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# Hidden asymptotics for the weak solutions of the strongly stratified Boussinesq system without rotation

Frédéric Charve\*

## Abstract

The asymptotics of the strongly stratified Boussinesq system when the Froude number goes to zero have been previously investigated, but the resulting limit system surprisingly did not depend on the thermal diffusivity  $\nu'$ . In this article we obtain richer asymptotics (depending on  $\nu'$ ) and we ask less assumptions on the initial data.

As for the rotating fluids system, the only way to reach these limits consists in considering non conventional initial data: to a function classically depending on the full space variable, we add a second one only depending on the vertical coordinate.

Thanks to a refined study of the structure of the limit system and to adapted Strichartz estimates, we obtain convergence in the context of weak Leray-type solutions providing explicit rates of convergence when possible. In the simpler case  $\nu = \nu'$  we are able to improve the Strichartz estimates and the convergence rates.

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**Keywords:** Geophysical incompressible fluids, Strichartz estimates, Besov and Sobolev spaces.

## 1 Introduction

### 1.1 Geophysical fluids: Primitive and Rotating fluids systems

It is commonly known that Geophysical fluids dynamics are greatly influenced by two concurrent "forces": the Coriolis force created by the rotation of the Earth around its axis, and the vertical stratification of the density induced by gravity. Both of these forces create rigidities whose influence on the dynamics is quantified by two quantities introduced by Physicists: the Rossby  $Ro$  and Froude  $Fr$  numbers. The smaller they are, the greater are the actions of these rigidities on the fluid dynamics.

Let us present a few models taking these forces into account. On one hand, the Primitive System (sometimes also called Primitive Equations) that we present here in the whole space and in the particular regime where both phenomena are of the same scale (that is we choose  $Ro = \varepsilon$  and  $Fr = \varepsilon F$  with  $F > 0$ ):

$$\begin{cases} \partial_t U_\varepsilon + v_\varepsilon \cdot \nabla U_\varepsilon - LU_\varepsilon + \frac{1}{\varepsilon} \mathcal{A}U_\varepsilon = \frac{1}{\varepsilon} (-\nabla \Phi_\varepsilon, 0), \\ \operatorname{div} v_\varepsilon = 0, \\ U_\varepsilon|_{t=0} = U_{0,\varepsilon}. \end{cases} \quad (PE_\varepsilon)$$

The unknowns are  $U_\varepsilon = (v_\varepsilon, \theta_\varepsilon) = (v_\varepsilon^1, v_\varepsilon^2, v_\varepsilon^3, \theta_\varepsilon)$ , where  $v_\varepsilon$  denotes the velocity of the fluid and  $\theta_\varepsilon$  the scalar potential temperature (linked to the density, temperature and salinity), and  $\Phi_\varepsilon$ , which

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is called the geopotential and contains in particular the pressure term and the centrifugal force. The diffusion operator  $L$  takes into account two heat regularization effects and is defined by

$$LU_\varepsilon \stackrel{\text{def}}{=} (\nu\Delta v_\varepsilon, \nu'\Delta\theta_\varepsilon),$$

where  $\nu, \nu' > 0$  respectively denote the kinematic viscosity and thermal diffusivity (both will be considered as viscosities in what follows). The last term  $\varepsilon^{-1}\mathcal{A}$  gathers the rotation and stratification effects and the matrix  $\mathcal{A}$  is defined by

$$\mathcal{A} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix}.$$

On the other hand, the Rotating fluids system only considers rotationnal effects and can be seen as deduced from the previous system only considering the velocity (so we get rid of the last line and column of  $\mathcal{A}$ ), it is written as follows:

$$\begin{cases} \partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon - \nu\Delta v_\varepsilon + \frac{e_3 \wedge v_\varepsilon}{\varepsilon} = -\nabla p_\varepsilon, \\ \operatorname{div} v_\varepsilon = 0, \\ v_\varepsilon|_{t=0} = v_0. \end{cases} \quad (RF_\varepsilon)$$

Both of these systems are variations of the famous Navier-Stokes system, but each of them shows better behaviour induced by the special structure brought by their respective limit systems as  $\varepsilon$  goes to zero: the QG/oscillating structure for  $(PE_\varepsilon)$ , and the 2D-3D structure for  $(RF_\varepsilon)$ . We refer to [18, 20, 23, 4, 12, 14] for more details. For a physical presentation of the models we refer to [21, 24, 25] as well as the introduction of [3]. We refer to [13, 14] for a small survey presenting recent articles devoted to Systems  $(RF_\varepsilon)$  and  $(PE_\varepsilon)$ .

We will use the same notations as in [12, 13]: for  $s \in \mathbb{R}$  and  $T > 0$  we define the spaces:

$$\begin{cases} \dot{E}_T^s = \mathcal{C}_T(\dot{H}^s(\mathbb{R}^3)) \cap L_T^2(\dot{H}^{s+1}(\mathbb{R}^3)), \\ \dot{B}_T^s = \mathcal{C}_T(\dot{B}_{2,1}^s(\mathbb{R})) \cap L_T^1(\dot{B}_{2,1}^{s+2}(\mathbb{R})), \end{cases}$$

endowed with the following norms (For  $(PE_\varepsilon)$ ,  $\nu_0 = \min(\nu, \nu')$ ):

$$\begin{cases} \|f\|_{\dot{E}_T^s}^2 \stackrel{\text{def}}{=} \|f\|_{L_T^\infty \dot{H}^s}^2 + \nu_0 \int_0^T \|f(\tau)\|_{\dot{H}^{s+1}}^2 d\tau, \\ \|f\|_{\dot{B}_T^s} \stackrel{\text{def}}{=} \|f\|_{L_T^\infty \dot{B}_{2,1}^s} + \nu' \int_0^T \|f(\tau)\|_{\dot{B}_{2,1}^{s+2}} d\tau, \end{cases}$$

where  $H^s(\mathbb{R}^3)$ ,  $\dot{H}^s(\mathbb{R}^3)$  and  $\dot{B}_{2,1}^s(\mathbb{R})$  respectively denote the inhomogeneous and homogeneous Sobolev spaces of index  $s \in \mathbb{R}$  and the homogeneous Besov space of indices  $(s, 2, 1)$ .

When  $T = \infty$  we simply write  $\dot{E}^s$  or  $\dot{B}^s$  and the corresponding norms are understood as taken over  $\mathbb{R}_+$  in time.

**Notation:** For a  $\mathbb{R}^3$  or  $\mathbb{R}^4$ -valued vector field, we will write  $f^h = (f^1, f^2)$  and will define  $f \cdot \nabla f = \sum_{i=1}^3 f^i \partial_i f$ . So that for instance we will indifferently write  $v_\varepsilon \cdot \nabla U_\varepsilon = U_\varepsilon \cdot \nabla U_\varepsilon$ .

## 1.2 Strongly stratified Boussinesq system without rotation

In this article we will focus on the following system, that only takes into account the stratification aspects. Here we will denote as  $Fr = \varepsilon$  the Froude number (this corresponds to  $\varepsilon F$  for the

Primitive system ( $PE_\varepsilon$ ) and the system is written as follows:

$$\begin{cases} \partial_t U_\varepsilon + v_\varepsilon \cdot \nabla U_\varepsilon - LU_\varepsilon + \frac{1}{\varepsilon} \mathcal{B} U_\varepsilon = \frac{1}{\varepsilon} (-\nabla \Phi_\varepsilon, 0), \\ \operatorname{div} v_\varepsilon = 0, \\ U_\varepsilon|_{t=0} = U_{0,\varepsilon}. \end{cases} \quad (S_\varepsilon)$$

The unknowns are the same as in the previous section  $U_\varepsilon = (v_\varepsilon, \theta_\varepsilon) = (v_\varepsilon^1, v_\varepsilon^2, v_\varepsilon^3, \theta_\varepsilon)$  and  $\Phi_\varepsilon$  (in what follows we will explicitly decompose it as the sum of the pressure term and another penalized pressure term that could be seen as an analogous of the centrifugal force). The diffusion operator  $L$  is also defined as previously with  $\nu, \nu' > 0$ . The last term  $\varepsilon^{-1} \mathcal{B}$  only takes into account stratification effects and the matrix  $\mathcal{B}$  is still skewsymmetric and defined by:

$$\mathcal{B} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We emphasize that System ( $S_\varepsilon$ ) is related to the following well-known Boussinesq system:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u - \kappa^2 \rho e_3 = -\nabla p, \\ \partial_t \rho + u \cdot \nabla \rho - \mu \Delta \rho = 0, \\ \operatorname{div} u = 0. \end{cases} \quad (1.1)$$

Indeed, as explained in [43] (we also refer to [34, 26]), observing that for  $\lambda > 0$ ,  $\bar{U}_\lambda(x) = (0, 0, 0, \lambda^2 x_3)$  is an explicit solution of (1.1) we can study solutions of (1.1) around  $\bar{U}_\lambda$  in the regime  $\kappa\lambda = \frac{1}{\varepsilon}$ . Performing the change of unknown function  $T = \frac{\kappa}{\lambda} \theta$ , leads to ( $S_\varepsilon$ ).

Rather than (1.1) (to which is devoted a huge literature), we will focus on ( $S_\varepsilon$ ) and begin with a brief review about recent works dedicated to this system or its inviscid version. Let us first mention that in [43], K. Widmayer considers a solution  $U_\varepsilon$  of ( $S_\varepsilon$ ) (in the setting  $\nu = \nu' = 0$ ) which belongs to  $\mathcal{C}([0, T], H^N)$  (for some  $T > 0$  and  $N \geq 6$ ) with  $\|U_\varepsilon\|_{L_T^\infty H^N} \leq C$  (uniformly in  $\varepsilon$ ). If in addition the initial data is independent of  $\varepsilon$  and satisfies  $\|U_0\|_{W^{5,1}} < \infty$ , then the solution can be decomposed into two parts  $U_\varepsilon = (U_\varepsilon^1, 0, 0) + U_\varepsilon^2$  such that for all  $t > 0$ :

$$\begin{cases} U_\varepsilon^2(t) \xrightarrow{\varepsilon \rightarrow 0} 0 \text{ in } W^{1,\infty}(\mathbb{R}^3), \\ U_\varepsilon^1(t) \xrightarrow{\varepsilon \rightarrow 0} \bar{u}(t) = (\bar{u}^1(t), \bar{u}^2(t)) \text{ in } L^2, \end{cases}$$

where  $\bar{u} : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^2$  solves the following two-dimensional incompressible Euler system (defining  $\mathbb{P}_0$  as the projector onto two-component functions with zero horizontal divergence):

$$\begin{cases} \partial_t \bar{u} + \bar{u} \cdot \nabla_h \bar{u} = -\nabla_h \bar{p}, \\ \operatorname{div}_h \bar{u} = 0, \\ \bar{u}|_{t=0} = \mathbb{P}_0(U_0)^h, \end{cases} \quad (1.2)$$

This result was later improved by R. Takada in [42] for a divergence-free initial data in  $H^{s+4}(\mathbb{R}^3)$  ( $s \geq 3$ ) with precise statement of the global existence of the solution  $U_\varepsilon$  and an explicit convergence rate ( $q \in [4, \infty]$ ):

$$\|U_\varepsilon - (\bar{u}, 0, 0)\|_{L_T^q W^{1,\infty}(\mathbb{R}^3)} \leq C_{T,q,s} \|U_0\|_{H^{s+4}(\mathbb{R}^3)} \varepsilon^{\frac{1}{q}}.$$

Concerning the viscous system, let us begin with [34]: S. Lee and R. Takada study, in the particular setting  $\nu = \nu'$ , the global existence of strong solutions when  $\varepsilon$  is small enough and for

more regular initial data (possibly large). More precisely, rewording their result with the notations we will introduce in the present article: they prove that if  $s \in ]\frac{1}{2}, \frac{5}{8}]$  there exist  $\delta_1, \delta_2 > 0$  such that for any initial data  $U_0$  such that  $\mathbb{P}_2 U_0 \in \dot{H}^{\frac{1}{2}}$ ,  $U_{0,osc} \stackrel{def}{=} (I_d - \mathbb{P}_2)U_0 \in \dot{H}^s$  with:

$$\|U_{0,osc}\|_{\dot{H}^s} \leq \delta_1 \varepsilon^{-\frac{1}{2}(s-\frac{1}{2})}, \text{ and } \|\mathbb{P}_2 U_0\|_{\dot{H}^{\frac{1}{2}}} \leq \delta_2,$$

there exists a unique global mild solution  $U_\varepsilon \in L^4(\dot{W}^{\frac{1}{2},3})$ . They also provide another result: if  $\|\mathbb{P}_2 U_0\|_{\dot{H}^{\frac{1}{2}}}$  is sufficiently small, there exists a global solution for small enough  $\varepsilon$  (in some sense it is an improvement of the Fujita-Kato theorem as the smallness is not required for the complete initial data but only for what we will call its "stratified part"  $\mathbb{P}_2 U_0$ ). As in the other works of their series (see [33, 30]) the main tool are Strichartz estimates obtained through the Littman theorem (see [35]). In [34] the difficulty is that, in contrast to the cases of Systems  $(RF_\varepsilon)$  and  $(PE_\varepsilon)$ , this theorem cannot be easily applied as the phase function  $|\xi_h|/|\xi|$  presents singularities and frequency truncations are necessary to obtain the result.

We point out that in [33, 30, 34] the aim is not to study the limit system, but this is the object of [40] in which S. Scrobogna adapts the ideas of [18, 5] to prove, in the general case  $\nu = \nu'$  that  $U_\varepsilon$  converges to  $(\tilde{v}^h, 0, 0)$  where  $\tilde{v}^h$  is the unique global solution of the two-component Navier-Stokes system:

$$\begin{cases} \partial_t \tilde{v}^h + \tilde{v}^h \cdot \nabla_h \tilde{v}^h - \nu \Delta \tilde{v}^h &= -\nabla_h \tilde{\pi}^0, \\ \operatorname{div}_h \tilde{v}^h &= 0, \\ \tilde{v}^h|_{t=0} &= \mathbb{P}_2 U_0, \end{cases} \quad (1.3)$$

We emphasize that  $\tilde{v}^h : \mathbb{R}_+ \times \mathbb{R}^3 \rightarrow \mathbb{R}^2$  depends on the full space variable and has two components. More precisely, still rewording the result from [40] using our own notations, if  $U_0 \in \dot{H}^{\frac{1}{2}}$  (inhomogeneous space) and if  $\mathbb{P}_2 U_0 \in \dot{H}^1$  then System (1.3) admits a unique global solution  $\tilde{v}^h \in \dot{E}^0 \cap \dot{E}^1$  and there exists  $\varepsilon_0 > 0$  such that for any  $\varepsilon < \varepsilon_0$ , System  $(S_\varepsilon)$  (with  $U_0$  as initial data) admits a unique global strong solution  $U_\varepsilon \in \dot{E}^{\frac{1}{2}}$  (in contrast to [34], and as in [5], no smallness condition is required neither from  $\mathbb{P}_2 U_0$  nor from  $U_{0,osc}$ ). Moreover  $U_\varepsilon$  converges towards  $(\tilde{v}^h, 0, 0)$  in the following sense: if we define  $W_\varepsilon$  as the unique global solution of the following linear system,

$$\begin{cases} \partial_t W_\varepsilon - L W_\varepsilon + \frac{1}{\varepsilon} \mathbb{P} \mathcal{B} W_\varepsilon = \begin{pmatrix} 0 \\ 0 \\ -\partial_3 (-\Delta_h)^{-1} \operatorname{div}_h (\tilde{v}^h \cdot \nabla_h \tilde{v}^h) \\ 0 \end{pmatrix}, \\ W_\varepsilon|_{t=0} = (I_d - \mathbb{P}_2)U_0, \end{cases} \quad (1.4)$$

then  $U_\varepsilon - (\tilde{v}^h, 0, 0) - W_\varepsilon$  goes to zero in  $\dot{E}^{\frac{1}{2}}$ . Let us precise that in the previous system, the external force term comes from the ideas in [5], which can be simply stated as follows: the system satisfied by  $U_\varepsilon - (\tilde{v}^h, 0, 0)$  features an external force term  $G$  which is independent of  $\varepsilon$  and prevents any convergence via simple energy methods. But if we "make it oscillate" by putting it (or at least a part of it) as an external force in the previous dispersive linear system then we can "eat" a sufficiently large part of  $G$  (considering  $U_\varepsilon - (\tilde{v}^h, 0, 0) - W_\varepsilon$ ) to make the convergence reachable again. In the case of  $(PE_\varepsilon)$  we pushed further this idea and obtained explicit convergence rates in terms of  $\varepsilon$  even for large ill-prepared initial data depending on  $\varepsilon$  and, as usual, the results are better when  $\nu = \nu'$  because the linearized system is nicer (see [6, 12, 13, 14]).

The previous result from [40] could be generalized asking less assumptions on the initial data:  $\tilde{v}_0$  less regular,  $U_{0,\varepsilon}$  dependent on  $\varepsilon$ , no low frequency assumption anymore, lower regularity assumptions ( $U_{0,\varepsilon} \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^{\frac{1}{2}+\delta}$ ) and for very large ill-prepared initial data (that is  $U_{0,\varepsilon,osc}$  large up to some size  $\varepsilon^{-\alpha}$ ). This would give a result close to [12, 13, 14], but in the present paper we would like to propose a *different kind of generalization*, inspired by the following surprising

observation:

*How is it that the limit system (1.3) in the result from [40] does not depend on  $\nu'$  ?*

Before focussing on this question, we would like to end this overview with some very interesting recent results devoted to an intermediate model, the three-scale limit. The models we have presented so far focus on regimes with either only rotation in  $(RF_\varepsilon)$ , or only stratification in  $(S_\varepsilon)$ , or both of them *but with comparable size* in  $(PE_\varepsilon)$  (the non-dispersive case when  $F = 1$  features tools and results completely different from what is done when  $F \neq 1$ , see for example [17, 11]). In [36, 37], the authors completely disconnect the two parameters (namely the Rossby number  $\varepsilon$  and the Froude number  $\delta$ ) and study the following *inviscid* system:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \frac{1}{\varepsilon} e_3 \times v + \frac{1}{\delta} \rho e_3 & = -\nabla \Phi, \\ \partial_t \rho + v \cdot \nabla \rho + \frac{1}{\delta} v_3 & = 0, \\ \operatorname{div} v = 0. \end{cases}$$

There are now obviously two kinds of penalization ( $\delta, \varepsilon \rightarrow 0$ ) and, in [36], P. Mu and S. Schochet study the case  $\mu = \delta/\varepsilon \rightarrow 0$  (stratification-dominant), whereas in [37], P. Mu and Z. Wei focus on the alternative case  $\nu = \varepsilon/\delta \rightarrow 0$  (rotation-dominant). In both cases the authors need suitable dispersion and Strichartz-type estimates, let us state the dispersive estimates from [36]: if  $p_\mu(\xi) = \sqrt{\xi_1^2 + \xi_2^2 + \mu^2 \xi_3^2}/|\xi|$  and  $\psi$  is some frequency truncation function, there exists  $C = C(\psi)$  such that if  $\varepsilon$  and  $\delta$  are sufficiently small, for all  $f \in L^1(\mathbb{R}^3)$  and all  $(t, x) \in \mathbb{R}^4$ , we have (denoting  $\kappa \stackrel{\text{def}}{=} \varepsilon^{\frac{1}{5}} \delta^{\frac{4}{5}}$ )

$$\left| \int_{\mathbb{R}^3} e^{ix \cdot \xi \pm \frac{1}{\delta} p_\mu(\xi)} \psi(\xi) \widehat{f}(\xi) d\xi \right| \leq \frac{C}{(1 + \frac{|t|}{\kappa})^{\frac{1}{4}}} \|f\|_{L^1(\mathbb{R}^3)}.$$

The authors also prove that the limit is  $U^S = (U_H^S, 0, 0)$ , where  $U_H^S$  solves (1.2), and they also provide a convergence rate: if  $k \geq 6$  is an integer, if  $q \in [8, \infty[$ , then for any divergence-free initial data  $U_0 \in H^k(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$  (inhomogeneous Sobolev space) and for any  $T > 0$  there exists  $\kappa_0$  and  $\mu_0$  such that for any  $\varepsilon, \delta > 0$  for which  $\kappa \leq \kappa_0$  and  $\mu \leq \mu_0$ , there exists a unique solution in  $\mathcal{C}_T([0, T], H^k(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T], H^{k-1}(\mathbb{R}^3))$  and for any  $p \in \mathbb{N} \cap [0, k-5]$ ,

$$\|U - U^S\|_{L^q([0, T], W^{k-5-p, \infty})} \leq C(\kappa^{\frac{1}{q}} + \mu^{\frac{3+p}{k+3/2}} \sqrt{-\ln \mu}).$$

In [37], similar dispersive estimates are obtained, for the phase  $p_\nu(\xi) = \sqrt{\nu^2(\xi_1^2 + \xi_2^2) + \xi_3^2}/|\xi|$  and now with  $\kappa = \varepsilon^{\frac{2}{3}} \delta^{\frac{1}{3}}$ . The authors show that when the initial data is taken as follows  $U_0(x) = (u_0^R(x_h), 0, 0) + w_0(x)$ , the limit is  $U^R(x) = (U_h^R(x_h), 0, \rho^R(x))$ , where  $U_h^R$  solves the 2D-Euler system with 3 components (which is to be related to the case of  $(RF_\varepsilon)$ , see [18, 14] and not to be confused with the 3D-Euler system with two components (1.2)) and  $\rho^R$  solves the following 2D-transport equation (in the full space variable  $x \in \mathbb{R}^3$ ):

$$\begin{cases} \partial_t \rho^R + U_h^R \cdot \nabla_h \rho^R = 0, \\ \rho^R|_{t=0} = w_0^A. \end{cases}$$

If the initial data is in  $H^{s+2}(\mathbb{R}^2) + H^s(\mathbb{R}^3)$  (with  $s \geq 3$ ), the authors also provide a convergence rate involving powers of  $\kappa$  and  $\nu$  (and some radius  $\sigma$ ).

Let us end this survey with [31] where the authors focus on the anisotropic case ( $\nu > 0$  and  $\nu' = 0$ ) in the torus. In this stratification-dominant case they obtain that, for well-prepared initial data (i.e. with small initial oscillating part) the limit of the global weak solution is  $U^S(x) = (U_h^S(x_h), 0, \rho^S(x))$  where  $U^S$  solves the 2D-Navier-Stokes system and  $\rho^S$  is independent of  $(t, x_h)$  (but is not specified explicitly). The authors also provide a convergence rate in terms

of  $\max(\varepsilon, \delta/\varepsilon)$ .

We can now precise the aim of the present article: in both cases  $\nu \neq \nu'$  and  $\nu = \nu'$ , we wish to go in a different direction and *look for ill-prepared initial data that allow a limit really depending on  $\nu'$* . We will specify this limit and provide, when possible, global-in-time estimates. As in [18, 37], it will obviously be necessary to get out of the setting of initial data depending on the full space variable and in the next section, which is devoted to the formal obtention of the limit system, we will explain in details how we are lead to modify the initial data.

