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# The MAXIMUM ZERO-SUM PARTITION Problem

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**Abstract.** We study the MAXIMUM ZERO-SUM PARTITION problem (or MZSP), defined as follows: given a multiset  $\mathcal{S} = \{a_1, a_2, \dots, a_n\}$  of integers  $a_i \in \mathbb{Z}^*$  such that  $\sum_{i=1}^n a_i = 0$ , find a maximum cardinality partition  $\{S_1, S_2, \dots, S_k\}$  of  $\mathcal{S}$  such that, for every  $1 \leq i \leq k$ ,  $\sum_{a_j \in S_i} a_j = 0$ . Solving MZSP is useful in genomics for computing evolutionary distances between pairs of species. Our contributions are a series of algorithmic results concerning MZSP, in terms of complexity, (in)approximability, with a particular focus on the fixed-parameter tractability of MZSP with respect to either (i) the size  $k$  of the solution, (ii) the number of negative (resp. positive) values in  $\mathcal{S}$  and (iii) the largest integer in  $\mathcal{S}$ .

## 1 Introduction

In this paper, we study the MAXIMUM ZERO-SUM PARTITION (or MZSP).

MAXIMUM ZERO-SUM PARTITION (MZSP)

**Instance :** A multiset  $\mathcal{S} = \{a_1, a_2, \dots, a_n\}$  of numbers  $a_i \in \mathbb{Z}^*$  s.t.  $\sum_{i=1}^n a_i = 0$ .

**Output :** A maximum cardinality partition  $\mathcal{S} = \{S_1, S_2, \dots, S_k\}$  of  $\mathcal{S}$  such that, for every  $1 \leq i \leq k$ ,  $\sum_{a_j \in S_i} a_j = 0$ .

This problem emerged in bioinformatics, where determining the distance between two genomes was studied [1]. The distance formula in that context depends on a parameter to be optimized, which directly relates to answering MZSP. However, the MZSP problem in itself was not central to the results in [1], and thus has not been studied *per se* in that paper. Hence, the goal of the present paper is to extensively study the MZSP problem from an algorithmic point of view.

*Definitions and Notations.* For any integer  $n$ ,  $\llbracket 1, n \rrbracket$  denotes the set of integers from 1 to  $n$ . Given a (multi)set  $S$  of integers and an integer  $p$ , we say that  $S$  *sums to  $p$*  when the sum of the elements of  $S$  is equal to  $p$ . When  $p = 0$ , we say that  $S$  is a *zero-sum (multi)set*. For any instance  $\mathcal{S}$  of MZSP, we let **neg** (resp. **pos**) denote the number of negative (resp. positive) integers in  $\mathcal{S}$  and  $m = \min\{\mathbf{neg}, \mathbf{pos}\}$ . We denote by  $n^*$  the number of distinct values in  $\mathcal{S}$ , by  $b = \sum_{i=1}^n \lceil \log_2(|a_i|) \rceil$  the

number of bits needed to encode  $\mathcal{S}$  (e.g. when  $\mathcal{S}$  is stored as a list, in which each element is binary encoded), and by  $B = \max_{i \in \llbracket 1, n \rrbracket} \{|a_i|\}$ . The cardinality of an optimal partition of  $\mathcal{S}$ , i.e. the size of the solution, is denoted by  $k$ . For example, if  $\mathcal{S} = \{-7, -7, -7, -1, -1, -1, 2, 3, 3, 4, 4, 4, 4\}$ , then  $n = 13$ ,  $\text{neg} = 6$ ,  $\text{pos} = 7$ ,  $m = 6$ ,  $n^* = 5$ ,  $B = 7$  and it can be seen that the optimal solution is  $k = 4$ : for instance,  $S_1 = S_2 = \{-7, 4, 3\}$ ,  $S_3 = \{-7, -1, 4, 4\}$  and  $S_4 = \{-1, -1, 2\}$  form a solution. UNARY MZSP denotes MZSP for which unary encoding of the input instance is used. For any positive integer  $p$ ,  $p$ -MZSP denotes the decision version of MZSP, in which, given  $p$  and a zero-sum integer multiset  $\mathcal{S}$ , we ask whether there exists a zero-sum partition  $S$  of  $\mathcal{S}$  such that  $|S| \geq p$ . We will also often use the  $O^*$  notation, as frequently done in parameterized complexity: for a given problem whose size of the input is  $n$  and parameter is  $k$ ,  $O^*(f(k))$  stands for  $O(f(k) \cdot \text{poly}(n))$ . In other words,  $O^*$  only describes the exponential part of the running time (in  $k$ ) and discards the polynomial factor (in  $n$ ).

*First observations.* Note that if we denote by  $-\mathcal{S}$  the multiset  $\mathcal{S}$  to which all signs have been switched, then  $-\mathcal{S}$  is a valid instance for MZSP, and both  $\mathcal{S}$  and  $-\mathcal{S}$  have the same optimum  $k$ . Consequently, any result that applies to  $\text{neg}$  (resp.  $\text{pos}$ ) applies to  $\text{pos}$  (resp.  $\text{neg}$ ), and thus to  $m$ . Note also that an  $m$ -size zero-sum partition of  $\mathcal{S}$  is necessarily optimal, since at least one positive (resp. negative) element of  $\mathcal{S}$  needs to be present in any  $S_i$  from the partition. In other words, we always have  $k \leq m$ . For any given  $p \in \mathbb{N}^*$ , a YES-instance for  $p$ -MZSP is also a YES-instance for  $p'$ -MZSP as long as  $p' \in \llbracket 1, p \rrbracket$ . Indeed, merging any two sets in a size- $p$  zero-sum partition of  $\mathcal{S}$  yields a size- $(p-1)$  zero-sum partition of  $\mathcal{S}$ . Finally, observe that if an integer  $a$  and its opposite  $-a$  both belong to  $\mathcal{S}$ , then there always exists an optimal solution  $S = \{S_1, S_2, \dots, S_k\}$  in which  $S_i = \{-a, a\}$  for some  $i \in \llbracket 1, n \rrbracket$ . Indeed, suppose  $-a$  and  $a$  both belong to  $\mathcal{S}$ , and observe an optimal solution  $S = \{S_1, S_2, \dots, S_k\}$  in which  $S_i \neq \{-a, a\}$  for every  $i \in \llbracket 1, k \rrbracket$ . Clearly, no  $S_i$  is such that  $\{-a, a\} \subset S_i$ , otherwise we could partition  $S_i$  into  $\{-a, a\}$  and  $S_i \setminus \{-a, a\}$ , both summing to zero, contradicting the optimality of  $S$ . Thus  $-a \in S_x$  and  $a \in S_y$  for some  $1 \leq x \neq y \leq k$ . Now consider the following partition  $S' = \{S'_1, S'_2, \dots, S'_k\}$  of  $\mathcal{S}$ : (i)  $S'_i = S_i$  for every  $i \in \llbracket 1, k \rrbracket$  such that  $i \neq x$  and  $i \neq y$ , (ii)  $S'_x = \{-a, a\}$  and (iii)  $S'_y = (S_x \cup S_y) \setminus \{-a, a\}$ . Every  $S'_i$ ,  $i \in \llbracket 1, k \rrbracket$ , sums to zero, and  $|S| = |S'| = k$ .

