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# Optimal bounds for the analytical traveling salesman problem

Jacek Graczyk<sup>1</sup> and Nicolae Mihalache<sup>2,3</sup>

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## Abstract

We supply two optimal lower bounds for the solution of the analytical traveling salesman problem in terms of Jones'  $\beta$ -numbers. We prove a linear lower bound for the length of every connecting curve in the general case. The four corners Cantor set shows that the linear bound is attained. For connected compact sets, we prove an exponential lower bound for their length. The estimate is optimal by the work of Bishop and Jones [2]. Finally, we bridge connected and disconnected cases by taking into account multiplicity of orthogonal projections. Applications to other areas of mathematics are briefly discussed.

*Keywords:* Traveling salesman, Jones  $\beta$ -numbers, meta-heuristic algorithms

*2010 MSC:* 30F40, 30E10, 26B15, 28A78, 28A75, 05C38

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## 1. Introduction

### 1.1. Analytical version of traveling salesman problem.

The classical traveling salesman problem (TSP) is to find the shortest itinerary between a finite collection of cities whenever a system of roads connecting them is given. As it is well known, TSP is NP-complete even if it is restricted to grid-graphs. The analytical version of the problem (ATSP) consists in finding the shortest rectifiable curve up to a universal multiplicative constant containing a given compact set  $K \subset \mathbb{R}^2$ . The cardinal of  $K$  may be infinite but the 1-dimensional Hausdorff measure of  $K$  is finite. Jones [8] introduced the technique of  $\beta$ -numbers to study ATSP in Euclidean spaces. The idea was to analyse planar sets in a similar way as functions in  $L^2$ -theory.

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The approach resulted in several substantial contributions, see [12] and the references therein. The main results of our paper have applications both in geometric analysis and dynamical systems [6, 5], see Section 1.2. The proofs are short and self-contained except for classical estimates of Jones [8], see formula (1). The main technical novelty is a direct estimate of length of a crossing curve that wiggles in many scales, compare [12] and [6]. The proof of Theorem 2 uses Theorem 1 to shortcut usual arguments. Theorem 3 provides sharp estimates in both connected and disconnected settings, giving insight into the stability problem of meta-heuristic algorithms for a time dependent TSP [10].

**Definition 1.1.** *Let  $K$  be a bounded set in  $\mathbb{R}^2$ ,  $x \in K$  and  $r > 0$ . We define  $\beta_K(x, r)$  by*

$$\beta_K(x, r) := \inf_L \sup_{z \in K \cap B(x, r)} \frac{\text{dist}(z, L)}{r},$$

where the infimum is taken over all lines  $L$  in  $\mathbb{R}^2$ .

It is natural to expect that a connected set which wiggles in many scales should be of a large length. In [8], a global parameter  $\beta_K^2$  was introduced

$$\beta_K^2 = \int_0^\infty \int_{\mathbb{R}^2} \beta_K^2(x, r) r^{-2} dx dr$$

to prove that the shortest curve  $\Gamma$  containing a bounded set  $K \subset \mathbb{R}^2$  is of length comparable to

$$\text{diam } K + \beta_K^2. \tag{1}$$

Bishop and Jones proved in [2] that if  $\beta^2(x, r)/r$  integrate to a finite value for almost all points in the sense of 1-dimensional Lebesgue measure then the set is contained in a countable union of rectifiable curves. In this context, one can ask what the length of such a curve could be. Indeed, this question has a long history, see, for example, [2] where the following upper bound for the length of wiggly curves was obtained:

**Fact 1.1.** *There exists  $C > 0$  such that if  $K \subset \mathbb{R}^2$  is a compact set of diameter 1 and if for all  $x \in K$ ,*

$$\int_0^1 \beta_K^2(x, t) \frac{dt}{t} \leq M,$$

then  $K$  lies on a rectifiable curve  $\Gamma$  of length at most  $Ce^{CM}$ .

*Linear growth.*

Let us denote by  $\mathcal{H}^1$  the 1-Hausdorff measure and by  $\pi^*$  the orthogonal projection on  $\mathbb{R}$ . We prove the following results.

**Theorem 1.** *Let  $K \subseteq [0, 1]^2$  be a compact set,  $W \subseteq K$  a Borel set and  $p := \mathcal{H}^1(\pi^*(W))$ . Assume that for all  $x \in W$ ,*

$$\int_0^{\text{diam } K} \beta_K^2(x, t) \frac{dt}{t} \geq M > 0. \quad (2)$$

*If  $\gamma \subseteq \mathbb{R}^2$  is a continuum containing  $K$ , then*

$$\mathcal{H}^1(\gamma) \geq c(Mp + \text{diam } K),$$

*where  $c$  is a universal constant.*

Observe that the lower bound of Theorem 1 differs substantially from the upper bound of Fact 1.1. Still, our estimate is optimal, as the canonical finite approximations of the planar  $\frac{1}{4}$ -Cantor set exhibit linear growth with  $M$  (comparable with the number of steps in the construction).

