# Estimation of multivariate generalized gamma convolutions, application to dependence structure modelling in insurance.

Véronique Maume-Deschamps

# Estimation of multivariate generalized gamma convolutions, application to dependence structure modelling in insurance.

Véronique Maume-Deschamps, joint work with Oskar Laverny, Esterina Masiello, Didier Rullière.

Partially funded by SCOR

ECMI 2023

Lyon 1

Math Institut Camille Jordan

SCOR The Art & Science of Risk

# Plan

# Plan

# Introduction

A (re-)insurer observes many (positive) random variables that represent losses from several portfolios of risks.
A major topic is the evaluation and modeling of the dependency between those risks. Indeed, the properties of this dependency has a huge impact on the behavior of the total loss, and therefore on the capital management of the company.

Risks can be heavy-tailed, skewed, and are usually dependant with each other, including in the extremes.

The infinite divisibility may be a wanted property.

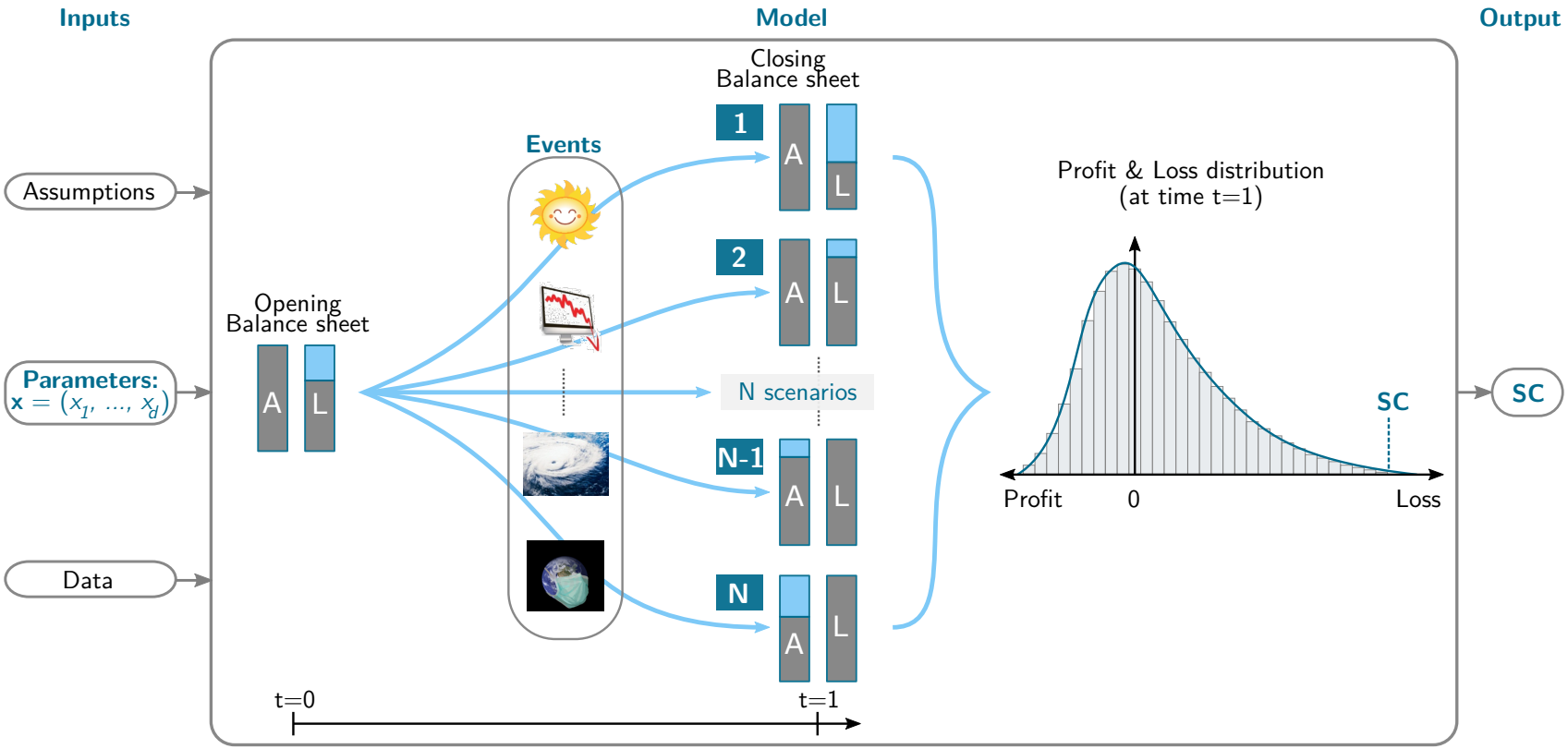
# Why infinite divisibility ?