In this paper, we study the MZSP problem under an algorithmic viewpoint, and, in particular, discuss its computational complexity, approximability and fixed-parameter tractability with respect to  $n$ ,  $n^*$ ,  $m$ ,  $B$  and  $k$  (see Table 1).

## 2 Computational Complexity of MZSP

**Theorem 1.** *MZSP is strongly NP-complete, even if each  $S_i$  in the solution  $S$  contains at most four elements.*

*Proof.* The proof is by reduction from 3-PARTITION, which has been proved to be strongly NP-complete [3], and whose definition is as follows.

Parameter	Results
$n$	Strongly NP-complete (Thm 1) No $2^{o(n)}b^{O(1)}$ algorithm unless ETH fails (Thm 4) FPT (Thm 5) No approximation within ratio $O(n^{1-\epsilon})$ (Thm 7)
$m$	NP-complete, even if bounded (Thm 2)
$k$	NP-complete, even if bounded (Thm 2)
$m$	W[1]-hard (Thm. 8)
Unary encoded instance	XP(Cor. 2)
$k$	W[1]-hard (Cor. 1)
Unary encoded instance	XP(Thm 9)
$n^*$	No $2^{o(n^*)}b^{O(1)}$ algorithm unless ETH fails (Thm 10) XP(Thm 11)
$B$	FPT(Thm 13)
$n^*+k$	FPT(Thm 12)

**Table 1.** Summary of our main results, in relation to parameters  $n, m, k, n^*, B$  and  $b$ .

3-PARTITION

**Instance** : An integer  $C$ , a multiset  $X = \{x_1, x_2, \dots, x_{3p}\}$  of integers such that (i)  $\sum_{i=1}^{3p} x_i = C \cdot p$  and (ii)  $\forall x_i \in X, \frac{C}{4} < x_i < \frac{C}{2}$ .

**Question** : Does there exist a partition  $\{X_1, X_2, \dots, X_p\}$  of  $X$  such that, for every  $i \in \llbracket 1, p \rrbracket, \sum_{x_j \in X_i} x_j = C$  ?

Let  $I = (C, X)$  be an instance of 3-PARTITION, and let  $\mathcal{S}$  be the multiset such that  $\mathcal{S} = \{x_1, \dots, x_{3p}, -C, \dots, -C\}$ , where  $-C$  appears  $p$  times in  $\mathcal{S}$ . Note that, by definition of 3-PARTITION, the sum of all elements in  $\mathcal{S}$  is equal to zero, hence  $\mathcal{S}$  is an instance of MZSP. We now show that  $I = (C, X)$  is a YES-instance for 3-PARTITION iff MZSP (with instance  $\mathcal{S}$ ) has a solution of cardinality  $p$ .

( $\Rightarrow$ ) If  $I$  is a YES-instance for 3-PARTITION, there exists  $t_j = (x_{i_{j,1}}, x_{i_{j,2}}, x_{i_{j,3}})$ ,  $j \in \llbracket 1, p \rrbracket$ , such that  $x_{i_{j,1}} + x_{i_{j,2}} + x_{i_{j,3}} = C$ . In particular, for every  $j \in \llbracket 1, p \rrbracket$ ,  $S_j = \{x_{i_{j,1}}, x_{i_{j,2}}, x_{i_{j,3}}, -C\}$  is a size- $p$  partition of  $\mathcal{S}$  in which every  $S_j$  sums to zero. Moreover, such partition is optimal: since  $\text{neg} = p$ , no zero-sum partition of  $\mathcal{S}$  can contain strictly more than  $p$  sets.

( $\Leftarrow$ ) Suppose there exists a solution of MZSP of cardinality  $p$ , say  $S = \{S_1, S_2, \dots, S_p\}$ . Since any zero-sum subset in  $S$  contains at least one negative element, every  $S_j, j \in \llbracket 1, p \rrbracket$ , contains *exactly* one negative element, namely  $-C$ . Since, in 3-PARTITION, every  $x_i$  satisfies  $\frac{C}{4} < x_i < \frac{C}{2}$ , exactly 3 such elements are required to sum to  $C$ . Thus, any  $S_j, j \in \llbracket 1, p \rrbracket$ , contains 3 elements of the form  $x_i$ , together with  $-C$ . Since each  $S_j$  sums to zero,  $\{x_1, \dots, x_{3p}\}$  can be partitioned in triplets, each summing to  $C$ , i.e.  $I = (C, X)$  is a YES-instance for 3-PARTITION.

In unary, 3-PARTITION and MZSP are both encoded in  $\Theta(p \cdot C)$  space. As 3-PARTITION is strongly NP-complete, MZSP is also strongly NP-complete.  $\square$

As discussed above, finding a solution  $S$  to MZSP where every zero-sum set  $S_i$  in  $S$  satisfies  $|S_i| = 2$  (if such a solution exists) is easy: look for two opposite values, combine them, and iterate. Theorem 1 proves that solving MZSP when every  $S_i$  contains 4 elements is (strongly) NP-complete, which rules out parameter “maximum size of an  $S_i$ ” for FPT considerations. Note that the case  $|S_i| = 3$  (or equivalently  $|S_i| \leq 3$ ) has been shown to be strongly NP-complete in [1].