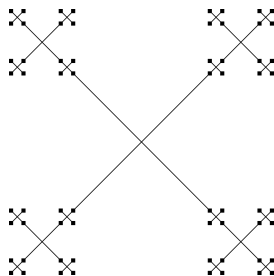


Figure 1: Squares represent cities, diagonals are connecting roads. For  $4^n$  cities, the shortest connecting curve has length of order  $n \sim M$ .

In Theorem 1,  $\mathcal{H}^1(\pi^*(W))$  can be replaced by the Favard length of  $W$  [12]. By Besicovitch theorem, cf Theorem 18.1 [11], purely unrectifiable compact of finite length projects in almost any direction to a set of measure zero. A compact  $K \supset \mathbb{C}$  is purely unrectifiable if it intersects every Lipschitz curve along a set of length 0. The planar  $\frac{1}{4}$ -Cantor set is an example of such set with at least two exceptional directions along which it projects into a closed interval. From the point of view of Theorem 1 the extremal case is when a compact of strictly positive and finite length has projections of zero length in every direction. This indeed can happen, see

Exemple 9.2 in [11]. A compact which is not  $\sigma$ -finite with respect to  $\mathcal{H}^1$  and projects into sets of null length in every direction was constructed in [9].

The current proof of Theorem 1 is simplified thanks to the idea from the dimension theory of real graphs which we learnt from C. Bishop [1, 6].

*Exponential growth.*

The following result shows that the exponential estimate of Fact 1.1 is optimal for connected compact sets.

**Theorem 2.** *There exists a constant  $c > 0$  such that for every continuum  $K \subseteq \mathbb{R}^2$  and  $M > 0$ , if for all  $x \in K$*

$$\int_0^{\text{diam } K} \beta_K^2(x, t) \frac{dt}{t} \geq M,$$

*then*

$$\mathcal{H}^1(K) \geq c e^{cM} \text{diam } K.$$

*Growth of length and clusters.*

We will derive a more general version of Theorems 1 and 2 by subdividing the wiggly portion  $W \subset K$ . In particular, the size of  $\pi^*(W)$  enters into the lower estimate of Theorem 3 counted with some multiplicity. The lower estimate in Theorem 1 depends on the length of  $\pi^*(W)$  which in many natural situations can be 0, for example when  $\#K < \infty$ . This problem is further discussed in Section 2.2, see formula (14).

For every  $x_0 \in \pi^*(W)$  and  $\varepsilon > 0$  define  $Z_\varepsilon(x_0)$  to be the cardinal of the largest  $\varepsilon$ -separated set contained in the vertical fiber  $W \cap \{x = x_0\}$ . We recall that a set is called  $\varepsilon$ -separated if every two points from it are at distance at least  $\varepsilon$ . Let  $Z_W(\alpha) := \int_{\pi^*(W)} Z_\varepsilon(x) dx$  with  $\varepsilon = e^{-\alpha}$ . Clearly, if a continuum  $\gamma$  contains  $K$  then  $\mathcal{H}^1(\gamma) \geq Z_\infty := \lim_{\alpha \rightarrow \infty} Z_W(\alpha)$ .

**Theorem 3.** *Let  $K \subseteq [0, 1]^2$  be a compact set,  $W \subseteq K$ . There is a universal constant  $c > 0$  so that for every  $q \in (0, 1)$ , if for all  $x \in W$ ,*

$$\int_0^1 \beta_K^2(x, t) \frac{dt}{t} > M > 0$$

*and  $\gamma \subseteq \mathbb{R}^2$  is a continuum containing  $K$ , then*

$$\mathcal{H}^1(\gamma) \geq c(1 - q)MZ_W(qM).$$

*If  $K$  is connected and  $W = K$  then*

$$\mathcal{H}^1(K) \geq c e^{c(1-q)M} Z_W(qM).$$

If the unit square  $[0, 1] \times [0, 1]$  is divided into  $\sim e^{2M}$  equal squares,  $M \gg 1$ , and  $K$  contains a circle of radius  $\sim e^{-M}$  in every square then Theorem 3 yields  $\mathcal{H}^1(\gamma) \geq c(1-q)Me^{qM}$  for any  $q \in (0, 1)$  while the shortest curve  $\gamma$  containing  $K$  is of length  $\mathcal{H}^1(\gamma) \lesssim e^M \sim Z_W(M)$ . By putting  $1 - q = \frac{C}{M}$ , we recover a sharp estimate  $\mathcal{H}^1(\gamma) \geq cZ_W(M - C) \sim Z_W(M)$ .