Let us now state a simplified version of the results that we prove in this article (the precise results are respectively Theorems 3.1 and 4.1):

**Theorem 1.1 (Existence)** *Let  $\tilde{\theta}_0 \in \dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R})$  and for all  $\varepsilon > 0$ , let  $U_{0,\varepsilon} \in L^2(\mathbb{R}^3)$ . Then for all  $\varepsilon > 0$ , System  $(S_\varepsilon)$  admits a global weak solution  $U_\varepsilon \in \dot{E}^0(\mathbb{R}^3)^3 \times (\dot{E}^0(\mathbb{R}^3) + \dot{B}^{-\frac{1}{2}}(\mathbb{R}))$  corresponding to the following initial data:*

$$U_{0,\varepsilon}(x_1, x_2, x_3) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{\theta}_0(x_3) \end{pmatrix}.$$

Moreover, there exists  $C = C_{\nu, \nu', \tilde{\theta}_0}$  such that  $\|U_\varepsilon\|_{\dot{E}^0(\mathbb{R}^3) + \dot{B}^{-\frac{1}{2}}(\mathbb{R})} \leq C(\|U_{0,\varepsilon}\|_{L^2} + 1)$

**Theorem 1.2 (Convergence)** *Let  $\delta > 0$ . Assume that  $\tilde{\theta}_0 \in \dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R})$  and that  $\tilde{v}_0^h \in H^{\frac{1}{2}+\delta}(\mathbb{R}^3)$  is a divergence-free two-component vectorfield ( $\operatorname{div}_h \tilde{v}_0^h = 0$ ). Assume in addition that for all  $\varepsilon > 0$ ,  $U_{0,\varepsilon} \in L^2(\mathbb{R}^3)$ , and that there exists a constant  $\mathbb{C}_0 \geq 1$  such that<sup>1</sup>:*

$$\begin{cases} \|\tilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}(\mathbb{R}^3)} \leq \mathbb{C}_0 & \text{and} & \|\tilde{\theta}_0\|_{\dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R})} \leq \mathbb{C}_0, \\ \sup_{\varepsilon > 0} \|U_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq \mathbb{C}_0, \\ \|U_{0,\varepsilon,S}^h - \tilde{v}_0^h\|_{L^2(\mathbb{R}^3)} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{cases} \quad (1.5)$$

Then  $U_\varepsilon$  converges (as  $\varepsilon$  goes to zero) to  $(\tilde{v}^h, 0, \tilde{\theta})$ , where  $\tilde{v}^h$  and  $\tilde{\theta}$  are the unique global solutions, respectively, of (1.3) (with initial data  $\tilde{v}_0$ ) and the following system:

$$\begin{cases} \partial_t \tilde{\theta} - \nu' \partial_3^2 \tilde{\theta} = 0, \\ \tilde{\theta}|_{t=0} = \tilde{\theta}_0. \end{cases}$$

The convergence is intended in the following sense: if  $D_\varepsilon = U_\varepsilon - (\tilde{v}^h, 0, \tilde{\theta})$ , then for all  $q \in ]2, 6[$ , we have  $\mathbb{P}_2 D_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$  in  $L_{loc}^2 L_{loc}^q$ , and there exists  $\varepsilon_1 = \varepsilon_1(\nu, \nu', q) > 0$  such that for all  $t \geq 0$ , there exists a constant  $D = D_{t,\delta,\nu,\nu',\mathbb{C}_0}$  such that for all  $\varepsilon \leq \varepsilon_1$ :

$$\|D_{\varepsilon,osc}\|_{L_t^2 L^q} = \|(I_d - \mathbb{P}_2)D_\varepsilon\|_{L_t^2 L^q} \leq D\varepsilon^{\frac{K(q)}{464}}, \quad \text{with} \quad K(q) \stackrel{def}{=} \frac{\min(\frac{6}{q} - 1, 1 - \frac{2}{q})^2}{(\frac{6}{q} - 1)}. \quad (1.6)$$

Moreover, when  $\nu = \nu'$ , the previous estimates can be upgraded into  $\|D_{\varepsilon,osc}\|_{L_t^2 L^q} \leq D\varepsilon^{\frac{K(q)}{368}}$  (now valid for all  $\varepsilon > 0$ ) and we have the following global-in-time estimates: there exists a constant  $C = C_{\nu,\delta,\mathbb{C}_0} > 0$  such that, for any  $\varepsilon > 0$ ,

$$\|D_{\varepsilon,osc}\|_{\tilde{L}^{\frac{4}{3}} \dot{B}_{8,2}^0 + \tilde{L}^1 \dot{B}_{8,2}^0} \leq C\varepsilon^{\frac{3}{16}}.$$

<sup>1</sup>where  $U_{0,\varepsilon,S} = \mathbb{P}_2 U_{0,\varepsilon}$  is what we call the "stratified part of  $U_{0,\varepsilon}$ ", see definitions below

The present article is structured as follows: in the next section, we will formally obtain the limit system and the natural decomposition it induces (namely the stratified part and the oscillating part), and study their properties (this decomposition determines what means to be well- or ill-prepared). The limit systems are studied in the end of Section 2.

Then we will decompose the solutions of  $(S_\varepsilon)$  and obtain a more classical system for which we have to slightly adapt the classical Leray theorem in the same spirit as what was done in [18, 19, 20] for the rotating fluids system. A precise existence theorem for weak solutions (for more general initial data depending on  $\varepsilon$ ) is given in the beginning of Section 3.

Using the Strichartz estimates that we put in the appendix, we can show the oscillating part  $(I_d - \mathbb{P}_2)D_\varepsilon$  goes to zero. From this, we obtain the convergence of the stratified part  $\mathbb{P}_2D_\varepsilon$ . The precise convergence theorem is stated in Section 4. This theorem also features far better convergence rate in the case when  $\nu = \nu'$  (using improved Strichartz estimates also proved in the appendix).

The study of these asymptotics for strong solutions (i.e. obtained with the Fujita-Kato theorem) with large regular ill-prepared initial data is dealt in the forthcoming companion work [15].

## 2 The limit system

In this section we will obtain the formal limit system when  $\varepsilon$  goes to zero. As pointed out in the introduction, a first attempt was made in [40] only with initial data depending on the full space variables. In our article thanks to a decomposition of the geopotential and to formal arguments in the spirit of [4], we will obtain a much more general limit system (see (2.22)). Studying its solutions and structure will help reformulate  $(S_\varepsilon)$  in a more suitable way.

### 2.1 Formal argument

Taking the divergence of the velocity part of  $(S_\varepsilon)$  leads to:

$$\Delta\Phi_\varepsilon = -\partial_3\theta_\varepsilon - \varepsilon \sum_{i=1}^3 \partial_i v_\varepsilon \cdot \nabla v_\varepsilon^i.$$

As  $v_\varepsilon$  is divergence-free we have:

$$\sum_{i=1}^3 \partial_i v_\varepsilon \cdot \nabla v_\varepsilon^i = \operatorname{div}(v_\varepsilon \cdot \nabla v_\varepsilon) = \sum_{i=1}^3 \partial_i (v_\varepsilon \cdot \nabla v_\varepsilon^i) = \sum_{i,j=1}^3 \partial_i \partial_j (v_\varepsilon^i v_\varepsilon^j),$$

from which we decompose the geopotential into  $\Phi_\varepsilon \stackrel{def}{=} P_\varepsilon^1 + \varepsilon P_\varepsilon^0$ , where:

$$\begin{cases} P_\varepsilon^1 = -\Delta^{-1} \partial_3 \theta_\varepsilon, \\ P_\varepsilon^0 = -\sum_{i,j=1}^3 \partial_i \partial_j \Delta^{-1} (v_\varepsilon^i v_\varepsilon^j). \end{cases} \quad (2.7)$$

We can then write in extension System  $(S_\varepsilon)$  as follows:

$$\begin{cases} \partial_t v_\varepsilon^1 + v_\varepsilon \cdot \nabla v_\varepsilon^1 - \nu \Delta v_\varepsilon^1 &= -\partial_1 P_\varepsilon^0 - \frac{1}{\varepsilon} \partial_1 P_\varepsilon^1, \\ \partial_t v_\varepsilon^2 + v_\varepsilon \cdot \nabla v_\varepsilon^2 - \nu \Delta v_\varepsilon^2 &= -\partial_2 P_\varepsilon^0 - \frac{1}{\varepsilon} \partial_2 P_\varepsilon^1, \\ \partial_t v_\varepsilon^3 + v_\varepsilon \cdot \nabla v_\varepsilon^3 - \nu \Delta v_\varepsilon^3 &= -\partial_3 P_\varepsilon^0 - \frac{1}{\varepsilon} (\partial_3 P_\varepsilon^1 + \theta_\varepsilon), \\ \partial_t \theta_\varepsilon + v_\varepsilon \cdot \nabla \theta_\varepsilon - \nu' \Delta \theta_\varepsilon &= \frac{1}{\varepsilon} v_\varepsilon^3, \\ \operatorname{div} v_\varepsilon &= 0. \end{cases} \quad (2.8)$$



If we assume that  $(v_\varepsilon, \theta_\varepsilon, P_\varepsilon^0, P_\varepsilon^1) \xrightarrow{\varepsilon \rightarrow 0} (\tilde{v}, \tilde{\theta}, \tilde{P}^0, \tilde{P}^1)$  in a sufficiently strong way such that the convergence also occurs for the derivatives and nonlinear terms, then taking into account the penalized terms from the right-hand-side, necessarily:

$$\begin{cases} \partial_1 \tilde{P}^1 = \partial_2 \tilde{P}^1 = 0, \\ \partial_3 \tilde{P}^1 + \tilde{\theta}_\varepsilon = 0, \\ \tilde{v}^3 = 0, \end{cases} \quad \text{which means that } \begin{cases} \tilde{P} \text{ and } \tilde{\theta} = -\partial_3 \tilde{P}^1 \text{ only depend on } x_3, \\ \tilde{v}^3 = 0. \end{cases} \quad (2.9)$$

Moreover, as  $v_\varepsilon$  is divergence-free and as  $\tilde{v}^3 = 0$ , we obtain:

$$\operatorname{div}_h \tilde{v}^h \stackrel{def}{=} \partial_1 \tilde{v}^1 + \partial_2 \tilde{v}^2 = 0 \quad \text{where we define } \tilde{v}^h \stackrel{def}{=} (\tilde{v}^1, \tilde{v}^2). \quad (2.10)$$

Next, if we ask in addition that:

$$\begin{cases} -\frac{1}{\varepsilon} \partial_1 P_\varepsilon^1 & \xrightarrow{\varepsilon \rightarrow 0} \tilde{X}, \\ -\frac{1}{\varepsilon} \partial_2 P_\varepsilon^1 & \xrightarrow{\varepsilon \rightarrow 0} \tilde{Y}, \\ -\frac{1}{\varepsilon} (\partial_3 P_\varepsilon^1 + \theta_\varepsilon) & \xrightarrow{\varepsilon \rightarrow 0} \tilde{Z}, \\ \frac{1}{\varepsilon} v_\varepsilon^3 & \xrightarrow{\varepsilon \rightarrow 0} \tilde{T}, \end{cases} \quad (2.11)$$

then we end up with the following limit system:

$$\begin{cases} \partial_t \tilde{v}^1 + \tilde{v}^h \cdot \nabla_h \tilde{v}^1 - \nu \Delta \tilde{v}^1 & = -\partial_1 \tilde{P}^0 + \tilde{X}, \\ \partial_t \tilde{v}^2 + \tilde{v}^h \cdot \nabla_h \tilde{v}^2 - \nu \Delta \tilde{v}^2 & = -\partial_2 \tilde{P}^0 + \tilde{Y}, \\ 0 & = -\partial_3 \tilde{P}^0 + \tilde{Z}, \\ \partial_t \tilde{\theta} - \nu' \partial_3^2 \tilde{\theta} & = \tilde{T}, \\ \operatorname{div}_h \tilde{v} & = 0. \end{cases} \quad (2.12)$$

Taking the limit in the expressions of  $P_\varepsilon^0$  and  $P_\varepsilon^1$  we obtain that:

$$\begin{cases} \Delta \tilde{P}^1 = \partial_3^2 \tilde{P}^1 = -\partial_3 \tilde{\theta}, \text{ which brings nothing new,} \\ \Delta \tilde{P}^0 = -\sum_{i=1}^2 \partial_i \tilde{v}^h \cdot \nabla_h \tilde{v}^i = -\sum_{i=1}^2 \partial_i (\tilde{v}^h \cdot \nabla_h \tilde{v}^i) = -\sum_{i,j=1}^2 \partial_i \partial_j (\tilde{v}^i \tilde{v}^j), \end{cases} \quad (2.13)$$

As the horizontal divergence  $\operatorname{div}_h \tilde{v}^h$  is zero, similarly as for  $(S_\varepsilon)$ , computing for (2.12)  $\partial_1(\widetilde{\text{line 1}}) + \partial_2(\widetilde{\text{line 2}}) + \partial_3(\widetilde{\text{line 3}})$ , we get:

$$\partial_1 \tilde{X} + \partial_2 \tilde{Y} + \partial_3 \tilde{Z} = 0. \quad (2.14)$$

Moreover, computing on  $(S_\varepsilon)$   $\partial_1(\text{line 2}) - \partial_2(\text{line 1})$ , we get that, introducing  $\omega_\varepsilon \stackrel{def}{=} \partial_1 v_\varepsilon^2 - \partial_2 v_\varepsilon^1$ :

$$\partial_t \omega_\varepsilon + v_\varepsilon \cdot \nabla \omega_\varepsilon + \operatorname{div}_h v_\varepsilon \cdot \omega_\varepsilon + \partial_1 v_\varepsilon^3 \cdot \partial_3 v_\varepsilon^2 - \partial_2 v_\varepsilon^3 \cdot \partial_3 v_\varepsilon^1 - \nu \Delta \omega_\varepsilon = 0. \quad (2.15)$$

Performing  $\varepsilon \rightarrow 0$  in the previous equation, using that  $\tilde{v}^3 = 0 = \operatorname{div}_h \tilde{v}^h$ , and defining  $\tilde{\omega} \stackrel{def}{=} \partial_1 \tilde{v}^2 - \partial_2 \tilde{v}^1$ , we get that:

$$\partial_t \tilde{\omega} + \tilde{v}^h \cdot \nabla_h \tilde{\omega} - \nu \Delta \tilde{\omega} = 0. \quad (2.16)$$

On the other hand, computing on (2.12)  $\partial_1(\widetilde{\text{line 2}}) - \partial_2(\widetilde{\text{line 1}})$ , we get that:

$$\partial_t \tilde{\omega} + \tilde{v}^h \cdot \nabla_h \tilde{\omega} - \nu \Delta \tilde{\omega} = \partial_1 \tilde{Y} - \partial_2 \tilde{X}, \quad (2.17)$$

so that, identifying the right-hand side from (2.16) and (2.17), we obtain that:

$$\partial_1 \tilde{Y} - \partial_2 \tilde{X} = 0. \quad (2.18)$$

Next, using the third line from (2.12) and formally solving in the Fourier variable the system:

$$\begin{cases} \partial_1 \tilde{X} + \partial_2 \tilde{Y} &= -\partial_3 \tilde{Z} = -\partial_3^2 \tilde{P}^0, \\ -\partial_2 \tilde{X} + \partial_1 \tilde{Y} &= 0, \end{cases}$$

we end up with:

$$(\tilde{X}, \tilde{Y}) = -\nabla_h \partial_3^2 \Delta_h^{-1} \tilde{P}^0, \quad (2.19)$$

so that:

$$-\nabla_h \tilde{P}^0 + (\tilde{X}, \tilde{Y}) = -\nabla_h \Delta_h^{-1} (\Delta_h \tilde{P}^0 + \partial_3^2 \tilde{P}^0) = -\nabla_h \Delta_h^{-1} \Delta \tilde{P}^0 = -\nabla_h \tilde{\pi}^0, \quad (2.20)$$

where from (2.13) we have that:

$$\tilde{\pi}^0 = \Delta_h^{-1} \Delta \tilde{P}^0 = -\sum_{i,j=1}^2 \Delta_h^{-1} \partial_i \partial_j (\tilde{v}^i \tilde{v}^j). \quad (2.21)$$

Gathering the previous informations, we can now write the formal limit system (2.12) as follows, still denoting  $\tilde{U} = (\tilde{v}^1, \tilde{v}^2, 0, \tilde{\Omega}_{QG}) = (\tilde{v}, 0, \tilde{\Omega}_{QG})$ :

$$\begin{cases} \partial_t \tilde{v}^h + \tilde{v}^h \cdot \nabla_h \tilde{v}^h - \nu \Delta \tilde{v}^h &= -\nabla_h \tilde{\pi}^0, & \text{and } \partial_t \tilde{\theta} - \nu' \partial_3^2 \tilde{\theta} = \tilde{T} \\ \operatorname{div}_h \tilde{v}^h = 0, \end{cases} \quad (2.22)$$

where  $\tilde{v}^3 = 0$ ,  $\tilde{\theta} = -\partial_3 \tilde{P}^1$ ,  $\tilde{Z} = \partial_3 \tilde{P}^0$ , and with  $\tilde{P}^1, \tilde{\theta}, \tilde{T}$  only depending on  $(t, x_3)$ .

**Remark 2.1** 1. Solving in the Fourier variable the following system:

$$\begin{cases} \partial_1 \tilde{v}^1 + \partial_2 \tilde{v}^2 &= 0, \\ -\partial_2 \tilde{v}^1 + \partial_1 \tilde{v}^2 &= \tilde{\omega}, \end{cases}$$

we get that  $\tilde{v}^h = (-\partial_2 \Delta_h^{-1} \tilde{\omega}, \partial_1 \Delta_h^{-1} \tilde{\omega}) = \nabla_h^\perp \Delta_h^{-1} \tilde{\omega}$ . Therefore the first system can be rewritten in terms of the vorticity  $\tilde{\omega} = \omega(\tilde{v}) = \partial_1 \tilde{v}^2 - \partial_2 \tilde{v}^1$  as follows:

$$\begin{cases} \partial_t \tilde{\omega} + \tilde{v}^h \cdot \nabla_h \tilde{\omega} - \nu \Delta \tilde{\omega} = 0, \\ \tilde{v}^h = \nabla_h^\perp \Delta_h^{-1} \tilde{\omega}. \end{cases} \quad (2.23)$$

2. We emphasize that there is no stretching term in (2.23), as in the 2D-Navier-Stokes case (full limit of the rotating fluids system, see [18]), or the Quasi-geostrophic system ( $\widetilde{QG}$ ) (limit of the Primitive system, see [4, 12]).

**Remark 2.2** 1. At this stage, nothing can help us precise the term  $\tilde{T}$ , we only know that  $\tilde{T} = \lim_{\varepsilon \rightarrow 0} \frac{v_\varepsilon^3}{\varepsilon}$ .

2. Moreover, nothing allows us to make a direct link between  $\tilde{T}(0, \cdot)$  and the initial velocity such as for instance:

$$\tilde{T}(0, x_3) = \lim_{\varepsilon \rightarrow 0} \frac{v_\varepsilon^3(0, x)}{\varepsilon}.$$

Indeed, we easily obtain in the case of the Primitive system ( $PE_\varepsilon$ ) (see [4]) that  $\frac{v_\varepsilon^3}{\varepsilon}$  has a limit when  $\varepsilon$  goes to zero:

$$\frac{v_\varepsilon^3}{\varepsilon F} \xrightarrow{\varepsilon \rightarrow 0} (\nu - \nu') \Delta \Delta_F^{-1} \widetilde{\theta}_{QG} + F[\partial_3 \Delta_F^{-1}, \widetilde{v}_{QG}^h \cdot \nabla_h] \widetilde{\Omega}_{QG},$$

whereas if, for instance, the initial data does not depend on  $\varepsilon$ , then for all  $x \in \mathbb{R}^3$  such that  $v_\varepsilon^3(x) \neq 0$ ,

$$\frac{v_\varepsilon^3(0, x)}{\varepsilon F} = \frac{v_0^3(x)}{\varepsilon F} \xrightarrow{\varepsilon \rightarrow 0} \pm\infty.$$

3. It seems that we have better considering  $\tilde{T}$  as a parameter, and in this article we will search for initial data corresponding to the case  $\tilde{T} = 0$ .
4. On the contrary, nothing forces  $\tilde{\theta}$  to be zero, and this term will be the key to obtain new asymptotics, in the spirit of [18, 20]: for the rotating fluids system, when the initial data is in  $L^2(\mathbb{R}^3)$  or in  $\dot{H}^s(\mathbb{R}^3)$  the only limit we can hope for is zero. But when Chemin, Desjardins, Gallagher and Grenier consider as an initial data the sum of a function of the previous type with some  $\bar{u} \in L^2(\mathbb{R}_h^2)$ , then richer asymptotics are at hand and they manage to reach as limit system the 2D-Navier-Stokes system with three components with  $\bar{u}$  as an initial data (as described by physicists in the Taylor-Proudman theorem). We show that similar phenomena will occur for the strongly stratified Boussinesq system.

## 2.2 The Stratified/osc structure, final form of the limit system

Thanks to the previous observations we will consider initial data satisfying:

$$\theta_{\varepsilon|t=0}(x) = \theta_{0,\varepsilon}(x) + \tilde{\theta}_{0,\varepsilon}(x_3),$$

so that our complete initial data is:

$$U_{\varepsilon|t=0}(x) = U_{0,\varepsilon}(x) + (0, 0, 0, \tilde{\theta}_{0,\varepsilon}(x_3)).$$

Moreover, the structure of the formal limit system suggests to introduce the following operators:

**Definition 2.1** For a  $\mathbb{R}^4$ -valued function, we introduce the following quantity, that we will call its vorticity:

$$\omega(f) = \partial_1 f^2 - \partial_2 f^1.$$

From this we define the stratified and oscillating (or oscillatory) parts of  $f$ , respectively denoted as  $f_S$  and  $f_{osc}$ , according to:

$$f_S = \begin{pmatrix} \nabla_h^\perp \Delta_h^{-1} \omega(f) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\partial_2 \Delta_h^{-1} \omega(f) \\ \partial_1 \Delta_h^{-1} \omega(f) \\ 0 \\ 0 \end{pmatrix}, \quad (2.24)$$

and, denoting  $\operatorname{div}_h f^h \stackrel{\text{def}}{=} \partial_1 f^1 + \partial_2 f^2$ ,

$$f_{osc} = f - f_S = \begin{pmatrix} \nabla_h \Delta_h^{-1} \operatorname{div}_h f^h \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_1 \Delta_h^{-1} \operatorname{div}_h f^h \\ \partial_2 \Delta_h^{-1} \operatorname{div}_h f^h \\ f^3 \\ f^4 \end{pmatrix}. \quad (2.25)$$

**Remark 2.3** For a  $\mathbb{R}^2$ -valued function  $f = (f^1, f^2) = f^h$ , we could introduce  $\mathbb{P}_2^h$  and  $f_S = \mathbb{P}_2^h f = \nabla_h^\perp \Delta_h^{-1} \omega(f)$  (this corresponds to  $\mathbb{P}_0$  in (1.2)), but with a slight notational abuse, we may also denote  $f_S = \mathbb{P}_2 f$  and  $f_{osc} = f - f_S = \nabla_h \Delta_h^{-1} \operatorname{div}_h f$ .

