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Sublinear measures, Menger curvature, and Hausdorff dimension



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ABSTRACT

We study metric properties of sets which are characterized by their approximation properties by smooth curves at scales of positive density and their relationships with known problems in complex dynamics and analysis. Three direct applications towards removability for bounded analytic functions, optimal bounds for Traveling Salesman problem, and Hausdorff dimension of unfolding attractors are discussed in the paper. The main technical ingredient is a construction, for every continuum K , of a finite Radon measure of bounded Menger curvature such that on every ball $B(x, r)$, $x \in K$, the measure is bounded by a universal constant multiple of $r \exp(-g(x, r))$, where $g(x, r) \geq 0$ is an explicit function. Theorem A is the main analytical result of the paper and provides a precise estimate on the growth of the function $g(x, r)$ in terms of Jones' β numbers. Its dimension theory counterpart is stated as Theorems 1 and 2 and yields sharp estimates of the Hausdorff dimension of compacta in terms of asymptotic properties of $\log g(x, r) / \log r$ for all $x \in K$ except for a set of finite linear measure.

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1. Introduction

We construct asymptotically sublinear Radon measures of bounded Menger curvature that vanish on rectifiable curves and are supported on planar continua. When the growth of sublinearity related to wiggleness in decreasing scales is close to extremal our methods yield sharp lower Hausdorff dimension estimates, Theorem 1 generalizes the results of [11]. On the other hand, if a compact is almost flat in most scales, its Hausdorff dimension is close to 1. Theorem 2 is a counterpart of the results based on porosity [33,5,44].

The construction uses Traveling Salesman techniques [30] and in its measure-theoretical part revisits some unpublished results of the second author [18,40]. The same construction is used in [24] to obtain optimal bounds for the analytical Traveling Salesman problem. This paper contains some applications of our results to Panlevé problem [17,47] (Section 1.2) and dynamical systems, Theorem 2 is a key ingredient in the proof of the conformal version of Yoccoz' problem [14] (Section 1.4). The metrical theorems and methods are formulated in Section 1.3. The exposition about the geometric dimension theory and Menger curvature is essentially self-contained. The main motivation of our paper comes from non-uniformly hyperbolic dynamical systems.

Constants. The letters C, C', c stand for absolute constants which may change at different occurrences. The notation $A \lesssim B$ means that there is a positive absolute constant C such that $A \leq CB$. Also, $A \simeq B$ is equivalent to $A \lesssim B$ and $A \gtrsim B$.

1.1. Sublinear measures and Menger curvature

A compact set K in \mathbb{C} is removable if for every open set $U \supset K$, every bounded analytic function $f : U \setminus K \rightarrow \mathbb{C}$ extends analytically over K . Panlevé problem consists in characterizing removable sets geometrically. A relatively simple calculation shows that every compact set of Hausdorff dimension strictly smaller than 1 is removable while every compact set of Hausdorff dimension strictly bigger than 1 is not removable. In the critical dimension 1, Panlevé problem was finally solved by G. David for compact sets of finite length (Vitushkin's conjecture): a compact set of finite length is removable if and only if it intersects every rectifiable curve in the plane along a set of zero length, [16,15].

Ahlfors in [1] defined *analytic capacity* for compact sets $K \subset \mathbb{C}$ to be $\sup_f |f'(\infty)|$, where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus K \rightarrow \mathbb{C}$ with $|f| < 1$, and proved that K is removable if and only if it has zero analytic capacity.

A positive Borel measure with a compact support K is of *linear growth* if there exists a positive constant C such that for every $r > 0$ and $x \in K$,

$$\mu(B(x, r)) \leq Cr, \tag{1}$$

and μ has a finite *Menger curvature* $c_K^2(\mu)$ if

$$c_K^2(\mu) = \iiint c(x, y, z)^2 d\mu(x)d\mu(y)d\mu(z) < \infty ,$$

where $c(x, y, z)$ is the inverse of the radius of the circumference passing through x, y, z . Shall any of the points x, y, z coincide then we set $c(x, y, z) = 0$.

X. Tolsa in [46] proved the following measure theoretic characterization of removable sets in the general case (without the hypothesis of finite length): a compact $K \subset \mathbb{C}$ has positive analytic capacity if and only if it supports a Borel measure of linear growth and finite curvature. The results of [46] resolve affirmatively Melnikov’s [37] and Vitushkin’s σ -finite conjectures.

1-Hausdorff measure \mathcal{H}^1 is of linear growth on every AD-regular set (Ahlfors-David), see Definition 2.4. To prove Theorem A, we will need the following property of AD-regular curves, called *local curvature condition* in [38,36]: for every ball B , $\text{diam } B \leq \text{diam } \Gamma$,

$$\int_{\Gamma \cap B} \int_{\Gamma \cap B} \int_{\Gamma \cap B} c(x, y, z)^2 d\mathcal{H}^1(x)d\mathcal{H}^1(y)d\mathcal{H}^1(z) \leq C(\Gamma) \text{diam } B, \tag{2}$$

where $C(\Gamma)$ depends solely on AD-regularity constant of Γ .

It is known that every planar compact $K \subset \mathbb{C}$ of $\dim_{\mathbb{H}}(K) > 1$ admits a linear measure of finite Menger curvature. Indeed, the existence of a finite linear measure comes directly from Frostman’s lemma. It has a bounded Menger curvature from the characterization of the Hausdorff dimension by the Newtonian capacity, cf. Chapter 8 in [35]. In [34], P. Mattila gives a weaker condition based on a gauge function that holds for compact sets of Hausdorff dimension 1 and suffices for non-removability. In particular, the examples in [34] show that the analytic and Newtonian capacity do not vanish simultaneously.

More than a decade ago, in the context of Melnikov’s conjecture, the second author proved that every continuum K in the plane admits a Radon measure of linear growth and finite curvature. The theorem was included in [40] but the proof presented there contains a gap (page 48, estimate (32) and what follows). Our approach to the problem relies on the technical Lemma 2.2 which both simplifies arguments and leads to stronger results with quantitative growth estimates. Its meaning is that in any scale wiggly compacts can be approximated only by relatively long curves. The current simple proof of this property borrows from the idea of lifting measures in the study of the dimension of the graphs of real functions that we learnt from C. Bishop [10]. The obtained bounds are optimal, it can be easily seen by connecting the canonical finite approximations of the planar 4-corner Cantor set. The asymptotically sublinear growth is used to derive various dimension estimates in the setting which is natural for non-uniformly hyperbolic dynamical systems.

The calculation of Menger curvature of the measure μ constructed in the proof of Theorem 3 (or Theorem 6) relies on a discretization method from Section 3. Since μ is obtained as a weak limit of measures μ_n defined on pieces of AD-regular curves, the discrete version of Menger curvature reduces calculations with respect to the limit

measure μ to estimates on AD-regular curves with respect to \mathcal{H}^1 , cf. (2). A general line of the calculation follows the idea of bounded distortion argument in dynamical systems.

We obtain, therefore, the following consequence of Theorems 6 and 3, Theorem 6 is formulated and proven in Appendix A, $\beta_K(x, r)$ is defined in Section 1.3.

Theorem A. *Let $K \subset \mathbb{C}$ be a continuum. There exist universal constants $C, C' > 0$ and a Radon measure μ supported on K such that*

(i) *for all $x \in K$ and $0 < r < R := \text{diam}(\text{supp } \mu)$*

$$\mu(B(x, r)) \leq Cr \exp \left(-C' \int_r^R \beta_K^2(x, t) \frac{dt}{t} \right).$$

(ii) $\mu(K) \geq C^{-1} R$ and $c_K^2(\mu) \leq C R$.

Moreover, if the set $F(K) = \{x \in K : \int_0^{\text{diam } K} \beta_K^2(x, t) \frac{dt}{t} < +\infty\}$ is of finite length and $\mathcal{H}^1(K \setminus F(K)) > 0$, then one can construct μ with the properties (i) and (ii) such that $\mu(F) = 0$.

Proof. If $\mathcal{H}^1(F(K)) < +\infty$ and $\mathcal{H}^1(K \setminus F(K)) > 0$, set $\mu := R\mu'$, where $R > 0$ and the measure μ' are provided by Theorem 3 applied to K , with $E := F(K)$. Otherwise, μ' is a rescaled back measure provided by Theorem 6 applied to K normalized to diameter 1. \square

Since *continua* (compact connected sets that contain at least two points) have always positive analytical capacity, the existence of measures of linear growth with bounded Menger curvature follows from the general result of [46]. Theorem A provides a precise estimate on the growth of these measures. Moreover, the second part of Theorem A has applications to analytical capacity which are discussed in the next section.

A set X is called *rectifiable* if there exists at most countably many Lipschitz curves Γ_j such that $\mathcal{H}^1(X \setminus \bigcup_j \Gamma_j) = 0$. Otherwise, X is *non-rectifiable*.

Definition 1.1. We say that a finite positive Borel measure μ with compact support in K is of *asymptotically sublinear growth* if μ is of linear growth on K , cf. (1), and there is a rectifiable set Γ such that

- $\mu(\Gamma) = 0$,
- $\mathcal{H}^1(K \setminus \Gamma) > 0$,
- for every $x \in K \setminus \Gamma$, $\lim_{r \rightarrow 0} \mu(B(x, r))/r = 0$.

The measure constructed in the first part of Theorem A is of linear growth, while μ from the second part is always asymptotically sublinear. Note that if $\mathcal{H}^1(F) = 0$, then also $\mathcal{H}^1(K \setminus F) > 0$.

1.2. Remarks about capacity

It is known that connected sets are not removable for bounded analytic functions. A natural problem is to understand how much removable sets differ metrically from connected ones. To this aim, we will introduce the concept of *connecting set*.

Definition 1.2. Let $K \subset \mathbb{R}^d$ be a compact set. We say that E is a connecting set for K if

- for any open neighborhood U of K the set $E \setminus U$ is contained in a finite union of rectifiable curves.
- $E \cup K$ is a *continuum*.

For any $A \in \mathbb{R}^d$ and $\delta > 0$, let

$$B(A, \delta) := \bigcup_{x \in A} B(x, \delta),$$

and $cc(A)$ be the set of connected components of A . Observe that if A is bounded, then $cc(\overline{B(A, \delta)}) = \{C_1, \dots, C_n\}$, where all C_j are non-degenerate *continua*.

Let L_1 be a segment of minimal length connecting two elements of $cc(\overline{B(A, \delta)})$. Having constructed k such segments, let L_{k+1} be a segment with endpoints in $\overline{B(A, \delta)}$, of minimal length, connecting two elements of

$$cc(A_k), \text{ where } A_k := \overline{B(A, \delta)} \cup \bigcup_{j=1}^k L_j.$$

Observe that A_{n-1} is a tree with vertices $cc(\overline{B(A, \delta)})$ and arcs L_1, \dots, L_{n-1} , so it is connected. We say that the segments L_1, \dots, L_{n-1} connect $\overline{B(A, \delta)}$.

If $\overline{B(A, \delta')}$ is connected for some $\delta' > \delta$, then for $j = 1, \dots, n - 1$,

$$L_j \subseteq \overline{B(A, \delta')} \setminus B(A, \delta). \tag{3}$$

Otherwise, let $x \in L_k \setminus \overline{B(A, \delta')}$, where $L_k = [a, b]$, $k \leq n - 1$, connects the components $C, C' \in cc(\overline{B(A, \delta)})$. Then

$$\text{dist}(C, C') = \text{dist}(a, b) = \text{dist}(a, x) + \text{dist}(x, b) > 2(\delta' - \delta).$$

By the minimality of L_k , and thus of $\text{dist}(C, C')$,

$$\#cc(\overline{B(A, \delta')}) = \#cc\left(\overline{B\left(\overline{B(A, \delta)}, \delta' - \delta\right)}\right) \geq \#cc(A_{k-1}) > 1.$$

Lemma 1.3. *For every compact set $K \subset \mathbb{R}^d$ there exists a connecting set.*

Proof. For $n \geq 0$, let $K_n := \overline{B(K, 2^{-n})}$. As described above, let G_0 be the union of finitely many segments connecting K_0 . We construct an increasing sequence $(G_n)_{n \geq 0}$ of finite unions of segments with the following properties.