The situation becomes more complicated if every circle in every square of size  $\sim e^{-M}$  is replaced by the boundaries of  $4^M$  squares of the canonical approximation  $K_M$  of the conveniently scaled  $\frac{1}{4}$ -Cantor set (and rotated by  $\tan^{-1}(2)$  so each of them projects onto some interval). Theorem 3, applied with  $q = \frac{1}{2}$  and  $2M$  instead of  $M$ , yields a correct estimate for a curve  $\gamma \supset K$ ,  $\mathcal{H}^1(\gamma) \gtrsim Me^M$ . However, if we rotate each copy of  $K_M$  of size  $\sim e^{-M}$  by a different angle, the estimate breaks down, as the length of their projection may drop by a factor of  $2^M$ . Notice that  $Z_W(q2M)$  only counts the number of copies of  $K_M$ , that is scales above  $e^{-M}$ . The correct estimate can be easily recovered if we notice that the hypothesis and the claims of Theorem 3 are bi-Lipschitz invariant.

*Technical remark..* The orthogonal projection  $\pi$  in the claim of our theorems can be replaced by the projection along any smooth flow without singular elements traversing a neighborhood of  $K$ . The universal constants will become dependent on the flow.

## 1.2. Applications

*Geometric analysis and complex dynamics..* In [6], a version of Theorem 1 plays a crucial role in the inductive construction of Radon sublinear measures with bounded Menger curvature [12].

A lower bound for ASTP is one of the key ingredients in the proof of generalizations of Bishop and Jones result about the Hausdorff dimension of uniformly wiggly continua [3]. For example, Theorem 1 of [6] states that if continuum  $K \subset \mathbb{C}$  is the union of two subsets  $K = W \cup E$ ,  $\mathcal{H}^1(E) < \infty$  and  $\mathcal{H}^1(W) > 0$ , then

$$\dim_{\mathbb{H}}(K) \geq 1 + C \inf_{x \in W} \liminf_{r \rightarrow 0} \frac{\int_r^{\text{diam} K} \beta_K^2(x, t) \frac{dt}{t}}{-\log r}, \quad (3)$$

where  $C$  is a universal constant. To get an upper bound for  $\dim_{\mathbb{H}}(W)$  it is enough to increase  $C$  and replace  $\inf_{x \in W}$  by  $\sup_{x \in K}$  in (3), see Theorem 2 in [6]. The upper estimate is based on the exponential upper bound of ATSP from [2] and is used to prove new geometric results and contribute to the understanding of Yoccoz' problem about generic continuity of the Hausdorff

dimension of unfolding attractors/repellers in dynamical systems [5]. In dynamical setting, often  $\dim_{\text{H}}(K) = \dim_{\text{H}}(W)$ , see [7], which yields sharp estimates for the Hausdorff dimension. The motivation for (3) comes from dynamical systems where one cannot expect that generic attractors/repellers wiggle in every scale as required in [3]. The approach in [6] based on averaging along scales is well suited to tackle metric problems of non-uniformly hyperbolic systems, when only some control in many but not all scales is possible.

*Real time algorithms.* Theorems 1 and 2 indicate a strong dependence of the solutions of ATSP on the initial geometry. The hypothesis that  $M > 0$  is needed to guarantee a 2-dimensional character of the problem. In real-life applications, the cities may depend on time, they can be added to or removed from the book of orders (“on demand” version of TSP or time-dependent vehicle routing problem (VRP) [10]). In this situation, our estimates predict a bifurcation of the solutions of TSP from linear in  $M$ , when the cities form hierarchical and well separated clusters, see Figure 1, to the exponential in  $M$  when the cities are uniformly distributed or clusters have “high multiplicity” in the sense of Theorem 3. This phenomenon underlines the importance of meta-heuristics for time-dependent TSP or VRP problems. A usage of local parameters like  $\int_0^1 \beta^2(x, t) \frac{dt}{t}$  in the discrete form, see Definition 2.1, can make decomposition algorithms more efficient.

