ,  $G_i \sim \mathcal{G}_{1,1}(\alpha_i, 1)$
- 2  $G_1, \dots, G_n$  are independent.

**Goal:** Estimate  $\nu$  from observations of the random vector  $\mathbf{X}$ .

**Pb:** It is a deconvolution problem, which is numerically hard (can also be interpreted as a solution-less moment problem for the Thorin measure, see MFK<sup>2</sup>).

<sup>2</sup> Justin Miles, Edward Furman, and Alexey Kuznetsov (2019). In: *Variance*

# The orthonormal Laguerre basis of $L_2(\mathbb{R}_+^d)$

Consider the following **orthonormal basis of  $L_2(\mathbb{R}_+^d)$**  called **Laguerre basis**<sup>3</sup>: let  $\mathbf{p} \in \mathbb{N}^d$ ,

$$\phi_{\mathbf{p}}(\mathbf{x}) = \prod_{i=1}^d \phi_{p_i}(x_i), \text{ with } \phi_p(x) = \sqrt{2} \sum_{k=0}^p \binom{p}{k} \frac{(-2x)^k}{k!} e^{-x}.$$

Any square integrable distribution density  $f$  can be expanded as :

$$f(\mathbf{x}) = \sum_{\mathbf{p} \in \mathbb{N}^d} a_{\mathbf{p}} \phi_{\mathbf{p}}(\mathbf{x}) \text{ where } a_{\mathbf{p}} = \int \phi_{\mathbf{p}}(\mathbf{x}) f(\mathbf{x}) \partial \mathbf{x}$$

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<sup>3</sup> Florian Dussap (2021). In: *Journal of Statistical Planning and Inference*

# Estimation

Consider a random sample  $\mathbf{X}_1, \dots, \mathbf{X}_N$  of  $\mathbf{X} \in \mathcal{G}_d(\alpha, \mathbf{s})$ . Let

$$\widehat{\mathbf{a}}_{\mathbf{p}} = \frac{1}{N} \sum_{i=1}^N \phi_{\mathbf{p}}(\mathbf{X}_i).$$

Consider the **integrated square error loss**:

$$L(\alpha, \mathbf{s}) = \sum_{\mathbf{k} \leq \mathbf{m}} (\widehat{a}_{\mathbf{k}} - a_{\mathbf{k}}(\alpha, \mathbf{s}))^2 \text{ and}$$

$$(\widehat{\alpha}, \widehat{\mathbf{s}}) = \operatorname{argmin}_{(\alpha, \mathbf{s})} L(\alpha, \mathbf{s}).$$

Remark that the theoretical Laguerre coefficients may be computed using the formula:

$$a_{\mathbf{k}} = \langle \phi_{\mathbf{k}}, f \rangle = \mathbb{E}(\phi_{\mathbf{k}}(\mathbf{X})) = \sqrt{2}^d \sum_{\mathbf{k} \leq \mathbf{p}} \binom{\mathbf{p}}{\mathbf{k}} \frac{(-2)^{|\mathbf{k}|}}{\mathbf{k}!} \mathbb{E}\left(\mathbf{X}^{\mathbf{k}} e^{-|\mathbf{X}|}\right).$$

# Consistency

We have the following consistency result <sup>4</sup> under the technical *well-behaved* assumption (which holds for univariate Gamma convolutions, independent multivariate Gamma convolution and invertible linear transformations of them e.g.).

## Theorem

If  $\mathbf{X}$  is drawn from an  $\epsilon$ -well-behaved density  $f_{\alpha, \mathbf{s}} \in \mathcal{G}_d(\alpha, \mathbf{s})$  then,

$$\|f_{(\widehat{\alpha}, \widehat{\mathbf{s}})} - f_{\alpha, \mathbf{s}}\|_2^2 \xrightarrow[N \rightarrow \infty, \mathbf{m} \rightarrow \infty]{a.s.} 0.$$

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<sup>4</sup> Oskar Laverny et al. (Jan. 2021). In: *Electronic Journal of Statistics*

## Dimensionality issues

The loss  $L(\alpha, \mathbf{s})$  is too costly to work with when  $d$  gets large. Indeed, the number of coefficients to compute is given by

$$D(m, d) = \sum_{i=1}^m \binom{i+d-1}{d-1}.$$

which is exponentially increasing in  $d$  and therefore unusable in high dimension (ok until  $d = 4$  or 5).