**Theorem 2.** *MZSP is NP-complete, even when  $k$  and  $m$  are bounded.*

*Proof.* We show NP-completeness of MZSP in the specific case  $m = k = 2$ , by reduction from PARTITION which is known to be NP-complete [8].

PARTITION

**Instance :** A multiset  $X = \{x_1, x_2, \dots, x_n\}$  of integers from  $\mathbb{N}^*$ .

**Question :** Does there exist a partition  $\{X_1, X_2\}$  of  $X$  s.t.  $\sum_{x_i \in X_1} x_i = \sum_{x_j \in X_2} x_j$  ?

Let  $X$  be an instance of PARTITION. We can always assume  $\sum_{i=1}^n x_i$  to be even, otherwise we have a NO-instance. Thus assume  $\sum_{i=1}^n x_i = 2N$ . The MZSP instance we build from  $X$  is  $\mathcal{S} = X \cup \{-N, -N\}$ . We show that  $X$  is a YES-instance for PARTITION iff MZSP yields a size-2 zero-sum partition for  $\mathcal{S}$ .

( $\Rightarrow$ ) Suppose there exists a partition  $\{X_1, X_2\}$  of  $X$  such that  $\sum_{x_i \in X_1} x_i = \sum_{x_j \in X_2} x_j$ . We thus have  $\sum_{x_i \in X_1} x_i = \sum_{x_j \in X_2} x_j = N$ , and  $\{X_1 \cup \{-N\}, X_2 \cup \{-N\}\}$  is a zero-sum partition of  $\mathcal{S}$ . Moreover, this partition is optimal since  $m = \text{neg} = 2$  and  $k \leq m$ .

( $\Leftarrow$ ) Suppose there exists a zero-sum partition of cardinality  $k = 2$  of  $\mathcal{S}$ , say  $\mathcal{S} = \{S_1, S_2\}$ . Because  $\text{neg} = 2$ , we know that  $S_1$  (resp.  $S_2$ ) contains exactly one negative integer; thus, both in  $S_1$  and  $S_2$ , this integer is  $-N$ . Assume  $S_1 = X_1 \cup \{-N\}$  and  $S_2 = X_2 \cup \{-N\}$ . In that case,  $\{X_1, X_2\}$  is a partition of  $X$ , and because both  $S_1$  and  $S_2$  sum to zero, we have  $\sum_{x_i \in X_1} x_i = \sum_{x_j \in X_2} x_j = N$ . Thus  $X$  is a YES-instance for PARTITION.  $\square$

Although we just showed that MZSP is strongly NP-complete in general, and remains NP-complete when  $k$  is bounded, we show there exists a pseudopolynomial algorithm that solves MZSP in the case  $k = 2$ .

**Theorem 3.** *2-MZSP can be solved in pseudopolynomial time.*

The following result gives two lower bounds on the time to solve MZSP, both based on the Exponential-Time Hypothesis (ETH, see e.g. [4] for a definition). Recall that  $b$  is the size of the input  $\mathcal{S}$ , assuming it is binary encoded.

**Theorem 4.** *Unless ETH fails, MZSP cannot be solved (i) in  $2^{o(n)} \cdot b^{O(1)}$  or (ii) in  $2^{o(\sqrt{b})}$ .*

We now show that the above ETH bound based on  $n$  is essentially tight.

**Theorem 5.** *MZSP is solvable in  $O^*(2^n)$ .*

*Proof.* We solve MZSP by dynamic programming. Given an instance  $\mathcal{S}$  of MZSP, we create a dynamic programming 1-dimensional table  $T$  indexed by the subsets of  $\mathcal{S}$ . We set  $T[\emptyset]$  to 0. Then, for increasing  $i \in \llbracket 1, n \rrbracket$ , and for every size- $i$  subset  $P_i$  of  $\mathcal{S}$ , we fill  $T[P_i]$  using the following rule:

$$T[P_i] = \begin{cases} \max_{a \in P_i} \{T[P_i \setminus \{a\}]\} & \text{if } P_i \text{ does not sum to } 0 \\ \max_{a \in P_i} \{T[P_i \setminus \{a\}]\} + 1 & \text{otherwise} \end{cases}$$

The optimal value  $k$  for MZSP is then found in  $T[\mathcal{S}]$ , and an optimal zero-sum partition of  $\mathcal{S}$  can be found by backtracking from that value. The space and time complexity of the above algorithm is  $O^*(2^n)$ , since it takes polynomial time to fill any of the  $2^n$  elements in  $T$ . It remains to show correctness. For this, for any subset  $P$  of  $\mathcal{S}$ , we denote by  $k_P$  the cardinality of a maximum zero-sum subpartition of  $P$ , where the term *subpartition* describes a partition of a subset of  $P$ . Our goal is to show that for any  $P$ ,  $T[P] = k_P$ . This is done by induction on  $i = |P|$ . When  $i = 0$ , this trivially holds as  $T[\emptyset]$  is set to 0. Suppose now that for some  $i \in \llbracket 0, n-1 \rrbracket$ , any  $P$  such that  $|P| = i$  satisfies  $T[P] = k_P$ . Let us now observe a set  $P$  of cardinality  $i+1$ . If  $P$  does not sum to zero, let us consider a maximum cardinality zero-sum subpartition of  $P$ , say  $(A_1, \dots, A_{k_P})$ . Since  $P$  does not sum to zero, there exists  $a \in P$  such that  $a \notin \bigcup_{i=1}^{k_P} A_i$ . Thus  $(A_1, \dots, A_{k_P})$  is a maximum cardinality zero-sum subpartition of  $P \setminus \{a\}$ , otherwise this would contradict the cardinality maximality of  $(A_1, \dots, A_{k_P})$ . Thus  $k_{P \setminus \{a\}} = k_P$ . Since  $T[P \setminus \{a\}] = k_{P \setminus \{a\}}$ , we have  $T[P] \geq k_P$  by definition of  $T[P]$ . Conversely, if  $a \in P$  and  $(A_1, \dots, A_{k_{P \setminus \{a\}}})$  is a zero-sum subpartition of  $P \setminus \{a\}$ , then  $(A_1, \dots, A_{k_{P \setminus \{a\}}})$  is a zero-sum subpartition of  $P$ . Thus  $k_{P \setminus \{a\}} \leq k_P$ , which means  $T[P \setminus \{a\}] \leq k_P$  and thus implies  $T[P] \leq k_P$ . Altogether, we have  $T[P] = k_P$ . Now if  $P$  sums to zero, let  $(A_1, \dots, A_{k_P})$  be a maximum cardinality zero-sum partition of  $P$ . We thus have  $P = \bigcup_{j=1}^{k_P} A_j$ . Let  $a \in A_{k_P}$  (note that  $a$  exists, since  $A_{k_P}$  is non-empty). Thus  $(A_1, \dots, A_{k_P-1})$  is a maximum zero-sum subpartition of  $P \setminus \{a\}$ , and hence  $k_P = k_{P \setminus \{a\}} + 1$ , which implies  $T[P] \geq k_P$ . Conversely, if  $a \in P$ , then  $a \in A_j$  for a given  $j \in \llbracket 1, k_P \rrbracket$ . Using similar arguments as previously, we can prove that  $k_P = k_{P \setminus \{a\}} + 1$ , and thus  $T[P] \leq k_P$ . Altogether, we have  $T[P] = k_P$ . We conclude that  $T[P] = k_P$  for any  $P \subseteq \mathcal{S}$ . In particular,  $T[\mathcal{S}]$  contains a maximum cardinality zero-sum partition of  $\mathcal{S}$ . By backtracking in  $T$ , the sought partition can be found in polynomial time, which solves MZSP.  $\square$