## 2. How to count squares

Let  $Q \subset \mathbb{C}$  be a square. Unless specified otherwise, squares are considered to contain only the left and top edges. Let  $|Q|$  denote the side length of  $Q$ . Let  $\Delta(Q)$  denote the set of all dyadic sub-squares of  $Q$  and  $\Delta_k(Q)$  denote the set of those with side length  $2^{-k}|Q|$ . For any  $\lambda > 0$ , let  $\lambda Q$  be the square with the same center as  $Q$  and with  $|\lambda Q| = \lambda|Q|$ . For any  $x \in Q_0$ , where  $Q_0 := [0, 1] \times [0, 1]$ , let

$$\Delta(x) = \{Q \in \Delta(Q_0) \mid x \in Q\}.$$

Let us also write  $\Delta$  for  $\Delta(Q_0)$  and  $\Delta_k$  for  $\Delta_k(Q_0)$ .

We need a discrete version of Definition 1.1.

**Definition 2.1.** *Let  $K$  be a compact set in the plane and  $Q$  a square such that  $Q \cap K \neq \emptyset$ . We define  $\beta_K(Q)$  by*

$$\beta_K(Q) := \inf_L \sup_{z \in K \cap 3Q} \frac{\text{dist}(z, L)}{|Q|},$$

where the infimum is taken over all lines  $L$  in the plane.

For a compact set  $K \subseteq Q_0$  is easy to check that

$$\beta_K^2(x) := \sum_{Q \in \Delta(x)} \beta_K(Q)^2 \simeq \int_0^1 \beta_K^2(x, t) \frac{dt}{t}. \quad (4)$$

We use the following reformulation of (1), if  $\gamma \subset Q_0$  is a rectifiable curve then

$$\beta_\infty^2(\gamma) := \sum_{Q \in \Delta} |Q| \beta_\gamma(Q)^2 \lesssim \mathcal{H}^1(\gamma). \quad (5)$$

### 2.1. Proof of Theorem 1

We can further relax the hypothesis of Theorem 1 using the following well known result (see for example [4]).

**Fact 2.1.** *There exists a universal constant  $C > 1$  such that any continuum  $K \subseteq \mathbb{R}^2$  with  $\mathcal{H}^1(K) < +\infty$  is contained in a curve  $\gamma$  such that*

$$\mathcal{H}^1(K) \leq \mathcal{H}^1(\gamma) \leq C \mathcal{H}^1(K).$$

To prove Theorem 1, by inequalities (1) and (5), it is sufficient to show that  $\beta_\infty(\gamma)$  is large enough. For all  $Q \in \Delta$ , let

$$S(Q) := \sum_{Q \subseteq Q' \in \Delta} \beta_K(Q')^2.$$

Denote  $\cup T = \cup_{Q \in T} Q$  when  $T$  is a collection of squares. For  $n \geq 1$ , let  $G_n := \{Q \in \Delta_n : S(Q) \geq M/2\}$ . Fix  $n \geq 1$  such that  $\mathcal{H}^1(\pi^*(\cup G_n)) \geq p/2$  and choose a minimal collection of squares  $G \subseteq G_n$  such that  $\pi^*(\cup G) = \pi^*(\cup G_n)$ . In particular,  $\mathcal{H}^1(\pi^*(\cup G)) = 2^{-n} \#G$  and for all  $Q \in \Delta_k, k \leq n$ , we have

$$\#G_Q \leq 2^n |Q|,$$

where  $G_Q := \{Q' \in G : Q' \subset Q\}$ . Therefore,

$$\begin{aligned} \beta_\infty(\gamma) &\geq \sum_{Q \in \Delta} \beta_K(Q)^2 |Q| \geq \sum_{G \ni Q' \subseteq Q \in \Delta} \beta_K(Q)^2 |Q| \\ &\geq 2^{-n} \sum_{G \ni Q' \subseteq Q \in \Delta} \#G_Q \beta_K(Q)^2 = 2^{-n} \sum_{Q' \in G} S(Q') \\ &\geq \frac{M}{2} 2^{-n} \#G = \frac{M}{2} \mathcal{H}^1(\pi^*(\cup G)) \geq \frac{Mp}{4}. \end{aligned}$$



## 2.2. Proof of Theorem 3

Let  $q \in (0, 1)$ . Without loss of generality  $qM \gtrsim 1$ , otherwise Theorem 3 is contained in Theorems 1 and 2. Also,  $(1 - q)M \gtrsim 1$ , otherwise we use the lower bound  $Z_\infty$  for  $\mathcal{H}^1(\gamma)$ . By definition, if  $W \subseteq K \subseteq [0, 1]^2$ , for all  $x \in [0, 1]$ ,  $Z_\varepsilon(x) \leq \frac{1}{\varepsilon} + 1$ , so  $Z_W(M) < e^M + 1$ . For every  $z \in W$ , let  $n(z) \in \mathbb{N}$  be minimal such that

$$\int_{2^{-n(z)}}^1 \beta_{K_r}^2(z, t) \frac{dt}{t} \geq \frac{1+q}{2} M. \quad (6)$$