Now we can completely precise the initial data and limit system that we will consider in this article:

$$U_{\varepsilon|t=0}(x) = U_{0,\varepsilon}(x) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{\theta}_{0,\varepsilon}(x_3) \end{pmatrix} = U_{0,\varepsilon,S}(x) + U_{0,\varepsilon,osc}(x) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{\theta}_{0,\varepsilon}(x_3) \end{pmatrix}. \quad (2.26)$$

And we will denote:

$$U_{0,\varepsilon} = \begin{pmatrix} v_{0,\varepsilon}^1 \\ v_{0,\varepsilon}^2 \\ v_{0,\varepsilon}^3 \\ \theta_{0,\varepsilon} \end{pmatrix} \quad \text{and} \quad U_\varepsilon = \begin{pmatrix} v_\varepsilon^1 \\ v_\varepsilon^2 \\ v_\varepsilon^3 \\ \theta_\varepsilon \end{pmatrix} = \begin{pmatrix} v_\varepsilon \\ \theta_\varepsilon \end{pmatrix} = \begin{pmatrix} v_\varepsilon^h \\ v_\varepsilon^3 \\ \theta_\varepsilon \end{pmatrix}.$$

The previous formal study suggests that the initial data in (2.22) are related to  $U_{0,\varepsilon,S}$  and  $\tilde{\theta}_{0,\varepsilon}$ , in the following way:

$$\begin{cases} U_{0,\varepsilon,S}^h(x) \xrightarrow{\varepsilon \rightarrow 0} \tilde{v}_0^h(x), & \text{or equivalently } U_{0,\varepsilon,S}(x) = \mathbb{P}_2 U_{0,\varepsilon}(x) \xrightarrow{\varepsilon \rightarrow 0} (\tilde{v}_0^h(x), 0, 0), \\ \tilde{\theta}_{0,\varepsilon}(x_3) \xrightarrow{\varepsilon \rightarrow 0} \tilde{\theta}_0(x_3), \end{cases} \quad (2.27)$$

so that our complete limit system is composed by the following two systems:

$$\begin{cases} \partial_t \tilde{v}^h + \tilde{v}^h \cdot \nabla_h \tilde{v}^h - \nu \Delta \tilde{v}^h = -\nabla_h \tilde{\pi}^0, \\ \operatorname{div}_h \tilde{v}^h = 0, \\ \tilde{v}^h|_{t=0} = \tilde{v}_0^h, \end{cases} \quad (2.28)$$

and

$$\begin{cases} \partial_t \tilde{\theta} - \nu' \partial_3^2 \tilde{\theta} = 0, \\ \tilde{\theta}|_{t=0} = \tilde{\theta}_0. \end{cases} \quad (2.29)$$

**Remark 2.4** We could simply choose  $\tilde{\theta}_{0,\varepsilon} = \tilde{\theta}_0$ , but for more generality we made the previous choice. The same can be done to generalize a little the result in [14] concerning rotating fluids.

The following proposition states properties induced by the stratified/oscillating structure.

**Proposition 2.1** With the notations from (2.24) and (2.25), there exist two pseudodifferential operators of order zero  $\mathcal{P}$  and  $\mathcal{Q}$  such that for any  $f$ ,

$$f_S = \mathcal{Q}f, \quad \text{and} \quad f_{osc} = \mathcal{P}f.$$

These operators satisfy:

1.  $\mathcal{Q} = \mathbb{P}_2$  and  $\mathcal{P} = I_d - \mathbb{P}_2$  (where the operators  $\mathbb{P}_k$  are defined in (5.92) and (5.91)).
2. For any  $s \in \mathbb{R}$ , we have  $((I_d - \mathbb{P}_2)f|_{\mathbb{P}_2f})_{H^s/\dot{H}^s} = 0 = (\mathcal{B}f|_{\mathbb{P}_2f})_{H^s/\dot{H}^s}$  (when defined).
3.  $(I_d - \mathbb{P}_2)f = f \iff \mathbb{P}_2f = 0 \iff \omega(f) = 0$ .
4.  $(I_d - \mathbb{P}_2)f = 0 \iff \mathbb{P}_2f = f \iff f^3 = f^4 = 0$  and  $\operatorname{div}_h f = 0 \iff$  there exists a scalar function  $\phi$  such that  $f = (-\partial_2\phi, \partial_1\phi, 0, 0) = (\nabla_h^\perp \phi, 0, 0)$ . Such a vector field is obviously divergence free (and horizontal divergence-free) and we will say that it is stratified. It also satisfies  $f = (f^h, 0, 0)$ .
5. If  $f$  is divergence-free, so is  $(I_d - \mathbb{P}_2)f$ .
6.  $\mathcal{B}\mathbb{P}_2f = 0$  (in  $\mathbb{R}^4$ ).
7.  $\mathbb{P}_2\mathbb{P} = \mathbb{P}\mathbb{P}_2 = \mathbb{P}_2$  and  $\mathbb{P}_2(I_d - \mathbb{P}) = (I_d - \mathbb{P})\mathbb{P}_2 = 0$  (in particular  $\mathbb{P}_2(\nabla q, 0) = 0$ ).
8. If  $f$  is a divergence-free vector field, then (we recall that we denote  $f \cdot \nabla f = \sum_{i=1}^3 f^i \partial_i f$ )

$$\omega(f \cdot \nabla f) = -\partial_3 f^3 \cdot \omega(f) + \partial_1 f^3 \cdot \partial_3 f^2 - \partial_2 f^3 \cdot \partial_3 f^1 + f \cdot \omega(f).$$

9. If  $f$  is a stratified vector field, then  $\omega(f \cdot \nabla f) = f \cdot \omega(f)$ .

The proof of this proposition is similar to what is done in [4, 5, 12].

**Remark 2.5** 1. Let us recall that in the case of the Primitive system the decomposition  $QG/osc$  (with its corresponding  $\mathcal{Q}/\mathcal{P}$ ) and  $\mathbb{P}_2/\mathbb{P}_{3,4}$  do not coincide except when  $\nu = \nu'$ , so that for instance the oscillating part still has a "  $\mathbb{P}_2$  " part (but we show it is very small in terms of  $\varepsilon$ ). In the present article, we have  $\mathcal{Q} = \mathbb{P}_2$  in the general case (but it is only when  $\nu = \nu'$  that  $\mathbb{P}_3$  and  $\mathbb{P}_4$  are orthogonal projectors of norm 1).

2. We chose to use the denomination Stratified rather than quasi-geostrophic.

### 2.3 Reformulation of the different systems

In order to properly study the solutions of  $(S_\varepsilon)$  with such unusual initial data, as in [18, 20] in the case of the rotating fluids, we need to rewrite System  $(S_\varepsilon)$  in a form where functions only depending on  $x_3$  do not appear in the initial data anymore. Doing this will move these functions in the transport terms which explains why a little adaptation of the proofs of the classical existence results such as the Leray or Fujita-Kato theorems is needed.

**Remark 2.6** When studying  $(PE_\varepsilon)$  or in [40], it was not necessary because the systems only feature functions of  $x \in \mathbb{R}^3$  and adapting the classical existence theorems requires no effort.

Let us first merge Systems (2.28) and (2.29). Introducing  $\tilde{v}^3 = 0$  and  $\tilde{P}^1$  such that  $\tilde{\theta}(x_3) = -\partial_3 \tilde{P}^1(x_3)$ , we can rewrite the limit systems as follows:

$$\begin{cases} \partial_t \tilde{v}^1 + \tilde{v} \cdot \nabla \tilde{v}^1 - \nu \Delta \tilde{v}^1 & = -\partial_1 \tilde{\pi}^0, \\ \partial_t \tilde{v}^2 + \tilde{v} \cdot \nabla \tilde{v}^2 - \nu \Delta \tilde{v}^2 & = -\partial_2 \tilde{\pi}^0, \\ \partial_t \tilde{v}^3 + \tilde{v} \cdot \nabla \tilde{v}^3 - \nu \Delta \tilde{v}^3 + \frac{1}{\varepsilon} \tilde{\theta} & = -\frac{1}{\varepsilon} \partial_3 \tilde{P}^1, \\ \partial_t \tilde{\theta} + \tilde{v} \cdot \nabla \tilde{\theta} - \nu' \Delta \tilde{\theta} - \frac{1}{\varepsilon} \tilde{v}^3 & = 0, \\ \operatorname{div} \tilde{v} & = 0. \end{cases} \quad (2.30)$$

If we set  $\tilde{U} \stackrel{def}{=} (\tilde{v}^h, 0, \tilde{\theta})$ , this system can be rewritten into:

$$\partial_t \tilde{U} + \tilde{v} \cdot \nabla \tilde{U} - L\tilde{U} + \frac{1}{\varepsilon} \mathcal{B}\tilde{U} = - \begin{pmatrix} \nabla_h \tilde{\pi}^0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{\varepsilon} \begin{pmatrix} \nabla \tilde{P}^1 \\ 0 \end{pmatrix}.$$

Denoting as  $\mathbb{P}$  the orthogonal Leray projector onto divergence-free vectorfields, we introduce  $\tilde{G} = \mathbb{P}(\nabla_h \tilde{\pi}^0, 0, 0)$  which satisfies the following result.

**Proposition 2.2** With the previous notations, there exists some  $\tilde{g}$  such that

$$\begin{pmatrix} \nabla_h \tilde{\pi}^0 \\ 0 \\ 0 \end{pmatrix} = \tilde{G} + \begin{pmatrix} \nabla \tilde{g} \\ 0 \end{pmatrix} \quad \text{with} \quad \tilde{G} = \mathbb{P} \begin{pmatrix} \partial_1 \tilde{\pi}^0 \\ \partial_2 \tilde{\pi}^0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_1 \partial_3^2 \Delta^{-1} \Delta_h^{-1} \tilde{q}_0 \\ \partial_2 \partial_3^2 \Delta^{-1} \Delta_h^{-1} \tilde{q}_0 \\ -\partial_3 \Delta^{-1} \tilde{q}_0 \\ 0 \end{pmatrix}, \quad (2.31)$$

where  $\tilde{q}_0$  is defined by ( $\tilde{\pi}^0$  has been introduced in (2.21))

$$\tilde{q}_0 = -\Delta_h \tilde{\pi}^0 = \sum_{i,j=1}^2 \partial_i \partial_j (\tilde{v}^i \tilde{v}^j) = \sum_{i=1}^2 \partial_i (\tilde{v}^h \cdot \nabla_h \tilde{v}^i).$$

Moreover, it is obvious that  $\omega(\tilde{G}) = 0 = \operatorname{div} \tilde{G}$  and  $\mathbb{P}_2 \tilde{G} = 0$ .

**Proof:** It is only about computing  $\tilde{G}$  using the fact that  $\mathbb{P} = I_d - \nabla \Delta^{-1} \operatorname{div}$ . ■

**Remark 2.7** Adding some gradient, we can also write that:

$$\tilde{G} = \mathbb{P} \begin{pmatrix} 0 \\ 0 \\ \partial_3 \Delta_h^{-1} \tilde{q}_0 \\ 0 \end{pmatrix},$$

which appears in the external force term from the auxiliary system (1.4) introduced in [40] when reproducing the method from [5], where the analogous term features two parts with different behaviours, namely  $G = G^b + G^l$ .

Thanks to this, we can finally reformulate our limit system as follows:

$$\begin{cases} \partial_t \tilde{U} + \tilde{U} \cdot \nabla \tilde{U} - L\tilde{U} + \frac{1}{\varepsilon} \mathcal{B}\tilde{U} = -\tilde{G} - \begin{pmatrix} \nabla \tilde{g} \\ 0 \end{pmatrix} - \frac{1}{\varepsilon} \begin{pmatrix} \nabla \tilde{P}^1 \\ 0 \end{pmatrix}, \\ \operatorname{div} \tilde{v} = 0, \\ \tilde{U}|_{t=0} = (\tilde{v}_0^h, 0, \tilde{\theta}_0). \end{cases} \quad (2.32)$$

Our first idea would be to study the system satisfied by  $U_\varepsilon - \tilde{U}$ , but as already mentioned, its initial data:

$$(U_\varepsilon - \tilde{U})|_{t=0}(x) = U_{0,\varepsilon,osc}(x) + (\mathbb{P}_2 U_{0,\varepsilon}^h(x) - \tilde{v}_0^h(x), 0, \tilde{\theta}_{0,\varepsilon}(x_3) - \tilde{\theta}_0(x_3)),$$

would feature functions depending only on  $x_3$ . As we wish to do minimal transformations in order to adapt the Leray theorem, we simply define the following function, that will help neutralize the  $x_3$ -only-dependent part:

$$\tilde{Z}_\varepsilon = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{K}_\varepsilon \end{pmatrix}, \quad \text{where } \tilde{K}_\varepsilon \text{ solves } \begin{cases} \partial_t \tilde{K}_\varepsilon - \nu' \partial_3^2 \tilde{K}_\varepsilon = 0, \\ \tilde{\theta}|_{t=0} = \tilde{\theta}_{0,\varepsilon} - \tilde{\theta}_0. \end{cases} \quad (2.33)$$

**Remark 2.8** Thanks to the estimates from Theorem 2.1 and (2.27),  $\tilde{K}_\varepsilon$  goes to zero as  $\varepsilon$  goes to zero.

Next, introducing  $\tilde{P}_\varepsilon^2$  such that  $\tilde{K}_\varepsilon = -\partial_3 \tilde{P}_\varepsilon^2$  (both of these functions only depend on  $x_3$ ),  $\tilde{Z}_\varepsilon$  satisfies:

$$\begin{cases} \partial_t \tilde{Z}_\varepsilon - L\tilde{Z}_\varepsilon + \frac{1}{\varepsilon} \mathcal{B}\tilde{Z}_\varepsilon = -\frac{1}{\varepsilon} (\nabla \tilde{P}_\varepsilon^2, 0), \\ \tilde{Z}_\varepsilon|_{t=0} = (0, 0, 0, \tilde{\theta}_{0,\varepsilon} - \tilde{\theta}_0). \end{cases} \quad (2.34)$$

Now we can properly rewrite the system we will study, let us finally set:

$$D_\varepsilon \stackrel{def}{=} U_\varepsilon - \tilde{U} - \tilde{Z}_\varepsilon = \begin{pmatrix} v_\varepsilon^h - \tilde{v}_0^h \\ v_\varepsilon^3 \\ \theta_\varepsilon - (\tilde{\theta} + \tilde{K}_\varepsilon) \end{pmatrix} = \begin{pmatrix} v_\varepsilon^h - \tilde{v}_0^h \\ v_\varepsilon^3 \\ \theta_\varepsilon - \tilde{\theta}_\varepsilon \end{pmatrix} = U_\varepsilon - \begin{pmatrix} \tilde{v}^h \\ 0 \\ \tilde{\theta}_\varepsilon \end{pmatrix}, \quad (2.35)$$

where the function  $\tilde{\theta}_\varepsilon \stackrel{def}{=} \tilde{\theta} + \tilde{K}_\varepsilon$  is nothing but the solution of:

$$\begin{cases} \partial_t \tilde{\theta}_\varepsilon - \nu' \partial_3^2 \tilde{\theta}_\varepsilon = 0, \\ \tilde{\theta}_\varepsilon|_{t=0} = \tilde{\theta}_{0,\varepsilon}. \end{cases} \quad (2.36)$$

Rewriting System  $(S_\varepsilon)$  as follows,

$$\begin{cases} \partial_t U_\varepsilon + U_\varepsilon \cdot \nabla U_\varepsilon - LU_\varepsilon + \frac{1}{\varepsilon} \mathcal{B}U_\varepsilon = - \begin{pmatrix} \nabla P_\varepsilon^0 \\ 0 \end{pmatrix} - \frac{1}{\varepsilon} \begin{pmatrix} \nabla P_\varepsilon^1 \\ 0 \end{pmatrix}, \\ U_\varepsilon|_{t=0} = U_{0,\varepsilon}. \end{cases} \quad (S_\varepsilon)$$

and subtracting Systems (2.32) and (2.34), we obtain that  $D_\varepsilon$  satisfies:

$$\partial_t D_\varepsilon - LD_\varepsilon + \frac{1}{\varepsilon} \mathcal{B}D_\varepsilon = - \left[ D_\varepsilon \cdot \nabla D_\varepsilon + D_\varepsilon \cdot \nabla \begin{pmatrix} \tilde{v}^h \\ 0 \\ \tilde{\theta}_\varepsilon \end{pmatrix} + \tilde{v}^h \cdot \nabla_h D_\varepsilon \right] + \tilde{G} - \begin{pmatrix} \nabla q_\varepsilon \\ 0 \end{pmatrix},$$

where

$$q_\varepsilon = -\tilde{g} + \frac{1}{\varepsilon}(\phi_\varepsilon - \tilde{P}^1 - \tilde{P}_\varepsilon^2) = P_\varepsilon^0 - \tilde{g} + \frac{1}{\varepsilon}(P_\varepsilon^1 - \tilde{P}^1 - \tilde{P}_\varepsilon^2).$$

Reformulating the second term from the right-hand side, we obtain the final form (2.37) of the system satisfied by  $D_\varepsilon = (V_\varepsilon, H_\varepsilon)$ :

$$\begin{cases} \partial_t D_\varepsilon - LD_\varepsilon + \frac{1}{\varepsilon} \mathcal{B}D_\varepsilon = - \left[ D_\varepsilon \cdot \nabla D_\varepsilon + \begin{pmatrix} D_\varepsilon \cdot \nabla \tilde{v}^h \\ 0 \\ D_\varepsilon^3 \cdot \partial_3 \tilde{\theta}_\varepsilon \end{pmatrix} + \tilde{v}^h \cdot \nabla_h D_\varepsilon \right] + \tilde{G} - \begin{pmatrix} \nabla q_\varepsilon \\ 0 \end{pmatrix}, \\ \operatorname{div} V_\varepsilon = 0, \\ D_\varepsilon|_{t=0} = U_{0,\varepsilon,osc} + (U_{0,\varepsilon,S} - (\tilde{v}_0^h, 0, 0)) = U_{0,\varepsilon,osc} + (U_{0,\varepsilon,S}^h - \tilde{v}_0^h, 0, 0). \end{cases} \quad (2.37)$$

## 2.4 Study of the limit system

We show in this section that the limit system has global regular solutions which specifies the necessary informations about  $\tilde{v}^h$  and  $\tilde{\theta}_\varepsilon$  required to study System (2.37).

Let us begin with System (2.29), which is only a one-dimensional heat equation (we refer for example to [2], Section 3.4.1, Lemma 5.10 and Proposition 10.3, see also Definition 5.1).

**Theorem 2.1** *Let  $s \in \mathbb{R}$ . For any  $\tilde{\theta}_0 \in \dot{H}^s(\mathbb{R})$  (respectively  $\tilde{\theta}_0 \in \dot{B}_{2,1}^s(\mathbb{R})$ ) there exists a unique global solution  $\tilde{\theta}$  of (2.29) and for all  $t \geq 0$ , we have:*

$$\|\tilde{\theta}\|_{\tilde{L}_t^\infty \dot{H}^s}^2 + \nu' \|\tilde{\theta}\|_{L_t^2 \dot{H}^{s+1}}^2 \leq 2 \|\tilde{\theta}_0\|_{\dot{H}^s}^2. \quad (2.38)$$

$$\text{(respectively } \|\tilde{\theta}\|_{\tilde{L}_t^\infty \dot{B}_{2,1}^s} + \nu' \|\tilde{\theta}\|_{L_t^1 \dot{B}_{2,1}^{s+2}} \leq \|\tilde{\theta}_0\|_{\dot{B}_{2,1}^s}.) \quad (2.39)$$

More generally for  $s \in \mathbb{R}$  and  $p, r \in [1, \infty]$ , there exists a constant  $C > 0$  such that if  $\tilde{\theta}_0 \in \dot{B}_{p,r}^s(\mathbb{R})$  then for all  $q \in [1, \infty]$

$$\|\tilde{\theta}\|_{\tilde{L}_t^q \dot{B}_{p,r}^{s+\frac{2}{q}}} \leq \frac{C}{(\nu')^{\frac{1}{q}}} \|\tilde{\theta}_0\|_{\dot{B}_{p,r}^s}. \quad (2.40)$$

**Remark 2.9** *Thanks to this result, the previously defined  $\tilde{\theta}$ ,  $\tilde{K}_\varepsilon$  and  $\tilde{\theta}_\varepsilon$  are global and satisfy similar estimates.*

On the other hand, thanks to (2.23) and Remark 2.1, we observe that System (2.28) (which is System (4.3) from [40]) is very close to the quasi-geostrophic system, and we can easily adapt Theorem 1 from [12] and state the following theorem which generalizes the results from Section 4 in [40] (as we need less initial regularity).

**Theorem 2.2** *Let  $\delta > 0$  and  $\tilde{v}_0^h \in H^{\frac{1}{2}+\delta}$  a  $\mathbb{R}^2$ -valued vectorfield such that  $\operatorname{div}_h \tilde{v}_0^h = 0$ . Then System (2.28) has a unique global solution  $\tilde{v}^h \in E^{\frac{1}{2}+\delta} = \dot{E}^0 \cap \dot{E}^{\frac{1}{2}+\delta}$  and there exists a constant  $C = C_{\delta,\nu} > 0$  such that for all  $t \geq 0$ , we have:*

$$\begin{aligned} \|\tilde{v}^h\|_{L^\infty H^{\frac{1}{2}+\delta}}^2 + \nu \|\nabla \tilde{v}^h\|_{L^2 H^{\frac{1}{2}+\delta}}^2 &\leq C_{\delta,\nu} \|\tilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}}^2 \max(1, \|\tilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}}^{\frac{1}{\delta}}) \\ &\leq C_{\delta,\nu} \max(1, \|\tilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}})^{2+\frac{1}{\delta}}, \end{aligned} \quad (2.41)$$

Moreover, we can also bound the term  $\tilde{G}$  introduced in (2.35): for all  $s \in [0, \frac{1}{2} + \delta]$ ,

$$\int_0^\infty \|\tilde{G}(\tau)\|_{\dot{H}^s} d\tau \leq C_{\delta,\nu} \max(1, \|\tilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}})^{2+\frac{1}{\delta}}. \quad (2.42)$$

**Proof:** Thanks to Remark 2.1 and the Biot-Savart law (which is similar to the one featured in the quasi-geostrophic system), the proof is very close to what we did in Section 2.1 from [12]. ■

### 3 Existence of global weak solutions

This section is devoted to prove the following result, which is the analogous of the Leray theorem for (2.37) and provides global weak solutions for any  $\varepsilon > 0$ .

**Theorem 3.1** *(Existence of Leray weak solutions) Assume  $\delta > 0$ ,  $\tilde{v}_0^h \in H^{\frac{1}{2}+\delta}(\mathbb{R}^3)$ ,  $\tilde{\theta}_{0,\varepsilon} \in \dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R})$  (for all  $\varepsilon > 0$ ) and that there exists a constant  $\mathbb{C}_0 \geq 1$  such that:*

$$\|\tilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}(\mathbb{R}^3)} \leq \mathbb{C}_0 \quad \text{and} \quad \sup_{\varepsilon > 0} \|\tilde{\theta}_{0,\varepsilon}\|_{\dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R})} \leq \mathbb{C}_0$$

Thanks to Theorems 2.1 and 2.2,  $\tilde{v}^h$  and  $\tilde{\theta}_\varepsilon$  globally exist (for all  $\varepsilon > 0$ ) and respectively belong to  $\dot{E}^0 \cap \dot{E}^{\frac{1}{2}+\delta}$  and  $\dot{B}^{-\frac{1}{2}}$ .

Moreover there exists a constant  $C_{\delta,\nu,\nu'} > 0$  such that for any fixed  $\varepsilon > 0$ , if  $U_{0,\varepsilon} \in L^2(\mathbb{R}^3)$ , then there exists a weak global solution of (2.37)  $(D_\varepsilon, q_\varepsilon)$  with  $D_\varepsilon \in \dot{E}^0$  and  $q_\varepsilon \in \dot{E}^1 + L^{\frac{4}{3}}(\mathbb{R}_+, L^2)$ , satisfying for all  $t \geq 0$ ,

$$\begin{aligned} \|D_\varepsilon(t)\|_{L^2}^2 + \nu_0 \int_0^t \|\nabla D_\varepsilon(\tau)\|_{L^2}^2 d\tau \\ \leq \left( \|U_{0,\varepsilon,osc}\|_{L^2}^2 + \|U_{0,\varepsilon,S}^h - \tilde{v}_0^h\|_{L^2}^2 + C_{\delta,\nu,\nu'} \mathbb{C}_0^{2+\frac{1}{\delta}} \right) e^{C_{\delta,\nu,\nu'} \mathbb{C}_0^{2+\frac{1}{\delta}}}. \end{aligned} \quad (3.43)$$

In what follows, we will state apriori estimates, give the Friedrichs scheme and quickly outline what is new to adapt the proof of the classical Leray theorem from the Navier-Stokes case and energy estimates.