The previous theorem is based on the fact that the number of distinct subsets in  $\mathcal{S}$  is upper bounded by  $O(2^n)$ . It is also possible to upper bound this number by a function of  $b$ , the number of bits needed to binary encode  $\mathcal{S}$ .

**Theorem 6.** MZSP is solvable in  $2^{O(\frac{b}{\log b})}$ .

*Proof.* Let us partition  $\mathcal{S}$  into  $S_p = \{a_i \in \mathcal{S} \text{ s.t. } |a_i| \leq \sqrt{b}\}$  and  $S_q = \{a_i \in \mathcal{S} \text{ s.t. } |a_i| > \sqrt{b}\}$ . Let us also denote, for any multiset  $E$ , by  $\mathcal{P}(E)$  the set of subsets of  $E$  (e.g.  $\mathcal{P}(\{2, 3, 3\}) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, \{3, 3\}, \{2, 3, 3\}\}$ ). In that case, we have  $|\mathcal{P}(S_p)| \leq (b+1)^{2\sqrt{b}}$ : indeed, by definition any element  $a \in S_p$

satisfies  $|a| \leq \sqrt{b}$ . Moreover,  $a$  appears at most  $b$  times in  $S_p$ , since any  $a$  needs at least one bit to be encoded, while  $b$  bits are enough to encode  $\mathcal{S}$ . Thus  $S_p$  contains at most  $2\sqrt{b}$  different numbers, each of them appearing at most  $b$  times in  $S_p$ . Hence  $|\mathcal{P}(S_p)| \leq (b+1)^{2\sqrt{b}} \leq 2^{2\sqrt{b}\log(b+1)} \leq 2^{2\frac{b}{\log b}}$ . On the other hand,  $S_q$  contains elements of size at least  $\sqrt{b}$ . Then  $S_q$  cannot be of cardinality greater than  $2\frac{b}{\log b}$ , otherwise encoding  $S_q$  would require more than  $b$  bits; thus  $|\mathcal{P}(S_q)| \leq 2^{2\frac{b}{\log b}}$ . Since  $S_p$  and  $S_q$  form a partition of  $\mathcal{S}$ , we have that  $|\mathcal{P}(\mathcal{S})| = |\mathcal{P}(S_p)| \cdot |\mathcal{P}(S_q)| \leq 2^{4\frac{b}{\log b}}$ . Since the dynamic programming algorithm from proof of Theorem 5 solves MZSP and has running time in  $O^*(|\mathcal{P}(\mathcal{S})|)$ , we conclude that MZSP can be solved in  $2^{O(\frac{b}{\log b})}$ , which proves the theorem.  $\square$

We now end this section by turning our attention to the inapproximability of MZSP, in Theorem 7 below.

**Theorem 7.** *Unless  $P=NP$ , MZSP cannot be approximated within ratio  $O(n^{1-\epsilon})$  for any  $\epsilon > 0$ .*

*Proof.* As for Theorem 2, we prove the result by reduction from PARTITION, which is known to be NP-complete [8]. Let  $X = \{x_1, x_2, \dots, x_\ell\}$  be an instance of PARTITION, and let  $\sum_{i=1}^{\ell} x_i = 2N$  with  $N \geq 1$ . We can indeed assume  $\sum_{i=1}^{\ell} x_i$  to be non-zero and even, otherwise the problem is trivially answered. The reduction is as follows: let  $q \geq 1$  be any integer, and let us recursively build a set  $\{p_0, p_1, \dots, p_q\}$  of integers. More precisely, we set  $p_0 = 1$ , and  $p_i = (2iN + 1)p_{i-1}$  for any  $i \in \llbracket 1, q \rrbracket$ . We note that for any  $0 \leq j < i \leq q$ ,  $p_j$  divides  $p_i$ . Based on  $X$  and on the values  $p_0, p_1, \dots, p_q$ , we now construct the multiset  $\mathcal{S} = \bigcup_{i=0}^q \{p_i X, -N p_i, -N \sum_{i=0}^q p_i\}$  where, for any  $i \in \llbracket 0, q \rrbracket$ ,  $p_i X$  denotes the values obtained by multiplying each element of  $X$  by  $p_i$ . It can be seen that  $\mathcal{S}$  sums to zero, and is thus a valid instance of MZSP. It can also be seen that the above reduction takes polynomial-time, as long as  $q$  remains polynomial in the input size of PARTITION. Intuitively, the above reduction consists in “copying”, a certain amount of times, an (expanded) instance  $X$  of PARTITION, so that the solution size of MZSP on  $\mathcal{S}$  increases, while maintaining the property that the different “expanded copies” of  $X$  in  $\mathcal{S}$  do not mutually interact.

Let us now prove correctness of our reduction, by showing the following:  
 (i)  $X$  is a YES-instance for PARTITION iff (ii) MZSP for  $\mathcal{S}$  yields a partition of cardinality  $q+2$  iff (iii) MZSP for  $\mathcal{S}$  yields a partition of cardinality 2.