We can find  $X \subset W$  such that  $r := \sup_{z \in X} n(z) + 10 < \infty$  and  $Z_X(qM) \geq \frac{1}{2} Z_W(qM)$ . Denote by  $Q_j(z)$  the square from  $\Delta_j$ ,  $j \leq r$ , containing  $z \in X$ ,  $G_j$  is the collection of all these squares  $Q_j(z)$ ,  $z \in X$ . Inequality (6) remains valid with  $2^{-n(z)}$  replaced by  $3|Q_r(z)|$ .

Let  $\gamma$  be a continuum containing  $K$  and  $W_j := \bigcup_{Q \in G_j} \gamma \cap 3Q$ . Since  $K_r := K \cup W_r \supset K$ , for every  $z \in W_r$ ,

$$\int_0^1 \beta_{K_r}^2(z, t) \frac{dt}{t} \geq \int_{3|Q(z)|}^1 \beta_K^2(z, t) \frac{dt}{t} \geq \frac{1+q}{2} M.$$

Let  $m$  be the largest integer such that  $2^{-m} \geq e^{-qM}$ , so  $r - m \gg 1$ . Then every  $Q$  from  $G_r$  is contained in some  $Q'$  from  $\Delta_m$ . If  $Q' \in \Delta_m$  then for every  $z \in W_r \cap Q'$ ,

$$\int_{|Q'|}^1 \beta_{K_r}^2(z, t) \frac{dt}{t} \leq qM$$

and

$$\int_0^{|Q'|} \beta_{K_r}^2(z, t) \frac{dt}{t} \geq \frac{1-q}{2} M. \quad (7)$$

If  $3Q \cap \gamma \neq \emptyset$ ,  $Q \in \Delta_m$ , then  $\gamma$  must traverse at least one dyadic square from  $\Delta_m$  contained in  $5Q$ . Let  $E(Q)$  be the set of all connected components of  $\gamma \cap 5Q$  that intersect  $3Q$ . For every  $\gamma_Q \in E(Q)$ ,  $\text{diam } \gamma_Q \geq |Q|$ . Since each square from  $G_m$  is counted at most 25 times,

$$\mathcal{H}^1(\gamma) \geq \frac{1}{25} \sum_{Q \in G_m} \mathcal{H}^1(\gamma \cap 5Q) \geq \frac{1}{25} \sum_{Q \in G_m} \sum_{\gamma_Q \in E(Q)} \mathcal{H}^1(\gamma_Q). \quad (8)$$

Let  $T(Q) := \partial Q \cup \partial 3Q \cup S$ , where  $S$  is a segment of length  $|Q|$  connecting  $\partial Q$  to  $\partial 3Q$ . Since  $T(Q) \cup \bigcup_{\gamma_Q \in E(Q)} \gamma_Q$  is connected and contains  $\gamma \cap 3Q$ , Theorem 1 and estimate (7) imply that

$$\sum_{\gamma_Q \in E(Q)} \mathcal{H}^1(\gamma_Q) + 17|Q| \gtrsim (1-q)M\mathcal{H}^1(\pi^*(Q \cap W_r)). \quad (9)$$

We have also a trivial estimate

$$\sum_{\gamma_Q \in E(Q)} \mathcal{H}^1(\gamma_Q) \geq |Q|,$$

that together with (9) yield

$$\sum_{\gamma_Q \in E(Q)} \mathcal{H}^1(\gamma_Q) \gtrsim (1-q)M\mathcal{H}^1(\pi^*(Q \cap W_r)). \quad (10)$$

Combining (8) and (10),

$$\begin{aligned} \frac{\mathcal{H}^1(\gamma)}{(1-q)M} &\gtrsim \sum_{Q \in G_m} \mathcal{H}^1(\pi^*(Q \cap W_r)) \\ &\gtrsim Z_{W_r}(m \log 2) \gtrsim Z_X(qM) \gtrsim Z_W(qM). \end{aligned} \quad (11)$$

If  $K$  is connected and  $W = K$  then we can apply Theorem 2 instead of Theorem 1 to get a better estimate than (10),

$$\sum_{\gamma_Q \in E(Q)} \mathcal{H}^1(\gamma_Q) \gtrsim e^{c(1-q)M}|Q|. \quad (12)$$