- for all $n \geq 0$, $G_n \cup K_n$ is a *continuum* and $G_n \cap K_n$ is a finite set;
- for all $n \geq 0$, $G_{n+1} \cup K_{n+1} \subseteq G_n \cup K_n$.

By intersection of a decreasing family of *continua*, it follows immediately that $\bigcup_{n \geq 0} G_n$ is a connecting set for K .

Assume that G_0, \dots, G_n satisfying the conditions above have been constructed. Fix $C \in cc(K_n)$ and let

$$cc(C \cap K_{n+1}) = \{C_1, \dots, C_m\}.$$

By (3), let $L_1, \dots, L_{m-1} \subseteq C \setminus B(K, 2^{-n-1})$ be connecting segments of the components of $C \cap K_{n+1}$. For every $P \in G_n \cap C$, let $L_P \subseteq C \setminus B(K, 2^{-n-1})$ be the shortest segment from P to $C \cap K_{n+1}$. Finally, let

$$L(C) := \bigcup_{j=1}^{m-1} L_j \cup \bigcup_{P \in G_n \cap C} L_P \text{ and } G_{n+1} := G_n \cup \bigcup_{C \in cc(K_n)} L(C).$$

One can easily check that G_{n+1} satisfies the conditions above. \square

Now, we are in position to formulate two corollaries to Theorem A.

Corollary 1.4. *Let $W \subset \mathbb{C}$ be a compact set such that for every $z \in W \setminus W'$,*

$$\int_0^{\text{diam } W} \beta_W^2(z, t) \frac{dt}{t} = +\infty$$

and $\mathcal{H}^1(W') < +\infty$. If there exists a connecting set $E \subset \mathbb{C}$ for W such that $\mathcal{H}^1(E \setminus W) < +\infty$ then either $\mathcal{H}^1(W \setminus W') = 0$ or W has positive analytic capacity and contains a non-rectifiable set which is not σ -finite with respect to \mathcal{H}^1 .

Proof. Let $K = W \cup E$. For every $z \in W \setminus W'$,

$$\int_0^{\text{diam } K} \beta_K^2(z, t) \frac{dt}{t} \geq \int_0^{\text{diam } W} \beta_W^2(z, t) \frac{dt}{t} = +\infty.$$

E is a connecting set thus for almost all $z \in E \setminus W$ with respect to \mathcal{H}^1 ,

$$\int_0^{\text{diam } E} \beta_E^2(z, t) \frac{dt}{t} < +\infty.$$

Since W is compact, for every $z \in E \setminus W$, there exists $B(z, r)$ such that $B(z, r) \cap W = \emptyset$. Combined with the above inequality, for a.e. $z \in E \setminus W$,

$$\int_0^{\text{diam } K} \beta_K^2(z, t) \frac{dt}{t} \leq \ln \frac{\text{diam } K}{r} + \int_0^r \beta_{E \setminus W}^2(z, t) \frac{dt}{t} < +\infty.$$

By the second part of Theorem A, we can distribute an asymptotically sublinear measure μ on K with a bounded Menger curvature provided $\mathcal{H}^1(W \setminus W') > 0$. The measure μ is concentrated on W , $\mu(E \setminus W) = 0$ since $\mathcal{H}^1((E \setminus W) \cup W') < +\infty$. W is compact and $\text{supp } \mu \subset W$. Moreover, $\text{supp } \mu$ is non-rectifiable and not of σ -finite \mathcal{H}^1 -measure because μ has vanishing \mathcal{H}^1 -density on W . By Theorem 1.2 of [46], W has a positive analytic capacity. \square

Corollary 1.5. *Let $W \subset \mathbb{C}$, $\mathcal{H}^1(W) > 0$, be a compact set with a connecting set E of finite length. Then W has a positive analytic capacity.*

Proof. Suppose that $W' = \{x \in W : \int_0^{\text{diam } W} \beta_W^2(x, t) \frac{dt}{t} < +\infty\}$ has positive length. Then there exist $M > 0$ and a compact subset $W'' \subset W'$, $\mathcal{H}^1(W'') > 0$ such that for all $x \in W''$, $\int_0^{\text{diam } W''} \beta_{W''}^2(x, t) \frac{dt}{t} < M$. By Theorem 1 in [11], see Theorem 5, W'' is contained in a rectifiable curve and thus is of positive analytic capacity. We can assume that $\mathcal{H}^1(W') = 0$ and apply Corollary 1.4 for W and E to conclude the proof. \square

As a direct consequence, we have the following necessary condition for removability of compact sets in the plane.

Theorem B. *Suppose that $W \subset \mathbb{C}$ is a removable compact set for bounded analytic functions. Let E be a connecting set for W . If B is an open ball such that $\mathcal{H}^1(B \cap W) > 0$ then $\mathcal{H}^1(B \cap E) = +\infty$.*

Proof. It is enough to notice that $(E \cap \overline{B}) \cup \partial B$ is a connecting set for $\overline{B} \cap W$. If $\mathcal{H}^1(\overline{B} \cap E) < +\infty$ then by Corollary 1.5, $\overline{B} \cap W$ would be of positive analytic capacity, a contradiction. \square

Theorem B is not new for sets which are σ -finite because the semiadditivity of analytic capacity [46] and Theorem 4 reduce it to the known results of [16].

About the connectedness hypothesis. One of the major questions is to find a good condition which would replace *connectedness*. Let π_θ be an orthogonal projection onto the line $\{re^{i\theta} : r \in \mathbb{R}\}$ and K a compact set, $K_B = K \cap B$. Is a version of Theorem A true under a local Favard condition,

$$\text{Fav}(K_B) = \frac{1}{\pi} \int_0^\pi \mathcal{H}^1(\pi_\theta(K_B)) \, d\theta \simeq r_B,$$

for every ball B centered at a point from K , $r_B \leq \text{diam } K$?

Favard length is closely related to the Buffon’s needle probability. See [32] for a counterexample relating the latter to analytic capacity.

1.3. Metric properties

1.3.1. Wiggleness and β -numbers

One way to quantify the concept of geometric oscillations is to use the theory of β -numbers, see Definition 1.6. A set K is called wiggly at x and scale $r > 0$ if one can draw a triangle contained in the ball $B(x, r)$ with vertices in K so that after rescaling to the unit ball, the triangle belongs to a compact family of triangles in \mathbb{R}^d , with $d \geq 2$.

In [12], it is proven that every connected compact planar set K which oscillates uniformly at every scale around every point of K has the Hausdorff dimension strictly bigger than 1. The dimension estimates of [12] are quantified in terms of β -numbers.

Definition 1.6. Let $K \subseteq \mathbb{R}^d$ with $d \geq 2$, $x \in \mathbb{R}^d$ and $r > 0$ such that $B(x, r) \cap K \neq \emptyset$. We define $\beta_K(x, r)$ by

$$\beta_K(x, r) := \inf_L \sup_{z \in K \cap B(x, r)} \frac{\text{dist}(z, L)}{r},$$

where the infimum is taken over all lines L in \mathbb{R}^d . If $B(x, r) \cap K = \emptyset$, we set $\beta_K(x, r) := 0$.

A bounded set K is called *uniformly wiggly* if

$$\beta_\infty(K) := \inf_{x \in K} \inf_{r \leq \text{diam } K} \beta(x, r) > 0.$$

Theorem 1.1 of [12] states that if $K \subset \mathbb{C}$ is a continuum and $\beta_\infty(K) > 0$ then $\dim_{\text{H}}(K) \geq 1 + c\beta_\infty^2(K)$, where c is a universal constant.

Mean wiggly sets and Hausdorff dimension. A direct consequence of the existence of measures of sublinear growth is Theorem 1, an integral and non-uniform version of the dimension results of [12]. Theorem 2 introduces the concept of almost flat sets and shows that the estimates of Theorem 1 are sharp. If non-wiggly portions of K have infinite

length then the assertion that $\dim_{\mathbb{H}}(K) > 1$ is generally not true. As an example take a unit segment in the plane accumulated by a smooth curve winding infinitely many times around it.

In a different context, a conceptually similar approach was adopted by P. Koskela and S. Rohde in their proof that mean porosity replaces porosity for upper estimates of the Hausdorff dimension [33].

Theorem 1 generalizes Theorem 1.1 of [12] in three directions. Firstly, it allows for an exceptional set $E \subset K$ of finite 1-dimensional Hausdorff measure. Secondly, it is enough to assume that the continuum K oscillates at every point $x \in K \setminus E$ and every scale $r > 0$ with some parameter $\beta_K(x, r) > 0$ which depends both on x and $r > 0$ so that

$$\liminf_{r \rightarrow 0} \frac{\int_r^{\text{diam } K} \beta_K^2(x, t) \frac{dt}{t}}{-\log r} \geq \beta_0^2 > 0. \tag{4}$$

This means that the uniform hypothesis of [12] that there exists $\beta_0 > 0$ so that for every $x \in K$ and every $r < \text{diam } K$,

$$\beta_K(x, r) \geq \beta_0$$

can be replaced by the integral condition (4) without affecting the main estimate on $\dim_{\mathbb{H}}(K)$ of [11] that $\dim_{\mathbb{H}}(K) \geq 1 + c\beta_0^2$, where c is a universal constant.

An additional feature of Theorem 1 is that the condition (4) can be further relaxed,

$$\liminf_{r \rightarrow 0} \frac{\int_r^{\text{diam } K} \beta_K^2(x, r) \frac{dt}{t}}{-\log r} > 0 \tag{5}$$

and still one obtains a non-uniform estimate that $\dim_{\mathbb{H}}(K) > 1$.

The main technical ingredient of the paper is Theorem 3. The construction of the measure μ in the proof of Theorem 3 is based on a combinatorial Lemma 2.2 and a geometric version of the corona type constructions, see [13,31] and the books [22,40].

Theorem 1. *Suppose that a continuum $K \subset \mathbb{C}$ is the union of two subsets $K = W \cup E$, $\mathcal{H}^1(E) < \infty$ and $\mathcal{H}^1(W) > 0$.*

- *If there exists $\beta_0 > 0$ such that for all x in W*

$$\liminf_{r \rightarrow 0} \frac{\int_r^{\text{diam } K} \beta_K^2(x, t) \frac{dt}{t}}{-\log r} \geq \beta_0^2,$$

then

$$\dim_{\mathbb{H}}(K) \geq 1 + c\beta_0^2,$$

where c is a universal constant.

- If for all $x \in W$

$$\liminf_{r \rightarrow 0} \frac{\int_r^{\text{diam } K} \beta_K^2(x, t) \frac{dt}{t}}{-\log r} > 0,$$

then

$$\dim_{\text{H}}(K) > 1.$$

The hypothesis about connectivity in Theorem 1 is implicitly weaker than formulated due to admission of an exceptional set E . Indeed, if $K \subset \mathbb{C}$ is not connected but it admits a connecting set F of finite length then the claims of Theorem 1 hold for $K \cup F$ and $\dim_{\text{H}}(K \cup F) = \dim_{\text{H}}(K)$.

Further examples showing that all hypotheses of Theorem 1 are essential are discussed in Section 1.3.3.

Almost flat sets. We show that the estimates of Theorem 1 are sharp. Recall that a set $N \subset \mathbb{C}$ is ε -porous at x at scale $r > 0$ if there is $z \in B(x, r)$ such that $B(z, \varepsilon r) \subset B(x, r) \setminus N$. The condition $\beta_N(x, r) \leq \alpha$ implies that N is $(1 - \alpha)/2$ -porous at scale r at x .

If one assumes that for every x in a bounded set N ,

$$\limsup_{r \rightarrow 0} \frac{\int_r^{\text{diam } N} \beta_N^2(x, t) \frac{dt}{t}}{-\log r} = 0, \tag{6}$$

then $\dim_{\text{H}}(N) \leq 1$ follows from the dimension results of D. Beliaev and S. Smirnov for mean porous sets, Corollary 1 in [5]. The condition $\beta_N(x, r) = 0$ is much stronger than the maximal porosity $1/2$ at x . It turns out that one can replace \limsup by \liminf in the inequality (6) and still obtain that $\dim_{\text{H}}(N) \leq 1$.