$((i) \Rightarrow (ii))$  Suppose  $X$  is a YES-instance for PARTITION. Then there exists  $P \subset X$  such that  $\sum_{a \in P} a = N$ . By construction, for every  $i \in \llbracket 0, q \rrbracket$ ,  $\sum_{a \in P} p_i a = N p_i$ . Hence, for every negative number  $-s$  in  $\mathcal{S}$  ( $s$  being either  $-N p_i$  for some  $i \in \llbracket 0, q \rrbracket$ , or  $-N \sum_{i=0}^q p_i$ ), it is possible to find a subset of  $\mathcal{S}$  summing to  $N p_i$ , and moreover any pair of such sets is mutually disjoint. Hence  $\mathcal{S}$  can be partitioned into zero-sum subsets, and the cardinality of such a partition is  $q+2$ .

$((ii) \Rightarrow (iii))$  If MZSP for  $\mathcal{S}$  yields a partition of cardinality  $q+2$ , and since  $q+2 \geq 2$  then, by merging any  $q+1$  sets in this partition, we obtain a size-2 zero-sum partition.

((iii)  $\Rightarrow$  (i)) Suppose there exists a zero-sum partition of MZSP for  $\mathcal{S}$ , of cardinality 2. In that case, there exists a non-empty zero-sum (multi)set  $P \subset \mathcal{S}$  that does not contain the negative integer  $-N \sum_{i=0}^q p_i$ . Let us denote  $i_0$  the smallest index  $i \in \llbracket 0, q \rrbracket$  such that  $-Np_i$  belongs to  $P$ . Note that for all  $i \in \llbracket 0, i_0 - 1 \rrbracket$ , we have  $p_i X \cap P = \emptyset$ : indeed, suppose by contradiction that this is not the case, and let  $A$  be the sum of the elements of  $\left(\bigcup_{i=0}^{i_0-1} p_i X\right) \cap P$ . Then we have  $A \leq (1 + p_1 + \dots + p_{i_0-1}) \cdot 2N$ , hence  $A \leq i_0 \cdot 2N \cdot p_{i_0-1}$ , which yields  $A < p_{i_0}$ . In particular,  $p_{i_0}$  does not divide  $A$ , since  $A \neq 0$ . As  $p_{i_0}$  divides every other element of  $P$ , we conclude that  $A = 0$ , which is the sought contradiction.

Now, let us consider  $P' = P \bmod p_{i_0+1}$ . From the above, the only elements from  $P$  that induce non zero elements in  $P'$  are the elements of  $p_{i_0} X \cap P$ , together with  $-Np_{i_0}$ . We thus conclude there exists a (multi)set  $K \subset X$  such that  $\sum_{a \in K} p_{i_0} a \equiv Np_{i_0} \pmod{p_{i_0+1}}$ . Since  $p_{i_0+1} > 2Np_{i_0}$ ,  $\sum_{a \in K} p_{i_0} a = Np_{i_0}$ , and thus  $\sum_{a \in K} a = N$ . In other words, we have a YES-instance for PARTITION.

Now we have proved correctness of our reduction, let us turn to proving our inapproximability result. Let  $\epsilon$  be any strictly positive value, and suppose that there exists an approximation algorithm  $\mathcal{A}$  for MZSP, of ratio  $\rho = O(n^{1-\epsilon})$  with  $n = |\mathcal{S}|$ . Take now an instance  $X$  of PARTITION, and recall that  $\ell = |X|$ . Let  $C$  be a constant such that  $\rho \leq Cn^{1-\epsilon}$  for sufficiently large  $n$ . We let  $q = \max(C^{\frac{1}{1-\epsilon}} - 1, (C^{\frac{1}{1-\epsilon}}(\ell+1) + 1)^{\frac{1}{\epsilon}-1} - 1)$ , and we proceed with the above mentioned reduction by building the MZSP instance  $\mathcal{S}$  based on  $X$  and on parameter  $q$ . We have that  $n = |\mathcal{S}| = (\ell + 1)(q + 1) + 1$ .

Then,  $\frac{q+1}{C^{\frac{1}{1-\epsilon}}} \geq 1$  and  $(q+1)^{\frac{1}{1-\epsilon}-1} - C^{\frac{1}{1-\epsilon}}(\ell+1) \geq 1$  which yields  $\frac{q+1}{C^{\frac{1}{1-\epsilon}}} \cdot ((q+1)^{\frac{1}{1-\epsilon}-1} - C^{\frac{1}{1-\epsilon}}(\ell+1)) \geq 1$  and  $C((\ell+1)(q+1)+1)^{1-\epsilon} \leq q+1$ . We thus conclude that  $Cn^{1-\epsilon} \leq q+1$ . We now apply  $\mathcal{A}$  on  $\mathcal{S}$ , and solve it polynomially within factor  $\rho \leq Cn^{1-\epsilon}$ . Thus we obtain  $\rho \leq q+1$ , hence  $\rho < q+2$ . However, we know from the above that if we have a YES-instance for PARTITION, then there exists a zero-sum partition of  $\mathcal{S}$  of cardinality  $q+2$ . In that case, the solution provided by the approximation algorithm  $\mathcal{A}$  is a zero-sum partition of  $\mathcal{S}$  of cardinality  $c \geq \frac{q+2}{\rho} > 1$ . Conversely, if  $\mathcal{A}$  provides a zero-sum partition of cardinality  $c > 1$ , then such zero-sum partition shows that  $X$  is a YES-instance for PARTITION.

Altogether, if there exists an approximation algorithm  $\mathcal{A}$  for MZSP of ratio  $\rho = O(n^{1-\epsilon})$ , it is possible to polynomially solve PARTITION: a contradiction, unless  $P=NP$ , to the fact that PARTITION is NP-complete [8].  $\square$

### 3 Parameterized Complexity of MZSP

*Parameters  $k$  and  $m$ .* We first consider fixed-parameterized complexity of MZSP with respect to the size  $k$  of the solution. On the way, we will also discuss parameter  $m = \min\{\text{neg}, \text{pos}\}$ , as we always have  $m \geq k$ . By Theorem 2, we know that, unless  $P=NP$ , MZSP is not FPT with respect to parameter  $k$  (resp.  $m$ ), since MZSP is NP-complete even in the case where both these values are constant. The following theorem and corollary show W[1]-hardness of UNARY MZSP with respect to the same parameters.