#### 3.1 A priori estimates

**Proposition 3.1** *Assume there exists a constant  $\mathbb{C}_0 \geq 1$  such that:*

$$\|\tilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}(\mathbb{R}^3)} \leq \mathbb{C}_0 \quad \text{and} \quad \sup_{\varepsilon > 0} \|\tilde{\theta}_{0,\varepsilon}\|_{\dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R})} \leq \mathbb{C}_0$$



Then for any fixed  $\varepsilon > 0$ , if we set  $\nu_0 = \min(\nu, \nu') > 0$ , there exists  $C_0 > 0$ , and  $C_{\delta, \nu, \nu'} > 0$  such that for any solution  $D_\varepsilon \in \dot{E}^0$  of System (2.37), and for all  $t \geq 0$ :

$$\begin{aligned} \|D_\varepsilon(t)\|_{L^2}^2 + \nu_0 \int_0^t \|\nabla D_\varepsilon(\tau)\|_{L^2}^2 d\tau &\leq \left( \|D_\varepsilon(0)\|_{L^2}^2 + \frac{1}{2} \int_0^t \|\tilde{G}(t')\|_{L^2} dt' \right) \\ &\quad \times \exp \left( C_0 \int_0^t \left[ \|\tilde{G}(t')\|_{L^2} + \frac{1}{\nu_0} \|\nabla \tilde{v}^h(t')\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{1}{\nu_0^{\frac{1}{3}}} \|\tilde{\theta}_\varepsilon(t')\|_{\dot{H}^1}^{\frac{4}{3}} \right] dt' \right) \\ &\leq \left( \|D_\varepsilon(0)\|_{L^2}^2 + C_{\delta, \nu, \nu'} C_0^{2+\frac{1}{\delta}} \right) e^{C_{\delta, \nu, \nu'} C_0^{2+\frac{1}{\delta}}}. \end{aligned} \quad (3.44)$$

**Proof:** taking the innerproduct in  $L^2$  of (2.37) with  $D_\varepsilon = (V_\varepsilon, H_\varepsilon)$ , and thanks to the fact that  $\operatorname{div} V_\varepsilon = 0 = \operatorname{div}_h \tilde{v}^h$ , we obtain that  $(D_\varepsilon \cdot \nabla D_\varepsilon | D_\varepsilon)_{L^2} = 0 = (\tilde{v}^h \cdot \nabla_h D_\varepsilon | D_\varepsilon)_{L^2}$  and:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|D_\varepsilon\|_{L^2}^2 + \nu_0 \|\nabla D_\varepsilon\|_{L^2}^2 &= -(D_\varepsilon \cdot \nabla \tilde{v}^h | D_\varepsilon^h)_{L^2} - (D_\varepsilon^3 \cdot \partial_3 \tilde{\theta}_\varepsilon | D_\varepsilon^4)_{L^2} + (\tilde{G} | D_\varepsilon)_{L^2} \\ &\stackrel{def}{=} A + B + C. \end{aligned} \quad (3.45)$$

The terms  $A$  and  $C$  are easy to bound and we immediately get that:

$$\begin{aligned} |A| &\leq \|D_\varepsilon \cdot \nabla \tilde{v}^h\|_{L^2} \|D_\varepsilon\|_{L^2} \leq \|D_\varepsilon\|_{L^6} \|\nabla \tilde{v}^h\|_{L^3} \|D_\varepsilon\|_{L^2} \leq \|D_\varepsilon\|_{\dot{H}^1} \|\nabla \tilde{v}^h\|_{\dot{H}^{\frac{1}{2}}} \|D_\varepsilon\|_{L^2} \\ &\leq \frac{\nu_0}{4} \|D_\varepsilon\|_{\dot{H}^1}^2 + \frac{C}{\nu_0} \|\nabla \tilde{v}^h\|_{\dot{H}^{\frac{1}{2}}}^2 \|D_\varepsilon\|_{L^2}^2, \end{aligned} \quad (3.46)$$

and

$$|C| \leq \|\tilde{G}\|_{L^2} \|D_\varepsilon\|_{L^2} \leq \frac{1}{2} \|\tilde{G}\|_{L^2} + \frac{1}{2} \|\tilde{G}\|_{L^2} \|D_\varepsilon\|_{L^2}^2. \quad (3.47)$$

The last term is not classical as it features functions of one variable. Thanks to the Minkowski and Young estimates (using twice  $(\frac{4}{3}, 4)$  in the last lines), the Sobolev interpolation estimates and the 1d-Sobolev injection  $\dot{H}^{\frac{1}{4}}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$ :

$$\begin{aligned} |B| &\leq \int_{\mathbb{R}^3} |D_\varepsilon^3(x_h, x_3)| \cdot |\partial_3 \tilde{\theta}_\varepsilon(x_3)| \cdot |D_\varepsilon^4(x_h, x_3)| dx \leq \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} |D_\varepsilon(x_h, x_3)|^2 |\partial_3 \tilde{\theta}_\varepsilon(x_3)| dx_3 \right) dx_h \\ &\leq \int_{\mathbb{R}^2} \|D_\varepsilon(x_h, \cdot)\|_{L^4(\mathbb{R})}^2 \|\partial_3 \tilde{\theta}_\varepsilon\|_{L^2(\mathbb{R})} dx_h \leq C \|\partial_3 \tilde{\theta}_\varepsilon\|_{L^2(\mathbb{R})} \int_{\mathbb{R}^2} \|D_\varepsilon(x_h, \cdot)\|_{\dot{H}^{\frac{1}{4}}(\mathbb{R})}^2 dx_h \\ &\leq C \|\tilde{\theta}_\varepsilon\|_{\dot{H}^1(\mathbb{R})} \int_{\mathbb{R}^2} \|D_\varepsilon(x_h, \cdot)\|_{L^2(\mathbb{R})}^{\frac{3}{2}} \|D_\varepsilon(x_h, \cdot)\|_{\dot{H}^1(\mathbb{R})}^{\frac{1}{2}} dx_h \\ &\leq C \|\tilde{\theta}_\varepsilon\|_{\dot{H}^1(\mathbb{R})} \left( \int_{\mathbb{R}^2} \|D_\varepsilon(x_h, \cdot)\|_{L^2(\mathbb{R})}^2 dx_h \right)^{\frac{3}{4}} \left( \int_{\mathbb{R}^2} \|\partial_3 D_\varepsilon(x_h, \cdot)\|_{L^2(\mathbb{R})}^2 dx_h \right)^{\frac{1}{4}} \\ &\leq C \|\tilde{\theta}_\varepsilon\|_{\dot{H}^1(\mathbb{R})} \|D_\varepsilon\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} \|\partial_3 D_\varepsilon\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \leq C \|\tilde{\theta}_\varepsilon\|_{\dot{H}^1(\mathbb{R})} \|D_\varepsilon\|_{L^2}^{\frac{3}{2}} \|\nabla D_\varepsilon\|_{L^2}^{\frac{1}{2}} \\ &\leq \frac{\nu_0}{4} \|D_\varepsilon\|_{\dot{H}^1}^2 + \frac{C}{\nu_0^{\frac{1}{3}}} \|\tilde{\theta}_\varepsilon\|_{\dot{H}^1(\mathbb{R})}^{\frac{4}{3}} \|D_\varepsilon\|_{L^2}^2. \end{aligned} \quad (3.48)$$

Gathering these estimates into (3.45) leads to:

$$\frac{d}{dt} \|D_\varepsilon\|_{L^2}^2 + \nu_0 \|\nabla D_\varepsilon\|_{L^2}^2 \leq \|\tilde{G}\|_{L^2} + \|D_\varepsilon\|_{L^2}^2 \left( \|\tilde{G}\|_{L^2} + \frac{C}{\nu_0} \|\nabla \tilde{v}^h\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{C}{\nu_0^{\frac{1}{3}}} \|\tilde{\theta}_\varepsilon\|_{\dot{H}^1(\mathbb{R})}^{\frac{4}{3}} \right),$$

which entails the first part of the result after integration in time and thanks to the Gronwall Lemma. Thanks to Theorem 2.2, we bound the terms involving  $\tilde{G}$  and  $\tilde{v}_0^h$  and to bound the last term, we use the estimates from Theorem 2.1 and more precisely (2.40).

We outline that it is not possible to directly use the estimate in  $\tilde{H}^s$  as the time integration exponent  $p$  cannot reach  $\frac{4}{3}$ . We cannot either use directly (2.40) with  $p = r = 2$  and  $q = \frac{4}{3}$  as (thanks to Proposition 5.2) we have:

$$\|\tilde{\theta}_\varepsilon\|_{\tilde{L}_t^{\frac{4}{3}} \tilde{H}^{s+\frac{3}{2}}} \leq \|\tilde{\theta}_\varepsilon\|_{L_t^{\frac{4}{3}} \tilde{H}^{s+\frac{3}{2}}},$$

and (2.40) only bounds the left-hand side of the estimates (we need to bound the right-hand side). To simplify, we use the estimates in the case  $(s, p, r, q) = (-\frac{1}{2}, 2, 1, \frac{4}{3})$ , and thanks to Proposition 5.2, we have for all  $t \geq 0$ :

$$\|\tilde{\theta}_\varepsilon\|_{L_t^{\frac{4}{3}} \tilde{H}^1} \leq \|\tilde{\theta}_\varepsilon\|_{L_t^{\frac{4}{3}} \dot{B}_{2,1}^1} \leq \|\tilde{\theta}_\varepsilon\|_{\tilde{L}_t^{\frac{4}{3}} \dot{B}_{2,1}^1} \leq \frac{C}{(\nu')^{\frac{3}{4}}} \|\tilde{\theta}_{0,\varepsilon}\|_{\dot{B}_{2,1}^{-\frac{1}{2}}} \leq \frac{CC_0}{(\nu')^{\frac{3}{4}}},$$

which concludes the proof. ■

### 3.2 The Friedrichs scheme

In order to properly write the scheme corresponding to System (2.37), the first step is to express the pressure  $q_\varepsilon$  in terms of  $D_\varepsilon = (V_\varepsilon, H_\varepsilon)$ :

$$\begin{aligned} q_\varepsilon &= -\frac{1}{\varepsilon} \partial_3 \Delta^{-1} H_\varepsilon - \Delta^{-1} \operatorname{div} \operatorname{div} (D_\varepsilon \otimes D_\varepsilon - (\tilde{v}^h, 0) \otimes (\tilde{v}^h, 0)) \\ &= -\frac{1}{\varepsilon} \partial_3 \Delta^{-1} H_\varepsilon - \Delta^{-1} \operatorname{div} \operatorname{div} (V_\varepsilon \otimes V_\varepsilon + V_\varepsilon \otimes (\tilde{v}^h, 0) + (\tilde{v}^h, 0) \otimes V_\varepsilon) \\ &= -\frac{1}{\varepsilon} \partial_3 \Delta^{-1} H_\varepsilon - \Delta^{-1} \left( \sum_{i,j=1}^3 \partial_i \partial_j (V_\varepsilon^i V_\varepsilon^j) + \sum_{i=1}^3 \sum_{j=1}^2 \partial_i \partial_j (V_\varepsilon^i \tilde{v}^j) + \sum_{i=1}^2 \sum_{j=1}^3 \partial_i \partial_j (\tilde{v}^i V_\varepsilon^j) \right), \end{aligned} \quad (3.49)$$

which leads to the following scheme (where for  $n \in \mathbb{N}$ ,  $J_n$  is the Fourier truncation operator on the ball centered at zero and with radius  $n$ ) for  $D_\varepsilon^n = (V_\varepsilon^n, H_\varepsilon^n)$ :

$$\begin{aligned} \partial_t D_\varepsilon^n - L J_n D_\varepsilon^n + \frac{1}{\varepsilon} \mathcal{B} J_n D_\varepsilon^n + \left( \begin{array}{l} J_n \operatorname{div} \left[ J_n V_\varepsilon^n \otimes J_n V_\varepsilon^n + J_n V_\varepsilon^n \otimes (\tilde{v}^h, 0) + (\tilde{v}^h, 0) \otimes J_n V_\varepsilon^n \right] \\ J_n \operatorname{div} \left[ J_n V_\varepsilon^n \cdot (J_n H_\varepsilon^n + \tilde{\theta}_\varepsilon) \right] + J_n \operatorname{div}_h \left( \tilde{v}^h \cdot J_n H_\varepsilon^n \right) \end{array} \right) \\ = J_n \tilde{G} - \left( \begin{array}{c} \nabla J_n q_\varepsilon^n \\ 0 \end{array} \right), \end{aligned} \quad (3.50)$$

with initial data  $D_\varepsilon^n|_{t=0} = J_n (U_{0,\varepsilon,osc} + (U_{0,\varepsilon,S}^h - \tilde{v}_0^h, 0, 0))$ , the pressure  $q_\varepsilon^n$  being defined according to (3.49).

### 3.3 Sketch of the proof

The rest of the proof of Theorem 3.1 is classical and we only give a sketch of it, pointing out what is different.

- The Friedrichs scheme is an ODE in  $L_n^2 = \{u \in L^2(\mathbb{R}^3), \text{ with } \operatorname{supp} \hat{u} \subset B(0, n)\}$  and its solution  $D_\varepsilon^n$  belongs to  $\mathcal{C}([0, T_{\varepsilon,n}^*], L_n^2(\mathbb{R}^3))$  for some lifespan  $T_{\varepsilon,n}^*$ .
- We have  $J_n D_\varepsilon^n = D_\varepsilon^n$  and, taking the divergence of the system, we show that  $\operatorname{div} D_\varepsilon^n = 0$ .

- Then the scheme can be rewritten in a form close to (2.37) and Proposition 3.1 implies that (3.43) is true for  $D_\varepsilon^n$ , which provides a uniform bound in  $n$  for  $\|D_\varepsilon^n\|_{L^\infty L^2}$  which implies  $T_{\varepsilon,n}^* = +\infty$  thanks to the blow-up criterion for the ODE.
- As  $D_\varepsilon^n$  is bounded in  $\dot{E}^0$ , we can extract a subsequence that weakly converges to some  $D_\varepsilon \in \dot{E}^0$  satisfying (3.43) with the same bound.
- All that remains is to show the weak limit solves (2.37), the only difficulty being to deal with the nonlinear terms limits, for which the classical argument is to use the Ascoli theorem. This is here that some adaptation is needed. In the Navier-Stokes case, it is shown that every term in the right-hand side is in  $L_{loc}^{\frac{4}{3}} \dot{H}^{-1}$ . It is easily proved for the terms involving  $\tilde{v}^h$  and all that remains is to check the term involving  $\tilde{\theta}_\varepsilon$ . With the same ideas as in the proof of Proposition 3.1:

$$\begin{aligned}
& \|J_n \operatorname{div} (J_n V_\varepsilon^n \cdot \tilde{\theta}_\varepsilon)\|_{\dot{H}^{-1}}^2 \leq \|V_\varepsilon^n \cdot \tilde{\theta}_\varepsilon\|_{L^2}^2 \\
& = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} |V_\varepsilon^n(x_h, x_3)|^2 |\tilde{\theta}_\varepsilon(x_3)|^2 dx_3 \right) dx_h \leq \int_{\mathbb{R}^2} \|V_\varepsilon^n(x_h, \cdot)\|_{L^4(\mathbb{R})}^2 \|\tilde{\theta}_\varepsilon\|_{L^4(\mathbb{R})}^2 dx_h \\
& \leq C \|\tilde{\theta}_\varepsilon\|_{\dot{H}^{\frac{1}{4}}}^2 \int_{\mathbb{R}^2} \|V_\varepsilon^n(x_h, \cdot)\|_{L^2(\mathbb{R})}^{\frac{3}{2}} \|\partial_3 V_\varepsilon^n(x_h, \cdot)\|_{L^2(\mathbb{R})}^{\frac{1}{2}} dx_h \\
& \leq C \|\tilde{\theta}_\varepsilon\|_{\dot{H}^{\frac{1}{4}}}^2 \|V_\varepsilon^n\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} \|\nabla V_\varepsilon^n\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}. \quad (3.51)
\end{aligned}$$

Thanks to the Young estimates (with  $(\frac{4}{3}, 4)$ ), we obtain that:

$$\|J_n \operatorname{div} (J_n V_\varepsilon^n \cdot \tilde{\theta}_\varepsilon)\|_{L_t^2 \dot{H}^{-1}} \leq C \|\tilde{\theta}_\varepsilon\|_{L_t^{\frac{8}{3}} \dot{H}^{\frac{1}{4}}} \|D_\varepsilon^n\|_{L_t^\infty L^2(\mathbb{R}^3)}^{\frac{3}{4}} \|\nabla D_\varepsilon^n\|_{L_t^2 L^2(\mathbb{R}^3)}^{\frac{1}{4}}.$$

The first term is bounded thanks to (2.40) and the fact that  $\tilde{\theta}_0 \in \dot{B}_{2,1}^{-\frac{1}{2}}$ :

$$\|\tilde{\theta}_\varepsilon\|_{L_t^{\frac{8}{3}} \dot{H}^{\frac{1}{4}}} \leq \|\tilde{\theta}_\varepsilon\|_{L_t^{\frac{8}{3}} \dot{B}_{2,1}^{\frac{1}{4}}} \leq \|\tilde{\theta}_\varepsilon\|_{\tilde{L}_t^{\frac{8}{3}} \dot{B}_{2,1}^{\frac{1}{4}}} \leq \frac{C}{(\nu')^{\frac{3}{8}}} \|\tilde{\theta}_{0,\varepsilon}\|_{\dot{B}_{2,1}^{-\frac{1}{2}}} \leq \frac{C \mathbb{C}_0}{(\nu')^{\frac{3}{8}}},$$

so that we have  $J_n \operatorname{div} (J_n V_\varepsilon^n \cdot \tilde{\theta}_\varepsilon) \in L^2(\mathbb{R}_+, \dot{H}^{-1}) \subset L_{loc}^{\frac{4}{3}}(\mathbb{R}, \dot{H}^{-1})$ .

- Concerning the regularity of the pressure term  $q_\varepsilon$ , only one term is different, namely  $-\frac{1}{\varepsilon} \partial_3 \Delta^{-1} D_\varepsilon^4$ , which obviously belongs to  $\dot{E}^1$ .

## 4 Convergence of $D_\varepsilon$

Let us first more precisely state the announced convergence result.

**Theorem 4.1** (*Convergence*) Assume  $\delta > 0$ ,  $\tilde{\theta}_0 \in \dot{B}_{2,1}^{-\frac{1}{2}}$  and  $\tilde{v}^h \in H^{\frac{1}{2}+\delta}$  (with  $\operatorname{div} \tilde{v}^h = 0$  or equivalently  $\tilde{v}^h = \mathbb{P}_2 \tilde{v}^h$ ). Assume in addition that for all  $\varepsilon > 0$ ,  $U_{0,\varepsilon} \in L^2$ ,  $\tilde{\theta}_{0,\varepsilon} \in \dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R})$  and that there exists a constant  $\mathbb{C}_0 \geq 1$  such that:

$$\left\{ \begin{array}{l} \|\tilde{v}_0^h\|_{H^{\frac{1}{2}+\delta}(\mathbb{R}^3)} \leq \mathbb{C}_0, \\ \sup_{\varepsilon>0} \|U_\varepsilon\|_{L^2} \leq \mathbb{C}_0, \\ \|U_{0,\varepsilon}^h - \tilde{v}_0^h\|_{L^2} \xrightarrow{\varepsilon \rightarrow 0} 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \|\tilde{\theta}_0\|_{\dot{B}_{2,1}^{-\frac{1}{2}}} \leq \mathbb{C}_0, \\ \sup_{\varepsilon>0} \|\tilde{\theta}_{0,\varepsilon}\|_{\dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R})} \leq \mathbb{C}_0, \\ \|\tilde{\theta}_{0,\varepsilon} - \tilde{\theta}_0\|_{\dot{B}_{2,1}^{-\frac{1}{2}}(\mathbb{R})} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{array} \right. \quad (4.52)$$

Then  $U_\varepsilon$  converges to  $(\tilde{v}^h, 0, \tilde{\theta})$  (where  $\tilde{v}^h$  and  $\tilde{\theta}$  are the global solutions of Systems (2.28) and (2.29)) in the following sense: if  $D_\varepsilon = U_\varepsilon - (\tilde{v}^h, 0, \tilde{\theta}_\varepsilon)$  (where  $\tilde{\theta}_\varepsilon$  is the global solution of (2.36)),

for all  $q \in ]2, 6[$ , there exists  $\varepsilon_1 = \varepsilon_1(\nu, \nu', q) > 0$  such that the stratified and oscillating parts of  $D_\varepsilon$  satisfy:

$$D_{\varepsilon, S} \xrightarrow{\varepsilon \rightarrow 0} 0 \text{ in } L_{loc}^2 L_{loc}^q,$$

and for all  $t \geq 0$ , there exists a constant  $D = D_{t, \delta, \nu, \nu', \mathbb{C}_0}$  such that, for all  $\varepsilon \leq \varepsilon_1$ ,

$$\|D_{\varepsilon, osc}\|_{L_t^2 L^q} \leq D\varepsilon^{\frac{K(q)}{464}}, \quad \text{with } K(q) \stackrel{def}{=} \frac{\min(\frac{6}{q} - 1, 1 - \frac{2}{q})^2}{(\frac{6}{q} - 1)}. \quad (4.53)$$

Moreover, when  $\nu = \nu'$ , the previous estimates can be upgraded into  $\|D_{\varepsilon, osc}\|_{L_t^2 L^q} \leq D\varepsilon^{\frac{K(q)}{368}}$  (now valid for all  $\varepsilon > 0$ ) and we can obtain global-in-time estimates with better convergence rate: there exists a constant  $C = C_{\nu, \delta, \mathbb{C}_0} > 0$  such that, for any  $\varepsilon > 0$ ,

$$\|D_{\varepsilon, osc}\|_{\tilde{L}^{\frac{4}{3}} \dot{B}_{8,2}^0 + \tilde{L}^1 \dot{B}_{8,2}^0} \leq C\varepsilon^{\frac{3}{16}}.$$

**Remark 4.1** Choosing  $\tilde{\theta}_{0, \varepsilon} = \tilde{\theta}_0$  for each  $\varepsilon > 0$  leads to the particular case described by the Theorem from the introduction.

In the rest of this section, we begin to prove the convergence of the oscillating part  $D_{\varepsilon, osc}$ , and then we use it to prove that the stratified part  $D_{\varepsilon, S}$  also goes to zero.

