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Stretch-width

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Abstract

We introduce a new parameter, called stretch-width, that we show sits strictly between clique-width and twin-width. Unlike the reduced parameters [BKW '22], planar graphs and polynomial subdivisions do not have bounded stretch-width. This leaves open the possibility of efficient algorithms for a broad fragment of problems within Monadic Second-Order (MSO) logic on graphs of bounded stretch-width. In this direction, we prove that graphs of bounded maximum degree and bounded stretch-width have at most logarithmic treewidth. As a consequence, in classes of bounded stretch-width, MAXIMUM INDEPENDENT SET can be solved in subexponential time $2^{\tilde{O}(n^{4/5})}$ on n -vertex graphs, and, if further the maximum degree is bounded, Existential Counting Modal Logic [Pilipczuk '11] can be model-checked in polynomial time. We also give a polynomial-time $O(\text{OPT}^2)$ -approximation for the stretch-width of symmetric 0, 1-matrices or ordered graphs.

Somewhat unexpectedly, we prove that exponential subdivisions of bounded-degree graphs have bounded stretch-width. This allows to complement the logarithmic upper bound of treewidth with a matching lower bound. We leave as open the existence of an efficient approximation algorithm for the stretch-width of unordered graphs, if the exponential subdivisions of all graphs have bounded stretch-width, and if graphs of bounded stretch-width have logarithmic clique-width (or rank-width).

2012 ACM Subject Classification Mathematics of computing → Discrete mathematics → Graph theory; Theory of computation → Graph algorithms analysis

Keywords and phrases Contraction sequences, reduced parameters, twin-width, clique-width, algorithms, algorithmic metatheorems

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1 Introduction

Various graph classes have bounded *twin-width*¹ such as, for instance, bounded clique-width graphs, proper minor-closed classes, proper hereditary subclasses of permutation graphs, and some expander classes [11]. Low twin-width, together with the witnessing *contraction sequences*, enables parameterized algorithms (that are unlikely in general graphs) for testing if a graph satisfies a first-order sentence [11, 7], and improved approximation algorithms for highly inapproximable packing and coloring problems [7, 4].

However one should not expect a large gain, in the low twin-width regime, as far as (non-parameterized) exact algorithms are concerned. This is because every graph obtained by subdividing (at least) $2\lceil \log n \rceil$ times each edge of an n -vertex graph G has twin-width at most 4 [3]. It was already observed in the 70's that a problem like MAXIMUM INDEPENDENT SET (MIS, for short) remains NP-complete in $2t$ -subdivisions [24]. Furthermore, known reductions [16] combined with the Sparsification Lemma [20], imply that unless the

¹ We refer the reader to Section 2 for the relevant definitions.

Exponential-Time Hypothesis² (ETH) fails [19] solving MIS in subcubic graphs requires time $2^{\Omega(n)}$. The previous remarks entail that, unless the ETH fails, solving MIS in subcubic graphs of twin-width at most 4 requires time $2^{\Omega(n/\log n)}$.

In contrast, on the significantly less general classes of bounded clique-width not only can MIS be solved in polynomial-time, but a fixed-parameter algorithm solving MSO_1 ³ model checking in time $f(w, |\varphi|)n^{O(1)}$ exists [14, 22], with w the clique-width of the input graph, φ the input sentence, and f some computable function.

In this paper, we start exploring the trade-off between class broadness and algorithmic generality in the zone delimited by bounded clique-width and bounded twin-width. It may seem like the *reduced parameters* [12], where a graph has *reduced* p at most k if it admits a contraction sequence in which all the red graphs have parameter p at most k , are exactly designed to tackle this endeavor. Indeed by definition, twin-width is *reduced* Δ , where Δ is the maximum degree, and it was shown that *reduced maximum connected component size* (under the name of *component twin-width*) is functionally equivalent to clique-width [10]. Between *maximum connected component size* and *maximum degree*, there are several parameters p , such as bandwidth, cutwidth, treewidth+ Δ , whose reduced parameters give rise to a strict [12] hierarchy between bounded clique-width and bounded twin-width. Unfortunately, even reduced bandwidth –the closest to clique-width among the above-mentioned reduced parameters– turns out to be too general in the following sense: the n -subdivision of any n -vertex graph has reduced bandwidth at most 2 [12]. This means, by the arguments of the second paragraph of this introduction, that solving MIS on graphs of bounded reduced bandwidth requires time $2^{\Omega(\sqrt{n})}$, unless the ETH fails, even among graphs of bounded degree. Actually, another fact leading to the same conclusion is that planar graphs have bounded reduced bandwidth [12].

We therefore introduce another parameter, that we call stretch-width⁴ and denote by stw , which, while inspired by reduced parameters, does not fully fit that framework. To a first approximation, *stretch-width* can be thought as *reduced bandwidth* where the bandwidth upper bound on the red graphs have to be witnessed by a single (and fixed) order on the vertex set. Observe indeed that the linear orders witnessing that all the red graphs of the sequence have low bandwidth can, in reduced bandwidth, be very different one from the other. We first show that the family of bounded stretch-width classes strictly contains the family of bounded clique-width classes. Using an upper bound of component twin-width by clique-width [2], we prove that:

► **Theorem 1.** *The stretch-width of any graph is at most twice its clique-width.*

Then we provide a separating class of bounded stretch-width and unbounded clique-width.