We may assume that  $K$  joins the left and the right edges of the unit square,  $\pi^*(K) \supset [0, 1]$ . In the estimates below,  $\gamma = K$  and the organization of squares through  $G_j$ ,  $j \leq r$ , is not used. Combining (8) and (12), similarly as it was done to obtain (11),

$$\mathcal{H}^1(\gamma) e^{-c(1-q)M} \gtrsim \sum_{Q \in \Delta_m: Q \cap K \neq \emptyset} |Q| \gtrsim Z_K(qM).$$

*Remark.* We can have a meaningful estimate of  $\mathcal{H}^1(\gamma)$  when every projection of  $W$  is of zero length. To this aim define  $P_r := \bigcup_{Q \in G_r} \partial Q$ . Every component of  $W_r$  that intersects  $Q \in G_r$  has length at least  $|Q| = \mathcal{H}^1(\pi^*(Q \cap P_r))$ . Hence, without loss of generality, we may assume that  $W_r \cap 3Q$  contains a horizontal segment of length  $|Q|$ . Let  $\hat{Q} \in G_m$ . Since each square  $Q \subset \hat{Q}$  from  $G_r$  is counted at most 9 times,

$$\mathcal{H}^1(\pi^*(Q \cap W_r)) \gtrsim \sum_{\hat{Q} \supset Q \in G_r} \mathcal{H}^1(\pi^*(3Q \cap W_r)) \gtrsim \mathcal{H}^1(\pi^*(\hat{Q} \cap P_r)). \quad (13)$$

Using (11) and adding (13) over all  $\hat{Q} \in G_m$ ,

$$\mathcal{H}^1(\gamma) \gtrsim (1-q)MZ_{P_r}(qM) > 0. \quad (14)$$

### 3. An inductive construction proves Theorem 2

The proof of Theorem 2 is by induction and relies on a construction à la Jones combined with an a priori estimate of Theorem 1.

Denote  $\mathbb{D} := B(0, 1)$  and by  $\lambda B(x, r) := B(x, \lambda r)$  for all  $x \in \mathbb{R}^2$  and  $\lambda, r > 0$ . The following technical result is the main inductive step.

**Proposition 1.** *There exists  $N \geq 1$  with the following property. Let  $S \subseteq 2\overline{\mathbb{D}}$  be a compact set containing 0 and let  $S' := S \cap \overline{\mathbb{D}}$ . If  $S' \cup \partial\mathbb{D}$  is connected and for all  $x \in S'$*

$$\int_0^1 \beta_S^2(x, t) \frac{dt}{t} \geq N,$$

*then there exists a collection of balls  $B_i := B(x_i, r_i)$ ,  $i = 1, \dots, n$ , with the following properties:*

- (i) *for all  $1 \leq i \leq n$ ,  $x_i \in S'$  and  $r_i < 10^{-3}$ ,*
- (ii)  *$S' \subseteq \bigcup_{i=1}^n 10B_i$  and the balls  $2B_i$  are disjoint,*
- (iii) *for all  $i$  and  $y \in S \cap 10B_i$ ,  $\int_{r_i}^1 \beta_S^2(y, t) \frac{dt}{t} \leq N$ ,*
- (iv)  *$\sum_{i=1}^n r_i > 10$ .*

*Proof of Theorem 2.* Theorem 2 is an immediate consequence of Proposition 1. Let  $x, y \in K$  be such that  $|x - y| = \text{diam } K$ . By rescaling, we can normalize  $K$  so that  $x = 0$  and  $y = 1$ . Denote  $S_0 := K$  and define  $S_1^{i_1} := S_0 \cap 2\overline{B_{i_1}}$ , where  $B_{i_1}$  are balls produced by Proposition 1 for  $S_0$ . For consistency put  $B_1^{i_1} := B_{i_1}$ . We continue by induction, normalizing and applying Proposition 1 to every already constructed compact  $S_1^{i_1}$ ,  $i_1 = 1, \dots, m_1$ , at least  $p := [M/N] - 1$  times, where  $[\alpha]$  is the integer part of  $\alpha$ . We end up with a collection  $(B_p^{k_p})_{k_p=1, \dots, m_p}$  of disjoint balls, centered at points of  $S_0$  with the sum of the radii at least  $10^p$ , by (iv) of Proposition 1. As  $S_0$  is a continuum, setting  $N \geq 10 \log 10$  and choosing  $c := \frac{\log 10}{N}$  proves Theorem 2.