## 4.1 Convergence of $D_\varepsilon$ in the case $\nu \neq \nu'$

### 4.1.1 Convergence of the oscillating part

We refer to (2.24) and (2.25) for the definition of  $D_{\varepsilon, S}$  and  $D_{\varepsilon, osc}$  and to Proposition 2.1 for properties of this orthogonal decomposition. Thanks to Propositions 2.1 (Points 6,7) and 2.2, we have  $\mathcal{B}D_\varepsilon = \mathcal{B}D_{\varepsilon, osc}$  and  $\mathbb{P}(\nabla q_\varepsilon, 0) = 0 = \mathbb{P}\tilde{G}$ , which entails that, applying  $\mathbb{P}(I_d - \mathbb{P}_2)$  to System (2.37),  $D_{\varepsilon, osc}$  solves:

$$\begin{cases} \partial_t D_{\varepsilon, osc} - (L - \frac{1}{\varepsilon} \mathbb{P}\mathcal{B})D_{\varepsilon, osc} = -\mathbb{P}(I_d - \mathbb{P}_2) \left[ D_\varepsilon \cdot \nabla D_\varepsilon + \begin{pmatrix} D_\varepsilon \cdot \nabla \tilde{v}^h \\ 0 \\ D_\varepsilon^3 \cdot \partial_3 \tilde{\theta}_\varepsilon \end{pmatrix} + \tilde{v}^h \cdot \nabla_h D_\varepsilon \right] + \tilde{G}, \\ D_{\varepsilon, osc}|_{t=0} = U_{0, \varepsilon, osc}. \end{cases} \quad (4.54)$$

Then, referring to Section 5.2 for the definition of  $\mathcal{P}_{r_\varepsilon, R_\varepsilon}$ , as in [18, 4] we truncate in frequency and split  $D_{\varepsilon, osc}$  into three parts:

$$D_{\varepsilon, osc} = \left( I_d - \chi\left(\frac{|D|}{R_\varepsilon}\right) \right) D_{\varepsilon, osc} + \chi\left(\frac{|D|}{R_\varepsilon}\right) \chi\left(\frac{|D_h|}{2r_\varepsilon}\right) D_{\varepsilon, osc} + \mathcal{P}_{r_\varepsilon, R_\varepsilon} D_{\varepsilon, osc}. \quad (4.55)$$

The first two terms are estimated as in [18, 4] using Lemma 5.1 and Propositions 2.1 and 3.43, for all  $t \geq 0$  and  $q \in ]2, 6[$ :

$$\begin{cases} \|(I_d - \chi(\frac{|D|}{R_\varepsilon}))D_{\varepsilon, osc}\|_{L_t^2 L^q} \leq R_\varepsilon^{-\frac{1}{2}(\frac{6}{q}-1)} \|D_{\varepsilon, osc}\|_{L_t^2 \dot{H}^1} \leq R_\varepsilon^{-\frac{1}{2}(\frac{6}{q}-1)} \|D_\varepsilon\|_{L_t^2 \dot{H}^1}, \\ \|\chi(\frac{|D|}{R_\varepsilon})\chi(\frac{|D_h|}{2r_\varepsilon})D_{\varepsilon, osc}\|_{L_t^2 L^q} \leq Ct^{\frac{1}{2}} (R_\varepsilon r_\varepsilon^2)^{\frac{1}{2}-\frac{1}{q}} \|D_{\varepsilon, osc}\|_{L_t^\infty L^2} \leq Ct^{\frac{1}{2}} (R_\varepsilon r_\varepsilon^2)^{\frac{1}{2}-\frac{1}{q}} \|D_\varepsilon\|_{L_t^\infty L^2}. \end{cases} \quad (4.56)$$

The third part is bounded thanks to the Strichartz estimates proved in Proposition 5.4: when  $\varepsilon \leq \varepsilon_1(\nu, \nu', m, M)$ , choosing  $(p, r) = (2, q)$  in (5.101) and defining  $F_{ext}$  as the right-hand side of System (4.54), we obtain that:

$$\|\mathcal{P}_{r_\varepsilon, R_\varepsilon} D_{\varepsilon, osc}\|_{L_t^2 L^q} \leq C_{\nu, \nu'} \frac{R_\varepsilon^{\frac{17}{4} - \frac{7}{2q}}}{r_\varepsilon^{\frac{19}{4} - \frac{3}{2q}}} \varepsilon^{\frac{1}{8}(1 - \frac{2}{q})} \left( \|\mathcal{P}_{r_\varepsilon, R_\varepsilon} U_{0, \varepsilon, osc}\|_{L^2} + \|\mathcal{P}_{r_\varepsilon, R_\varepsilon} F_{ext}\|_{L_t^1 L^2} \right), \quad (4.57)$$

All that remains is then to bound the external force terms. Three of them can be bounded using Lemma 5.1:

$$\|\mathcal{P}_{r_\varepsilon, R_\varepsilon}(fg)\|_{L_t^1 L^2} \leq CR_\varepsilon^{\frac{3}{2}} \|fg\|_{L_t^1 L^1} \leq CR_\varepsilon^{\frac{3}{2}} t^{\frac{1}{2}} \|f\|_{L_t^\infty L^2} \|g\|_{L_t^2 L^2},$$

so that we immediately obtain:

$$\begin{aligned} & \|\mathcal{P}_{r_\varepsilon, R_\varepsilon} (D_\varepsilon \cdot \nabla D_\varepsilon + D_\varepsilon \cdot \nabla \tilde{v}^h + \tilde{v}^h \cdot \nabla_h D_\varepsilon)\|_{L_t^1 L^2} \\ & \leq CR_\varepsilon^{\frac{3}{2}} t^{\frac{1}{2}} \left( \|D_\varepsilon\|_{L_t^\infty L^2} + \|\tilde{v}^h\|_{L_t^\infty L^2} \right) \|\nabla D_\varepsilon\|_{L_t^2 L^2} + \|D_\varepsilon\|_{L_t^\infty L^2} \|\nabla \tilde{v}^h\|_{L_t^2 L^2}. \end{aligned} \quad (4.58)$$

The last term is bounded as we did for the Friedrichs scheme:

$$\begin{aligned} \|\mathcal{P}_{r_\varepsilon, R_\varepsilon} (D_\varepsilon^3 \cdot \partial_3 \tilde{\theta}_\varepsilon)\|_{L^2}^2 & \leq \int_{\mathbb{R}^2} \|D_\varepsilon^3(x_h, \cdot)\|_{L^4(\mathbb{R})}^2 \|\partial_3 \tilde{\theta}_\varepsilon\|_{L^4(\mathbb{R})}^2 dx_h \\ & \leq C \|\tilde{\theta}_\varepsilon\|_{\dot{H}^{\frac{5}{4}}(\mathbb{R})}^2 \int_{\mathbb{R}^2} \|D_\varepsilon^3(x_h, \cdot)\|_{\dot{H}^{\frac{1}{4}}(\mathbb{R})}^2 dx_h \leq C \|\tilde{\theta}_\varepsilon\|_{\dot{H}^{\frac{5}{4}}(\mathbb{R})}^2 \|D_\varepsilon\|_{L^2(\mathbb{R}^3)}^{\frac{3}{2}} \|\nabla D_\varepsilon\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}, \end{aligned} \quad (4.59)$$

and

$$\|\mathcal{P}_{r_\varepsilon, R_\varepsilon} (D_\varepsilon^3 \cdot \partial_3 \tilde{\theta}_\varepsilon)\|_{L_t^1 L^2} \leq \|\tilde{\theta}_\varepsilon\|_{L_t^{\frac{8}{5}} \dot{H}^{\frac{5}{4}}} \|D_\varepsilon\|_{L_t^\infty L^2}^{\frac{3}{4}} \|\nabla D_\varepsilon\|_{L_t^2 L^2}^{\frac{1}{4}}. \quad (4.60)$$

Thanks to (2.40), as  $\tilde{\theta}_0 \in \dot{B}_{2,1}^{-\frac{1}{2}}$ :

$$\|\tilde{\theta}_\varepsilon\|_{L_t^{\frac{8}{5}} \dot{H}^{\frac{5}{4}}} \leq \|\tilde{\theta}_\varepsilon\|_{L_t^{\frac{8}{5}} \dot{B}_{2,1}^{\frac{5}{4}}} \leq \|\tilde{\theta}_\varepsilon\|_{L_t^{\frac{8}{5}} \dot{B}_{2,1}^{\frac{5}{4}}} \leq \frac{C}{(\nu')^{\frac{7}{8}}} \|\tilde{\theta}_{0, \varepsilon}\|_{\dot{B}_{2,1}^{-\frac{1}{2}}} \leq \frac{C \mathbb{C}_0}{(\nu')^{\frac{7}{8}}}, \quad (4.61)$$

so that, thanks to Theorem 2.2 and (3.43) we can write that:

$$\|\mathcal{P}_{r_\varepsilon, R_\varepsilon} F_{ext}\|_{L_t^1 L^2} \leq C_{\delta, \nu, \nu'} e^{C_{\delta, \nu, \nu'} \mathbb{C}_0^{2 + \frac{1}{8}}} \left( \|U_{0, \varepsilon, osc}\|_{L^2}^2 + \|U_{0, \varepsilon, S}^h - \tilde{v}_0^h\|_{L^2}^2 + \mathbb{C}_0^{2 + \frac{1}{8}} \right). \quad (4.62)$$

Gathering (4.56), (4.57) and (4.62), and thanks to the assumptions on the initial data, we finally obtain that for all  $t \geq 0$  and  $q \in ]2, 6[$  (replacing  $r_\varepsilon, R_\varepsilon$  by their expressions in  $\varepsilon$ ):

$$\begin{aligned} \|D_{\varepsilon, osc}\|_{L_t^2 L^q} & \leq C_{\delta, \nu, \nu'} e^{C_{\delta, \nu, \nu'} \mathbb{C}_0^{2 + \frac{1}{8}}} \left( \|U_{0, \varepsilon, osc}\|_{L^2}^2 + \|U_{0, \varepsilon, S}^h - \tilde{v}_0^h\|_{L^2}^2 + \mathbb{C}_0^{2 + \frac{1}{8}} \right) \\ & \quad \times \left( R_\varepsilon^{-\frac{1}{2}(\frac{6}{q} - 1)} + t^{\frac{1}{2}} (R_\varepsilon r_\varepsilon^2)^{\frac{1}{2} - \frac{1}{q}} + \frac{R_\varepsilon^{\frac{23}{4} - \frac{7}{2q}}}{r_\varepsilon^{\frac{19}{4} - \frac{3}{2q}}} \varepsilon^{\frac{1}{8}(1 - \frac{2}{q})} (1 + t^{\frac{1}{2}}) \right) \\ & \leq D_{t, \delta, \nu, \nu', \mathbb{C}_0} \left( 1 + \|U_{0, \varepsilon, osc}\|_{L^2}^2 + \|U_{0, \varepsilon, S}^h - \tilde{v}_0^h\|_{L^2}^2 \right) \\ & \quad \times \left( \varepsilon^{\frac{M}{2}(\frac{6}{q} - 1)} + \varepsilon^{(2m - M)(\frac{1}{2} - \frac{1}{q})} + \varepsilon^{\frac{1}{8}(1 - \frac{2}{q}) - M(\frac{23}{4} - \frac{7}{2q}) - m(\frac{19}{4} - \frac{3}{2q})} \right) \\ & \leq D_{t, \delta, \nu, \nu', \mathbb{C}_0} \left( 1 + \|U_{0, \varepsilon, osc}\|_{L^2}^2 + \|U_{0, \varepsilon, S}^h - \tilde{v}_0^h\|_{L^2}^2 \right) \varepsilon^{J(m, M, q)} \\ & \leq D_{t, \delta, \nu, \nu', \mathbb{C}_0} \varepsilon^{J(m, M, q)}, \end{aligned} \quad (4.63)$$

where we define

$$J(m, M, q) \stackrel{def}{=} \min \left( \frac{M}{2} \left( \frac{6}{q} - 1 \right), (2m - M) \left( \frac{1}{2} - \frac{1}{q} \right), \frac{1}{8} \left( 1 - \frac{2}{q} \right) - M \left( \frac{23}{4} - \frac{7}{2q} \right) - m \left( \frac{19}{4} - \frac{3}{2q} \right) \right). \quad (4.64)$$

Let us recall that  $m, M$  also have to satisfy:

$$\begin{cases} 3M + m < 1, \\ M < 2m. \end{cases} \quad (4.65)$$

In order to provide a more explicit convergence rate, we can observe that the third term in the minimum becomes positive when  $m, M$  are small enough, but we have to care that, due to the first two terms, they are not too small. Introducing  $A, B$  as below (both of them are positive as  $q \in ]2, 6[$ ),

$$A = \frac{6}{q} - 1 \quad \text{and} \quad B = 1 - \frac{2}{q},$$

we rewrite the third term in the minimum as a function of  $A$  and  $B$ , which leads to:

$$J(m, M, q) = A \min \left( \frac{M}{2}, \left( m - \frac{M}{2} \right) \frac{B}{A}, \left( \frac{1}{8} - \frac{31}{4}M - \frac{27}{4}m \right) \frac{B}{A} - 2(m + M) \right). \quad (4.66)$$

Choosing  $m = M$  (and taking into account (4.65)), we are reduced, given  $A, B > 0$ , to choose  $m$  small but with the view that the following quantity has to be as large as possible:

$$J(m, m, q) = A \min \left( \frac{m}{2}, \frac{mB}{2A}, \left( \frac{1}{8} - \frac{29}{2}m \right) \frac{B}{A} - 4m \right). \quad (4.67)$$

Observing that:

$$m \leq \frac{1}{232} \implies \begin{cases} \frac{1}{8} - \frac{29}{2}m \geq \frac{1}{16}, \\ J(m, m, q) \geq A \min \left( \frac{m}{2}, \frac{mB}{2A}, \frac{1}{16} \frac{B}{A} - 4m \right), \end{cases}$$

we obtain that if in addition  $m = M \leq \frac{1}{128} \frac{B}{A}$ , then  $\frac{1}{16} \frac{B}{A} - 4m \geq \frac{1}{32} \frac{B}{A}$  and

$$\begin{aligned} J(m, m, q) &\geq A \min \left( \frac{m}{2}, \frac{mB}{2A}, \frac{1}{32} \frac{B}{A} \right) \\ &\geq A \min \left( 1, \frac{B}{A} \right) \min \left( \frac{m}{2}, \frac{1}{32} \right) \geq A \min \left( 1, \frac{B}{A} \right) \frac{m}{2} = \min(A, B) \frac{m}{2}. \end{aligned} \quad (4.68)$$

Finally, if we choose  $m_0 = M_0 = \min \left( \frac{1}{232}, \frac{1}{128} \frac{B}{A} \right)$ , we obtain:

$$\begin{aligned} J(m_0, M_0, q) &\geq \frac{1}{16} \min \left( \frac{1}{29}, \frac{1}{16} \frac{B}{A} \right) \min(A, B) \\ &\geq \frac{1}{16 \times 29} \min \left( 1, \frac{B}{A} \right) \min(A, B) \geq \frac{\min(A, B)^2}{464A}, \end{aligned} \quad (4.69)$$

so that, plugging into (4.63), we end-up with the fact that for all  $t > 0$ , all  $q \in ]2, 6[$  and all  $\varepsilon \leq \varepsilon_1(\nu, \nu', q)$ :

$$\|D_{\varepsilon, osc}\|_{L_t^2 L^q} \leq D_{t, \delta, \nu, \nu', C_0} \varepsilon^{\frac{\min(\frac{6}{q} - 1, 1 - \frac{2}{q})^2}{464(\frac{6}{q} - 1)}}. \blacksquare$$

**Remark 4.2** We emphasize that  $\varepsilon_1 = \varepsilon_1(\nu, \nu', m, M) \stackrel{def}{=} \left( \frac{\sqrt{2}}{|\nu - \nu'|} \right)^{\frac{1}{1 - (3M + m)}}$  turns into  $\varepsilon_1(\nu, \nu', q)$  as  $m, M$  are fixed only depending on  $q$ .

### 4.1.2 Convergence of $D_{\varepsilon,S}$

Thanks to the assumptions on the initial data, the estimates (3.43) can be bounded uniformly in  $\varepsilon$  as follows: for all  $t \geq 0$ , and all  $\varepsilon > 0$

$$\|D_\varepsilon(t)\|_{L^2}^2 + \nu_0 \int_0^t \|\nabla D_\varepsilon(\tau)\|_{L^2}^2 d\tau \leq C_{\delta,\nu,\nu'} \mathbb{C}_0^{2+\frac{1}{\delta}} e^{C_{\delta,\nu,\nu'} \mathbb{C}_0^{2+\frac{1}{\delta}}}. \quad (4.70)$$

Therefore we can extract from  $(D_\varepsilon)_{\varepsilon>0}$  a subsequence that weakly converges to some  $\tilde{D} \in \dot{E}^0$  also satisfying (4.70). To simplify, let us also denote  $(D_\varepsilon)_{\varepsilon>0}$  this subsequence. We showed in the previous section that for all  $q \in ]2, 6[$  and all  $t > 0$ ,  $\|D_{\varepsilon,osc}\|_{L_t^2 L^q} \xrightarrow{\varepsilon \rightarrow 0} 0$ , which in particular implies that  $D_{\varepsilon,osc} \xrightarrow{\varepsilon \rightarrow 0} 0$  and  $D_{\varepsilon,S} \xrightarrow{\varepsilon \rightarrow 0} \tilde{D}$ .

Applying the operator  $\mathbb{P}_2$  to System (2.37), we obtain that  $D_{\varepsilon,S} = (D_{\varepsilon,S}^h, 0, 0)$  satisfies (using once more Proposition 2.1):

$$\begin{aligned} \partial_t D_{\varepsilon,S} - \nu \Delta D_{\varepsilon,S} &= -\mathbb{P}_2 \left[ D_{\varepsilon,S} \cdot \nabla D_{\varepsilon,S} + D_{\varepsilon,S} \cdot \nabla \begin{pmatrix} \tilde{v}^h \\ 0 \\ 0 \end{pmatrix} + \tilde{v}^h \cdot \nabla_h D_{\varepsilon,S} \right. \\ &\quad \left. + D_{\varepsilon,S} \cdot \nabla D_{\varepsilon,osc} + D_{\varepsilon,osc} \cdot \nabla D_{\varepsilon,S} + D_{\varepsilon,osc} \cdot \nabla D_{\varepsilon,osc} + D_{\varepsilon,osc} \cdot \nabla \begin{pmatrix} \tilde{v}^h \\ 0 \\ 0 \end{pmatrix} + \tilde{v}^h \cdot \nabla_h D_{\varepsilon,osc} \right], \end{aligned} \quad (4.71)$$

with initial data:  $D_{\varepsilon,S}|_{t=0} = (U_{0,\varepsilon,S}^h - \tilde{v}_0^h, 0, 0)$ .

Now, as  $D_{\varepsilon,S} \xrightarrow{\varepsilon \rightarrow 0} \tilde{D}$  (weakly) and  $D_{\varepsilon,osc} \xrightarrow{\varepsilon \rightarrow 0} 0$  (strongly in  $L_{loc}^2 L^q$ ), every term involving  $D_{\varepsilon,osc}$  weakly converges to zero. All that remains is to show  $D_{\varepsilon,S} \cdot \nabla D_{\varepsilon,S}$  weakly converges to  $\tilde{D} \cdot \nabla \tilde{D}$  which is similar to what is done in the classical Navier-Stokes case. We obtain that  $\tilde{D}$  satisfies the following system:

$$\begin{cases} \partial_t \tilde{D} - \nu \Delta \tilde{D} = -\mathbb{P}_2 \left[ \tilde{D} \cdot \nabla \tilde{D} + \tilde{D} \cdot \nabla \begin{pmatrix} \tilde{v}^h \\ 0 \\ 0 \end{pmatrix} + \tilde{v}^h \cdot \nabla_h \tilde{D} \right], \\ \tilde{D}|_{t=0} = 0. \end{cases} \quad (4.72)$$

Thanks to Proposition 3.1 (with zero initial data and external force  $\tilde{G}$  replaced by zero), we obtain that  $\tilde{D} = 0$ . To finish the proof, we have to upgrade the convergence of  $D_{\varepsilon,S}$  towards zero: we repeat the arguments from [4] and manage to finally obtain that for all  $q \in ]2, 6[$ :

$$D_{\varepsilon,S} \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{in } L_{loc}^2 L_{loc}^q.$$

### 4.2 Convergence of $D_{\varepsilon,osc}$ and $D_{\varepsilon,S}$ in the case $\nu = \nu'$

In the case  $\nu = \nu'$ , we can use the previous arguments but we do not need anymore any smallness condition on  $\varepsilon$  and (taking advantage of the fact that the  $\mathbb{P}_k$  are orthogonal) we can save  $R_\varepsilon/r_\varepsilon$  in the Strichartz estimates (5.101) and in the final estimates (4.63), which modifies  $J(m, M, q)$  into:

$$K(m, M, q) \stackrel{def}{=} \min \left( \frac{M}{2} \left( \frac{6}{q} - 1 \right), (2m - M) \left( \frac{1}{2} - \frac{1}{q} \right), \frac{1}{8} \left( 1 - \frac{2}{q} \right) - M \left( \frac{19}{4} - \frac{7}{2q} \right) - m \left( \frac{15}{4} - \frac{3}{2q} \right) \right), \quad (4.73)$$

which allows a slight improvement in the final convergence rate (following the same steps as before): for all  $t > 0$ , all  $q \in ]2, 6[$  and all  $\varepsilon > 0$ :

$$\|D_{\varepsilon,osc}\|_{L_t^2 L^q} \leq D_{t,\delta,\nu,\nu',\mathbb{C}_0} \varepsilon^{\frac{\min(\frac{6}{q}-1, 1-\frac{2}{q})^2}{368(\frac{6}{q}-1)}}.$$

Nevertheless, when  $\nu = \nu'$  we can obtain much better Strichartz estimates (see Proposition 5.6 in the appendix) which do not require frequency truncations. So the convergence for  $D_{\varepsilon,osc}$  is directly given by this proposition, the only restrictions in choosing the coefficients  $(d, p, r, \theta)$  being guided by the possible bounds for the external force term. These new Strichartz estimates will allow global in time estimates and a better convergence rate.