Theorem 2. *Let N be a set in \mathbb{R}^d with $d \geq 2$ such that for all $x \in N$,*

$$\liminf_{r \rightarrow 0} \frac{\int_r^1 \beta_N^2(x, t) \frac{dt}{t}}{-\log r} \leq \beta_0^2,$$

then for each $d \geq 2$ there is a universal constant $C_d > 0$ such that

$$\dim_{\text{H}}(N) \leq 1 + C_d \beta_0^2.$$

Proof. Without loss of generality we may assume that $\text{diam } N = 1$. Fix $\epsilon > 0$ and $n_0 > 1$. For every $n \geq n_0$, denote

$$X_n = \left\{ x \in N : \int_{2^{-n}}^1 \beta_N^2(x, t) \frac{dt}{t} \leq (\epsilon + \beta_0^2)n \log 2 \right\}.$$

By definition, $N = \bigcup_{n \geq n_0} X_n$. By Besicovitch’s covering theorem, we can find $M(d)$ subcollections \mathcal{G}_i such that every two balls from the same subcollection \mathcal{G}_i are disjoint, every ball is of radius 2^{-n} , and X_n is covered by the balls from $\mathcal{G}(n) = \bigcup_{i \leq M} \mathcal{G}_i$. Let Z_i be the set of the centers of the balls from \mathcal{G}_i . It is a finite set with the property that

$$\int_0^{\text{diam } Z_i} \beta_{Z_i}^2(x, t) \frac{dt}{t} \leq \int_{2^{-n}}^1 \beta_N^2(x, t) \frac{dt}{t} \leq (\epsilon + \beta_0^2)n \log 2.$$

By Theorem 5, for every $i \leq M(d)$, the set Z_i is contained in a curve Γ_i of length $\mathcal{H}^1(\Gamma_i) \leq Ce^{C(\epsilon + \beta_0^2)n \log 2}$, where $C > 1$ only depends on the ambient dimension d . Using the fact that the balls from \mathcal{G}_i are disjoint, we have that

$$\sum_{B \in \mathcal{G}_i} (\text{diam } B)^{1+\alpha} \leq 2^{(-n+1)\alpha+1} \mathcal{H}^1(\Gamma_i) \leq C_1(\alpha)2^{-n(\alpha-C(\epsilon+\beta_0^2))}.$$

Hence, the sum of the diameters to the power $1 + \alpha$ of the balls from $\mathcal{G}(n)$ is smaller than

$$M(d)C_1(\alpha)2^{-n(\alpha-C(\epsilon+\beta_0^2))}.$$

Let $\mathcal{G} = \bigcup_{n \geq n_0} \mathcal{G}(n)$. Assuming that α is bigger than $2C(\epsilon + \beta_0^2)$, we obtain that

$$\sum_{B \in \mathcal{G}} (\text{diam } B)^{1+\alpha} \leq \sum_{n=n_0}^{\infty} M(d)C_1(\alpha)2^{-n(\alpha-C(\epsilon+\beta_0^2))} \leq C'(d, \alpha)2^{-n_0 C \beta_0^2},$$

where C' only depends on α and the ambient dimension d . Passing with n_0 to $+\infty$, we infer that $\mathcal{H}^{1+\alpha}(N) < +\infty$ which completes the proof. \square

Generalizations are possible. Theorem 1 can be generalized to \mathbb{R}^d , with $d \geq 2$. However, there are technical complications which would make the current paper too long. One of them is that the localization principle which is one of the premises of the current proof of Theorem 1 is no longer true beyond \mathbb{R}^2 : one can not replace a continuum K by $K \cap B \cup \partial B$ in \mathbb{R}^3 because $\mathcal{H}^1(\partial B) = +\infty$ and Theorem 1 allows only exceptional sets of finite length.

There are other natural directions for generalizations related to the concept of non-porosity when K is almost topologically dense in some scales. This should lead to new sharp estimates of $d - \dim_{\mathbb{H}}(K)$. Also, getting sharp estimates for $\dim_{\mathbb{H}}(K)$ without the connectivity hypothesis is possible.

Finally, an approach of [19] can be revisited where a uniform non-flatness of compacts with respect to n -dimensional planes is studied. Under some additional topological assumptions which replace connectedness, it is proved that the Hausdorff dimension of uniformly non-flat compacts in \mathbb{R}^d is strictly bigger than n .

1.3.2. *Discrete versions of wiggleness*

The integral condition (4) has a discrete counterpart. Let K be a set in \mathbb{R}^d and fix $\lambda \in (0, 1)$. For every $x \in K$ and $t > 0$,

$$\beta_K(x, \lambda t) \leq \lambda \beta_K(x, t). \tag{7}$$

Hence, if $r \in A_m = [\lambda^{m+1}, \lambda^m]$, $m \in \mathbb{N}$, then the limit

$$\liminf_{r \rightarrow 0} \frac{1}{-\log r} \int_r^1 \beta_K^2(x, t) \frac{dt}{t} \tag{8}$$

is equivalent to

$$\liminf_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} \beta^2(x, \lambda^i) \simeq \liminf_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} \sup_{t \in A_i} \beta^2(x, t). \tag{9}$$

We say that a set $K \subset \mathbb{R}^d$ is *mean wiggly* at a point $x \in K$ with parameters $\lambda, \kappa, \beta_0 \in (0, 1)$ if there is $r(x) > 0$ such that for $\lambda^n < r(x)$ the number of wiggly scales λ^m , $m < n$, is greater than κn . Here, a wiggly scale is defined by the condition that $\beta_K(x, \lambda^m) \geq \beta_0$.

In particular, the limit (9) is positive iff K is mean wiggly with some positive parameters at x . In view of (7), λ, κ, β_0 are not independent and without loss of generality we may set $\lambda = \frac{1}{2}$.

The hypothesis of Theorem 1 can be reformulated in terms of mean density of wiggly scales. A direct calculation leads to the following corollary.

Corollary 1.7. *Suppose that $K \subset \mathbb{C}$ is a continuum and $K = W \cup E$ where W and E satisfy the following:*

- $\mathcal{H}^1(E) < \infty$ and $\mathcal{H}^1(W) > 0$.
- K is mean wiggly at every point $x \in W$.

Then

$$\dim_{\mathbb{H}}(K) > 1.$$

If additionally, the parameters of the wiggleness κ, β_0 at $x \in W$ are uniform, that is do not depend on $x \in W$, then

$$\dim_{\mathbb{H}}(K) \geq 1 + c' \beta_0^2 \kappa,$$

where c' is a universal constant.

Almost flat sets. A set $K \subset \mathbb{R}^d$ is *almost flat* at a point $x \in K$ with parameters $\kappa, \beta_0 \in (0, 1)$ if there is a sequence of positive integers (N_i) such that for every $i \in \mathbb{N}$ the number of flat scales $2^{-m}, m < N_i$, is greater than κN_i . A flat scale is defined by the condition $\beta(x, 2^{-m}) \leq \beta_0$.

The following corollary is an immediate consequence of Theorem 2.

Corollary 1.8. *Suppose that the set $K \subset \mathbb{R}^d, d \geq 2$, is almost flat at every point $x \in K$ with the parameters κ, β_0 that do not depend on $x \in K$. Then,*

$$\dim_{\mathbb{H}}(K) \leq 1 + c'_d(1 - \kappa + \beta_0^2 \kappa),$$

where for each $d \geq 2, c'_d$ is a universal constant.

Corollary 1 in [5], (see also [44,35]), implies that if K is almost flat at every point $x \in K$ with the parameters κ, β_0 then

$$\dim_{\mathbb{H}}(K) < d - \kappa + \frac{C}{|\log \beta_0|},$$

where C is a universal constant.

1.3.3. Various examples

Example 1. With the notations from Theorem 1, we describe an example such that: $K = E \cup W$ is a *continuum*, E has finite positive 1-measure (denoted by $\mathcal{H}^1(E)$), W is uniformly wiggly and $\dim_{\mathbb{H}}(W) = 1, \mathcal{H}^1(W) = 0$. This shows that the hypothesis $\mathcal{H}^1(W) > 0$ is needed. Note also that E is a connecting set for W .

This is a modified version of the four corners Cantor set. Let $S_0 = [0, 1]^2$ be the unit square. Let S_1^1, \dots, S_1^4 the disjoint sub-squares of side-length $0 < \frac{1}{4}a_1 < \frac{1}{2}$ which have each a common corner with S_0 . We repeat the construction inside each square of n -th generation to obtain four squares of side length $|S_{n+1}| = \frac{1}{4}a_{n+1}|S_n|$. We set

$$W = \bigcap_{n \geq 1} \bigcup_{i=1}^{4^n} S_n^i.$$

Observe that $|S_n| = 4^{-n} \prod_{i=1}^n a_i$. We let $a_n \nearrow 1$ and show that $\dim_{\mathbb{H}}(W) = 1$. We may define a measure μ supported on W such that $\mu(S_n^i) = 4^{-n}$. Then

$$\lim_{n \rightarrow +\infty} \frac{\log \mu(S_n)}{\log |S_n|} = \liminf_{n \rightarrow +\infty} \frac{n \log 4}{n \log 4 + \log a_1 + \dots + \log a_n} = 1.$$

As $\frac{|S_n|}{|S_{n+1}|}$ is bounded, Billingsley's lemma shows that $\dim_{\mathbb{H}}(W) = 1$. The same bound gives a lower bound of $\beta_W(x, r)$ for all $x \in W$ and $r > 0$.

The set W can be covered by the 4^n squares of n -th generation. Therefore it is enough to have $\lim_{n \rightarrow +\infty} \prod_{i=1}^n a_i = 0$ to obtain that $\mathcal{H}^1(W) = 0$.

Let E be the union of all diagonals of all squares. We may easily compute that

$$\mathcal{H}^1(E) = 2\sqrt{2} \left(1 + \sum_{n \geq 0} 4^n |S_n| \right).$$

We set $a_n = \frac{n^2}{(n+1)^2}$ so $\mathcal{H}^1(E) < +\infty$. Note that this sequence satisfies the previous conditions.

The set $K = W \cup E$ is a continuum with the desired properties. K is also locally connected.

Example 2. If in the previous example we set $a_n = 1$ for every $n \geq 0$ (the standard four corners Cantor set) then $\mathcal{H}^1(W)$ is finite, $\mathcal{H}^1(E) = \infty$, and $\dim_{\text{H}}(W \cup E) = 1$. The continuum $K = W \cup E$ is locally connected.

This shows that the hypothesis $\mathcal{H}^1(E) < \infty$ is needed in Theorem 1.

Example 3. Another example in the same direction, this time with W a *continuum* itself, is the following.

Let $W = [0, 1]$ and E the union of vertical segments of length 2^{-n} , with center $k2^{-n} \in W$, for all $n > 0$ and $0 < k < 2^n$, k odd. The continuum $K = W \cup E$ is locally connected and $\dim_{\text{H}}(K) = 1$ as a countable union of segments. It is not hard to see that $\beta_K(x, r)$ has a uniform lower bound for all $x \in W$ and all $0 < r < 1$ (but $\beta_W(x, r) = 0$).

1.4. Applications to dynamical systems

In the area of dynamical systems one cannot expect that generic systems are uniformly hyperbolic and that geometry of invariant fractals can be estimated at every scale as it is required in [12,23]. However, there are various results showing that non-uniformly hyperbolic systems are typical in ambient parameter spaces [29,6,7,42,2,27,45,3]. Our estimates of the Hausdorff dimension rely on some control of wiggleness in many but not all scales and fit into a general scheme of studying metric properties of attractors for generic dynamical systems.

Mean wiggleness, see Section 1.3.2, together with transversality sheds new light on a surprising amount of an upper semi-continuity of the Hausdorff dimension of attractors/repellers under a perturbation. The underlying mechanism is simple, once wiggleness is created through a *transversal intersection* in a large scale, it is stable and propagates through iterates to most points and scales of an attractor/repeller [27,26,8].

Our techniques apply directly to three different problems in dynamical systems.