**Theorem 8.** UNARY MZSP parameterized by  $m$  is  $W[1]$ -hard.

Theorem 8 implies the following corollary, as we always have  $m \geq k$ .

**Corollary 1.** UNARY MZSP parameterized by  $k$  is  $W[1]$ -hard.

*Proof.* (of Theorem 8) Let us assume that integers are encoded in unary. We reduce from UNARY BIN-PACKING, which is known to be  $W[1]$ -hard with respect to parameter “size of the solution” [6], to UNARY MZSP. We first recall the definition of BIN-PACKING (presented here in its decision version):

BIN-PACKING

**Instance :** a multiset of strictly positive integers  $P = \{w_1, \dots, w_n\}$ , an integer  $W$ , an integer  $t$ .

**Question :** Does there exist a partition  $\{J_1, \dots, J_t\}$  of  $P$  such that  $\sum_{w_j \in J_i} w_j \leq W$  for every  $i \in \llbracket 1, t \rrbracket$  ?

As mentioned above, UNARY BIN-PACKING is BIN-PACKING in which all integers are assumed to be encoded in unary; besides, UNARY BIN-PACKING, parameterized by the number  $t$  of bins, is known to be  $W[1]$ -hard [6]. Let  $I = (P, W, t)$  be an instance of UNARY BIN-PACKING. Moreover, assume  $\sum_{i=1}^n w_i = tW$ , since UNARY BIN-PACKING remains  $W[1]$ -hard parameterized by the number  $t$  of bins under this condition [6]. Let us now construct the following instance of MZSP:  $\mathcal{S} = \{w_1, \dots, w_n, \underbrace{-W, \dots, -W}_t\}$ . Note that we have  $m = t$ . We now show that

MZSP admits a  $t$ -size zero-sum partition iff  $I = (P, W, t)$  is a YES-instance for UNARY BIN-PACKING.

( $\Rightarrow$ ) Suppose  $(P, W, t)$  is a YES-instance for UNARY BIN-PACKING. Thus there exists a partition  $\{J_1, \dots, J_t\}$  of  $P$  such that  $\sum_{w_j \in J_i} w_j \leq W$  for every  $i \in \llbracket 1, t \rrbracket$ . However, since we assume  $\sum_{i=1}^n w_i = tW$ , we conclude that every  $J_i$ ,  $i \in \llbracket 1, t \rrbracket$ , is such that  $\sum_{w_j \in J_i} w_j = W$ . Hence,  $\{A_1, A_2, \dots, A_t\}$ , where  $A_i = J_i \cup \{-W\}$  for every  $i \in \llbracket 1, t \rrbracket$ , is a  $t$ -size zero-sum partition of  $\mathcal{S}$ .

( $\Leftarrow$ ) Conversely, suppose there exists a  $t$ -size zero-sum partition  $\{A_1, A_2, \dots, A_t\}$  of  $\mathcal{S}$ . Since  $\mathcal{S}$  contains exactly  $t$  negative numbers, this implies that every  $A_i$ ,  $i \in \llbracket 1, t \rrbracket$ , contains exactly one occurrence of  $-W$ . Thus  $\{J_1, J_2, \dots, J_t\}$ , where  $J_i = A_i \setminus \{-W\}$  for every  $i \in \llbracket 1, t \rrbracket$ , is a partition of  $P$ . Moreover, since each  $A_i$  sums to zero, we know that each  $J_i$  sums to  $W$ , which ensures that  $(P, W, t)$  is a YES-instance for UNARY BIN-PACKING.

The above reduction is a valid parameterized reduction, since parameter  $t$  for UNARY BIN-PACKING is strictly equal to parameter  $m$  for UNARY MZSP. Moreover, the instance  $\mathcal{S}$  of UNARY MZSP that we built satisfies  $k = m$ .  $\square$

Theorem 8 suggests that, even encoded in unary, MZSP admits no FPT algorithm parameterized by  $k$ . What we can show, in the following theorem, is that MZSP encoded in unary is in XP when parameterized by  $k$ .

**Theorem 9.** UNARY MZSP is in XP parameterized by  $k$ .

Since we always have  $m \geq k$ , Theorem 9 implies the following corollary.

**Corollary 2.** UNARY MZSP is in XP when parameterized by  $m$ .

*Parameter number of distinct values  $n^*$ .* We now discuss parameter  $n^*$ , for which we first provide a complexity lower bound based on ETH. Recall that  $b$  is the (binary encoded) size of the input instance  $\mathcal{S}$ .

**Theorem 10.** *Unless ETH fails, MZSP cannot be solved in  $2^{o(n^*)}b^{O(1)}$ .*

*Proof.* The proof is based on a combination of (i) a reduction from 3-SAT to SUBSET-SUM presented in [7] and inspired from [10] and (ii) the reduction from SUBSET-SUM to MZSP from proof of Theorem 4. More precisely, starting from any instance of 3-SAT with  $n$  variables and  $m$  clauses, an instance of SUBSET-SUM containing  $n' = 2n + 2m$  integers, among which  $2n + m$  are pairwise distinct, is constructed. Moreover, in proof of Theorem 4, the instance  $\mathcal{S}$  of MZSP built from SUBSET-SUM contains  $n'' = n' + 2 = 2n + 2m + 2$  integers, among which  $n^* = 2n + m + 2$  are pairwise distinct. Under ETH, 3-SAT cannot be solved in  $2^{o(n)}$ . If we combine this information with the sparsification method [5] (which allows to consider only 3-SAT instances for which  $m = O(n)$ ) and the above argument, we conclude that, under ETH, MZSP cannot be solved in  $2^{o(n^*)}b^{O(1)}$ .  $\square$

Concerning parameterized complexity with respect to  $n^*$ , we suspect MZSP to be W[1]-hard parameterized by  $n^*$ , but the question remains open. Meanwhile, we are able to prove (see Theorem 11) that the problem is in XP with respect to  $n^*$ . The rationale for this result is that the multisets that constitute any maximum zero-sum partition of  $\mathcal{S}$  are few, and that we can efficiently compute them. In order to prove Theorem 11, we need to introduce several definitions, and first prove two propositions (Propositions 1 and 2).