To do this we remark that, as in [14], among the various external force terms, we can observe two distinct regularities, which suggests us to decompose  $D_{\varepsilon,osc} = D_{\varepsilon,osc}^1 + D_{\varepsilon,osc}^2$  which respectively solve:

$$\begin{cases} \partial_t D_{\varepsilon,osc}^1 - (L - \frac{1}{\varepsilon} \mathbb{P} \mathcal{B}) D_{\varepsilon,osc}^1 = -\mathbb{P}(I_d - \mathbb{P}_2) \begin{pmatrix} 0 \\ 0 \\ 0 \\ D_\varepsilon^3 \cdot \partial_3 \tilde{\theta}_\varepsilon \end{pmatrix} + \tilde{G}, \\ D_{\varepsilon,osc}^1|_{t=0} = U_{0,\varepsilon,osc}, \end{cases} \quad (4.74)$$

and

$$\begin{cases} \partial_t D_{\varepsilon,osc}^2 - (L - \frac{1}{\varepsilon} \mathbb{P} \mathcal{B}) D_{\varepsilon,osc}^2 = -\mathbb{P}(I_d - \mathbb{P}_2) \left[ D_\varepsilon \cdot \nabla D_\varepsilon + \begin{pmatrix} D_\varepsilon \cdot \nabla \tilde{v}^h \\ 0 \\ 0 \end{pmatrix} + \tilde{v}^h \cdot \nabla_h D_\varepsilon \right], \\ D_{\varepsilon,osc}^2|_{t=0} = 0. \end{cases} \quad (4.75)$$

The initial data and external force terms in the first system can be respectively bounded in  $L^2$  and  $L^1(\mathbb{R}_+, L^2)$  (globally in time, we refer to (4.60) and Theorem 2.2), whereas the external force terms in the second system can be bounded in  $L^1(\mathbb{R}_+, \dot{H}^{-\frac{1}{2}})$  according to the following product law: if  $\operatorname{div} f = 0$ ,

$$\|f \cdot \nabla g\|_{\dot{H}^{-\frac{1}{2}}} \leq \|fg\|_{\dot{H}^{\frac{1}{2}}} \leq \|f\|_{\dot{H}^1} \|g\|_{\dot{H}^1}.$$

Thanks to Proposition 5.6, for any  $(p_1, p_2, r_1, r_2)$  satisfying:

$$\frac{2}{r_1} + \frac{1}{p_1} = 1 \quad \text{and} \quad \frac{2}{r_2} + \frac{1}{p_2} = \frac{5}{4}, \quad (4.76)$$

we have (thanks to Theorem 2.2, (4.60), (3.43) and (4.61)) that for all  $t \geq 0$ :

$$\begin{aligned} \|D_{\varepsilon,osc}^1\|_{\tilde{L}_t^{p_1} \dot{B}_{r_1,2}^0} &\leq C_{p_1,r_1,\nu} \varepsilon^{\frac{1}{4}(1-\frac{2}{r_1})} \left( \|U_{0,\varepsilon,osc}\|_{L^2} + \int_0^t (\|D_\varepsilon^3 \cdot \partial_3 \tilde{\theta}_\varepsilon\|_{L^2} + \|\tilde{G}\|_{L^2}) dt' \right) \\ &\leq C_{p_1,r_1,\nu} \varepsilon^{\frac{1}{4}(1-\frac{2}{r_1})} \left( \|U_{0,\varepsilon,osc}\|_{L^2} + \|\tilde{\theta}_\varepsilon\|_{L_t^{\frac{8}{7}} \dot{H}^{\frac{5}{4}}} \|D_\varepsilon\|_{L_t^\infty L^2}^{\frac{3}{4}} \|\nabla D_\varepsilon\|_{L_t^2 L^2}^{\frac{1}{4}} + \|\tilde{G}\|_{L_t^1 L^2} \right) \\ &\leq C_{p_1,r_1,\nu} \varepsilon^{\frac{1}{4}(1-\frac{2}{r_1})} \left( \|U_{0,\varepsilon,osc}\|_{L^2} + C_{\delta,\nu} \mathbb{C}_0^{2+\frac{1}{8}} \right) e^{C_{\delta,\nu} \mathbb{C}_0^{2+\frac{1}{8}}} \\ &\leq C_{p_1,r_1,\nu,\delta,\mathbb{C}_0} \varepsilon^{\frac{1}{4}(1-\frac{2}{r_1})} (\|U_{0,\varepsilon,osc}\|_{L^2} + 1), \end{aligned} \quad (4.77)$$

and

$$\begin{aligned} \|D_{\varepsilon,osc}^2\|_{\tilde{L}_t^{p_2} \dot{B}_{r_2,2}^0} &\leq C_{p_2,r_2,\nu} \varepsilon^{\frac{1}{4}(1-\frac{2}{r_2})} \int_0^t (\|D_\varepsilon\|_{\dot{H}^1}^2 + 2\|D_\varepsilon\|_{\dot{H}^1} \|\tilde{v}^h\|_{\dot{H}^1}) dt' \\ &\leq C_{p_2,r_2,\nu,\delta,\mathbb{C}_0} \varepsilon^{\frac{1}{4}(1-\frac{2}{r_2})} (\|U_{0,\varepsilon,osc}\|_{L^2}^2 + 1). \end{aligned} \quad (4.78)$$



If we wish the largest possible exponent for  $\varepsilon$ , we need  $r_1, r_2$  to be as large as possible. Let us recall that  $p_2, r_2$  also satisfy (see Proposition 5.6)  $p_2 \leq 4/(1 - 2/r_2)$  which is satisfied as (4.76) (and the fact that  $p_2 \geq 1$ ) leads to:

$$\frac{8}{5} \leq r_2 \leq 8.$$

Choosing  $r_1 = r_2 = 8$ , we finally obtain  $(p_1, p_2) = (\frac{4}{3}, 1)$  (the conditions from Proposition 5.6 are satisfied) and:

$$\|D_{\varepsilon, osc}\|_{\tilde{L}^{\frac{4}{3}} \dot{B}_{8,2}^0 + \tilde{L}^1 \dot{B}_{8,2}^0} \leq C_{\nu, \delta, C_0} \varepsilon^{\frac{3}{16}} (\|U_{0, \varepsilon, osc}\|_{L^2} + 1) \leq C_{\nu, \delta, C_0} \varepsilon^{\frac{3}{16}},$$

which concludes the proof. ■

## 5 Appendix

### 5.1 Notations, Sobolev spaces and Littlewood-Paley decomposition

For a complete presentation of the Sobolev spaces and the Littlewood-Paley decomposition, we refer to [2]. We will use the same notations as in the appendix of [12]. Let us first mention the following lemma:

**Proposition 5.1** ([2] Chapter 2) *We have the following continuous injections:*

$$\begin{cases} \text{For any } p \geq 1, & \dot{B}_{p,1}^0 \hookrightarrow L^p, \\ \text{For any } p \in [2, \infty[, & \dot{B}_{p,2}^0 \hookrightarrow L^p, \\ \text{For any } p \in [1, 2], & \dot{B}_{p,p}^0 \hookrightarrow L^p. \end{cases}$$

Sometimes it is more convenient to work in slightly modification of the classical  $L_t^p \dot{B}_{q,r}^s$  Spaces: the Chemin-Lerner time-space Besov spaces As explained in the following definition, the integration in time is performed before the summation with respect to the frequency decomposition index:

**Definition 5.1** [2] *For  $s, t \in \mathbb{R}$  and  $a, b, c \in [1, \infty]$ , we define the following norm*

$$\|u\|_{\tilde{L}_t^a \dot{B}_{b,c}^s} = \left\| \left( 2^{js} \|\dot{\Delta}_j u\|_{L_t^a L^b} \right)_{j \in \mathbb{Z}} \right\|_{l^c(\mathbb{Z})}.$$

The space  $\tilde{L}_t^a \dot{B}_{b,c}^s$  is defined as the set of tempered distributions  $u$  such that  $\lim_{j \rightarrow -\infty} S_j u = 0$  in  $L^a([0, t], L^\infty(\mathbb{R}^d))$  and  $\|u\|_{\tilde{L}_t^a \dot{B}_{b,c}^s} < \infty$ .

We refer once more to [2] (Section 2.6.3) for more details and will only recall the following proposition:

**Proposition 5.2** *For all  $a, b, c \in [1, \infty]$  and  $s \in \mathbb{R}$ :*

$$\begin{cases} \text{if } a \leq c, & \forall u \in L_t^a \dot{B}_{b,c}^s, & \|u\|_{\tilde{L}_t^a \dot{B}_{b,c}^s} \leq \|u\|_{L_t^a \dot{B}_{b,c}^s} \\ \text{if } a \geq c, & \forall u \in \tilde{L}_t^a \dot{B}_{b,c}^s, & \|u\|_{\tilde{L}_t^a \dot{B}_{b,c}^s} \geq \|u\|_{L_t^a \dot{B}_{b,c}^s}. \end{cases}$$

## 5.2 Truncations

In this section we precise the truncation operator that we will abundantly use: let us choose a function  $\chi \in C_0^\infty(\mathbb{R}, \mathbb{R})$  taking values into  $[0, 1]$  and such that:

$$\begin{cases} \text{supp } \chi \subset B(0, 1), \\ \chi \equiv 1 \text{ near } B(0, \frac{1}{2}). \end{cases}$$

Given  $0 < r < R$  we will denote by  $\mathcal{C}_{r,R}$  the following set (where  $\xi = (\xi_h, \xi_3)$ ):

$$\mathcal{C}_{r,R} = \{\xi \in \mathbb{R}^3, \quad |\xi| \leq R \text{ and } |\xi_h| \geq r\}. \quad (5.79)$$

If we define  $f_{r,R}(\xi) = \chi(\frac{|\xi|}{R})(1 - \chi(\frac{|\xi_h|}{2r}))$ , then:

$$\begin{cases} \text{supp } f_{r,R} \subset \mathcal{C}_{r,R}, \\ f_{r,R} \equiv 1 \text{ on } \mathcal{C}_{2r, \frac{R}{2}}. \end{cases} \quad (5.80)$$

Now we introduce the following frequency truncation operator on  $\mathcal{C}_{r,R}$  ( $\mathcal{F}^{-1}$  is the inverse Fourier transform and  $|D|^s$  the classical derivation operator:  $|D|^s f = \mathcal{F}^{-1}(|\xi|^s \widehat{f}(\xi))$ ):

$$\begin{aligned} \mathcal{P}_{r,R} u &= f_{r,R}(D)u = \chi\left(\frac{|D|}{R}\right)\left(1 - \chi\left(\frac{|D_h|}{2r}\right)\right)u \\ &= \mathcal{F}^{-1}\left(f_{r,R}(\xi)\widehat{u}(\xi)\right) = \mathcal{F}^{-1}\left(\chi\left(\frac{|\xi|}{R}\right)\left(1 - \chi\left(\frac{|\xi_h|}{2r}\right)\right)\widehat{u}(\xi)\right), \end{aligned} \quad (5.81)$$

Thanks to (5.80), we have:

$$f_{\frac{r}{2}, 2R}(D)f_{r,R}(D)u = f_{r,R}(D)u. \quad (5.82)$$

In what follows (and as in [8, 12]) we will use it for particular radii  $r_\varepsilon = \varepsilon^m$  and  $R_\varepsilon = \varepsilon^{-M}$ , where  $m$  and  $M$  will be precised later. Let us end with the following anisotropic Bernstein-type result (easily adapted from [4], see [28] for more general anisotropic estimates):

**Lemma 5.1** *There exists a constant  $C > 0$  such that for all function  $f$ ,  $\alpha > 0$ ,  $1 \leq q \leq p \leq \infty$  and all  $0 < r < R$ , we have*

$$\|\chi\left(\frac{|D|}{R}\right)\chi\left(\frac{|D_h|}{r}\right)f\|_{L^p} \leq C(Rr^2)^{\frac{1}{q} - \frac{1}{p}} \|\chi\left(\frac{|D|}{R}\right)\chi\left(\frac{|D_h|}{r}\right)f\|_{L^q}. \quad (5.83)$$

Moreover if  $f$  has its frequencies located in  $\mathcal{C}_{r,R}$ , then

$$\||D|^\alpha f\|_{L^p} \leq CR^\alpha \|f\|_{L^p}. \blacksquare$$

## 5.3 About the linearized system

Consider the linearized system ( $f_0, F_{ext}$  being divergence-free, the second form is obtained using the Leray projector  $\mathbb{P}$ ):

$$\begin{cases} \partial_t f - (L - \frac{1}{\varepsilon}\mathcal{B})f = F_{ext}, \\ \text{div } f = 0, \\ f|_{t=0} = f_0. \end{cases} \iff \begin{cases} \partial_t f - (L - \frac{1}{\varepsilon}\mathbb{P}\mathcal{B})f = F_{ext}, \\ f|_{t=0} = f_0. \end{cases} \quad (5.84)$$

Applying the Fourier transform turns the equation into (as in [4, 40]):

$$\partial_t \widehat{f} - \mathbb{B}(\xi, \varepsilon)\widehat{f} = \widehat{F_{ext}},$$

where

$$\mathbb{B}(\xi, \varepsilon) = L - \widehat{\frac{1}{\varepsilon} \mathbb{P} \mathcal{B}} = \begin{pmatrix} -\nu(\xi_2^2 + \xi_3^2) & \nu\xi_1\xi_2 & \nu\xi_1\xi_3 & \frac{\xi_1\xi_3}{\varepsilon|\xi|^2} \\ \nu\xi_1\xi_2 & -\nu(\xi_1^2 + \xi_3^2) & \nu\xi_2\xi_3 & \frac{\xi_2\xi_3}{\varepsilon|\xi|^2} \\ \nu\xi_1\xi_3 & \nu\xi_2\xi_3 & -\nu(\xi_1^2 + \xi_2^2) & -\frac{\xi_1^2 + \xi_2^2}{\varepsilon|\xi|^2} \\ 0 & 0 & \frac{1}{\varepsilon} & -\nu'|\xi|^2 \end{pmatrix}.$$

**Remark 5.1** Note that in [40, 34, 36] the authors consider the matrix  $L - \widehat{\frac{1}{\varepsilon} \mathbb{P} \mathcal{B}}$ . Doing this will gather our first two eigenvalues (see below) into the same double eigenvalue.

### 5.3.1 Eigenvalues, projectors

We begin with the eigenvalues and eigenvectors of matrix  $\mathbb{B}(\xi, \varepsilon)$ . The methods are similar to those in [4, 5, 6, 8, 10]. As there are some differences we will give precise results and skip details. First, the characteristic polynomial is:

$$\det(XI_4 - \mathbb{B}(\xi, \varepsilon)) = X(X + \nu|\xi|^2) \left( X^2 + (\nu + \nu')|\xi|^2 X + \nu\nu'|\xi|^4 + \frac{\xi_1^2 + \xi_2^2}{\varepsilon^2|\xi|^2} \right),$$

whose roots are much simpler to obtain compared to System  $(PE_\varepsilon)$ . The discriminant of the degree 2 factor is:

$$D = (\nu - \nu')^2|\xi|^4 - 4\frac{\xi_1^2 + \xi_2^2}{\varepsilon^2|\xi|^2}.$$

Which is nonpositive if, and only if,  $|\nu - \nu'| \varepsilon |\xi|^3 \leq 2|\xi_h|$ . If  $\xi \in \mathcal{C}_{r,R}$  it suffices to ask that  $|\nu - \nu'| \varepsilon R^3 \leq 2r$ . This asks to split the discussion into two cases.

- When  $\nu \neq \nu'$ , if we choose  $(r_\varepsilon, R_\varepsilon) = (\varepsilon^m, \varepsilon^{-M})$  ( $m, M$  being precised later) then  $D < 0$  on  $\mathcal{C}_{r_\varepsilon, R_\varepsilon}$  as soon as  $|\nu - \nu'| \varepsilon R_\varepsilon^3 \leq 2r_\varepsilon$ , which is equivalent to:

$$\varepsilon^{1-(3M+m)} \leq \frac{2}{|\nu - \nu'|}. \quad (5.85)$$

So, when

$$3M + m < 1 \quad \text{and} \quad 0 < \varepsilon \leq \varepsilon_0 = \varepsilon_0(\nu, \nu', m, M) \stackrel{\text{def}}{=} \left( \frac{2}{|\nu - \nu'|} \right)^{\frac{1}{1-(3M+m)}}, \quad (5.86)$$

we obtain the following eigenvalues:

$$\lambda_1(\varepsilon, \xi) = 0, \quad \lambda_2(\varepsilon, \xi) = -\nu|\xi|^2, \quad \lambda_{3,4}(\varepsilon, \xi) = \lambda_\pm(\varepsilon, \xi),$$

with, for  $\eta \in \{-1, 1\}$ :

$$\lambda_\eta(\varepsilon, \xi) = -\frac{\nu + \nu'}{2}|\xi|^2 + i\eta \frac{|\xi_h|}{\varepsilon|\xi|} \sqrt{1 - \frac{(\nu - \nu')^2 \varepsilon^2 |\xi|^6}{4|\xi_h|^2}}. \quad (5.87)$$

The eigenvectors are as follows:

$$V_1(\varepsilon, \xi) = \begin{pmatrix} (\varepsilon\nu\nu'|\xi|^2 + \frac{1}{\varepsilon|\xi|^2})\xi_1 \\ (\varepsilon\nu\nu'|\xi|^2 + \frac{1}{\varepsilon|\xi|^2})\xi_2 \\ \varepsilon\nu\nu'|\xi|^2\xi_3 \\ \nu\xi_3 \end{pmatrix}, \quad V_2(\varepsilon, \xi) = \frac{1}{|\xi_h|} \begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \\ 0 \end{pmatrix},$$

and for  $\eta \in \{-1, 1\}$ ,  $V_{3,4}(\varepsilon, \xi) = V_{\pm}(\varepsilon, \xi)$ :

$$V_{\eta}(\varepsilon, \xi) = \begin{pmatrix} \frac{\xi_1 \xi_3}{\sqrt{2}|\xi||\xi_h|} \\ \frac{\xi_2 \xi_3}{\sqrt{2}|\xi||\xi_h|} \\ -\frac{|\xi_h|}{\sqrt{2}|\xi|} \\ \frac{(\nu-\nu')\varepsilon|\xi|^3}{2\sqrt{2}|\xi_h|} + i\frac{\eta}{\sqrt{2}}\sqrt{1 - \frac{(\nu-\nu')^2\varepsilon^2|\xi|^6}{4|\xi_h|^2}} \end{pmatrix}. \quad (5.88)$$

- When  $\nu = \nu'$ ,  $D < 0$  on  $\mathbb{R}^3$  and the eigenvalues become:

$$\lambda_1(\varepsilon, \xi) = 0, \quad \lambda_2(\varepsilon, \xi) = -\nu|\xi|^2, \quad \lambda_{3,4}(\varepsilon, \xi) = \lambda_{\pm}(\varepsilon, \xi),$$

with, for  $\eta \in \{-1, 1\}$ :

$$\lambda_{\eta}(\varepsilon, \xi) = -\nu|\xi|^2 + i\eta\frac{|\xi_h|}{\varepsilon|\xi|}, \quad (5.89)$$

with the corresponding eigenvectors:

$$V_1(\varepsilon, \xi) = \begin{pmatrix} (\varepsilon\nu^2|\xi|^2 + \frac{1}{\varepsilon|\xi|^2})\xi_1 \\ (\varepsilon\nu^2|\xi|^2 + \frac{1}{\varepsilon|\xi|^2})\xi_2 \\ \varepsilon\nu^2|\xi|^2\xi_3, \\ \nu\xi_3 \end{pmatrix}, \quad V_2(\varepsilon, \xi) = \frac{1}{|\xi_h|} \begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \\ 0 \end{pmatrix}, \quad V_{\eta}(\varepsilon, \xi) = \begin{pmatrix} \frac{\xi_1 \xi_3}{\sqrt{2}|\xi||\xi_h|} \\ \frac{\xi_2 \xi_3}{\sqrt{2}|\xi||\xi_h|} \\ -\frac{|\xi_h|}{\sqrt{2}|\xi|} \\ i\frac{\eta}{\sqrt{2}} \end{pmatrix} \quad (5.90)$$

In both cases, the first eigenvector is not orthogonal to  $(\xi, 0)$  so does not correspond to a divergence-free quantity and will then play no role in the study of the linearized system. We can easily see that in both cases  $V_2$  is orthogonal to  $V_3$  and  $V_4$ , but  $V_3$  and  $V_4$  are orthogonal only when  $\nu = \nu'$ .

As in [4], any  $\mathbb{R}^4$ -valued divergence-free function  $f$  can have its Fourier transform decomposed into the family  $(V_2(\varepsilon, \xi), V_3(\varepsilon, \xi), V_4(\varepsilon, \xi))$  according to:

$$\widehat{f}(\xi) = \sum_{k=2}^4 a_k(\varepsilon, \xi)V_k(\varepsilon, \xi).$$

Denoting  $\mathcal{P}_i(\varepsilon, \xi) = a_k(\varepsilon, \xi)V_k(\varepsilon, \xi)$  we introduce the corresponding projectors: for  $k \in \{2, 3, 4\}$  we define the Fourier projector  $\mathbb{P}_k = \mathbb{P}_k(\varepsilon, D)$  as:

$$\mathbb{P}_k f = \mathcal{F}^{-1}\left(\mathcal{P}_k(\varepsilon, \xi)(\widehat{f}(\xi))\right). \quad (5.91)$$

Due to the fact that  $V_2$  is orthogonal to the other two eigenvectors, we have:

$$\mathbb{P}_2 f = \mathcal{F}^{-1}\left((\widehat{f}(\xi) \cdot V_2(\varepsilon, \xi))V_2(\varepsilon, \xi)\right), \quad (5.92)$$

and it is only in the case  $\nu = \nu'$  that similar properties also stand for  $k = 3, 4$ :

$$\mathbb{P}_k f = \mathcal{F}^{-1}\left((\widehat{f}(\xi) \cdot V_k(\varepsilon, \xi))V_k(\varepsilon, \xi)\right).$$

We gather in the following proposition the properties we will use to obtain the Strichartz estimates.

**Proposition 5.3** *If  $\nu \neq \nu'$ , for all  $m, M > 0$  with  $3M + m < 1$ , there exists  $\varepsilon_1 > 0$  such that for all  $\varepsilon < \varepsilon_1 = \left(\frac{\sqrt{2}}{|\nu - \nu'|}\right)^{\frac{1}{1-(3M+m)}}$ , if  $r_\varepsilon = \varepsilon^m$  and  $R_\varepsilon = \varepsilon^{-M}$ , then for all  $\xi \in \mathcal{C}_{r_\varepsilon, R_\varepsilon}$ , the matrix  $\mathbb{B}(\xi, \varepsilon) = L - \widehat{\frac{1}{\varepsilon}}\mathbb{P}\mathcal{B}$  is diagonalizable and its eigenvalues satisfy:*

$$\begin{cases} \lambda_1(\varepsilon, \xi) = 0, \\ \lambda_2(\varepsilon, \xi) = -\nu|\xi|^2, \\ \lambda_3(\varepsilon, \xi) = -\frac{\nu+\nu'}{2}|\xi|^2 + i\frac{|\xi_h|}{\varepsilon|\xi|} - i\varepsilon D(\varepsilon, \xi), \\ \lambda_4(\varepsilon, \xi) = \overline{\lambda_3(\varepsilon, \xi)}, \end{cases} \quad (5.93)$$

where  $D(\varepsilon, \xi)$  satisfies for all  $\xi \in \mathcal{C}_{r_\varepsilon, R_\varepsilon}$  (with  $k \in \{1, 2\}$ ):

$$\begin{cases} |D(\varepsilon, \xi)| \leq (\nu - \nu')^2 \frac{1}{4\sqrt{2}} \frac{|\xi|^5}{|\xi_h|} \leq C_0(\nu - \nu')^2 \frac{R_\varepsilon^5}{r_\varepsilon} = C_0(\nu - \nu')^2 \varepsilon^{-(5M+m)}, \\ |\partial_{\xi_k} D(\varepsilon, \xi)| \leq (\nu - \nu')^2 \frac{9}{2\sqrt{2}} \frac{|\xi|^5}{|\xi_h|^2} \leq C_0(\nu - \nu')^2 \frac{R_\varepsilon^5}{r_\varepsilon^2} = C_0(\nu - \nu')^2 \varepsilon^{-(5M+2m)}, \\ |\partial_{\xi_3} D(\varepsilon, \xi)| \leq (\nu - \nu')^2 \frac{15}{4\sqrt{2}} \frac{|\xi|^4}{|\xi_h|} \leq C_0(\nu - \nu')^2 \frac{R_\varepsilon^4}{r_\varepsilon} = C_0(\nu - \nu')^2 \varepsilon^{-(4M+m)}, \end{cases}$$

Moreover, the projectors  $\mathbb{P}_k = \mathbb{P}_k(\varepsilon, D)$  satisfy that, for any divergence-free  $\mathbb{R}^4$ -valued vectorfield  $f$ , we have:

$$\begin{cases} \mathbb{P}_2 f = (\nabla_h^\perp \Delta_h^{-1} \omega(f), 0, 0), \quad \text{with } \omega(f) = \partial_1 f^2 - \partial_2 f^1, \\ \|\mathbb{P}_2 f\|_{\dot{H}^s} \leq \|(f^1, f^2)\|_{\dot{H}^s} \leq \|f\|_{\dot{H}^s}, \quad \text{for any } s \in \mathbb{R}. \end{cases} \quad (5.94)$$

and

$$\begin{cases} (Id - \mathbb{P}_2)f = (\nabla_h \Delta_h^{-1} \operatorname{div}_h f^h, f^3, f^3), \quad \text{with } \operatorname{div}_h f^h = \partial_1 f^1 + \partial_2 f^2, \\ \|(Id - \mathbb{P}_2)f\|_{\dot{H}^s} \leq \|f\|_{\dot{H}^s}, \quad \text{for any } s \in \mathbb{R}. \end{cases} \quad (5.95)$$

Finally for  $k = 3, 4$ ,

$$\|\mathbb{P}_k \mathcal{P}_{r_\varepsilon, R_\varepsilon} f\|_{\dot{H}^s} \leq \sqrt{2} \frac{R_\varepsilon}{r_\varepsilon} \|\mathcal{P}_{r_\varepsilon, R_\varepsilon} f\|_{\dot{H}^s} = \sqrt{2} \varepsilon^{-(m+M)} \|\mathcal{P}_{r_\varepsilon, R_\varepsilon} f\|_{\dot{H}^s}. \quad (5.96)$$

If  $\nu = \nu'$ , there is no need anymore of a frequency truncation or an expansion for the last two eigenvalues (no  $\varepsilon_1$  either is necessary), and the  $\mathbb{P}_k$  ( $k \in \{2, 3, 4\}$ ) are orthogonal so for any divergence-free  $\mathbb{R}^4$ -valued vectorfield  $f$ , we have:

$$\|\mathbb{P}_k f\|_{\dot{H}^s} \leq \|f\|_{\dot{H}^s}, \quad \text{for any } s \in \mathbb{R}.$$

**Proof:** to obtain the developpment of  $\lambda_j(\varepsilon, \xi)$  ( $j \in \{3, 4\}$ ) we use (5.87) and the following Taylor expansion of order one:

$$\sqrt{1+x} = 1 + \frac{x}{2} \int_0^1 \frac{du}{\sqrt{1+xu}},$$

so that we immediately get:

$$D(\varepsilon, \xi) = (\nu - \nu')^2 \frac{|\xi|^5}{8|\xi_h|} \int_0^1 \frac{du}{\sqrt{1 - u \frac{(\nu - \nu')^2 \varepsilon^2 |\xi|^6}{4|\xi_h|^2}}}. \quad (5.97)$$

To bound  $D$  and its derivatives, we impose a little stronger condition than (5.86) on  $\varepsilon$ , namely we ask that:

$$\left(\varepsilon \frac{|\nu - \nu'| |\xi|^3}{2|\xi_h|}\right)^2 \leq \left(\varepsilon \frac{|\nu - \nu'| R_\varepsilon^3}{2r_\varepsilon}\right)^2 \leq \frac{1}{2}, \quad \text{which leads to } \varepsilon \leq \left(\frac{\sqrt{2}}{|\nu - \nu'|}\right)^{\frac{1}{1-(3M+m)}}. \quad (5.98)$$

To bound the projectors norms, the methods are similar to what we did in [4] and we also use (5.98) in the cases  $i = 3, 4$ , as we obtain:

$$|\mathcal{F}(\mathbb{P}_i \mathcal{P}_{r_\varepsilon, R_\varepsilon} f)(\xi)| \leq \frac{|\xi|}{\sqrt{2}|\xi_h|} \frac{1}{\sqrt{1 - \frac{(\nu - \nu')^2 \varepsilon^2 |\xi|^6}{4|\xi_h|^2}}} |\mathcal{F}(\mathcal{P}_{r_\varepsilon, R_\varepsilon}(f^3, f^4))|,$$

which gives (5.96).