Unfolding of attractors. One of the objectives in dynamical systems is to understand how non-linear attractors unfold within one-parameter families. A natural quantity which

captures the unfolding is Hausdorff dimension. It is known, by [49], that $\dim_{\mathbb{H}}(\mathcal{J}_c) > 1$ for all $c \in \mathcal{M} \setminus \{0, -2\}$, where $\dim_{\mathbb{H}}(\mathcal{J}_c)$ denotes the Hausdorff dimension of the Julia set \mathcal{J}_c of the quadratic polynomial $z^2 + c$ and \mathcal{M} is the Mandelbrot set. At $c = 0$ or -2 , the Julia set is respectively the unit circle or the segment $[-2, 2]$.

D. Ruelle proved in [43] that the function $c \in \mathcal{M} \mapsto \dim_{\mathbb{H}}(\mathcal{J}_c)$ is real analytic at every hyperbolic $c \in \mathbb{C}$ and its asymptotic behavior at $c = 0$ is given by

$$\dim_{\mathbb{H}}(\mathcal{J}_c) = 1 + \frac{|c|^2}{4 \log 2} + O(|c|^3). \tag{10}$$

J-C Yoccoz posed a directly related question in the strongly dissipative planar setting corresponding to $c = -2$ [9,48]: does Hausdorff dimension of the Hénon attractor (the closure of the unstable manifold) tend to 1 for almost all parameters in the sense of 2-dimensional Lebesgue measure when the attractor unfolds from the initial interval? A conformal version of Yoccoz’ problem is about unfolding Julia sets in the logistic family at $c = -2$. The technique of the current paper yields Hölder type dependence of $\dim_{\mathbb{H}}(\mathcal{J}_c)$ [14].

Siegel disks. About two decades ago, M. Herman raised the following question, is it true that for almost all rotation numbers Siegel disks for rational functions are Hölder domains with the boundary of Hausdorff dimension strictly bigger than 1 [25]? There are no known examples of Siegel disks with Hölder or John property other than quasidisks. The uniform wiggleness techniques of [23] do not extend to rotation numbers of unbounded type. However, a typical rotation number satisfies an ergodic estimate for its continued fraction coefficients a_n , $n \in \mathbb{N}$, $\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n \ln a_i = Q$, where Q is an absolute constant. Theorem 1 transforms the second part of Herman’s question: is the boundary of a typical Siegel disk mean wiggly? A diophantine condition on the rotation number is necessary as there exist Siegel disks with smooth boundaries [4].

Bowen alternative. We begin with an observation that every Collet-Eckmann rational map satisfies the TCE condition for rational maps, [41], which asserts that every $z \in \mathcal{J}_c$ has infinitely many neighborhoods W_n which can be mapped properly by f_c^n onto $B(f_c^n(z), \epsilon)$ with a bounded degree d and the set $\{n \in \mathbb{N} : \exists W_n\}$ has a density $\sigma > 0$ in \mathbb{N} . The degree d , the large scale parameter ϵ and the density σ do not depend on z . It can be easily verified, using compactness and Fatou’s theorem [21], that TCE connected Julia sets of rational functions are mean wiggly unless they are a conformal circle/line. It is not known if all connected Julia sets which are not analytic are mean wiggly. The problem is directly related to the remaining part of Bowen’s alternative in rational dynamics: every rational map with only elliptic Fatou components with unbounded rotation numbers has the Julia set of the Hausdorff dimension bigger than 1 [49,50,23].

2. Density of wiggleness

We want to construct a probability measure μ which captures wiggleness of a continuum K and allows for effective lower bounds of $\dim_{\mathbb{H}}(K)$. Let $F(K)$ be the set of points of K at which K is flat (or not wiggly).

$$F(K) := \left\{ x \in K : \int_0^1 \beta_K(x, t)^2 \frac{dt}{t} < +\infty \right\}. \tag{11}$$

Observe that for all $\alpha \geq 1$, $x \in K$ and $t > 0$, $\beta_K(x, \alpha t) \geq \alpha^{-1} \beta_K(x, t)$. We may conclude that for all $x \in F(K)$,

$$\lim_{t \rightarrow 0} \beta_K(x, t) = 0.$$

We want to construct μ which vanishes on $F(K)$. In this case, μ will be generally scale dependent, as wiggly parts of K can be contained in a ball of arbitrary small radius.

The main ingredient of the proofs of Theorem 1 and of Theorem A (when $\mathcal{H}^1(F(K)) < +\infty$ and $\mathcal{H}^1(K \setminus F(K)) > 0$), is the following result.

Theorem 3. *There exist two universal constants $C, C' > 0$ such that the following holds. Let $K \subset \mathbb{C}$ be a continuum and $F(K) \subseteq E \subseteq K$ with E a Borel set, $\mathcal{H}^1(E) < +\infty$ and $\mathcal{H}^1(K \setminus E) > 0$. There exist $y \in K$, $R > 0$ and a Radon probability measure μ supported on $B(y, R) \cap K$, such that $\mu(E) = 0$ and for all $x \in K$ and $r > 0$*

$$\mu(B(x, r)) \leq R^{-1} C r \exp \left(-C' \int_r^R \beta_K^2(x, t) \frac{dt}{t} \right)$$

and

$$c_K^2(\mu) \leq C R^{-2}.$$

The proof of Theorem 3 relies on a combinatorial result, Lemma 2.2, that replaces a *stopping time* argument, usually needed in corona type constructions, by a direct estimate of length of a crossing curve which is wiggly at many scales.

As $K \setminus E$ may have a much smaller diameter than K and we want $\mu(E) = 0$, we generally have $R \ll \text{diam } K$. In any case $R \leq \text{diam } K$.

As a preparation, we need to state some results and prove a few facts about the length (or 1-Hausdorff measure) of wiggly continua in the plane.

Let $Q \subset \mathbb{C}$ be a square (with sides parallel to the axis). Unless specified otherwise, squares are considered to contain only the left and top edges. Let $|Q|$ denote the side length of Q . Let $\Delta(Q)$ denote the set of all dyadic sub-squares of Q and $\Delta_k(Q)$ those

with side length $2^{-k}|Q|$. For any $\lambda > 0$, let λQ be the square with the same center as Q and with $|\lambda Q| = \lambda|Q|$. For a set S of squares, $|S| := \sum_{Q \in S} |Q|$ and $\#S$ is its cardinal.

For any $x \in Q_0$, where Q_0 is a fixed square, let

$$\Delta(x) = \{Q \in \Delta(Q_0) \mid x \in Q\}.$$

Let us also write Δ for $\Delta(Q_0)$ and Δ_k for $\Delta_k(Q_0)$.

Definition 2.1. Let K be a compact set in the plane and Q a square such that $Q \cap K \neq \emptyset$. We define $\beta_K(x, r)$ by

$$\beta_K(Q) := \inf_L \sup_{z \in K \cap 3Q} \frac{\text{dist}(z, L)}{|Q|},$$

where the infimum is taken over all lines L in the plane. For squares that do not intersect K we set $\beta_K(Q) := 0$.

For a compact set $K \subseteq Q_0$ is easy to check that

$$\beta_\infty^K(x) := \int_0^{\text{diam } K} \beta_K(x, t)^2 \frac{dt}{t} \simeq \sum_{Q \in \Delta(x)} \beta_K(Q)^2. \tag{12}$$

We will use the following fact, due to the second author (see [30] for a proof, [39] for a generalization in \mathbb{R}^d). If $\gamma \subset Q_0$ is a rectifiable curve, then

$$\beta_\infty(\gamma) := \sum_{Q \in \Delta} \beta_\gamma(Q)^2 |Q| \lesssim \mathcal{H}^1(\gamma). \tag{13}$$

It is well known that any planar continuum of finite 1-measure is contained in a rectifiable curve of comparable length, see Theorem 4.

Let us denote by π^* the projection on the real axis \mathbb{R} .

Lemma 2.2. *There exists $C > 0$ such that for all $M > 0$ and $A \subseteq \mathbb{C}$, if $\gamma \supseteq A$ is a curve, then*

$$\mathcal{H}^1(\gamma) \geq C M \mathcal{H}^1(\pi^*(\{x \in A : \beta_\infty^A(x) \geq M\})).$$

Proof. As we may assume that $\gamma \subseteq Q_0$, by inequality (13), it is sufficient to show that $\beta_\infty(\gamma)$ is large enough. As the inequality to prove is invariant under homothety, we may assume that $Q_0 = [0, 1]^2$.

Let $p := \mathcal{H}^1(\pi^*(\{x \in A : \beta_\infty^A(x) \geq M\}))$ and for all $Q \in \Delta$,

$$S(Q) := \sum_{Q \subseteq Q' \in \Delta} \beta_A(Q')^2.$$

If X is a collection of squares from Δ then we put $\cup X := \bigcup_{Q \in X} Q$.

For $n \geq 1$, let $G_n := \{Q \in \Delta_n : S(Q) \geq M/2\}$. Fix $n \geq 1$ such that $\mathcal{H}^1(\pi^*(\cup G_n)) \geq p/2$. Let $G \subseteq G_n$ be a minimal collection of squares such that

$$\pi^*(\cup G) = \pi^*(\cup G_n).$$

In particular, $\mathcal{H}^1(\pi^*(\cup G)) = 2^{-n} \#G$ and for all $Q \in \Delta_k, k \leq n$, we have

$$\#G_Q \leq 2^n |Q|,$$

where $G_Q = \{Q' \in G : Q' \subset Q\}$. Using the above inequality,

$$\begin{aligned} \beta_\infty(\gamma) &\geq \sum_{Q \in \Delta} \beta_A(Q)^2 |Q| \geq \sum_{Q \in \Delta, G_Q \neq \emptyset} \beta_A(Q)^2 |Q| \\ &\geq 2^{-n} \sum_{Q \in \Delta, G_Q \neq \emptyset} \#G_Q \beta_A(Q)^2 = 2^{-n} \sum_{Q' \in G} S(Q') \\ &\geq \frac{M}{2} 2^{-n} \#G = \frac{M}{2} \mathcal{H}^1(\pi^*(\cup G)) \geq \frac{Mp}{4}. \quad \square \end{aligned}$$

We can further relax the hypothesis using the following well known result, see Theorem 1.8 in [20].

Theorem 4. *There exists a universal constant $C > 1$ such that any continuum $K \subseteq \mathbb{R}^n$ with $\mathcal{H}^1(K) < +\infty$ is contained in a curve γ such that*

$$\mathcal{H}^1(K) \leq \mathcal{H}^1(\gamma) \leq C\mathcal{H}^1(K).$$

Another direction in which Lemma 2.2 can be relaxed, is by performing a change of coordinates.

Corollary 2.3. *Let $\varepsilon \in [0, 1)$ and $L > 1$. There exists $M > 0$ such that every compact set $K \subseteq \overline{B(0, 1)}$ such that $K \cup \partial B(0, 1) \cup \{0\}$ is connected and $\mathcal{H}^1(\|E\|) \leq \varepsilon$, where $E := \{x \in K : \beta_\infty^K(x) < M\}$, satisfies*

$$\mathcal{H}^1(K) > L.$$

Proof. Let $A := B(0, 1) \setminus \overline{(B(0, (1 - \varepsilon)/2))}$ and

$$\varphi : A \setminus \mathbb{R}_- \mapsto \left(\log \frac{1 - \varepsilon}{2}, 0 \right) \times (-\pi, \pi)$$

be defined by $\varphi(z) = \log z$. We may assume that $K \cap \mathbb{R}_- = \emptyset$, otherwise we replace φ by a lift and consider its image into $(\log \frac{1 - \varepsilon}{2}, 0) \times [-\pi, \pi]$.