Improvement of the method by Oskar Laverny<sup>5</sup> by using an approximation of an integrated loss function and random projections (tested for  $d = 1000$ ).

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<sup>5</sup> Oskar Laverny (2022). In: *arXiv preprint arXiv:2203.13741*



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# Multivariate log-Normal

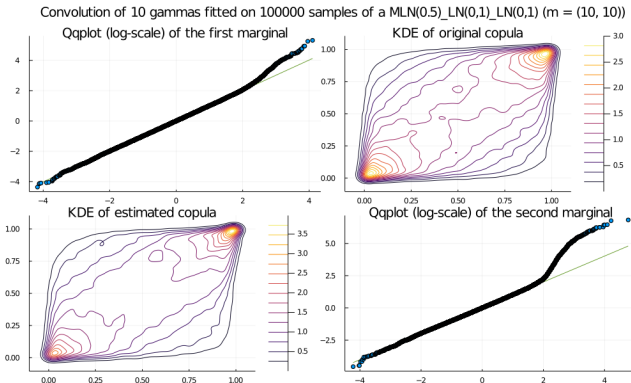


Figure: MLN case:  $n = 10$

# Multivariate log-Normal

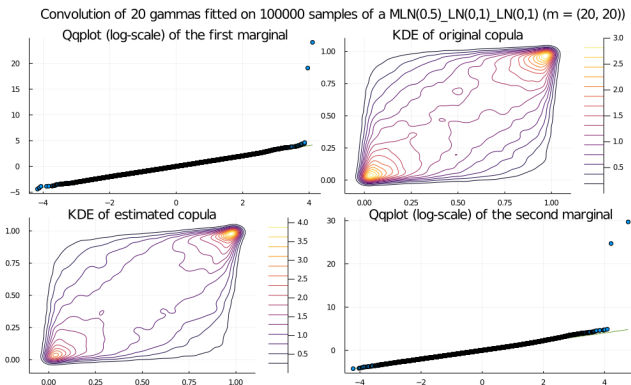


Figure: MLN case:  $n = 20$

# Tail dependency: A Clayton copula

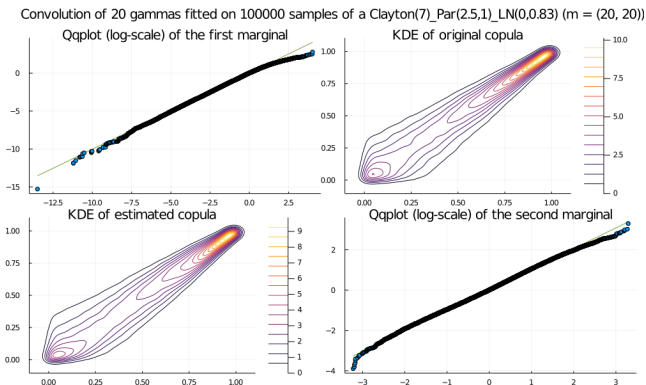


Figure: Clayton case:  $n = 20$

# Conclusions

- The univariate Thorin class is wide: LN, Pareto,  $\alpha$ -stable, (some) Weibull, ...
- The Multivariate analogue provides an asymmetrical dependence structure which include tail dependency, and can take a lot of different shapes.
- Deconvolution is a hard inverse problem, and estimation of these distributions is complicated.
- The final additive risk-factor model gives easy interpretation of parameters and easy aggregation schemes. Should be shortly **used by SCOR in some parts of their internal model.**
- Improvement to go to high dimension in Oskar' work <sup>6</sup> and **the open-source Julia package <sup>7</sup> provides the implementation.**

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<sup>6</sup> Oskar Laverny (2022). In: *arXiv preprint arXiv:2203.13741*

<sup>7</sup> Oskar Laverny (Mar. 2021). Version v0.1

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Miles, Justin, Edward Furman, and Alexey Kuznetsov (2019). "Risk Aggregation: A General Approach via the Class of Generalized Gamma Convolutions". In: *Variance*.

Thank you

Thanks for your attention.