**Definition (ID r.v. X)**

1. $X$ is $n$-divisible if and only if there exists $X_1, ...X_n$ i.i.d. such that $X = X_1 + ... + X_n$
2. $X$ is *ID* iff this can be done for all $n$.
3. $X_i$ is called a *piece* of $X$.

- Dependencies are largely unobservables in reinsurance;
- Experts are providing information as divisibility parameters: "20% of this is driven by the same risk factors as 10% of that.." *this means comonotonic pieces accross marginals.*

# Company's Solvency




**SC (Solvency Capital):** regulatory capital calculated using a risk measure that the re-insurer must have to absorb potential losses. For example, in the Solvency II directive of the European Union, it is the $VaR(99.5\%)$ of the P&L distribution.

# Risk factors modelisation

In the internal modeling point of view, the distributions of several losses $X_1, \ldots, X_d$ are supposed to be known, but the dependence structure between them must still be evaluated and taken into account to assess the variability and behavior of the total loss: $\sum_{i=1}^{d} X_i$.

It might be achieved by a risk factors decomposition: $X_i = \sum_{j=1}^{k} X_{i,j}$, where $X_{i,j}, X_{\ell,j}$ are either independent or comonotonic.
Divisibility allows to interpret the $X_{i,j}$ as pieces of $X_i$.
$\Longrightarrow$ multivariate generalized Gamma convolutions give an answer to that question.

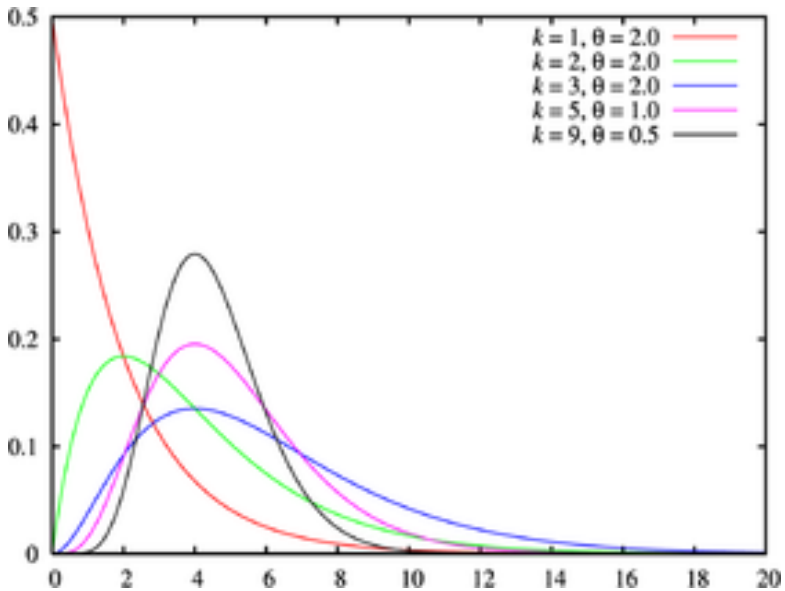
# Plan

# Recall Gamma distributions

Well known class of infinite divisible distributions.

$X \sim \mathcal{G}(k, \theta)$ if its density function is:

$$\frac{x^{k-1} e^{k/\theta}}{\Gamma(k)\theta^k}.$$



# Multivariate Gamma convolutions classes $\mathcal{G}_{d,n}$ and $\mathcal{G}_d$

$\mathbf{x}, \mathbf{t} \in \mathbb{R}^d$, **Recall:** Cumulant generating function:
$K(\mathbf{t}) = \ln\left(\mathbb{E}\left(e^{\langle \mathbf{t}, \mathbf{X}\rangle}\right)\right)$.

Multivariate Thorin Classes [1]

$$\mathbf{X} \sim \mathcal{G}_{d,n}(\alpha, \mathbf{s}) \Leftrightarrow K(\mathbf{t}) = -\sum_{i=1}^{n} \alpha_i \ln\left(1 - \langle \mathbf{s_i}, \mathbf{t}\rangle\right)$$

$$\mathbf{X} \sim \mathcal{G}_d(\nu) \Leftrightarrow K(\mathbf{t}) = -\int \ln\left(1 - \langle \mathbf{s}, \mathbf{t}\rangle\right) \nu(\partial \mathbf{s}).$$

**Example**

*All gammas, log-Normals, Paretos, $\alpha$-stables, and sums and/or products of these are in $\mathcal{G}_1$.*

No estimation procedures were available before our work.