We want to apply Lemma 2.2 to $K' := \varphi(K) \cup \{\log \frac{1-\varepsilon}{2}, 0\} \times [-\pi, \pi]$. Despite the fact that φ is not linear, we argue that there are uniform constants $\kappa, \sigma > 0$ such that for all $x \in K \cap A$

$$\beta_\infty^{K'}(\varphi(x)) \geq \kappa \beta_\infty^K(x) - \sigma.$$

Remark that φ is conformal. In general, we only need that the change of coordinates φ is locally invertible and of class $\mathcal{C}^{1+\alpha}$. Indeed, by the compactness of K , there exists a constant $\lambda > 0$ (that only depends on φ and K) such that for all $x, y \in K$

$$\text{dist}(\varphi(y), \varphi(x) + d\varphi_x(y - x)) \leq \lambda \text{dist}(x, y)^{1+\alpha},$$

where $d\varphi_x$ is the Jacobian of φ at x . On top of the linear distortion introduced by the Jacobian (uniformly bounded by the compactness of K , say by a constant $C > 0$), we have the following

$$\beta_{K'}(\varphi(x), t \|d\varphi_x\|) \gtrsim C \beta_K(x, t) - \lambda t^\alpha.$$

We obtain the existence of κ and σ by integrating over t and this completes the proof because φ preserves the length up to a multiplicative constant which depends solely on ε . \square

Length estimates and Ahlfors-David regularity.

Definition 2.4. A set $E \subset \mathbb{R}^d$ with $d \geq 2$ is *Ahlfors-David regular* if there exists $C > 1$ such that for all $x \in E$ and $0 < R < \text{diam } E$,

$$C^{-1}R \leq \mathcal{H}^1(E \cap B(x, R)) \leq CR.$$

We will use the following result due to the second author and Bishop (Theorem 1 in [11]).

Theorem 5. *There exists $C > 0$ such that if $K \subset \mathbb{R}^d$ with $d \geq 2$ is a compact set of diameter 1 and if for all $x \in K$,*

$$\beta_\infty^K(x) \leq M,$$

then K lies on a rectifiable curve Γ of length at most Ce^{CM} and which is Ahlfors-David regular with constant depending only on M .

The fact that Γ is Ahlfors-David regular is not stated in [11] but is an immediate consequence of the bound for the length.

Proof. As a limit of rectifiable curves with uniform bounded length containing K , we may assume that Γ realizes the infimum of the length of such curves. We consider a square Q centered at $x \in K$ and apply the theorem to the set $|Q|^{-1}(\overline{Q} \cap K)$. We obtain a curve Γ' of length at most Ce^{CM} . We connect the endpoints of $|Q|\Gamma'$ to ∂Q and take the union with ∂Q to obtain γ' . By the choice of Γ and our construction,

$$\mathcal{H}^1(\Gamma \cap Q) \leq \mathcal{H}^1(\gamma') \leq |Q|(5 + Ce^{CM}). \quad \square$$

2.1. Proof of Theorem 3 without the curvature estimate

We recall that $F = F(K)$ denotes the subset of K of points at which K is flat, see definition (11). Take a Borel set $E \subset K$ containing $F(K)$ with $\mathcal{H}^1(E) < +\infty$. Assume that $\mathcal{H}^1(K \setminus E) > 0$. The density theorem for \mathcal{H}^1 (see Theorem 6.2 in [35]) implies that for almost all points $x \in K \setminus E$ with respect to \mathcal{H}^1 ,

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^1(B(x, r) \cap E)}{2r} = 0.$$

Therefore, for any $\varepsilon > 0$ there is a ball $B = B(y, R)$, $y \in K$, $R > 0$ such that

$$\mathcal{H}^1(E \cap B) < \varepsilon R \text{ and } 2R \leq \text{diam } K.$$

We will construct a measure μ with the density properties as claimed in Theorem 3, supported on $B \cap K$, such that $\mu(E) = 0$. Let $B_0 := B$ and $0 < \varepsilon < \frac{1}{100}$.

The proof has three steps.

1. For any ball $B = B(x_B, R_B)$, $x_B \in K$, $2R_B \leq \text{diam } K$ and such that $\mathcal{H}^1(B \cap E) < \varepsilon R_B$ we construct a measure μ_B supported on B .
2. We construct the probability measure μ on B_0 .
3. We show that μ has the desired scaling properties and that $\mu(E) = 0$.

Step 1. Let $B = B(x_B, R_B)$ with $R_B \leq \text{diam } K$, $x_B \in K$, such that $\mathcal{H}^1(B \cap E) < \varepsilon R_B$. Let $K_B := K \cap \frac{1}{2}\overline{B}$ and observe that $K_B \cup \frac{1}{2}\partial B$ is a *continuum* (where $\lambda B(z, r) = B(z, \lambda r)$, $\lambda \overline{B}(z, r) = \overline{B}(z, \lambda r)$ and $\lambda \partial B(z, r) = \partial B(z, \lambda r)$, for any $z \in \mathbb{R}^d$ and $\lambda, r > 0$). For any $x \in K_B$, let

$$t_B(x) := \inf \left\{ r \in (0, R_B) : \int_r^{R_B} \beta_{K_B}^2(x, t) \frac{dt}{t} \leq M \right\},$$

where M is a large constant that will be specified later. Observe that $t_B(x) = 0$ if and only if $\int_0^{R_B} \beta_{K_B}^2(x, t) \frac{dt}{t} \leq M$. Let

$$Z(B) := \overline{\{x \in K_B : t_B(x) = 0\}}.$$

One can easily check that for any $x \in Z(B)$,

$$\int_0^{R_B} \beta_{Z(B)}^2(x, t) \frac{dt}{t} \leq \int_0^{R_B} \beta_{K_B}^2(x, t) \frac{dt}{t} \leq 4M,$$

therefore $Z(B) \subseteq F(K) \subseteq E$. We obtain that $\mathcal{H}^1(Z(B)) < \varepsilon R_B$.

We show that there exists $C_M > 0$, that depends only on M , and a measure μ_B supported on a small neighborhood of K_B such that

1. $R_B \leq \mu_B(\mathbb{C}) = \mu_B(B) \leq C_M R_B$,
 2. $\mu_B(B(x, r)) \leq C_M r$, for all $x \in \mathbb{C}$ and $r > 0$.
- (14)

By the standard covering lemma (see [28], page 2), there exists a countable set $X'(B) \subset K_B \setminus Z(B)$ such that

$$K_B \setminus Z(B) \subseteq \bigcup_{x \in X'(B)} B(x, 10t_B(x)),$$

and the balls $B(x, 2t_B(x))$, $x \in X'(B)$, are pairwise disjoint.

From now on we make a standing assumption (it is enough to take $M \geq 6 \log 10$) that for all $x \in X'(B)$,

$$t_B(x) \leq 10^{-6} R_B. \tag{15}$$

This ensures that for all $x \in X'(B)$,

$$B(x, 10t_B(x)) \subseteq B. \tag{16}$$

The following lemma provides a key estimate which allows to distribute the measure μ on the balls $B(x, t_B(x))$ centered at $X(B) \subseteq X'(B)$ and prove the scaling properties of μ in Step 3.

Lemma 2.5. *If M is sufficiently large, then there exists $X(B) \subseteq X'(B)$ such that for each $x \in X(B)$, $\mathcal{H}^1(B(x, t_B(x)) \cap E) < \frac{\varepsilon}{2} t_B(x)$,*

$$\sum_{x \in X(B)} t_B(x) \geq 10R_B,$$

and

$$\sum_{x \in X(B)} \mathcal{H}^1(B(x, t_B(x)) \cap E) < \frac{\varepsilon}{2} R_B.$$

It is not hard to check that the set $Y = Z(B) \cup X(B)$ is compact. Also, one can check that for every $x \in Y$, $\int_0^{R_B} \beta_Y^2(x, t) \frac{dt}{t} \leq 100 M$. By Theorem 5, the set Y is contained in an Ahlfors-David regular curve Γ_B with length comparable to R_B (and the multiplicative constant depending only on M). As the balls $B(x, 2t_B(x))$ are pairwise disjoint, we may assume that Γ_B contains a cross

$$G(x) := [x - t_B(x), x + t_B(x)] \cup [x - it_B(x), x + it_B(x)] \subseteq B$$

for every $x \in X'(B)$. $G(x) \subseteq B$ by relation (16). We define

$$\mu_B := \mathcal{H}^1|_{G_B}, \text{ where } G_B := \bigcup_{x \in X'(B)} G(x) \subseteq B.$$

By the properties of Γ_B it is easy to check that μ_B satisfies inequalities (14).

The following proof concludes Step 1.

Proof of Lemma 2.5. We use a geometric construction to show that if M is sufficiently large, then

$$\sum_{x \in X'(B)} t_B(x) \geq 23R_B. \tag{17}$$

Assuming (17), as $t_B(x) \leq 10^{-6}R_B$ for all $x \in X'(B)$, there is a partition of $X'(B) = X_1(B) \cup X_2(B)$ such that for each $i \in \{1, 2\}$,

$$\sum_{x \in X_i(B)} t_B(x) \geq 11R_B.$$

As $\mathcal{H}^1(B \cap E) < \varepsilon R_B$, using relation (16), there is $i_0 \in \{1, 2\}$ such that

$$\sum_{x \in X_{i_0}(B)} \mathcal{H}^1(B(x, t_B(x)) \cap E) < \frac{\varepsilon}{2} R_B.$$

Let $X(B) = \{x \in X_{i_0}(B) : \mathcal{H}^1(B(x, t_B(x)) \cap E) < \frac{\varepsilon}{2} t_B(x)\}$ and suppose that

$$\sum_{x \in X_{i_0}(B) \setminus X(B)} t_B(x) \geq R_B.$$

Then

$$\sum_{x \in X_{i_0}(B) \setminus X(B)} \mathcal{H}^1(B(x, t_B(x)) \cap E) \geq \frac{\varepsilon}{2} R_B,$$

which contradicts the choice of i_0 , as balls $B(x, t_B(x))$ with $x \in X_{i_0}(B)$ are disjoint.

In the sequel, we prove inequality (17). For every $x \in X'(B)$, let

$$H(x) := \partial B(x, 2t_B(x)) \cup \partial B(x, 10t_B(x)) \cup [x - 10t_B(x), x + 10t_B(x)]. \tag{18}$$

Then $H(x)$ is connected and $\mathcal{H}^1(H(x)) \leq 100t_B(x)$. Let

$$Z'(B) := Z(B) \setminus \bigcup_{x \in X'(B)} B(x, 10t_B(x)).$$

Observe that

$$K_B \subset Z'(B) \cup \bigcup_{x \in X'(B)} B(x, 10t_B(x)).$$

Let us define

$$S(B) := Z'(B) \cup \bigcup_{x \in X'(B)} H(x). \tag{19}$$

As $\mathcal{H}^1(Z'(B)) \leq 10^{-2}R_B$ and $\mathcal{H}^1(H(x)) \leq 100t_B(x)$, it is enough to show that $\mathcal{H}^1(S(B)) \geq 2301R_B$ in order to prove inequality (17). We may assume that for any $x \neq x' \in X'(B)$, $B(x, 10t_B(x)) \not\subset B(x', 10t_B(x'))$. The reader can therefore check that $S(B)$ is compact and that

$$S'(B) := \frac{1}{2}B \cap S(B) \cup \frac{1}{2}\partial B$$

is a *continuum*. We show that for every $x \in \lambda B \cap S'(B) \setminus Z'(B)$, with $\lambda := 1/2 - 1/10$,

$$\beta_\infty^{S'(B)}(x) \geq 10^{-3}M, \tag{20}$$

provided M is large enough. Having established the estimate (20), we can conclude the proof of inequality (17) by Corollary 2.3 applied to (a rescaled version of) $S'(B) \subseteq \frac{1}{2}\overline{B}$ with the exceptional set

$$Z'(B) \cup \left(\frac{1}{2}\overline{B} \setminus \lambda B \cap S'(B) \right),$$

$\varepsilon = 50^{-1} + 5^{-1}$, $L = 4602$ and M sufficiently large.