*Basic continuity properties of  $\beta$ -numbers.* Before proving Proposition 1, we gather some basic facts about  $\beta$ -numbers. Let  $x, y \in \mathbb{R}^2$ ,  $A \subseteq \mathbb{R}^2$  and  $r, r' > 0$  such that  $B(x, r) \subseteq B(y, r')$ . Then

$$\frac{r}{r'} \beta_A(x, r) \leq \beta_A(y, r').$$

As  $\beta$  numbers are bounded by 1, for all  $\alpha > 1$  we have

$$\int_r^{\alpha r} \beta_A^2(x, t) \frac{dt}{t} \leq \log \alpha. \quad (15)$$

Combining the above inequalities, if  $|x - y| \leq r$  and  $R > r$ ,

$$\int_r^R \beta_A^2(x, t) \frac{dt}{t} \leq 3 + 2 \int_r^R \beta_A^2(y, t) \frac{dt}{t}. \quad (16)$$

**Lemma 3.1.** *Let  $A \subseteq \bigcup_{i=1}^n B(x_i, r_i)$ ,  $x \in A$  such that  $2r_i \leq r$  for all  $i = 1, \dots, n$ . Then*

$$\frac{\sqrt{3}}{4} \beta_A(x, r) \leq \beta_D(x, r), \quad (17)$$

where  $D = \{x\} \cup \bigcup_{i=1}^n \partial B(x_i, r_i)$ .

*Proof.* The closure  $H$  of the  $r\beta_D(x, r)$ -neighborhood of some line  $L$  includes  $B(x, r) \cap D \ni x$ . If  $\beta_A(x, r) > \beta_D(x, r)$ , then  $A \not\subseteq H$ . Otherwise, there is nothing to prove. As  $H$  is convex, it contains  $\text{Conv}(B(x, r) \cap D)$ , the convex hull of  $B(x, r) \cap D$ . Fix  $y \in A \setminus H$  and let  $q$  be such that  $y \in B_q := B(x_q, r_q)$ .

As  $y \in B_q$  but  $y \notin \text{Conv}(B(x, r) \cap \partial B_q)$ ,  $\partial B(x, r) \cap \partial B_q = \{a, b\} \subset H$ . There are two points  $a', b' \in \partial B(x, r) \cap \partial H$  in the closed arc  $\overline{(ab)}$  of  $\partial B(x, r)$ , contained in  $\overline{B_q}$ . As  $\text{dist}(a, b) \leq 2r_q \leq r$ , we have

$$\beta_D(x, r) \geq \frac{\text{dist}(x, [a', b'])}{2r} \geq \frac{\text{dist}(x, [a, b])}{2r} \geq \frac{\sqrt{3}}{4}.$$

The trivial bound  $\beta_A(x, r) \leq 1$  completes the proof. □

*Proof of Proposition 1.* Let  $T > 2 \log 10$  be a large number that will be specified later and let  $N := 2T + 6$ . For all  $x \in S'$  we define

$$r(x) := \inf \left\{ r \in (0, 1) : \int_{10r}^1 \beta_S^2(x, t) \frac{dt}{t} \leq T \right\}.$$

By the hypothesis, for all  $x \in S'$ ,  $r(x) > 0$ . By compactness and a standard covering lemma, there exist  $x_i \in S'$ ,  $i = 1, \dots, n$  such that

$$S' \subseteq \bigcup_{i=1}^n B(x_i, 10r(x_i)),$$

and the balls  $B(x_i, 2r(x_i))$ ,  $i = 1, \dots, n$ , are pairwise disjoint. We denote  $r_i = r(x_i)$  and  $B_i = B(x_i, r_i)$  and prove the proposition for this collection of balls. By the lower bound for  $T$ , inequality (15) and the above construction, the first two claims are satisfied. By the choice of  $N$  and inequalities (15) and (16), the third claim is also satisfied.