► **Theorem 2.** *There is an infinite family of graphs G with bounded stretch-width and clique-width in $\Omega(\log |V(G)|/\log \log |V(G)|)$.*

After establishing Theorem 6, we even get a simpler construction with bounded degree and stretch-width, but treewidth and clique-width in $\Omega(\log |V(G)|)$. As was done for twin-width [9], we give an effective characterization of bounded stretch-width for symmetric 0, 1-matrices (or ordered graphs).

² the assumption that there is a $\lambda > 1$ such that n -variable 3-SAT cannot be solved in time $\lambda^n n^{O(1)}$

³ Monadic Second-Order logic, when the second-order variables can only be vertex subsets

⁴ We refer a reader who would already want a formal definition to the start of Section 3.

► **Theorem 3.** *A class \mathcal{C} of symmetric 0,1-matrices has bounded stretch-width if and only if there is an integer k such that no matrix of \mathcal{C} has a k -wide division.*

The k -wide division (see definition in Section 4) is a scaled-down version of the k -rich division that analogously characterizes matrices of bounded twin-width [9]. Theorem 3 yields a polynomial-time approximation algorithm for the stretch-width of symmetric 0,1-matrices. More precisely:

► **Theorem 4.** *Given an integer k and a symmetric $n \times n$ 0,1-matrix M , there is an $n^{O(1)}$ -time algorithm that outputs a sequence witnessing that $stw(M) = O(k^3)$ or correctly reports that $stw(M) > k$.*

Compared to the approximation algorithm for the twin-width of a matrix, this is better both in terms of running time (polynomial vs fixed-parameter tractable) and approximation factor (quadratic vs exponential).

Conveniently for the sought algorithmic applications, planar graphs and n^c -subdivisions of n -vertex graphs (for any constant c) both have *unbounded* stretch-width (whereas they have bounded reduced bandwidth if $c \geq 1$). We indeed establish the following upper bound on treewidth, implying that graphs of bounded maximum degree and bounded stretch-width have at most logarithmic treewidth.

► **Theorem 5.** *There is a c such that for every graph G , $tw(G) \leq c\Delta(G)^4 stw(G)^2 \log |V(G)|$.*

We match Theorem 5 with a lower bound. There are graphs with bounded $\Delta + stw$ and treewidth growing as a logarithm of their number of vertices. This is because, as we prove, very long subdivisions of bounded-degree graphs have bounded stretch-width.

► **Theorem 6.** *Every $(\geq n^{2^m})$ -subdivision of every n -vertex m -edge graph G of maximum degree d has stretch-width at most $32(4d + 5)^3$.*

By $(\geq s)$ -subdivision of G , we mean every graph obtained by subdividing each edge of G at least s times. In particular, for every natural k , the n -vertex $k^2 2^{2k(k-1)}$ -subdivision of the $k \times k$ grid has bounded maximum degree (by 4) and stretch-width (by 296352), whereas it has treewidth $k = \Omega(\sqrt{\log n})$. A more careful argument and reexamination of Theorem 6 show that, for some constant c , the n -vertex 2^{ck} -subdivision of the $k \times k$ grid has bounded $\Delta + stw$, and treewidth $k = \Omega(\log n)$ matching the upper bound of Theorem 5.

The proofs of Theorems 5 and 6 involve the notion of *overlap graph* of a graph G whose vertex set is totally ordered by \prec , with one vertex per edge of G , and an edge between two “overlapping edges” of G , that is, two edges ab and cd such that $a \prec c \prec b \prec d$. Using Theorem 3, we show that finding a vertex ordering such that the overlap graph has no large biclique allows to bound the stretch-width.

► **Lemma 7.** *For every ordered graph (G, \prec) and every integer t , if the overlap graph of (G, \prec) has no $K_{t,t}$ subgraph then $stw(G) < 32(2t + 1)^3$.*

Theorem 6 is then derived by designing a long subdivision process that, for ordered graphs of maximum degree d , reduces the bicliques in the overlap graph to a size at most linear in d .

Theorem 5 has direct algorithmic implications for classes of bounded stretch-width.

► **Proposition 8.** *There is an algorithm that solves MAX INDEPENDENT SET in graphs of bounded stretch-width with running time $2^{\tilde{O}(n^{4/5})}$.*

Pilipczuk [23] showed that any problem expressible in Existential Counting Modal Logic (ECML) admits a single-exponential fixed-parameter algorithm in treewidth. In particular, ECML model checking can be solved in polynomial time in any class with logarithmic treewidth. This logic allows existential quantifications over vertex and edge sets followed by an *arithmetic formula* and a *counting modal formula* that shall be satisfied from every vertex v . The arithmetic formula is a quantifier-free expression that may involve the cardinality of the vertex and edge sets, as well as integer parameters. Counting modal formulas enrich quantifier-free Boolean formulas with $\diamond^S \varphi$, whose semantics is that the current vertex v has a number of neighbors satisfying φ in a prescribed ultimately periodic set S of non-negative integers.