We still have to prove inequality (20). Let $y \in \lambda B \cap S'(B) \setminus Z'(B)$. We can find $x_0 \in X'(B) \cap \frac{1}{2}B$ such that

$$t_B(x_0) = \sup\{t_B(x) : x \in X'(B), |x - y| \leq 50t_B(x)\}.$$

We show that for $t \in (10^3t_B(x_0), R_B/10)$,

$$20 \beta_{S'(B)}(y, t) \geq \beta_{K_B}(x_0, t/10). \tag{21}$$

Observe that if for some $x \in X'(B) \cap \frac{1}{2}B$, $z \in B(x, 10t_B(x)) \cap B(y, t)$, then either $B(x, 10t_B(x)) \subseteq B(y, t)$ (including the case $x = x_0$) or

$$t > |z - y| > 40t_B(x), \text{ therefore } \text{dist}(y, \partial B(x, 10t_B(x))) \geq t/2. \tag{22}$$

By the choice of x_0 and t , $B(x_0, t/10) \subset B(y, t)$, and therefore

$$10 \beta_{K_B}(y, t) \geq \beta_{K_B}(x_0, t/10).$$

It is enough to prove that

$$2 \beta_{S'(B)}(y, t) \geq \beta_{K_B}(y, t). \tag{23}$$

Let L be a line minimizing $\beta_{S'(B)}(y, t)$ and take a point

$$z \in B(y, t) \cap (K_B \setminus S'(B)) \subseteq \frac{1}{2}B.$$

In particular, $z \in K_B \setminus Z'(B)$, so there exists $x \in X'(B)$ such that $z \in B(x, 10t_B(x))$. By inequality (22), there are two possibilities.

1. $\text{dist}(y, \partial B(x, 10t_B(x))) \geq t/2$ thus $\beta_{S'(B)}(y, t) \geq 1/2$; inequality (23) follows by the trivial bound $\beta_{K_B}(y, t) \leq 1$;
2. $B(x, 10t_B(x)) \subseteq B(y, t)$, and then $\partial B(x, 10t_B(x)) \subset S'(B)$ is contained in a $t\beta_{S'(B)}(y, t)$ neighborhood of L , so $\text{dist}(z, L) < t\beta_{S'(B)}(y, t)$.

Inequality (23) is now established.

For each $y \in S'(B) \setminus Z'(B)$, we integrate inequality (21) and obtain

$$\begin{aligned} \int_{10^3 t_B(x_0)}^{R_B/10} \beta_{S(B)}^2(y, t) \frac{dt}{t} &\geq 20^{-2} \int_{10^2 t_B(x_0)}^{R_B/100} \beta_{K_B}^2(x_0, t) \frac{dt}{t} \\ &\geq 20^{-2}(M - 4 \log 10) \\ &\geq 10^{-3}M, \end{aligned}$$

if M is large enough. This proves inequality (20). \square

Step 2. We define a sequence $(\mu_n)_{n \geq 0}$ of probability measures supported respectively on a decreasing sequence of compact neighborhoods of K , having K as their intersection. The measure μ is then a weak limit of this sequence.

For each $n \geq 0$, we define a collection of disjoint balls \mathcal{F}_n as follows. Let $\mathcal{F}_0 := \{B_0\}$ and

$$\mathcal{F}_{n+1} := \{B(x, t_B(x)) : x \in X(B), B \in \mathcal{F}_n\}.$$

Let $B_1, B_2 \in \mathcal{F}_n$, $B'_1, B'_2 \in \mathcal{F}_{n+1}$ such that $B'_1 = B(x, t_{B_1}(x))$ with $x \in X(B_1)$ and respectively $B'_2 = B(y, t_{B_2}(y))$ with $y \in X(B_2)$. As $2B_1$ and $2B_2$ are disjoint and $t_B(x) \leq 10^{-4}R_{B_1}$, $t_B(y) \leq 10^{-4}R_{B_2}$ we obtain that

$$B'_1 \cap B'_2 = \emptyset, B_1 \cap B'_2 = \emptyset \text{ and } B'_1 \cap B_2 = \emptyset.$$

We will use the measures μ_B constructed at the previous step to define inductively the sequence $(\mu_n)_{n \geq 0}$. Let

$$\mu_0 := \frac{\mu_{B_0}}{\mu_{B_0}(\mathbb{C})}.$$

Assume that μ_0, \dots, μ_n have been defined with the following properties.

- (P1) $\text{supp } \mu_n$ is contained in the closure of the disjoint union of balls in \mathcal{F}_n which is contained in a 10^{-n} -neighborhood of K .
- (P2) $\mathcal{H}^1(E \cap \text{supp } \mu_n) < 2^{-n} \varepsilon R_{B_0}$ by Lemma 2.5.
- (P3) $\mu_n(B) \leq 10^{-k} R_{B_0}^{-1} R_B$ if $B \in \mathcal{F}_k$ for some $k \leq n$.

We define μ_{n+1} in the following way. For any ball $B \in \mathcal{F}_{n+1}$,

$$\mu_{n+1|B} := \frac{\mu_n(B)}{\mu_B(B)} \mu_B. \tag{24}$$

Observe that $\mu_{n+1}(B) = \mu_n(B)$ and that by Lemma 2.5, for every $x \in X(B)$ we have

$$\frac{\mu_{n+1}(B(x, t_B(x)))}{\mu_{n+1}(B)} = \frac{t_B(x)}{\sum_{y \in X(B)} t_B(y)} \leq \frac{t_B(x)}{10R_B}. \tag{25}$$

Properties (P1), (P2) and (P3) for μ_{n+1} are direct consequences of the inequality (15) and Lemma 2.5 and the choice of $X(B) \subseteq X'(B)$ for any ball $B \in \mathcal{F}_{n+1}$.

Step 3. Observe that if $B \in \mathcal{F}_n$ is a ball of radius R_B and $R = R_{B_0}$, then

$$\mu(2B) = \mu(B) \leq 10^{-n} R^{-1} R_B. \tag{26}$$

We want to show that for every $x \in K$ and $r > 0$,

$$\mu(B(x, r)) \leq CR^{-1}r \exp \left(-C' \int_r^R \beta_K^2(x, t) \frac{dt}{t} \right), \tag{27}$$

where $C, C' > 0$ are universal constants.

It is enough to prove this bound for $x \in \text{supp } \mu$. Otherwise, we have either $\text{supp } \mu \cap B(x, r) = \emptyset$, so $\mu(B(x, r)) = 0$, or for some $y \in \text{supp } \mu \cap B(x, r)$, $B(x, r) \subset B(y, 2r)$, so

$$\mu(B(x, r)) \leq \mu(B(y, 2r)) \leq 2CR^{-1}r \exp \left(-C' \int_{2r}^R \beta_K(y, t)^2 \frac{dt}{t} \right).$$

As $|x - y| < r$, a simple computation shows that

$$1 + \int_r^R \beta_K^2(x, t) \frac{dt}{t} \simeq 1 + \int_{2r}^R \beta_K(y, t)^2 \frac{dt}{t},$$

with universal constants.

Fix $x \in \text{supp } \mu$ and $r > 0$. If $r \geq R$ there is nothing to prove. We have $\{x\} = \cap_{n \geq 0} B_n$, where $B_n \in \mathcal{F}_n$ for all $n > 0$. Let

$$N := \max\{n : B(x, r) \subseteq 2B_n\}.$$

A direct computation leads to

$$\int_r^R \beta_K^2(x, t) \frac{dt}{t} \lesssim (N + 4)M. \tag{28}$$

On the other hand, by inequality (26),

$$\mu(2B_N) \leq 10^{-N} R^{-1} R_{B_N}.$$

If $r \simeq R_{B_N}$, the last two inequalities and the properties (14) imply the conclusion (27).

Suppose that $r < 10^{-6} R_{B_N}$. For any $B \in \mathcal{F}_{N+1}$ which intersects $B(x, r)$, we have $R_B \leq 4r$, otherwise $B(x, r) \subseteq 2B$ which contradicts the choice of N . Thus $B \subseteq B(x, 10r)$. Denote $\mathcal{K} := \{B \in \mathcal{F}_{N+1} : B \cap B(x, r) \neq \emptyset\}$. Using the inequality (26) and the fact that Γ_{B_N} constructed in Step 1 is Ahlfors-David regular, we can estimate

$$\begin{aligned} \mu(B(x, r)) &\leq \sum_{B \in \mathcal{K}} \mu(B) \leq 10^{-N-1} R^{-1} \sum_{B \in \mathcal{K}} R_B \\ &\leq c 10^{-N-1} R^{-1} \mathcal{H}^1(\Gamma_{B_N} \cap B(x, 10r)) \\ &\leq c' 10^{-N} R^{-1} r, \end{aligned}$$

where $c, c' > 0$ are universal constants, as M is fixed. Combined with the inequality (28), this proves the conclusion (27).

By (P2), $\mathcal{H}^1(E \cap \text{supp } \mu) = 0$. The inequality (27) implies that

$$\mu(B(x, r)) \leq CR^{-1}r$$

for every ball $B(x, r)$, $x \in K$, $r \leq R$ and thus the measure μ vanishes on $E \supset F$.

2.2. Proof of Theorem 1

For simplicity, suppose that $\text{diam } K = 1$. Theorem 3 supplies a Radon probability measure μ with $\mu(W) = 1$ supported on a ball of radius $R > 0$.

For every $x \in W$ we define a measurable function $\beta(x) : W \mapsto [0, +\infty)$ by

$$\liminf_{r \rightarrow 0} \frac{\int_r^1 \beta_K^2(x, t) \frac{dt}{t}}{-\log r} = \beta^2(x).$$

For every $x \in W$ we have that

$$\begin{aligned} \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} &\geq \liminf_{r \rightarrow 0} \frac{-\log r + C' \int_r^R \beta_K^2(x, t) \frac{dt}{t}}{-\log r} \\ &= 1 + C' \liminf_{r \rightarrow 0} \frac{\int_r^1 \beta_K^2(x, t) \frac{dt}{t}}{-\log r} = 1 + C' \beta^2(x). \end{aligned}$$

The mass distribution principle implies that

$$\dim_H(K) \geq \dim_H(W') \geq 1 + C' \text{essup}_\mu \beta^2(x),$$

where $\text{essup}_\mu \beta^2(x)$ is an essential supremum of $\beta^2(x)$ with respect to μ . The proof of the theorem follows as $\text{essup}_\mu \beta(x) > 0$ if $\beta(x) > 0$ and $\text{essup}_\mu \beta(x) \geq \beta_0^2$ if $\beta(x) \geq \beta_0$ for every $x \in W$.

3. Menger curvature

In this section we complete the proof of Theorem 3. Let μ be the measure constructed in the first part of the proof. We show that there is a universal constant $C > 0$ such that

$$c_K^2(\mu) = \int \int \int_{B_0 B_0 B_0} c(x, y, z)^2 d\mu(x) d\mu(y) d\mu(z) \leq C \frac{\mu(B_0)}{(\text{diam } B_0)^2},$$

where $c(x, y, z)$ is the inverse of the radius of the circumcircle of the triangle $T(x, y, z)$ with vertices $x, y, z \in \mathbb{C}$, that is the Menger curvature of the triple x, y, z , [36]. We set $c(x, y, z) = 0$ shall any two points among x, y and z coincide.

By elementary geometry,

$$c(x, y, z) = \frac{4 \text{ area } T(x, y, z)}{|x - y||x - z||y - z|} = \frac{2 d(x, L_{y,z})}{|x - y||x - z|} \tag{29}$$

where $L_{y,z}$ is the line through y and z .

Discrete Menger curvature. Let us define a discrete version of c_K^2 . For every $n \in \mathbb{N}$ and every triple B_1, B_2, B_3 of balls, define $c(B_1, B_2, B_3)$ to be 0 if $B_1 = B_2 = B_3$ and $\sup_{x \in B_1, y \in B_2, z \in B_3} c(x, y, z)$ otherwise. We put

$$c_n^2(B) := \sum_{B_1, B_2, B_3 \in \mathcal{F}_n(B)} c(B_1, B_2, B_3)^2 \mu_n(B_1) \mu_n(B_2) \mu_n(B_3),$$

where $\mathcal{F}_n(B)$ stands for all balls from \mathcal{F}_n contained in some ball B .