Let us suppose that  $\mathcal{S}$  is a multiset containing  $n^*$  distinct values, denoted  $a_1, \dots, a_{n^*}$ . We introduce several notions: given any multiset  $\mathcal{M}$  built from  $a_1, \dots, a_{n^*}$ , we call *multiplicity multiset* of  $\mathcal{M}$  the multiset  $\{u_1, u_2, \dots, u_{n^*}\}$  representing the multiplicities of each  $a_i$  in  $\mathcal{M}$ : more precisely, for any  $i \in \llbracket 1, n^* \rrbracket$ ,  $u_i \in \mathbb{N}$  is the number of times  $a_i$  appears in  $\mathcal{M}$ . With this notation, we can define a partial order  $\leq$  on multiplicity multisets as follows: let  $u = \{u_1, u_2, \dots, u_{n^*}\}$  and  $v = \{v_1, v_2, \dots, v_{n^*}\}$  be two multiplicity multisets; we write  $u \leq v$  whenever  $u_i \leq v_i$  for every  $i \in \llbracket 1, n^* \rrbracket$ . Now let  $s = \{s_1, s_2, \dots, s_{n^*}\}$  be the multiplicity multiset of  $\mathcal{S}$ . We first define two sets, respectively named  $K$  and  $D$ :  $K$  is the set of irreducible multiplicity multisets of  $\mathcal{S}$  leading to zero-sum subsets of  $\mathcal{S}$ , and  $D$  is the set of all zero-sum subsets of  $\mathcal{S}$ . In the following, for simplicity we write  $u = 0$  for any vector  $u$  whenever all its coordinates are equal to 0. Formally,  $K$  and  $D$  are defined as follows:

$$K = \left\{ u \in \mathbb{N}^{n^*} \mid \sum_{i=1}^{n^*} u_i a_i = 0, u \neq 0 \text{ and } \forall v \in \mathbb{N}^{n^*}, v \leq u \text{ and } \sum_{i=1}^{n^*} v_i a_i = 0 \Rightarrow v = 0 \text{ or } v = u \right\} \text{ and}$$

$$D = \left\{ u \in \mathbb{N}^{n^*} \mid u \leq s, u \neq 0 \text{ and } \sum_{i=1}^{n^*} u_i a_i = 0 \right\}.$$

Any maximum zero-sum partition of  $\mathcal{S}$  is induced by elements of  $K \cap D$  only. We define a third set  $Z$  as follows:  $Z = \left\{ u \in \mathbb{N}^{n^*} \mid u \neq 0 \text{ and } \sum_{i=1}^{n^*} u_i a_i = 0 \right\}$ . Note that both  $K$  and  $D$  are included in  $Z$ . We are now interested in two properties, related to the above definitions. We begin with Proposition 1.

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**Algorithm 1** XP algorithm for solving MZSP, parameterized by  $n^*$ 


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1: Compute  $D$ 
2: Initialize  $T$ 
3: for every  $i \in \llbracket 1, n \rrbracket$  do
4:   for every  $u \in D$  do
5:     for every  $v \in D$  do
6:       if  $v + u \in D$  then
7:          $T[v + u] = \max(T[v] + 1, T[v + u])$ 
8:       end if
9:     end for
10:  end for
11: end for
12: return  $T[s]$ 

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**Proposition 1.** *Let  $U$  be a subset of  $Z$  such that  $K \cap D \subseteq U$ . There exists a size- $p$  zero-sum partition of  $\mathcal{S}$  iff there exists  $\ell \geq p$  elements of  $U$ , say  $u^1, u^2, \dots, u^\ell$  such that  $\sum_{i=1}^{\ell} u^i = s$ .*

For solving  $p$ -MZSP, it thus suffices to compute  $K \cap D$ , and to test whether it is possible to reach  $s$ , using  $p$  or more elements of  $K \cap D$ . Before that, we discuss the maximum cardinalities of  $K$  and  $D$  (see Proposition 2), which will be useful to evaluate the time complexity to generate these sets.

**Proposition 2.**  $D \subseteq \llbracket 0, n \rrbracket^{n^*}$  and  $K \subseteq \llbracket 0, n^*B - 1 \rrbracket^{n^*}$ .

In order to compute  $D$  (resp.  $K$ ), it thus suffices to generate each element of  $\llbracket 0, n \rrbracket^{n^*}$  (resp.  $\llbracket 0, n^*B - 1 \rrbracket^{n^*}$ ), and check for each of them whether it belongs to  $D$  (resp.  $K$ ). For each element of  $\llbracket 0, n \rrbracket^{n^*}$ , checking its membership to  $D$  can be achieved in  $O(n^2 \log B)$ , thus  $D$  can be computed in  $O(n^{n^*+2} \log(B))$ . Concerning  $K$ , testing if an element of  $\llbracket 0, n^*B - 1 \rrbracket^{n^*}$  sums to zero can be done in  $O(n^* \log(n^*B))$ , and by dynamic programming, we can check if it is irreducible in  $O((n^*B)^{n^*})$ ; thus  $K$  can be computed in  $O((n^*B)^{2n^*})$ .

The set  $K$  (and its computation) will be useful later for proving Theorem 13. In the following, we first focus on set  $D$ , whose cardinality is denoted  $c_D$ . Indeed, starting from  $D$ , Algorithm 1 shows that MZSP can be solved in  $O(nc_D^2 n^* \log n + n^{n^*+2} \log B)$ . Since, by Proposition 2 above,  $c_D$  is in  $O(n^{n^*})$ , this shows that MZSP is XP relatively to parameter  $n^*$ , as stated by the following theorem.

**Theorem 11.** *MZSP is in XP when parameterized by  $n^*$ .*

*Proof.* We provide an algorithm that runs in time  $O(nc_D^2 n^* \log n + n^{n^*+2} \log B)$ . The proof derives from Algorithm 1, in which  $T$  is an array indexed by the elements of  $D$  and which is initialized as follows:  $T[0] = 0$ , and for every other vector  $v \in D$ ,  $T[v] = -\infty$ . We also recall that  $s$  is the multiplicity multiset of  $\mathcal{S}$ .