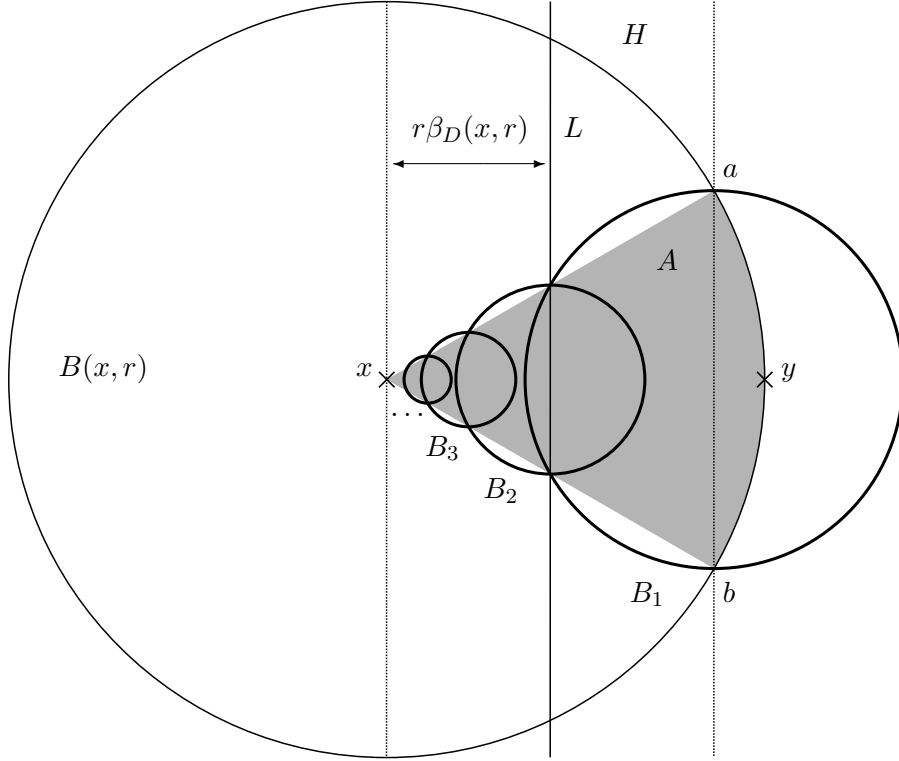


Figure 2: Extremal case for inequality (17)

We only need to prove that we can choose some universal  $T > 10$  such that  $\sum_{i=1}^n r_i > 10$ . For every  $i = 1, \dots, n$  let

$$H_i := \partial(10B_i) \cup [x_i - 10r_i, x_i + 10r_i],$$

where  $[x_i - 10r_i, x_i + 10r_i]$  is the horizontal diameter of  $10B_i$ . Observe that for all  $i$

$$\mathcal{H}^1(H_i) < 100r_i. \quad (18)$$

We may assume that the cover  $(10B_i)_{i=1}^n$  of  $S'$  is minimal, that is, if  $i \neq j$ , then  $10B_i \not\subseteq 10B_j$ . Let  $H := \bigcup_{i=1}^n H_i$ . Observe that  $\Gamma := H \cup \partial\mathbb{D}$  is a *continuum*. Let  $W := H \cap \frac{1}{2}\overline{\mathbb{D}}$ . We show that for all  $x \in W$

$$\int_0^1 \beta_H^2(x, t) \frac{dt}{t} \geq \frac{3}{32}T - 1. \quad (19)$$

As  $\Gamma$  is a *continuum*, there exists a line  $L \ni 0$  such that  $\mathcal{H}^1(\pi_L(W)) \geq \frac{1}{2} - 10^{-3}$ . By Theorem 1, this means that if  $T$  is large enough,

$$\mathcal{H}^1(\Gamma) \geq 1000 + 2\pi.$$

By inequality (18), we obtain  $\sum_{i=1}^n r_i > 10$  and the proof is complete.

To show inequality (19), let  $x \in W$  and choose  $1 \leq k \leq n$  such that  $B_k$  is the largest ball such that  $x \in 30B_k$ . By inequality (17),

$$\int_0^1 \beta_H^2(x, t) \frac{dt}{t} \geq \int_{30r_k}^{\frac{1}{3}} \beta_H^2(x, t) \frac{dt}{t} \geq \frac{3}{16} \int_{30r_k}^{\frac{1}{3}} \beta_{S'}^2(x, t) \frac{dt}{t}.$$

By inequalities (16) and (15) and the choice of  $r_k$ ,

$$\int_{30r_k}^{\frac{1}{3}} \beta_{S'}^2(x, t) \frac{dt}{t} \geq \frac{1}{2} \int_{30r_k}^{\frac{1}{3}} \beta_{S'}^2(x_k, t) \frac{dt}{t} - \frac{3}{2} \geq \frac{1}{2}(T - 2 \log 3 - 3),$$

because  $r_k < 10^{-3}$ ,  $x_k \in \frac{2}{3}\mathbb{D}$ , hence

$$\int_{30r_k}^{\frac{1}{3}} \beta_{S'}^2(x_k, t) \frac{dt}{t} = \int_{30r_k}^{\frac{1}{3}} \beta_S^2(x_k, t) \frac{dt}{t}.$$

□

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