[1] Lennart Bondesson (Nov. 2009). en. In: *Journal of Statistical Planning and Inference*

# Motivation: interpretability of $\mathcal{G}_{d,n}$ models

$\mathbf{X} \in \mathcal{G}_{d,n}$ has an additive risk-factor structure: there exists gamma random variables $G_1, \dots G_n$ such that:

$$\begin{pmatrix} X_1 \\ \dots \\ X_d \end{pmatrix} = \begin{pmatrix} s_{1,1} & \dots & \dots & s_{1,n} \\ \dots & \dots & \dots & \dots \\ s_{d,1} & \dots & \dots & s_{d,n} \end{pmatrix} \cdot \begin{pmatrix} G_1 \\ \dots \\ G_n \end{pmatrix}$$

1. For all $i \in 1, \dots, n$, $G_i \sim \mathcal{G}_{1,1}(\alpha_i, 1)$
2. $G_1, \dots, G_n$ are independent.

**Goal:** Estimate $\nu$ from observations of the random vector $\mathbf{X}$.
**Pb:** It is a deconvolution problem, which is numerically hard (can also be interpreted as a solution-less moment problem for the Thorin measure, see MFK [2]).

[2] Justin Miles, Edward Furman, and Alexey Kuznetsov (2019). In: *Variance*

# The orthonormal Laguerre basis of $L_2(\mathbb{R}_+^d)$

Consider the following orthonormal basis of $L_2(\mathbb{R}_+^d)$ called Laguerre basis [3]: let $\mathbf{p} \in \mathbb{N}^d$,

$$\phi_{\mathbf{p}}(\mathbf{x}) = \prod_{i=1}^{d} \phi_{p_i}(x_i), \text{ with } \phi_p(x) = \sqrt{2} \sum_{k=0}^{p} \binom{p}{k} \frac{(-2x)^k}{k!} e^{-x}.$$

Any square integrable distribution density $f$ can be expended as :

$$f(\mathbf{x}) = \sum_{\mathbf{p} \in \mathbb{N}^d} a_{\mathbf{p}} \phi_{\mathbf{p}}(\mathbf{x}) \text{ where } a_{\mathbf{p}} = \int \phi_{\mathbf{p}}(\mathbf{x}) f(\mathbf{x}) \partial \mathbf{x}$$

[3] Florian Dussap (2021). In: *Journal of Statistical Planning and Inference*

# Estimation

Consider a random sample $\mathbf{X}_1, \ldots, \mathbf{X}_N$ of $\mathbf{X} \in \mathcal{G}_d(\alpha, \mathbf{s})$. Let

$$\widehat{\mathbf{a_p}} = \frac{1}{N} \sum_{i=1}^{N} \phi_{\mathbf{p}}(\mathbf{X}_i).$$

Consider the integrated square error loss:
$L(\alpha, \mathbf{s}) = \sum_{\mathbf{k} \leq \mathbf{m}} \left(\widehat{a_{\mathbf{k}}} - a_{\mathbf{k}}(\alpha, \mathbf{s})\right)^2$ and

$$(\widehat{\alpha}, \widehat{\mathbf{s}}) = \text{argmin}_{(\alpha, \mathbf{s})} L(\alpha, \mathbf{s}).$$

Remark that the theoretical Laguerre coefficients may be computed using the formula:

$$a_{\mathbf{k}} = \langle \phi_{\mathbf{k}}, f \rangle = \mathbb{E}\left(\phi_{\mathbf{k}}(\mathbf{X})\right) = \sqrt{2}^d \sum_{\mathbf{k} \leq \mathbf{p}} \binom{\mathbf{p}}{\mathbf{k}} \frac{(-2)^{|\mathbf{k}|}}{\mathbf{k}!} \mathbb{E}\left(X^{\mathbf{k}} e^{-|\mathbf{X}|}\right).$$

# Consistency

We have the following consistency result [4] under the thechnical *well-behaved* asumption (which holds for univariate Gamma convolutions, independent multivariate Gamma convolution and invertible linear transformations of them e.g.).