### 5.3.2 Strichartz estimates when $\nu \neq \nu'$

Consider the following system (in the case  $\nu = \nu'$ , we have  $L = \nu\Delta$ ):

$$\begin{cases} \partial_t f - (\nu\Delta - \frac{1}{\varepsilon} \mathbb{P}\mathcal{B})f = F_{ext}, \\ f|_{t=0} = f_0. \end{cases} \quad (5.99)$$

The aim of this section is to prove the following Strichartz estimates:

**Proposition 5.4** *For any  $d \in \mathbb{R}$ ,  $r \geq 2$ ,  $q \geq 1$  and  $p \in [1, \frac{8}{1-\frac{2}{r}}]$ , there exists a constant  $C_{\nu, \nu', p, r} > 0$  such that for any  $\varepsilon \in ]0, \varepsilon_1]$  (where  $\varepsilon_1 = (\sqrt{2}/|\nu - \nu'|)^{\frac{1}{1-(3M+m)}}$ ) and any  $f$  solving (5.84) with initial data  $f_0$  and external force  $F_{ext}$  such that  $\operatorname{div} f_0 = \operatorname{div} F_{ext} = 0$  and  $\omega(f_0) = \omega(F_{ext}) = 0$ , then for  $k = 3, 4$ ,*

$$\begin{aligned} & \| |D|^d \mathbb{P}_k \mathcal{P}_{r_\varepsilon, R_\varepsilon} f \|_{\tilde{L}_t^p \dot{B}_{r, q}^0} \\ & \leq C_{\nu, \nu', p, r} \frac{R_\varepsilon^{\frac{17}{4} - \frac{7}{2r}}}{r_\varepsilon^{\frac{15}{4} + \frac{2}{p} - \frac{3}{2r}}} \varepsilon^{\frac{1}{8}(1 - \frac{2}{r})} \left( \| \mathcal{P}_{r_\varepsilon, R_\varepsilon} f_0 \|_{\dot{B}_{2, q}^d} + \| \mathcal{P}_{r_\varepsilon, R_\varepsilon} F_{ext} \|_{L^1 \dot{B}_{2, q}^d} \right). \end{aligned} \quad (5.100)$$

**Remark 5.2** *The condition  $\varepsilon \leq \varepsilon_1$  is needed only for us to use the estimates from Proposition 5.3.*

**Proof:** As in [4, 12, 14], we first assume  $F_{ext} = 0$  (the inhomogeneous case is dealt reproducing the same steps on the Duhamel term), and we will prove that under the previous assumptions,

$$\| \mathbb{P}_k \mathcal{P}_{r_\varepsilon, R_\varepsilon} f \|_{L_t^p L^r} \leq C_{\nu, \nu', p, r} \frac{R_\varepsilon^{\frac{17}{4} - \frac{7}{2r}}}{r_\varepsilon^{\frac{15}{4} + \frac{2}{p} - \frac{3}{2r}}} \varepsilon^{\frac{1}{8}(1 - \frac{2}{r})} \left( \| \mathcal{P}_{r_\varepsilon, R_\varepsilon} f_0 \|_{L^2} + \| \mathcal{P}_{r_\varepsilon, R_\varepsilon} F_{ext} \|_{L^1 L^2} \right), \quad (5.101)$$

which will give the result if applied to  $\Delta_j |D|^d f$ . The proof is close to the one of Proposition 51 from [12], but also features improvements coming from [13] (as explained in this article, using the Riesz-Thorin theorem allows to upgrade the condition  $r > 4$  into  $r \geq 2$ ). We will skip details and point out what is new. Let  $\mathcal{A}$  be the set:

$$\mathcal{A} \stackrel{def}{=} \{ \psi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3, \mathbb{R}), \quad \| \psi \|_{L^p(\mathbb{R}_+, L^r(\mathbb{R}^3))} \leq 1 \}.$$

We follow the steps from [13]: for  $k \in \{3, 4\}$ , thanks to Plancherel and (5.81),

$$\begin{aligned} \| \mathbb{P}_k \mathcal{P}_{r_\varepsilon, R_\varepsilon} f \|_{L^p L^r} &= \sup_{\psi \in \mathcal{A}} \int_0^\infty \int_{\mathbb{R}^3} \mathbb{P}_k \mathcal{P}_{r_\varepsilon, R_\varepsilon} f(t, x) \psi(t, x) dx dt \\ &= C \sup_{\psi \in \mathcal{A}} \int_0^\infty \int_{\mathbb{R}^3} e^{-\frac{\nu + \nu'}{2} t |\xi|^2 + i \frac{t}{\varepsilon} \frac{|\xi_h|}{|\xi|} - i t \varepsilon D(\varepsilon, \xi)} \mathcal{F}(\mathbb{P}_k \mathcal{P}_{r_\varepsilon, R_\varepsilon} f_0)(\xi) f_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}(\xi) \widehat{\psi}(t, \xi) d\xi dt \\ &\leq C \| \mathbb{P}_k \mathcal{P}_{r_\varepsilon, R_\varepsilon} f_0 \|_{L^2} \sup_{\psi \in \mathcal{A}} \left[ \int_0^\infty \int_0^\infty \int_{\mathbb{R}^3} f_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}(\xi) \widehat{\psi}(t, \xi) \right. \\ &\quad \left. \times e^{-\frac{\nu + \nu'}{2} (t+t') |\xi|^2 + i \frac{t-t'}{\varepsilon} \frac{|\xi_h|}{|\xi|} - i (t-t') \varepsilon D(\varepsilon, \xi)} f_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}(\xi) \widehat{\psi}(t', \xi) dx dt dt' \right]^{\frac{1}{2}}. \end{aligned} \quad (5.102)$$

Dispatching in both terms the heat semi-group, and using once more Plancherel and the Hölder inequality, we obtain:

$$\begin{aligned} \|\mathbb{P}_k \mathcal{P}_{r_\varepsilon, R_\varepsilon} f\|_{L^p L^r} &\leq C \|\mathbb{P}_k \mathcal{P}_{r_\varepsilon, R_\varepsilon} f_0\|_{L^2} \\ &\times \sup_{\psi \in \mathcal{A}} \left[ \int_0^\infty \int_0^\infty \|e^{\frac{\nu+\nu'}{4}(t+t')\Delta} \mathcal{P}_{\frac{r_\varepsilon}{2}, 2R_\varepsilon} \psi(t, \cdot)\|_{L^{\bar{r}}} \|L_{\varepsilon, t, t'} \psi(t', \cdot)\|_{L^r} dt dt' \right]^{\frac{1}{2}}, \end{aligned} \quad (5.103)$$

where we have defined for some  $g$ :

$$\begin{aligned} (L_{\varepsilon, t, t'} g)(x) &= \int_{\mathbb{R}^3} e^{ix \cdot \xi} e^{-\frac{\nu+\nu'}{4}(t+t')|\xi|^2 + i\frac{t-t'}{\varepsilon} \frac{|\xi_{h_1}|}{|\xi|} - i(t-t')\varepsilon D(\varepsilon, \xi)} f_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}(\xi) \widehat{g}(\xi) d\xi \\ &= K_{\varepsilon, t, t'} \star g, \end{aligned} \quad (5.104)$$

with

$$K_{\varepsilon, t, t'}(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} e^{-\frac{\nu+\nu'}{4}(t+t')|\xi|^2 + i\frac{t-t'}{\varepsilon} \frac{|\xi_{h_1}|}{|\xi|} - i(t-t')\varepsilon D(\varepsilon, \xi)} f_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}(\xi) d\xi.$$

To estimate the first term in (5.103), we cannot directly use the well-known smoothing effect of the heat flow described in Lemma 2.3 in [2] (see Section 2.1.2) as the frequencies are not supported in an annulus but in the set  $\mathcal{C}_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}$ , which needs us to adapt the result as stated in the following proposition (see (5.79) for the definition of the set  $\mathcal{C}_{r, R}$ ):

**Proposition 5.5** *Let  $0 < r < R$  be fixed. There exists a constant  $C$  such that for any  $p \in [1, \infty]$ ,  $t \geq 0$  and any function  $u$  we have:*

$$\text{Supp } \widehat{u} \subset \mathcal{C}_{r, R} \Rightarrow \|e^{t\Delta} u\|_{L^p} \leq C \frac{R^3}{r^4} e^{-\frac{t}{2}r^2} \|u\|_{L^p}.$$

We refer to the end of the appendix for the proof of this proposition, which allows us to write that:

$$\|e^{\frac{\nu+\nu'}{4}(t+t')\Delta} \mathcal{P}_{\frac{r_\varepsilon}{2}, 2R_\varepsilon} \psi(t, \cdot)\|_{L^{\bar{r}}} \leq C \frac{R_\varepsilon^3}{r_\varepsilon^4} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2} \|\psi(t, \cdot)\|_{L^{\bar{r}}}. \quad (5.105)$$

We start to deal with the second term through the same steps: with a view to use the Riesz-Thorin theorem, we bound the following norms:

$$\begin{cases} \|L_{\varepsilon, t, t'}\|_{L^2 \rightarrow L^2} \leq C_0 e^{-\frac{\nu+\nu'}{16}(t+t')r_\varepsilon^2}, \\ \|L_{\varepsilon, t, t'}\|_{L^1 \rightarrow L^\infty} \leq \|K_{\varepsilon, t, t'}\|_{L^\infty}. \end{cases} \quad (5.106)$$

Thanks to the definition of the kernel, we easily obtain that

$$\|K_{\varepsilon, t, t'}\|_{L^\infty} \leq C_0 R_\varepsilon^3 e^{-\frac{\nu+\nu'}{16}(t+t')r_\varepsilon^2}.$$

In order to bound this norm with a negative power of  $\frac{|t-t'|}{\varepsilon}$ , we can use the classical argument developed by Chemin, Desjardins, Gallagher and Grenier in [18, 19, 20] which allows us to ask that  $x_2 = 0$  (thanks to the invariance of the integrated functions under rotations around the  $x_3$ -axis) and obtain that:

$$\|K_{\varepsilon, t, t'}\|_{L^\infty} = \sup_{x_1, x_3 \in \mathbb{R}} |K_{\varepsilon, t, t'}(x_1, 0, x_3)|,$$

so that we introduce  $a(\xi) = \frac{|\xi_{h_1}|}{|\xi|}$  and the operator  $\mathcal{L}$ , defined as follows:

$$\mathcal{L}f = \begin{cases} \frac{1}{1 + \frac{t-t'}{\varepsilon} \alpha(\xi)^2} (f(\xi) + i\alpha(\xi) \partial_{\xi_2} f(\xi)) & \text{if } t > t', \\ \frac{1}{1 + \frac{t'-t}{\varepsilon} \alpha(\xi)^2} (f(\xi) - i\alpha(\xi) \partial_{\xi_2} f(\xi)) & \text{else,} \end{cases} \quad (5.107)$$

where the function  $\alpha$  is defined as:

$$\alpha(\xi) = -\partial_{\xi_2} a(\xi) = -\frac{\xi_2 \xi_3^2}{|\xi_h| |\xi|^3}.$$

Let us assume  $t > t'$  for instance: as  $x_2 = 0$ ,  $\mathcal{L}(e^{ix \cdot \xi + i \frac{t-t'}{\varepsilon}} a(\xi))$  and performing an integration by parts, we have

$$K_{\varepsilon, t, t'}(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi + i \frac{t-t'}{\varepsilon}} a(\xi) ({}^t \mathcal{L}) \left( e^{-\frac{\nu+\nu'}{4}(t+t')|\xi|^2 - i(t-t')\varepsilon D(\varepsilon, \xi)} f_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}(\xi) \right) d\xi,$$

where the transposed operator satisfies:

$${}^t \mathcal{L} f = \left( \frac{1}{1 + \frac{t-t'}{\varepsilon} \alpha(\xi)^2} - i \partial_{\xi_2} \alpha(\xi) \frac{1 - \frac{t-t'}{\varepsilon} \alpha(\xi)^2}{(1 + \frac{t-t'}{\varepsilon} \alpha(\xi)^2)^2} \right) f(\xi) - \frac{i \alpha(\xi)}{1 + \frac{t-t'}{\varepsilon} \alpha(\xi)^2} \partial_{\xi_2} f(\xi).$$

Now following the steps (it is about applying  ${}^t \mathcal{L}$ , using the estimates we provided in Proposition 5.3 for  $D$ ,  $\partial_{\xi_2} D$  and getting rid of  $t, t'$  using the exponential  $e^{-\frac{\nu+\nu'}{4}(t+t')|\xi|^2}$  (the details are omitted but close to what is done for instance in [4, 6]) we end-up with:

$$|K_{\varepsilon, t, t'}(x)| \leq \frac{C_0}{r_\varepsilon^2} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2} \int_{\mathcal{C}_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}} \frac{d\xi}{1 + \frac{|t-t'|}{\varepsilon} \frac{\xi_2^2 \xi_3^4}{|\xi_h|^2 |\xi|^6}}. \quad (5.108)$$

Unfortunately, due to the definition of  $\mathcal{C}_{r, R}$  we cannot follow anymore the steps from the classical method, as we cannot bound from below on  $\mathcal{C}_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}$  the term  $\frac{\xi_2^2 \xi_3^4}{|\xi_h|^2 |\xi|^6}$  (neither  $\xi_2$  nor  $\xi_3$  can be).

The method seems useless and the first idea would be to make  $|\xi_2|$  easily bounded from below and change the set  $\mathcal{C}_{r, R}$  into:

$$\mathcal{C}'_{r, R} = \{\xi \in \mathbb{R}^3, |\xi| \leq R, \text{ and } |\xi_2| \geq r\},$$

but in this case we would not be able to perform the key argument with the operator  $\mathcal{L}$  as we cannot assume  $x_2 = 0$  anymore as the function  $\chi(\frac{|\xi|}{2R_\varepsilon}) \left( 1 - \chi(\frac{|\xi_2|}{r_\varepsilon}) \right)$  is not invariant under rotations around the  $x_3$ -axis.

It seems at first sight that we cannot continue any further but reproducing the previous arguments choosing this time  $x_1 = 0$  and defining the analogous of  $\mathcal{L}$  with  $\alpha$  replaced by  $\beta = -\partial_{\xi_1} a(\xi) = -\frac{\xi_1 \xi_3^2}{|\xi_h| |\xi|^3}$  and  $\partial_{\xi_2}$  replaced by  $\partial_{\xi_1}$ , we end-up with:

$$|K_{\varepsilon, t, t'}(x)| \leq \frac{C_0}{r_\varepsilon^2} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2} \int_{\mathcal{C}'_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}} \frac{d\xi}{1 + \frac{|t-t'|}{\varepsilon} \frac{\xi_1^2 \xi_3^4}{|\xi_h|^2 |\xi|^6}}. \quad (5.109)$$

So gathering (5.109) and (5.108), we can then write (using in the last line that  $\max(a, b) \geq (a+b)/2$ ):

$$\begin{aligned} |K_{\varepsilon, t, t'}(x)| &\leq \frac{C_0}{r_\varepsilon^2} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2} \int_{\mathcal{C}_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}} \min \left( \frac{1}{1 + \frac{|t-t'|}{\varepsilon} \frac{\xi_2^2 \xi_3^4}{|\xi_h|^2 |\xi|^6}}, \frac{1}{1 + \frac{|t-t'|}{\varepsilon} \frac{\xi_1^2 \xi_3^4}{|\xi_h|^2 |\xi|^6}} \right) d\xi \\ &= \frac{C_0}{r_\varepsilon^2} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2} \int_{\mathcal{C}_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}} \frac{d\xi}{\max \left( 1 + \frac{|t-t'|}{\varepsilon} \frac{\xi_2^2 \xi_3^4}{|\xi_h|^2 |\xi|^6}, 1 + \frac{|t-t'|}{\varepsilon} \frac{\xi_1^2 \xi_3^4}{|\xi_h|^2 |\xi|^6} \right)} \\ &= \frac{C_0}{r_\varepsilon^2} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2} \int_{\mathcal{C}_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}} \frac{d\xi}{1 + \frac{|t-t'|}{\varepsilon} \max(\xi_1^2, \xi_2^2) \frac{\xi_3^4}{|\xi_h|^2 |\xi|^6}} \\ &\leq \frac{C_0}{r_\varepsilon^2} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2} \int_{\mathcal{C}_{\frac{r_\varepsilon}{2}, 2R_\varepsilon}} \frac{d\xi}{1 + \frac{|t-t'|}{\varepsilon} \frac{\xi_3^4}{2|\xi|^6}} \end{aligned} \quad (5.110)$$



Now, the previous difficulties have disappeared and we can bound the kernel according to:

$$\begin{aligned}
|K_{\varepsilon,t,t'}(x)| &\leq \frac{C_0}{r_\varepsilon^2} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2} \int_{\frac{r_\varepsilon}{2} \leq |\xi_h| \leq 2R_\varepsilon} \int_{-\sqrt{(4R_\varepsilon^2-|\xi_h|^2)}}^{\sqrt{(4R_\varepsilon^2-|\xi_h|^2)}} \frac{d\xi_3}{1 + \frac{|t-t'|}{\varepsilon} \frac{\xi_3^4}{2|\xi|^6}} d\xi_h \\
&\leq \frac{C_0}{r_\varepsilon^2} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2} \int_{|\xi_h| \leq 2R_\varepsilon} \int_{-2R_\varepsilon}^{2R_\varepsilon} \frac{d\xi_3}{1 + \frac{|t-t'|}{\varepsilon} \frac{\xi_3^4}{2^7 R_\varepsilon^6}} d\xi_h \leq C_0 \frac{R_\varepsilon^2}{r_\varepsilon^2} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2} \int_{-2R_\varepsilon}^{2R_\varepsilon} \frac{d\xi_3}{1 + \frac{|t-t'|}{\varepsilon} \frac{\xi_3^4}{2^7 R_\varepsilon^6}}.
\end{aligned} \tag{5.111}$$

Performing the change of variable  $z = \left(\frac{|t-t'|}{\varepsilon}\right)^{\frac{1}{4}} \frac{1}{2^{\frac{7}{4}} R_\varepsilon^{\frac{3}{2}}} \xi_3$ , the integral satisfies:

$$\begin{aligned}
\int_{-2R_\varepsilon}^{2R_\varepsilon} \frac{d\xi_3}{1 + \frac{|t-t'|}{\varepsilon} \frac{\xi_3^4}{2^7 R_\varepsilon^6}} &= \left(\frac{\varepsilon}{|t-t'|}\right)^{\frac{1}{4}} 2^{\frac{7}{4}} R_\varepsilon^{\frac{3}{2}} \int_{-\left(\frac{|t-t'|}{\varepsilon}\right)^{\frac{1}{4}} \frac{1}{2^{\frac{3}{4}} R_\varepsilon^{\frac{1}{2}}}}^{\left(\frac{|t-t'|}{\varepsilon}\right)^{\frac{1}{4}} \frac{1}{2^{\frac{3}{4}} R_\varepsilon^{\frac{1}{2}}}} \frac{dz}{1+z^4} \\
&\leq \left(\frac{\varepsilon}{|t-t'|}\right)^{\frac{1}{4}} 2^{\frac{7}{4}} R_\varepsilon^{\frac{3}{2}} \int_{\mathbb{R}} \frac{dz}{1+z^4}, \tag{5.112}
\end{aligned}$$

so that we finally obtain that:

$$\|K_{\varepsilon,t,t'}\|_{L^\infty} \leq C_0 \frac{R_\varepsilon^{\frac{7}{2}}}{r_\varepsilon^2} \frac{\varepsilon^{\frac{1}{4}}}{|t-t'|^{\frac{1}{4}}} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2}. \tag{5.113}$$

Using this together with (5.106), we obtain, doing as in the previous section that for any  $r \in [2, \infty]$ :

$$\|L_{\varepsilon,t,t'}\psi(t', \cdot)\|_{L^r} \leq C_0 \frac{R_\varepsilon^{\frac{7}{2}(1-\frac{2}{r})}}{r_\varepsilon^{2(1-\frac{2}{r})}} \frac{\varepsilon^{\frac{1}{4}(1-\frac{2}{r})}}{|t-t'|^{\frac{1}{4}(1-\frac{2}{r})}} e^{-\frac{\nu+\nu'}{32}(t+t')r_\varepsilon^2} \|\psi(t', \cdot)\|_{L^{\bar{r}}}.$$

Gathering this estimates together with (5.105), and thanks to (5.96), we can properly bound (5.103) and obtain that:

$$\|\mathbb{P}_k \mathcal{P}_{r_\varepsilon, R_\varepsilon} f\|_{L^p L^r} \leq C_0 \|\mathcal{P}_{r_\varepsilon, R_\varepsilon} f_0\|_{L^2} \times \sup_{\psi \in \mathcal{A}} \frac{R_\varepsilon^{1+\frac{3}{2}+\frac{7}{4}(1-\frac{2}{r})}}{r_\varepsilon^{1+2+(1-\frac{2}{r})}} \varepsilon^{\frac{1}{8}(1-\frac{2}{r})} \left[ \int_0^\infty \int_0^\infty \frac{h(t)h(t')}{|t-t'|^{\frac{1}{4}(1-\frac{2}{r})}} dt dt' \right]^{\frac{1}{2}},$$

with  $h(t) = e^{-\frac{\nu+\nu'}{16}tr_\varepsilon^2} \|\psi(t, \cdot)\|_{L^{\bar{r}}}$ . As in the previous section, the last term is bounded using the Hardy-Littlewood-Sobolev estimates like in [12, 13]: if  $m \geq 1$  is such that:

$$\frac{2}{m} + \frac{1}{4} \left(1 - \frac{2}{r}\right) = 2,$$

then

$$\int_0^\infty \int_0^\infty \frac{h(t)h(t')}{|t-t'|^{\frac{1}{4}(1-\frac{2}{r})}} dt dt' \leq C \|h\|_{L^m}^2,$$

so that if  $\beta \geq 1$  is defined as  $\frac{1}{\beta} + \frac{1}{\bar{p}} = \frac{1}{m}$ , that is  $\frac{1}{\beta} = \frac{1}{p} - \frac{1}{8} \left(1 - \frac{2}{r}\right)$  (this is here that we need the condition  $p \leq \frac{8}{1-\frac{2}{r}}$ ), we have:

$$\|h\|_{L^m} \leq \|e^{-\frac{\nu+\nu'}{16}tr_\varepsilon^2}\|_{L^\beta} \|\psi\|_{L_t^{\bar{p}} L_x^{\bar{r}}}.$$

This finally entails that:

$$\|\mathbb{P}_k \mathcal{P}_{r_\varepsilon, R_\varepsilon} f\|_{L^p L^r} \leq C_{\nu, \nu', p, r} \frac{R_\varepsilon^{\frac{17}{4} - \frac{7}{2r}}}{r_\varepsilon^{\frac{15}{4} + \frac{2}{p} - \frac{3}{2r}}} \varepsilon^{\frac{1}{8}(1-\frac{2}{r})} \|\mathcal{P}_{r_\varepsilon, R_\varepsilon} f_0\|_{L^2},$$

which concludes the proof.  $\blacksquare$

**Remark 5.3** Our dispersive estimates could be compared with Lemma 5.5 of [40]: it seems that we do not obtain as strong a result, but actually it seems to us that the rate  $\tau^{-\frac{1}{2}}$  (with  $\tau = |t - t'|/\varepsilon$ ) stated in that lemma (we obtain the rate  $\tau^{-\frac{1}{4}}$  in (5.113)) cannot be obtained with this method.