Clearly, Algorithm 1 runs in  $O(nc_D^2 n^* \log n)$  where  $c_D = |D|$ , since  $nc_D^2$  additions on vectors are realized, each taking  $O(n^* \log n)$  time. To this complexity,  $O(n^{n^*+2} \log B)$  should be added for the precomputation of  $D$ . We now show

that Algorithm 1 is correct. At the end of the algorithm, for any  $v \in D$ ,  $T[v]$  represents the largest number of elements of  $D$  that we can sum to obtain  $v$ . Let us denote  $w_v$  this value (thus  $T[v] = w_v$ ). For any  $i \in \llbracket 1, n \rrbracket$ , let  $\mathcal{P}(i)$  be the following property: for every  $v \in D$ ,  $T[v] = w_v$  if  $w_v \in \llbracket 0, i \rrbracket$ . Our goal is to prove, by induction on  $i$ , that  $\mathcal{P}(i)$  holds for any  $i \in \llbracket 1, n \rrbracket$ . First,  $\mathcal{P}(0)$  is true since  $w_v = 0$  implies  $v = 0$ . Now let  $i \geq 0$ , and let us assume  $\mathcal{P}(i)$  holds. Let  $v \in D$ . If  $w_v \leq i$ , then by induction hypothesis, we have  $T[v] = w_v$ . If not, then there exists  $u \in D$  such that  $w_{v-u} = i$ , and hence  $T[v-u] = i$ . Hence, by construction of  $T$ ,  $T[v] = i + 1 = w_v$ , which consequently proves that  $\mathcal{P}(i+1)$  holds. By induction,  $\mathcal{P}(i)$  holds for any  $i \in \llbracket 1, n \rrbracket$ . In particular,  $T[s]$  represents the largest number of elements from  $D$  that can be summed in order to obtain  $s$ . By Proposition 1, there exists  $\ell$  elements of  $D$  whose sum is  $s$  iff there exists a zero-sum partition of  $\mathcal{S}$ , of cardinality  $\ell$ . Thus the cardinality of a maximum zero-sum partition of  $\mathcal{S}$  is  $T[s]$ , which shows correctness of Algorithm 1.  $\square$

As mentioned before, we conjecture MZSP to be W[1]-hard parameterized by  $n^*$ . In contrast, we have the following result.

**Theorem 12.** *MZSP is FPT when parameterized by  $n^* + k$ .*

*Parameter maximum absolute value  $B$ .* Recall that  $B$  is the greatest integer (in absolute value) in an instance  $\mathcal{S}$  of MZSP.

**Theorem 13.** *MZSP is FPT when parameterized by  $B$ .*

*Proof.* In order to prove the result, we will provide an ILP model for our problem. We will then show that the number of variables of our ILP is a function of  $n^*$  and  $B$  only, which, combined with the fact that ILP is FPT parameterized by its number of variables [2, 9] and the fact that  $n^* \leq 2B$ , allows us to conclude. Given an integer  $k$ , we are interested in solving  $k$ -MZSP, which asks whether a size- $k$  zero-sum partition of  $\mathcal{S}$  exists. Let  $(\mathcal{S}, k)$  be an instance of  $k$ -MZSP. Let us number the  $n^*$  distinct values in  $\mathcal{S}$   $a_1, \dots, a_{n^*}$  and let  $s_1, \dots, s_{n^*}$  be their respective multiplicities in  $\mathcal{S}$ . Let  $c_K = |K|$ , where  $K$  is the set defined previously, and let us compute  $K$  – recall that  $c_K$ , by Proposition 2 and the discussion that follows, satisfies  $c_K = O((n^*B)^{n^*})$ , and that  $K$  can be computed in  $O((n^*B)^{2n^*})$ . Our ILP based on the following  $c_K$  variables  $x_u$ ,  $u \in K$ , where  $x_u$  represents the number of times element  $u$  appears in a zero-sum partition of  $\mathcal{S}$  of cardinality at least  $k$ . The ILP formulation of the problem is as follows.

<p><math>k</math>-MZSP (ILP model):</p> <div style="text-align: center; margin-top: 10px;"> <p><b>c.1</b> <math>\forall u \in K \quad x_u \geq 0</math></p> <p><b>c.2</b> <math>\forall i \in \llbracket 1, n^* \rrbracket \quad \sum_{u \in K} u_i x_u = s_i</math></p> <p><b>c.3</b> <math>\sum_{u \in K} x_u \geq k</math></p> </div>
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We now show correctness of our ILP model, by proving that there exists a zero-sum partition of MZSP of cardinality  $k$  iff the above ILP formulation admits a solution.

( $\Rightarrow$ ) Suppose  $\mathcal{S}$  admits a size- $k$  zero-sum partition. Then, by Proposition 1, we know there exist  $\ell \leq k$  elements of  $K$  which sum to  $s$ , that we will call  $u^1, \dots, u^\ell$ . For  $u \in K$ , let  $x_u$  denote the number of times  $u$  appears in  $(u_1, \dots, u_\ell)$ . Then, by definition,  $\forall u \in K, x_u \geq 0$ ,  $\sum_{u \in K} x_u u = s$  and  $\sum_{u \in K} x_u = \ell \geq k$ . Thus our ILP formulation admits a solution.

( $\Leftarrow$ ) Conversely, suppose there exists  $x_u$  for  $u \in K$ , which is a solution to the above ILP formulation. Let us build  $(u^i)_{i \in [1, \ell]}$ , where element  $u$  appears exactly  $x_u$  times. Then, from C.2,  $\sum_{i=1}^{\ell} u^i = s$ . Moreover, from C.3,  $\ell \geq k$ . Thus, from Proposition 1, there exists a size- $k$  zero-sum partition of  $\mathcal{S}$ .

Since  $n^* \leq 2B$ , and since ILP, parameterized by the number  $x$  of variables, is FPT and can be solved in  $O^*(x^{2.5x+o(x)})$  [2, 9], and since here  $x = c_K = O((n^*B)^{n^*})$ , the result follows.  $\square$

## 4 Conclusion

We provided diverse algorithmic results concerning the MZSP problem: hardness, (in)approximability and fixed-parameterized complexity considerations with respect to parameters  $n, m, k, n^*$  and  $B$ . Some questions about MZSP remain unanswered. In particular, we conjecture MZSP to be W[1]-hard parameterized by the number  $n^*$  of distinct values in  $\mathcal{S}$ ; (dis)proving it remains open.

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