**Theorem**

*If* $\mathbf{X}$ *is drawn from an* $\epsilon$*-well-behaved density* $f_{\alpha,\mathbf{s}} \in \mathcal{G}_d(\alpha, \mathbf{s})$ *then,*

$$\| f_{(\widehat{\alpha},\widehat{\mathbf{s}})} - f_{\alpha,\mathbf{s}} \|_2^2 \xrightarrow[\substack{N\to\infty \\ \mathbf{m}\to\infty}]{a.s} 0.$$

[4] Oskar Laverny et al. (Jan. 2021). In: *Electronic Journal of Statistics*

# Dimensionality issues

The loss $L(\alpha, \mathbf{s})$ is too costly to work with when $d$ gets large. Indeed, the number of coefficents to compute is given by

$$D(m, d) = \sum_{i=1}^{m} \binom{i+d-1}{d-1}.$$

which is exponentially increasing in $d$ and therefore unusable in high dimension (ok until $d = 4$ or 5.

Improvement of the method by Oskar Laverny [5] by using an approximation of an integrated loss function and random projections (tested for $d = 1000$).

[5] Oskar Laverny (2022). In: *arXiv preprint arXiv:2203.13741*

# Plan

# Multivariate log-Normal

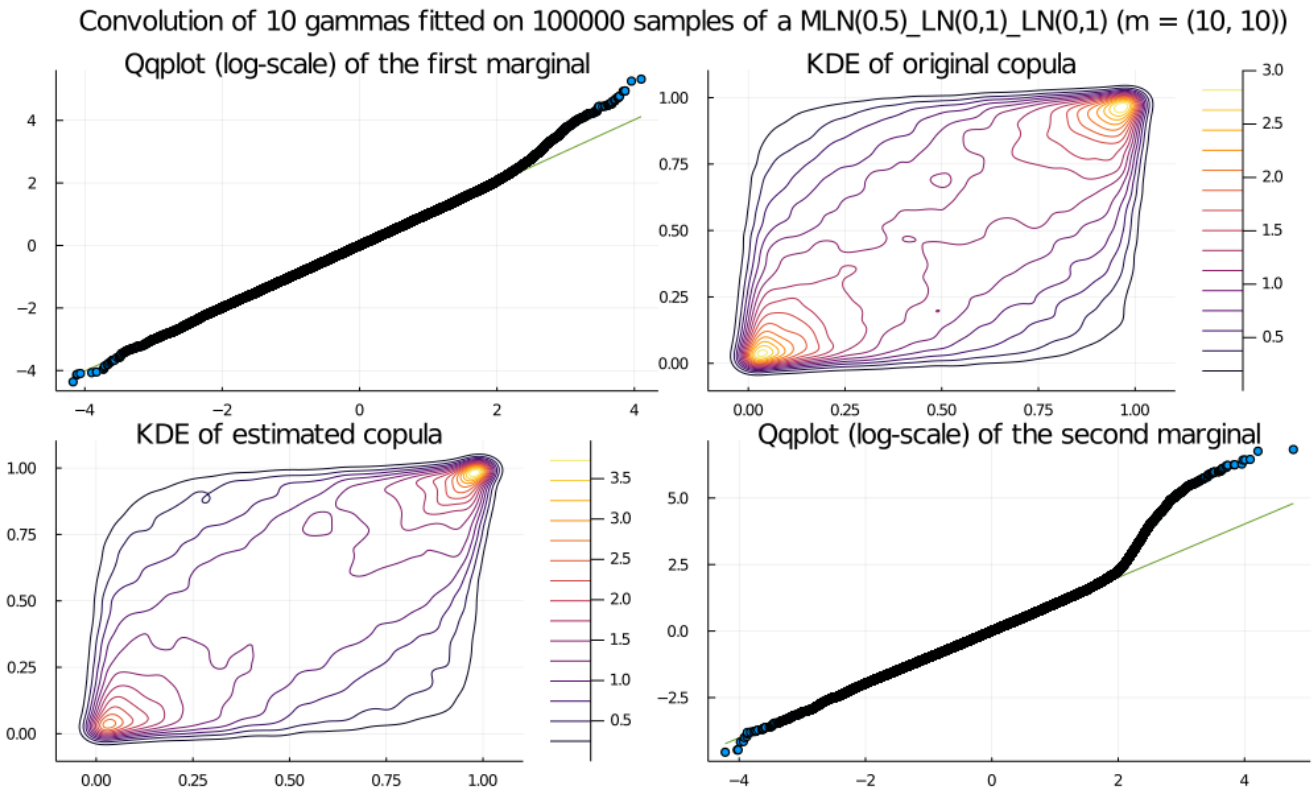