For every x, y, z from the support of μ , choose the largest $n(x, y, z)$ such that $x, y, z \in B \in \mathcal{F}_n(x, y, z)$. In particular, the balls $B_x, B_y, B_z \in \mathcal{F}_{n(x, y, z)+1}$ containing x, y, z , respectively, can not all coincide, and

$$c(x, y, z) \leq c(B_x, B_y, B_z) < +\infty.$$

Consequently, since for every $B \in \mathcal{F}_n$, $\mu(B) = \mu_n(B)$,

$$\begin{aligned} c_K^2(\mu) &\leq \sum_{n \geq 0} \sum_{B \in \mathcal{F}_n} \sum_{\substack{B_1, B_2, B_3 \in \mathcal{F}_{n+1}(B) \\ \#\{B_1, B_2, B_3\} > 1}} \int \int \int_{B_1 B_2 B_3} c(x, y, z)^2 d\mu(x) d\mu(y) d\mu(z) \\ &\leq \sum_{n=0}^{\infty} \sum_{B \in \mathcal{F}_n} \sum_{B_1, B_2, B_3 \in \mathcal{F}_{n+1}(B)} c(B_1, B_2, B_3)^2 \mu(B_1) \mu(B_2) \mu(B_3) \\ &= \sum_{n=0}^{\infty} \sum_{B \in \mathcal{F}_n} c_{n+1}^2(B). \end{aligned} \tag{30}$$

By the definition of μ_n , on every $B' \in \mathcal{F}_n$, μ_n is equal to an affinely normalized restriction of \mathcal{H}^1 to the cross $G_{B'}$. We recall that $\Gamma_B \subset B$ is an Ahlfors-David regular curve passing through all crosses $G_{B'}$ in B . The Ahlfors-David regularity constant does not depend on B .

Our objective is to prove that the discrete $c(B_1, B_2, B_3)^2$ and its continuous counterpart,

$$\begin{aligned} c(B_1, B_2, B_3)^2 &:= \int \int \int_{B_1 B_2 B_3} c(x, y, z)^2 \frac{d\mu_n(x)}{\mu_n(B_1)} \frac{d\mu_n(y)}{\mu_n(B_2)} \frac{d\mu_n(z)}{\mu_n(B_3)} \\ &= \frac{1}{64} \int \int \int_{G_{B_1} G_{B_2} G_{B_3}} c(x, y, z)^2 \frac{d\mathcal{H}^1(x)}{R_1} \frac{d\mathcal{H}^1(y)}{R_2} \frac{d\mathcal{H}^1(z)}{R_3}, \end{aligned}$$

are universally comparable.

The last integral allows to define $c(B_1, B_2, B_3)^2$ for any triple of balls B_1, B_2, B_3 .

Observe that if balls $2B_1, 2B_2, 2B_3$ are pairwise disjoint then for every $i, j = 1, 2, 3$, $z_i, z'_i \in B_i$,

$$|z_i - z_j| \simeq |z'_i - z'_j| \tag{31}$$

and if we define

$$T(B_1, B_2, B_3)^2 := \frac{1}{64} \int_{G_{B_1}} \int_{G_{B_2}} \int_{G_{B_3}} \text{area } T(x, y, z)^2 \frac{d\mathcal{H}^1(x)}{R_1} \frac{d\mathcal{H}^1(y)}{R_2} \frac{d\mathcal{H}^1(z)}{R_3}$$

then by (29) and (31) for any $z_i \in B_i, i = 1, 2, 3$,

$$\prod_{i \neq j} |z_i - z_j| c(B_1, B_2, B_3) \simeq 4T(B_1, B_2, B_3). \tag{32}$$

We start with a distortion lemma.

Lemma 3.1. *Let $2B_1, 2B_2, 2B_3$ and $2B'_1, 2B'_2, 2B'_3$ be two triples of pairwise disjoint balls such that (i) $R'_i \simeq R_i$ and (ii) $\text{dist}(B_i, B'_i) \lesssim R'_i, i = 1, 2, 3$, then*

$$c(B_1, B_2, B_3) \simeq c(B'_1, B'_2, B'_3).$$

Proof. Without loss of generality, we may assume that $B'_2 = B_2$ and $B'_3 = B_3$. Choose any two points $y \in B_2$ and $z \in B_3$ and define

$$c(B_1, y, z)^2 = \frac{1}{64} \int_{G_{B_1}} c(x, y, z)^2 \frac{d\mathcal{H}^1(x)}{R_1}.$$

By (31), (32) and Fubini theorem, it is enough to prove that

$$T(B_1, y, z) \simeq T(B'_1, y, z)$$

where

$$T(B_1, y, z)^2 = \int_{G_{B_1}} \text{area } T(x, y, z)^2 \frac{d\mathcal{H}^1(x)}{R_1}.$$

A further reduction is possible because $R_1 \simeq R'_1$. If we put

$$L_{y,z}(B_1)^2 = \int_{G_{B_1}} d(x, L_{y,z})^2 d\mathcal{H}^1(x)$$

then $T(B_1, y, z)^2 = \frac{|y-z|^2}{4R_1} L_{y,z}(B_1)^2$ and the proof will follow once we show

$$L_{y,z}(B_1) \simeq L_{y,z}(B'_1).$$

Let $d(B_1, L_{y,z}) = \sup_{x \in B_1} d(x, L_{y,z})$. An explicit calculation gives

$$\int_{G_{B_1}} d(x, L_{y,z})^2 d\mathcal{H}^1(x) \simeq R_1 d(B_1, L_{y,z})^2$$

which ends the proof because (i) and (ii) imply $d(B_1, L_{y,z}) \simeq d(B'_1, L_{y,z})$. \square

A geometric intuition behind Proposition 1 is that when $2B_1, 2B_2, 2B_3$ are pairwise disjoint then $\mathfrak{c}(B_1, B_2, B_3)$ is equal to $1/r$, where r is the radius of the smallest Apollonius circle tangent to B_1, B_2, B_3 . If we vary the center of the Apollonius circle and its radius r , we have 3 independent parameters, and (33) should stay true as claimed below. Any other case can be reduced to the “disjoint case” through the distortion Lemma 3.1 and a simple calculation.

Proposition 1. *For every $B \in \mathcal{F}_{n-1}$ and every triple $B_1, B_2, B_3 \in \mathcal{F}_n(B)$,*

$$\mathfrak{c}(B_1, B_2, B_3) \lesssim c(B_1, B_2, B_3)$$

Proof. Recall that μ_n on $B \in \mathcal{F}_n$ is the restriction of \mathcal{H}^1 to the corresponding cross G_B up to a normalizing factor $\alpha(B)$.

Suppose that B_1, B_2, B_3 are contained in $B \in \mathcal{F}_{n-1}$. The proof is based on the observation that

$$\mathfrak{c}(z_1, z_2, z_3) \gtrsim \mathfrak{c}(B_1, B_2, B_3) \tag{33}$$

on large portions of $B_i, i = 1, 2, 3$, with respect to μ_n .

The proof falls into several cases.

1.) B_1, B_2, B_3 are disjoint. By construction, $2B_1, 2B_2, 2B_3$ are disjoint in this case. Choose $\xi_i \in \overline{B_i}, i = 1, 2, 3$, such that

$$\mathfrak{c}(\xi_1, \xi_2, \xi_3) = \mathfrak{c}(B_1, B_2, B_3).$$

By (31) and (32), it is enough to prove that

$$\text{area } T(\xi_1, \xi_2, \xi_3) \lesssim T(B_1, B_2, B_3). \tag{34}$$

Fix $\kappa = 100$.

Case I. Suppose that κB_1 , intersects the line L_{ξ_2, ξ_3} . Let us fix $y \in B_2$ and $z \in B_3$. We use estimates (29) and (31),

$$\int_{G_{B_1}} \text{area } T(x, y, z)^2 d\mathcal{H}^1(x) \gtrsim |\xi_2 - \xi_3|^2 \int_{G_{B_1}} d(x, L_{y,z})^2 d\mathcal{H}^1(x).$$

The integral on the right hand side is the smallest when G_{B_1} intersects the line $L_{y,z}$. An explicit calculation yields

$$\int_{G_{B_1}} d(x, L_{y,z})^2 d\mathcal{H}^1(x) \gtrsim R_1^3 .$$

Since κB_1 intersects L_{ξ_2, ξ_3} ,

$$\kappa R_1 |\xi_2 - \xi_3| \gtrsim \text{area } T(\xi_1, \xi_2, \xi_3)$$

and the proof follows.

Case II. We may assume now that every ball κB_i is disjoint from the line L_{ξ_j, ξ_k} , $i \neq j \neq k \neq i$. Suppose that $R_1 \leq R_2 \leq R_3$. By the choice of κ , R_3 is very small with comparison to the distance of B_3 to L_{ξ_1, ξ_2} . Therefore,

$$\begin{aligned} \int_{G_{B_3}} \text{area } T(\xi_1, \xi_2, z)^2 d\mathcal{H}^1(z) &\gtrsim R_3 d(\xi_3, L_{\xi_1, \xi_2})^2 |\xi_1 - \xi_2|^2 \\ &\gtrsim R_3 \text{area } T(\xi_1, \xi_2, \xi_3)^2 . \end{aligned}$$

Let us fix $x \in G_{B_1}$ and take a pencil of all lines passing by x and $y \in G_{B_2}$. The lines are all inside an angle with the vertex x and an aperture $\theta(x)$. We can replace $d\mathcal{H}^1(y)$ by $|\xi_1 - \xi_2|d\theta$ in the integral defining $T(B_1, B_2, B_3)$, possibly affecting its value by a multiplicative constant $\simeq 1$. Indeed, the lines $L_{x,y}$, $y \in G_{B_2}$, can be parameterized by $\theta \in [0, \theta(x)]$ and the map $y \mapsto \theta(y)$ is at most 2 to 1 for all $y \in G_{B_2}$.

If $x = \xi_1$ then the set of θ s such that for every $z \in B_3$,

$$d(z, L_{\xi_1, y}) \simeq d(\xi_3, L_{\xi_1, \xi_2}),$$

is of size comparable to $\theta(\xi_1)$. Indeed, if θ_0 corresponds to a line passing through ξ_1 and the center of B_3 and θ_{ξ_2} corresponds to L_{ξ_1, ξ_2} then

$$\theta(\xi_1) \lesssim |\theta_{\xi_2} - \theta_0|$$

because $d(z, L_{\xi_1, y})$ on $B_2 \times B_3$ takes its maximum, up to a multiplicative constant $\simeq 1$, for $y = \xi_2$ and $z = \xi_3$. Hence, for $x = \xi_1$,

$$\int_{G_{B_2}} \int_{G_{B_3}} \text{area } T(x, y, z)^2 d\mathcal{H}^1(y) d\mathcal{H}^1(z) \gtrsim R_2 R_3 \text{area } T(\xi_1, \xi_2, \xi_3)^2 . \tag{35}$$

Finally, we will vary x inside B_1 . If there is $y_x \in G_{B_2}$ such that L_{x, y_x} is parallel to L_{ξ_1, ξ_2} then the distance between B_3 and L_{x, y_x} is at least $(\kappa - 2)R_3$. Therefore, for any $z \in B_3$,

$$d(z, L_{x,y_x}) \geq d(z, L_{\xi_1,\xi_2}) - 2R_1 \simeq d(\xi_3, L_{\xi_1,\xi_2})$$

and, as in the case $x = \xi_1$, we can argue that there is a set of θ s of size comparable to $\theta(x)$ such that $\text{area } T(x, y(\theta), z) \simeq \text{area } T(\xi_1, \xi_2, \xi_3)$. If

$$\mathcal{H}^1\{x \in G_{B_1} : \exists y_x\} \gtrsim R_1,$$

Proposition 3.1 follows by integrating (35) over G_{B_1} with respect to $d\mathcal{H}^1(x)$. Otherwise, we can take a reflection B'_1 of B_1 with respect to L_{ξ_1,ξ_2} and observe that y_x exists for $x \in G_{B'_1}$ of \mathcal{H}^1 -measure comparable to R_1 . The proof of Proposition 1 follows from the distortion Lemma 3.1 applied to the triples B_1, B_2, B_3 and $\frac{1}{2}B'_1, B_2, B_3$.