We refer the interested reader to the forthcoming companion article [15] where the method presented in [40] is completely detailed (with our choices of  $r, R$  we need to precisely estimate the dependency in  $r, R$ ) and improved to obtain dispersive estimates leading to anisotropic Strichartz estimates and involving the rate  $\tau^{-\frac{1}{4}}$ . Had we used this method for isotropic estimates, we would have ended with the same conditions and estimates as in Proposition 5.4 but with the larger coefficient  $R_\varepsilon^{\frac{7-\frac{9}{r}}{r} - (\frac{13}{2} + \frac{2}{p} - \frac{7}{r})}$  instead of  $R_\varepsilon^{\frac{17}{4} - \frac{7}{2r} - (\frac{15}{4} + \frac{2}{p} - \frac{3}{2r})}$ .

### 5.3.3 Strichartz estimates when $\nu = \nu'$

We prove in this section the following Strichartz estimates, that are close to their counterparts from [13]:

**Proposition 5.6** For any  $d \in \mathbb{R}$ ,  $r \geq 2$ ,  $q \geq 1$ ,  $\theta \in [0, 1]$  and  $p \in [1, \frac{4}{\theta(1-\frac{2}{r})}]$ , there exists a constant  $C = C_{p,r,\theta}$  such that for any  $f$  solving (5.99) for initial data  $f_0$  and external force  $F_{ext}$  both with zero divergence and vorticity (that in the kernel of  $\mathbb{P}_2$ ), then

$$\| |D|^d f \|_{\tilde{L}_t^p \dot{B}_{r,q}^0} \leq \frac{C_{p,r,\theta}}{\nu^{\frac{1}{p} - \frac{\theta}{4}(1-\frac{2}{r})}} \varepsilon^{\frac{\theta}{4}(1-\frac{2}{r})} \left( \|f_0\|_{\dot{B}_{2,q}^{\sigma_1}} + \|F_{ext}\|_{\tilde{L}_t^1 \dot{B}_{2,q}^{\sigma_1}} \right), \quad (5.114)$$

where  $\sigma_1 = d + \frac{3}{2} - \frac{3}{r} - \frac{2}{p} + \frac{\theta}{2}(1 - \frac{2}{r})$ .

**Proof:** As the result is close to Proposition 4 from [13] we will refer to this article for details and only point out what is different: namely the eigenvalues of the hessian and the singularity of the phase near  $\{(0, 0)\} \times \mathbb{R}$ .

As in [13], the classical non stationary phase argument from [18, 12] is enhanced thanks to the Riesz-Thorin theorem (as in [23]) together with the Littman theorem (as in [33, 34, 30]) that we recall here:

**Theorem 5.1** (Littman [35, 41]) Assume that  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function compactly supported in  $K$  and  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function such that for any  $\xi \in K$ , the Hessian  $D^2\phi(\xi)$  has at least  $k$  nonzero eigenvalues. Then there exists a constant  $A$  such that for any  $\lambda \in \mathbb{R}$  and  $x \in \mathbb{R}^n$ ,

$$\left| \int_{\mathbb{R}^n} e^{ix \cdot \xi + i\lambda\phi(\xi)} \psi(\xi) d\xi \right| \leq A \sqrt{|x|^2 + \lambda^2}^{-\frac{k}{2}} \leq A |\lambda|^{-\frac{k}{2}}.$$

As in [13], we emphasize that the use of the Littman theorem only improves the coefficients: indeed, had we not used this result (but all the other arguments from [12] enhanced with the Riesz-Thorin theorem), we would have ended with the following alternative (and very close) proposition:

**Proposition 5.7** For any  $d \in \mathbb{R}$ ,  $r > 2$ ,  $q \geq 1$ ,  $\theta \in [0, \frac{2}{3}[$  and  $p \in [1, \frac{8}{\theta(1-\frac{2}{r})}]$ , there exists a constant  $C = C_{p,r,\theta}$  such that for any  $f$  solving (5.99) for initial data  $f_0$  and external force  $F_{ext}$  both with zero divergence and vorticity (that in the kernel of  $\mathbb{P}_2$ ), then

$$\| |D|^d f \|_{\tilde{L}_t^p \dot{B}_{r,q}^0} \leq \frac{C_{p,r,\theta}}{\nu^{\frac{1}{p} - \frac{\theta}{8}(1-\frac{2}{r})}} \varepsilon^{\frac{\theta}{8}(1-\frac{2}{r})} \left( \|f_0\|_{\dot{B}_{2,q}^{\sigma_2}} + \|F_{ext}\|_{\tilde{L}_t^1 \dot{B}_{2,q}^{\sigma_2}} \right), \quad (5.115)$$

where  $\sigma_2 = d + \frac{3}{2} - \frac{3}{r} - \frac{2}{p} + \frac{\theta}{4}(1 - \frac{2}{r})$ .

Let us continue the proof of Proposition 5.6 and first assume that  $F_{ext} = 0$ . As explained in [12, 13], as  $\operatorname{div} f_0 = 0 = \omega(f_0)$ , we have  $f_0 = \mathbb{P}f_0 = \mathbb{P}_3f_0 + \mathbb{P}_4f_0$ , and:

$$f(t) = \mathcal{F}^{-1} \left( e^{-\nu t|\xi|^2 + i\frac{t}{\varepsilon} \frac{|\xi h_1|}{|\xi|}} \mathcal{P}_3(\xi, \varepsilon) \widehat{f}_0(\xi) + e^{-\nu t|\xi|^2 - i\frac{t}{\varepsilon} \frac{|\xi h_1|}{|\xi|}} \mathcal{P}_4(\xi, \varepsilon) \widehat{f}_0(\xi) \right).$$

Thanks to the orthogonality properties when  $\nu = \nu'$  (the projectors are of norm 1), we are reduced to the study of:

$$f(t) = \mathcal{F}^{-1} \left( e^{-\nu t|\xi|^2 + i\frac{t}{\varepsilon} \frac{|\xi h_1|}{|\xi|}} \widehat{f}_0(\xi) \right),$$

If  $\varphi$  is the truncation function introduced in the Littlewood-Paley decomposition supported in the annulus centered at zero of radii  $\frac{3}{4}$  and  $\frac{8}{3}$  (see appendices from [12, 13] or [2] for a complete presentation), we denote by  $\varphi_1$  another smooth truncation function, with support in a slightly larger annulus than  $\operatorname{supp} \varphi$  (for instance the annulus centered at zero and of radii  $\frac{1}{2}$  and 3) and equal to 1 on  $\operatorname{supp} \varphi$ .

With the same set  $\mathcal{A}$  and the same steps as in the previous section, for any  $j \in \mathbb{Z}$  and  $r \geq 1$ :

$$\begin{aligned} \|\dot{\Delta}_j f\|_{L^p L^r} &= \sup_{\psi \in \mathcal{A}} \int_0^\infty \int_{\mathbb{R}^3} \dot{\Delta}_j f(t, x) \psi(t, x) dx dt \\ &= C \sup_{\psi \in \mathcal{A}} \int_0^\infty \int_{\mathbb{R}^3} e^{-\nu t|\xi|^2 + i\frac{t}{\varepsilon} \frac{|\xi h_1|}{|\xi|}} \widehat{\Delta}_j f_0(\xi) \varphi_1(2^{-j}\xi) \widehat{\psi}(t, \xi) d\xi dt \\ &\leq C \sup_{\psi \in \mathcal{A}} \|\dot{\Delta}_j f_0\|_{L^2} \left[ \int_0^\infty \int_0^\infty \|L_j(\frac{t-t'}{\varepsilon}) \psi(t, \cdot)\|_{L^r} \|e^{\nu(t+t')\Delta} \varphi_1(2^{-j}D) \overline{\psi(t', \cdot)}\|_{L^{\bar{r}}} dt dt' \right]^{\frac{1}{2}}, \end{aligned} \tag{5.116}$$

with  $L_j(\sigma)$  defined as follows:

$$L_j(\sigma)g = \int_{\mathbb{R}^3} e^{ix \cdot \xi + i\sigma \frac{|\xi h_1|}{|\xi|}} \varphi_1(2^{-j}|\xi|) \widehat{g}(\xi) d\xi = K_j(\sigma) * g, \tag{5.117}$$

where

$$K_j(\sigma)(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi + i\sigma \frac{|\xi h_1|}{|\xi|}} \varphi_1(2^{-j}|\xi|) d\xi. \tag{5.118}$$

Thanks to the frequency truncation (remember that  $\operatorname{supp} \varphi_1 \subset \mathcal{C}(0, \frac{1}{2}, 3)$ ) and the classical estimates for the heat kernel in this case (we refer for example to Lemma 2.3 from [2]) we easily get that:

$$\|e^{\nu(t+t')\Delta} \varphi_1(2^{-j}D) \overline{\psi(t', \cdot)}\|_{L^{\bar{r}}} \leq C' e^{-\frac{\nu}{4}(t+t')2^{2j}} \|\psi(t')\|_{L^{\bar{r}}}. \tag{5.119}$$

To bound the other term we will, as in [13], successively bound its  $L^2 \rightarrow L^2$  and  $L^1 \rightarrow L^\infty$  norms, and conclude thanks to the Riesz-Thorin theorem. Thanks to the Plancherel formula there exists a constant  $C$  (only depending on  $\varphi_1$ ) such that :

$$\|L_j(\sigma)g\|_{L^2} \leq C \|g\|_{L^2}. \tag{5.120}$$

Thanks to the Young estimates:

$$\|L_j(\sigma)g\|_{L^\infty} \leq \|K_j(\sigma)\|_{L^\infty} \|g\|_{L^1}, \tag{5.121}$$

and performing the change of variable  $\xi = 2^j \eta$ , we get  $K_j(\sigma)(x) = 2^{3j} K_0(\sigma)(2^j x)$  and:

$$\|K_j(\sigma)\|_{L^\infty} \leq 2^{3j} \|K_0(\sigma)\|_{L^\infty}, \tag{5.122}$$

And as in [13, 33, 30, 34] we need bound the  $L^\infty$ -norm of  $K_0$  (see (5.118)). First we rewrite it as follows, denoting  $b(\xi) = \frac{|\xi_h|}{|\xi|}$ :

$$K_0(\sigma)(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi + i\sigma b(\xi)} \varphi_1(|\xi|) d\xi, \quad (5.123)$$

Compared to the case of the Primitive system (see [30, 13] the problem is here that the derivatives of  $b$  present singularities when  $\xi_h = (0, 0)$ . Using the Littman theorem then requires frequency cut-offs, as done in Proposition 3.1 from [34]:

**Proposition 5.8** [34], *With the previous notations, there exists a constant  $C > 0$  such that for all  $x \in \mathbb{R}^3$  and all  $\sigma \neq 0$ ,*

$$\|K_0(\sigma)\|_{L^\infty} \leq C(1 + |\sigma|)^{-\frac{1}{2}} \leq C \min(1, |\sigma|^{-\frac{1}{2}}). \quad (5.124)$$

Had we not the singularity problem, we would simply perform the frequency truncations (as in [33, 30, 34, 13]) to split the integral into several parts, on which the Littman theorem can be applied. The decomposition corresponds to zones where we precisely know how many nonzero eigenvalues are featured by the Hessian  $D^2b$  which writes in our case:

$$D^2b(\xi) = \frac{1}{|\xi_h|^3 |\xi|^5} \begin{pmatrix} \xi_3^2 (\xi_2^2 |\xi|^2 - 3\xi_1^2 |\xi_h|^2) & -\xi_1 \xi_2 \xi_3^2 (|\xi|^2 + 3|\xi_h|^2) & \xi_1 \xi_3 |\xi_h|^2 (3|\xi_h|^2 - |\xi|^2) \\ -\xi_1 \xi_2 \xi_3^2 (|\xi|^2 + 3|\xi_h|^2) & \xi_3^2 (\xi_1^2 |\xi|^2 - 3\xi_2^2 |\xi_h|^2) & \xi_2 \xi_3 |\xi_h|^2 (3|\xi_h|^2 - |\xi|^2) \\ \xi_1 \xi_3 |\xi_h|^2 (3|\xi_h|^2 - |\xi|^2) & \xi_2 \xi_3 |\xi_h|^2 (3|\xi_h|^2 - |\xi|^2) & -|\xi_h|^4 (3|\xi_h|^2 - 2|\xi|^2) \end{pmatrix}, \quad (5.125)$$

To ease the understanding of what is done in [33, 30, 34] (where the proofs are done only reasoning with the determinant) we compute the eigenvalues of this matrix (as we did in [13]):

$$\left\{ \frac{\xi_3^2}{|\xi_h| |\xi|^3}, -\frac{|\xi_h| \pm \sqrt{|\xi|^2 + 3\xi_3^2}}{2|\xi|^3} \right\}. \quad (5.126)$$

Notice that the behaviour of the eigenvalues is a little different: if  $\xi_3 = 0$ , then only one eigenvalue is nonzero (and is equal to  $-\frac{1}{|\xi_h|^2}$ ) and if  $\xi_3 \neq 0$  then none of the eigenvalues is zero. The idea is basically to split the integral into:

$$\begin{aligned} K_0(\sigma)(x) &= \int_{\mathbb{R}^3} e^{ix \cdot \xi + i\sigma b(\xi)} \chi(|\xi_3|) \varphi_1(|\xi|) d\xi + \int_{\mathbb{R}^3} e^{ix \cdot \xi + i\sigma b(\xi)} (1 - \chi(|\xi_3|)) \varphi_1(|\xi|) d\xi \\ &= K_{0,1}(\sigma)(x) + K_{0,2}(\sigma)(x), \end{aligned} \quad (5.127)$$

Thanks to the study of  $D^2b$ , at least 1 eigenvalue is nonzero for the first term, and the three of them are non zero for the second one, and if we could directly apply Theorem 5.1 we would end-up with:

$$\|K_{0,1}(\sigma)\|_{L^\infty} \leq C \min(1, |\sigma|^{-\frac{1}{2}}), \quad \|K_{0,2}(\sigma)\|_{L^\infty} \leq C \min(1, |\sigma|^{-\frac{3}{2}}).$$

But this is not so simple as function  $b$  is not regular on  $\text{supp } \varphi_1$ . In [34] the authors overcome this difficulty with frequency truncations.

**Remark 5.4** *We can complete Remark 9 from [13]: in the case of the rotating fluids, we decompose the frequency space into three zones, on two of them at least two eigenvalues are nonzero, on the third zone, every eigenvalue are nonzero. In the case of the Primitive system, on one zone at least one eigen value is nonzero, at least two of them in the second zone and all of them in the third zone. For the Stratified Boussinesq system, singularities arise and on the first zone, at least one eigenvalue is non zero, all of them in the other zone. We also refer to [26] who revisit Strichartz estimates for the linearized systems of the three models thanks to restriction theory (without resorting to dispersive estimates).*

Gathering (5.124), (5.121) and (5.122) leads to

$$\|L_j(\sigma)g\|_{L^\infty} \leq C_F 2^{3j} \min(1, |\sigma|^{-\frac{1}{2}}) \|g\|_{L^1},$$

Doing as in [13], gathering the previous estimate with (5.120) and thanks to the Riesz-Thorin theorem, we obtain that for all  $r \in [2, \infty]$  and  $\theta \in [0, 1]$ :

$$\|L_j(\sigma)g\|_{L^r} \leq C_r \frac{2^{3j(1-\frac{\theta}{r})}}{|\sigma|^{\frac{\theta}{2}(1-\frac{2}{r})}} \|g\|_{L^{\bar{r}}}, \quad (5.128)$$

so that, using also (5.119), we can bound (5.116) and obtain:

$$\|\dot{\Delta}_j f\|_{L^p L^r} \leq C_r 2^{\frac{3j}{2}(1-\frac{2}{r})} \varepsilon^{\frac{\theta}{4}(1-\frac{2}{r})} \|\dot{\Delta}_j f_0\|_{L^2} \sup_{\psi \in \mathcal{B}} \left[ \int_0^\infty \int_0^\infty \frac{h(t)h(t')}{|t-t'|^{\frac{\theta}{2}(1-\frac{2}{r})}} dt dt' \right]^{\frac{1}{2}},$$

with  $h(t) = e^{-\frac{\nu}{4}t^2} \|\psi(t, \cdot)\|_{L^{\bar{r}}}$ . The last term is bounded exactly like in [13] using the Hardy-Littlewood-Sobolev estimates (this is here that we need the condition  $p \leq \frac{4}{\theta(1-\frac{2}{r})}$ ) and we finally obtain:

$$\|\dot{\Delta}_j f\|_{L^p L^r} \leq \frac{C_{p,r,\theta}}{\nu^{\frac{1}{p}-\frac{\theta}{4}(1-\frac{2}{r})}} \varepsilon^{\frac{\theta}{4}(1-\frac{2}{r})} 2^{j(\frac{3}{2}-\frac{3}{r}-\frac{2}{p}+\frac{\theta}{2}(1-\frac{2}{r}))} \|\dot{\Delta}_j f_0\|_{L^2}$$

which leads to the result in the homogeneous case. The inhomogeneous case (i.e. when  $F_{ext} \neq 0$ ) easily follows applying the previous steps to the Duhamel term. ■

**Remark 5.5** 1. In the case  $\nu = \nu'$ , there is a difference between the power of  $|t-t'|/\varepsilon$  in the dispersive estimates produced with the Littman method compared to the non-stationnary phase method: respectively  $-1/2$  and  $-1/4$ . This is also noticed in [13, 14] in the case of the rotating fluids (respectively  $-1$  and  $-1/2$ ). In [13] we also point out that for the primitive system both exponents are the same.

2. When  $\nu \neq \nu'$  or when anisotropic estimates are needed the non-stationnary phase method is preferred to the Littman method. We refer to [15] for more details.

### 5.3.4 Proof of Proposition 5.5

We adapt here the proof of Lemma 2.3 from [2]. As in [9] we only present here what is new. Thanks to (5.81) we can write that:

$$e^{t\Delta} u = \mathcal{F}^{-1} \left( f_{\frac{\varepsilon}{2}, 2R}(\xi) e^{-t|\xi|^2} \widehat{u}(\xi) \right) = g(t, \cdot) \star u,$$

with

$$g(t, x) = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} f_{\frac{\varepsilon}{2}, 2R}(\xi) e^{-t|\xi|^2} d\xi (2\pi)^{-3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} \chi\left(\frac{|\xi|}{2R_\varepsilon}\right) \left(1 - \chi\left(\frac{|\xi h|}{r_\varepsilon}\right)\right) e^{-t|\xi|^2} d\xi,$$

so that  $\|e^{t\Delta} u\|_{L^p} \leq \|g(t, \cdot)\|_{L^1} \|u\|_{L^p}$  and in order to bound the  $L_x^1$ -norm of  $g$  we use the same method as in [2] but will only perform twice the integrations by parts (we are in  $\mathbb{R}^3$ ):

$$g(t, x) = (2\pi)^{-3} (1 + |x|^2)^{-2} \int_{\mathbb{R}^3} e^{ix \cdot \xi} (I_d - \Delta_\xi)^2 \left( \chi\left(\frac{|\xi|}{2R_\varepsilon}\right) \left(1 - \chi\left(\frac{|\xi h|}{r_\varepsilon}\right)\right) e^{-t|\xi|^2} \right) d\xi.$$

The rest of the proof is then a matter of computing the derivatives of  $f_{\frac{\varepsilon}{2}, 2R}(\xi) e^{-t|\xi|^2}$  using the fact that:

$$\begin{cases} \Delta(fg) = (\Delta f)g + 2\nabla f \cdot \nabla g + f\Delta g, \\ \nabla(e^{-t|\xi|^2}) = -2te^{-t|\xi|^2} \xi, \quad \text{and} \quad \Delta(e^{-t|\xi|^2}) = -2t(3 - 2t|\xi|^2)e^{-t|\xi|^2}. \end{cases}$$

To simplify the notations, let us put  $K = f_{\frac{r}{2}, 2R}$ . We have:

$$\begin{aligned} (I_d - \Delta)^2(K(\xi)e^{-t|\xi|^2}) &= \left[ (I_d - \Delta)^2 K(\xi) + 2t \left( 3K(\xi) - 10\Delta K(\xi) + 2\xi \cdot \nabla K(\xi) \right) \right. \\ &\quad + 4t^2 \left( 15K(\xi) + 20\xi \cdot \nabla K(\xi) + 4\xi \cdot (\xi \cdot \nabla) \nabla K(\xi) \right) \\ &\quad \left. + 8t^3 \left( -10|\xi|^2 K(\xi) - 4|\xi|^2 \xi \cdot \nabla K(\xi) \right) + 16t^4 |\xi|^4 K(\xi) \right] e^{-t|\xi|^2}. \end{aligned} \quad (5.129)$$

We can get rid of the powers of  $t$  with estimates such as  $t^\alpha |\xi|^{2\alpha} e^{-t|\xi|^2} \leq C' e^{-\frac{t}{2}|\xi|^2} \leq C' e^{-\frac{t}{2}r^2}$  and the proof follows from the fact that there exists a constant  $C > 0$  such that for all  $\xi \in \mathcal{C}_{\frac{r}{2}, 2R}$ , we have

$$\begin{cases} \|\nabla K\| \leq C\left(\frac{1}{R} + \frac{1}{r}\right), \\ |\Delta K| \leq C\left(\frac{1}{R^2} + \frac{1}{Rr} + \frac{1}{r^2}\right), \\ \|(\xi \cdot \nabla) \nabla K\| \leq C\left(\frac{1}{R} + \frac{1}{r} + \frac{R}{r^2}\right), \\ |\Delta^2 K| \leq C\left(\frac{1}{R^4} + \frac{1}{R^3 r} + \frac{1}{R^2 r^2} + \frac{1}{Rr^3} + \frac{1}{r^4}\right), \end{cases}$$

so that, integrating on  $\mathcal{C}_{\frac{r}{2}, 2R}$ , we finally obtain:

$$|g(t, x)| \leq \frac{C}{(1 + |x|^2)^2} \frac{R^3}{r^4} e^{-\frac{t}{2}r^2} \quad \text{and} \quad \|g(t, \cdot)\|_{L^1} \leq C \frac{R^3}{r^4} e^{-\frac{t}{2}r^2},$$

which concludes the proof. ■

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