Figure: MLN case: $n = 10$

# Multivariate log-Normal

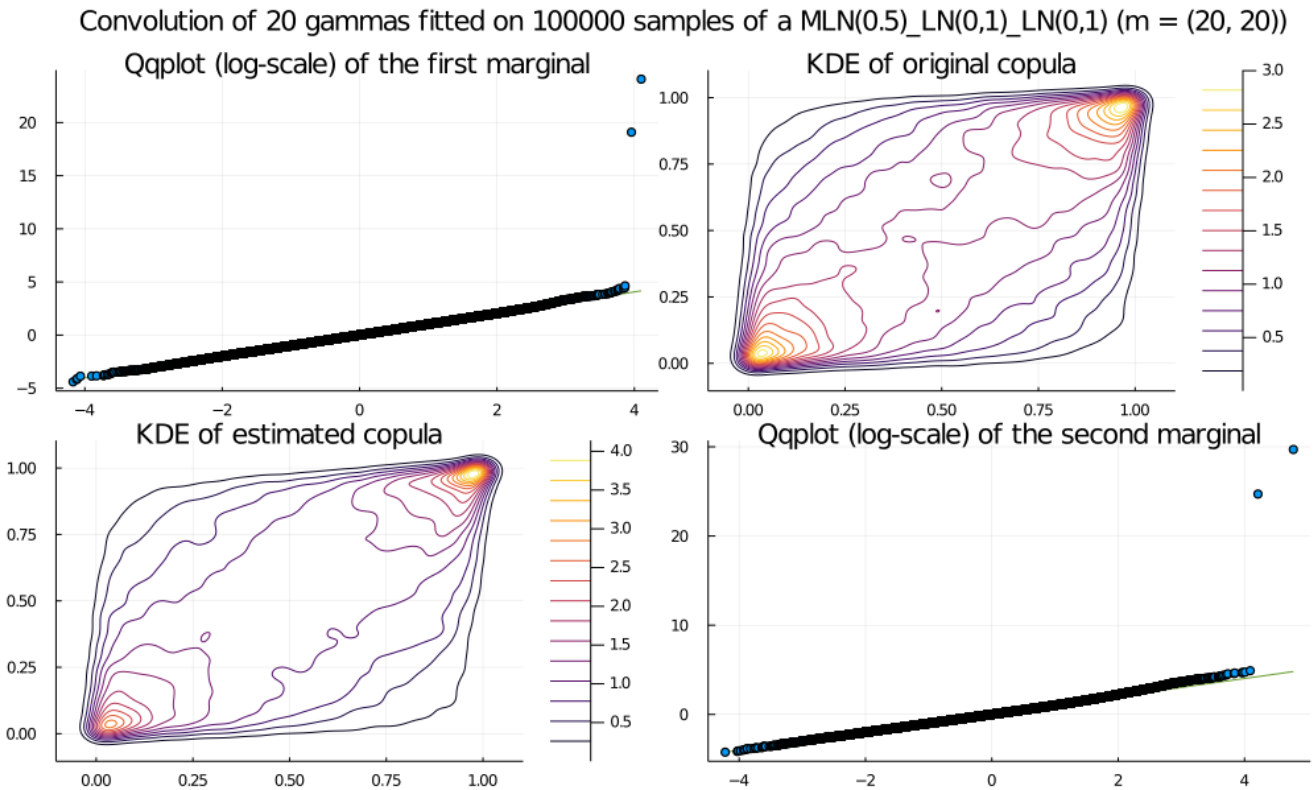



Figure: MLN case: $n = 20$

# Tail dependency: A Clayton copula

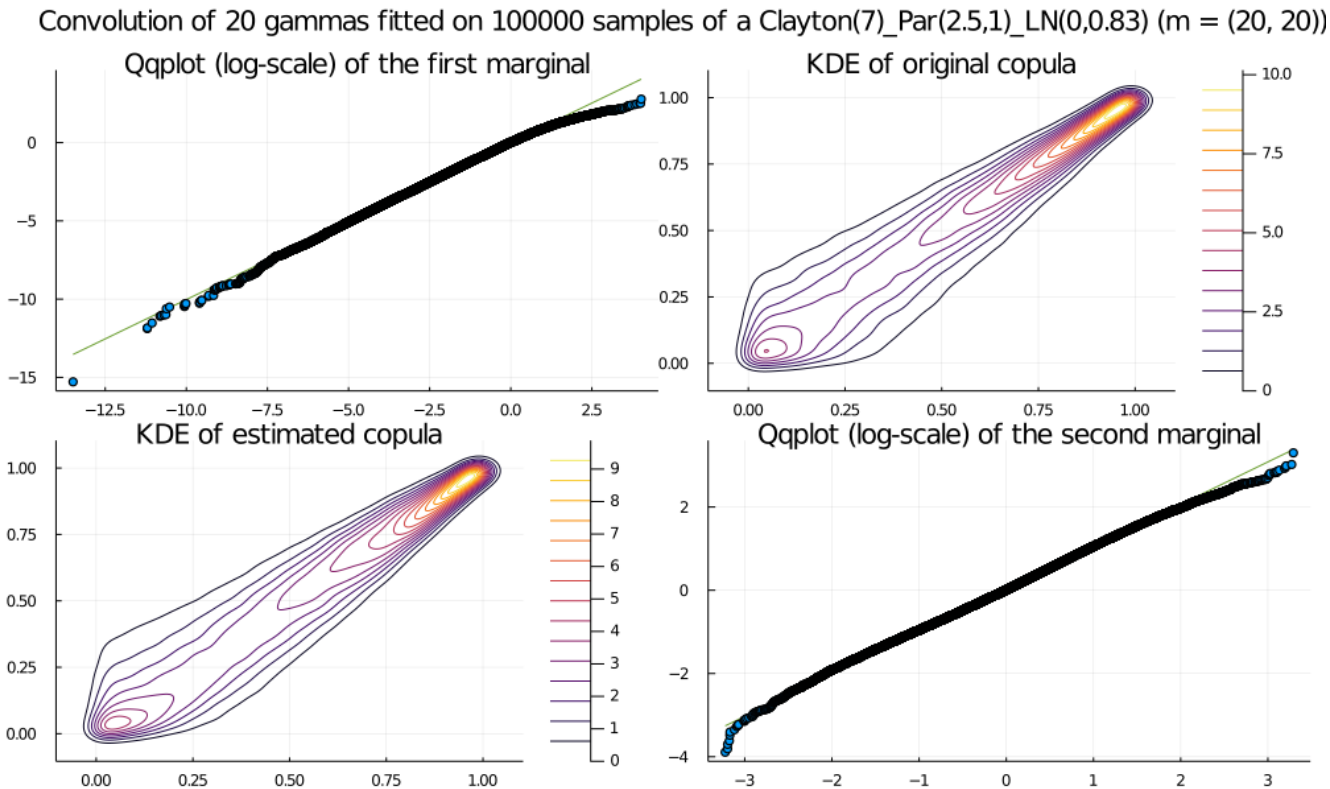



Figure: Clayton case: $n = 20$

# Conclusions

- The univariate Thorin class is wide: LN, Pareto, $\alpha$-stable, (some) Weibull, ...
- The Multivariate analogue provides an asymmetrical dependence structure which include tail dependency, and can take a lot of different shapes.
- Deconvolution is a hard inverse problem, and estimation of these distributions is complicated.
- The final additive risk-factor model gives easy interpretation of parameters and easy aggregation schemes. Should be shortly used by SCOR in some parts of their internal model.
- Improvement to go to high dimension in Oskar' work [6] and the open-source Julia package [7] provides the implementation.

[6] Oskar Laverny (2022). In: *arXiv preprint arXiv:2203.13741*

[7] Oskar Laverny (Mar. 2021). Version v0.1

# References I



Bondesson, Lennart (Nov. 2009). "On Univariate and Bivariate Generalized Gamma Convolutions". en. In: *Journal of Statistical Planning and Inference* 139.11, pp. 3759–3765.


Dussap, Florian (2021). "Anisotropic multivariate deconvolution using projection on the Laguerre basis". In: *Journal of Statistical Planning and Inference* 215, pp. 23–46.


Laverny, Oskar (Mar. 2021). *ThorinDistributions.jl*. Version v0.1.


— (2022). "Estimation of high dimensional Gamma convolutions through random projections". In: *arXiv preprint arXiv:2203.13741*.


Laverny, Oskar et al. (Jan. 2021). "Estimation of Multivariate Generalized Gamma Convolutions through Laguerre Expansions.". In: *Electronic Journal of Statistics* 15.2.


Miles, Justin, Edward Furman, and Alexey Kuznetsov (2019). "Risk Aggregation: A General Approach via the Class of Generalized Gamma Convolutions". In: *Variance*.

# Thank you

Thanks for your attention.