2.) $B_1 = B_2 \neq B_3$. Points $\xi_i \in \overline{B_i}$, $i = 1, 3$, are chosen so that $|\xi_1 - \xi_3|$ is equal to the distance $\text{dist}(B_1, B_3)$ between B_1 and B_3 . A direct calculation shows that

$$c(B_1, B_1, B_3) \simeq |\xi_1 - \xi_3|^{-1}. \tag{36}$$

We can find two balls B'_1 and B'_2 contained in B_1 so that $2B'_1, 2B'_2$ are disjoint, $R'_1 = R'_2 = \frac{1}{4}R_1$ and such that the line through their centers is orthogonal to the line passing by the centers of B_1 and B_3 . Since $|\xi_1 - \xi_3| \geq R_1 + R_3$, we have that

$$\text{dist}(B'_1, B_3) \simeq \text{dist}(B'_2, B_3) \simeq |\xi_1 - \xi_3|. \tag{37}$$

By 1.), we can apply Proposition 1 to B'_1, B'_2, B_3 . Using (36) and (37),

$$\begin{aligned} c(B_1, B_1, B_3)^2 &\geq c(B'_1, B'_2, B_3)^2 \gtrsim c(B'_1, B'_2, B_3)^2 \\ &\geq (2R_1 + |\xi_1 - \xi_3|)^{-2} \gtrsim c(B_1, B_1, B_3)^2. \end{aligned}$$

The proof of Proposition 1 is now complete. \square

Menger curvature is bounded. We are ready to conclude the estimates.

If $B \in \mathcal{F}_{n-1}$ then $\mu(B) = \alpha_n(B)\text{diam } B$ and

$$\alpha_n(B) \leq 10^{-n+1}(\text{diam } B_0)^{-1}.$$

The first inequality in (38) follows from Proposition 1, we recall that G_{B_i} stands for the cross inside balls $B_i \subset B$, $i = 1, 2, 3$.

$$\begin{aligned} c_n^2(B) &\lesssim \sum_{B_1, B_2, B_3 \in \mathcal{F}_n(B)} \int_{G_{B_1}} \int_{G_{B_2}} \int_{G_{B_3}} c(x, y, z)^2 d\mu_n(x) d\mu_n(y) d\mu_n(z) \\ &\lesssim \sum_{B_1, B_2, B_3 \in \mathcal{F}_n(B)} \int_{G_{B_1}} \int_{G_{B_2}} \int_{G_{B_3}} c(x, y, z)^2 d\mu_n(x) d\mu_n(y) d\mu_n(z) \end{aligned}$$

$$\begin{aligned}
 &\leq \int_{\Gamma_B} \int_{\Gamma_B} \int_{\Gamma_B} c(x, y, z)^2 d\mu_n(x) d\mu_n(y) d\mu_n(z) \\
 &\leq \alpha_n(B)^3 \int_{\Gamma_B} \int_{\Gamma_B} \int_{\Gamma_B} c(x, y, z)^2 d\mathcal{H}^1(x) d\mathcal{H}^1(y) d\mathcal{H}^1(z) \\
 &\leq \alpha_n(B)^3 C(\Gamma_B) \text{diam } B \\
 &\leq \alpha_n(B)^2 C(\Gamma_B) \mu(B),
 \end{aligned} \tag{38}$$

where the constant $C(\Gamma_B)$ comes from estimate (2) and is independent from B and n .

Combining (30) and (38),

$$\begin{aligned}
 c_K^2(\mu) &\leq \sum_{n=0}^{\infty} \sum_{B \in \mathcal{F}_n} c_{n+1}^2(B) \\
 &\lesssim \sum_{n=0}^{\infty} \sum_{B \in \mathcal{F}_n} \alpha_n(B)^2 \mu(B) \\
 &\leq \sum_{n=0}^{\infty} 10^{-2n} \mu(B_0) (\text{diam } B_0)^{-2} \\
 &\lesssim \mu(B_0) (\text{diam } B_0)^{-2}.
 \end{aligned} \tag{39}$$

4. Appendix A

Universal version of μ . If we do not require that the measure μ from Theorem 3 has the support disjoint from F then the construction can be modified to obtain a uniform scaling property of μ .

The following result proves Theorem A when $\mathcal{H}^1(F(K)) = +\infty$ or $\mathcal{H}^1(K \setminus F(K)) = 0$.

Theorem 6. *Let $K \subset \mathbb{C}$ be a continuum of diameter 1. There exist universal constants $C, C' > 0$ and a Radon probability measure μ supported on K such that $c_K^2(\mu) \leq C$ and for all $x \in K$ and $r > 0$*

$$\mu(B(x, r)) \leq Cr \exp \left(-C' \int_r^1 \beta_K^2(x, t) \frac{dt}{t} \right).$$

Proof. The proof of Theorem 6 is very similar to the proof of Theorem 3 and is based on Lemma 2.2 and the constructions from the proof of Theorem 45 in [40], compare [18]. As in the proof of Theorem 3 we have three steps.

1. For any ball B centered on K we construct a measure μ_B .
2. We construct the probability measure μ on K .
3. We show that μ has the desired scaling properties.

The estimates from the third step are very much the same as in the proof of Theorem 3 (with B_0 replaced by K , $R := 1/2$, and the claims about $\mathcal{H}^1(E \cup \text{supp } \mu_j)$ and $\mathcal{H}^1(E \cap \text{supp } \mu)$ dropped). The first and the second steps are slightly different as they must account for the existence of non-wiggly parts of K . We use the notation from the proof of Theorem 3.

Step 1. Let $B = B(x_B, R_B)$ be a ball centered on K with $R_B \leq 1/2$. Let $K_B := K \cap \overline{B}$ and observe that $K_B \cup \partial B$ is a connected compact. We say that $B \in \mathcal{G}$ (respectively that $B \in \mathcal{B}$) if $\mathcal{H}^1(Z(B)) \geq \frac{R_B}{100}$ (respectively if $\mathcal{H}^1(Z(B)) < \frac{R_B}{100}$).

Assume first that $B \in \mathcal{G}$. Since $\int_0^{R_B} \beta_{Z(B)}^2(x, t) \frac{dt}{t} \leq 4M$ for any $x \in Z(B)$, by Theorem 5, the set $Z(B)$ is contained in an Ahlfors-David regular curve Γ_B whose regularity constant depends only on M and of length comparable to R_B . We set

$$\mu_B := \mathcal{H}^1|_{Z(B)}.$$

By the properties of Γ_B , there exists a constant $C_M > 0$ that depends only on M such that the conditions (14) are satisfied.

If $B \in \mathcal{B}$ then we can repeat the construction of μ_B from the first step of Theorem 3. The only modification is that we define μ on $G'_B := \bigcup_{x \in X'(B)} G(x)$ rather than on $G_B := \bigcup_{x \in X(B)} G(x)$, as it was the case in the proof of the Theorem 3. We put

$$\mu_B := \mathcal{H}^1|_{G'_B}$$

By the properties of Γ_B it is easy to check that μ_B satisfies the inequalities (14).

Step 2. We construct a sequence $(\mu_n)_{n \geq 0}$ of probability measures supported respectively on a decreasing sequence of compact neighborhoods of K , having K as their intersection. The measure μ is a weak limit of this sequence.

For each $n \geq 0$, we define a collection of disjoint balls \mathcal{F}_n as follows. Let $x_0, y_0 \in K$ be such that $|x_0 - y_0| = 1 = \text{diam } K$. Let $\mathcal{F}_0 := \{B(x_0, 1)\}$ and

$$\mathcal{F}_{n+1} := \{B(x, t_B(x)) : x \in X(B), B \in \mathcal{F}_n \cap \mathcal{B}\}.$$

Remarks. 1) If $\mathcal{F}_n \subseteq \mathcal{G}$ then $\mathcal{F}_{n+1} = \emptyset$.

2) Let $B_1, B_2 \in \mathcal{F}_n, B'_1, B'_2 \in \mathcal{F}_{n+1}$. As in the second step of the proof of Theorem 3 we obtain that

$$B'_1 \cap B'_2 = \emptyset, B_1 \cap B'_2 = \emptyset \text{ and } B'_1 \cap B_2 = \emptyset.$$

Let $\mathcal{G}_n := \mathcal{F}_n \cap \mathcal{G}$ and $\mathcal{B}_n := \mathcal{F}_n \cap \mathcal{B}$. We will use the measures μ_B constructed in the previous step to define inductively the sequence $(\mu_n)_{n \geq 0}$. Let

$$\mu_0 := \frac{\mu_B}{\mu_B(\mathbb{C})},$$

where $B = B(x_0, 1)$. Assume that μ_0, \dots, μ_n have been defined with the following properties.

(P1) $\text{supp } \mu_n$ is contained in the disjoint union of balls

$$\mathcal{F}_n \cup \bigcup_{k=0}^{n-1} \mathcal{G}_k,$$

which is contained in a 10^{-n} -neighborhood of K .

(P2) $\mu_n(B) \leq 10^{-k} R_B$ if $B \in \mathcal{F}_k$ for some $k \leq n$ (R_B is the radius of B).

Define μ_{n+1} in the following way. For any ball $B \in \mathcal{G}_k, k \leq n$,

$$\mu_{n+1|B} := \mu_n|_B.$$

For any ball $B \in \mathcal{F}_{n+1}$,

$$\mu_{n+1|B} := \frac{\mu_n(B)}{\mu_B(B)} \mu_B.$$

Properties (P1) and (P2) for μ_{n+1} are direct consequences of the inequalities (17) and (15), and the fact that $Z(B) \subset K$ for any ball B .

Menger curvature. For every x, y, z from the support of μ , choose the largest $n(x, y, z)$ such that $x, y, z \in B \in \mathcal{F}_{n(x,y,z)}$. The same argument as that leading to (30) applies to the current setting, the only difference is that we want to distinguish balls B from \mathcal{B}_n and \mathcal{G}_n , and using bound (39),

$$\begin{aligned} c_K^2(\mu) &\leq \sum_{n=0}^{+\infty} \sum_{B \in \mathcal{B}_n} c_{n+1}^2(B) + \sum_{n \geq 0} \sum_{B \in \mathcal{G}_n} \int_B \int_B \int_B c(x, y, z)^2 d\mu(x) d\mu(y) d\mu(z) \\ &\lesssim \mu(K) + \sum_{n \geq 0} \sum_{B \in \mathcal{G}_n} c_{Z(B)}^2(\mu_n), \end{aligned} \tag{40}$$

where

$$c_{Z(B)}^2(\mu_n) = \int_{Z(B)} \int_{Z(B)} \int_{Z(B)} c(x, y, z)^2 d\mu_n(x) d\mu_n(y) d\mu_n(z).$$

Let $B \in \mathcal{G}_n$ and $B' \in \mathcal{F}_{n-1}$ so that $B \subset B'$. Then $\mu_n = \alpha_{n-1}(B') \mu_B$ and $\alpha_{n-1}(B') \leq 10^{1-n} R_{B'}$, where $R_{B'}$ is the radius of $R_{B'}$. Every $Z(B)$ is contained in AD-regular curve Γ_B of length comparable to R_B and $\mu_B = \mathcal{H}^1|_{Z(B)}$. Hence, by estimate (2),

$$\begin{aligned} c_{Z(B)}^2(\mu_n) &\leq \alpha_{n-1}(B')^3 \int_{\Gamma_B} \int_{\Gamma_B} \int_{\Gamma_B} c(x, y, z)^2 d\mathcal{H}^1(x) d\mathcal{H}^1(y) d\mathcal{H}^1(z) \\ &\lesssim \alpha_{n-1}(B')^3 C(\Gamma_B) R_B \\ &\lesssim 10^{-n} \mu(B). \end{aligned} \tag{41}$$

Combining (40) and (41),

$$c_K^2(\mu) \lesssim \mu(K) + \sum_{n=1}^{+\infty} 10^{-n} \sum_{B \in \mathcal{B}_n} \mu(B) \lesssim \mu(K)$$

which completes the proof. \square

Corollary 4.1. *Let μ be supplied either by Theorem 3 or Theorem 6. Then μ is absolutely continuous with respect to 1-dimensional Hausdorff measure \mathcal{H}^1 .*

Proof. In both cases, for each measure μ , there exists a constant $M > 0$ such that for every ball $B(x, r)$, $x \in K$, $r < \text{diam } K$,

$$\mu(B(x, r)) < Mr. \quad (42)$$

A standard argument shows that $\mathcal{H}^1(A) = 0 \implies \mu(A) = 0$. \square

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