The logic ECML+C gives to ECML the power of also using in the arithmetic formula the number of connected components in subgraphs induced by some vertex or edge sets. There is a Monte-Carlo polynomial-time algorithm for ECML+C in graphs of treewidth at most a logarithm function in their number of vertices [23]. Most NP-hard graph problems, such as MAXIMUM INDEPENDENT SET, MINIMUM DOMINATING SET, STEINER TREE, etc. are expressible in ECML+C; see [23, Appendix D] for the ECML+C formulation of several examples.

► **Corollary 9.** *Problems definable in ECML (resp. ECML+C) can be solved in polynomial time (resp. randomized polynomial time) in bounded-degree graphs of bounded stretch-width.*

Perspectives. Proposition 8 and Corollary 9 constitute some preliminary pieces of evidence of the algorithmic amenability of classes of bounded stretch-width. We ask several questions. How can the running time of Proposition 8 be improved? (As far as we know, there could be a polynomial-time algorithm for any problem defined in ECML on graphs of bounded stretch-width.) As for twin-width, an approximation algorithm for stretch-width of (unordered) graphs remains open. Lemma 7 gives some hope that this question might be easier than its twin-width counterpart, especially among *sparse* graphs.

Can we lift the bounded-degree requirement in Theorem 6, that is, is there a function f and a constant c , such that the stretch-width of any $(\geq f(n))$ -subdivision of any n -vertex graph is at most c ? Our separating example showing that bounded stretch-width is strictly more general than bounded clique-width (Theorem 2) yields graphs of essentially logarithmic clique-width. Is that true in general?

► **Conjecture 10.** *For every class \mathcal{C} of bounded stretch-width, there is a constant c such that for every n -vertex graph $G \in \mathcal{C}$ the clique-width of G is at most $c \log n$.*

We ask the same question with *rank-width* instead of *clique-width*, which would be more algorithmically helpful. One interpretation of Theorem 5 is that graphs of bounded maximum degree and bounded stretch-width have logarithmic treewidth. Whether the *bounded-degree* constraint can be relaxed to the mere absence of large bipartite complete subgraphs is related to Conjecture 10. A positive answer to Conjecture 10 would indeed imply this relaxation, as Gurski and Wanke have shown that graphs without $K_{t,t}$ subgraphs have treewidth at most their clique-width times $3t$ [18]. A natural future work would consist of using the witness of low stretch-width to get improved algorithms compared to those attained with a witness of low twin-width.

Related work. Our work is in line with twin-width [11], and the reduced parameters [12]. Theorem 1 closely follows a similar proof in the sixth paper of the twin-width series [10], while Theorem 3 is inspired by the fourth paper [9], and notably the so-called *rich divisions*.

Finding the right logic for a given width parameter, or the right width parameter for a given logic has been a common goal ever since Courcelle's and Courcelle-Makowsky-Rotics's theorems [13, 14] relating treewidth with MSO_2 , and clique-width with MSO_1 . Recent developments (all from 2023) include an efficient model checking of the new logic A&C DN (an extension of Existential MSO_1) on classes of bounded mim-width [5], the new parameter flip-width [27], which could lead to an efficient first-order (FO) model checking in a very general class, and efficient model checking algorithms for FO extensions with disjoint-paths predicates in proper minor-closed classes [17], and in proper topological-minor-closed classes [25].

Classes with logarithmic treewidth, although not a priori defined as such, are somewhat rare. To our knowledge, the first such example is the class of triangle-free graphs with no theta (see [26] for the lower bound, and [1], for the upper bound). Another example consists of graphs without $K_{t,t}$ subgraph and bounded induced cycle packing number [6]. We add a new family: graphs of bounded maximum degree and bounded stretch-width. Note that these three families are pairwise incomparable.

2 Preliminaries

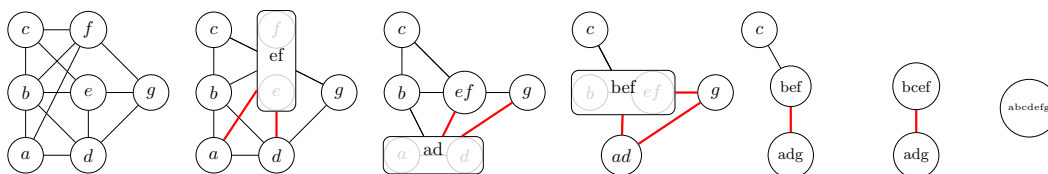
For $i \leq j$ two integers, we denote the set of integers that are at least i and at most j by $[i, j]$, and $[i]$ is a short-hand for $[1, i]$. We use the standard graph-theoretic notations. In particular, for a graph G , we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. If $S \subseteq V(G)$, the *subgraph of G induced by S* , denoted $G[S]$ is the graph obtained from G by removing the vertices not in S .

2.1 Contraction sequences and twin-width

Twin-width is a graph parameter introduced by Bonnet, Kim, Thomassé, and Watrigant [11]. A possible definition involves the notions of *trigraphs*, *red graphs*, and *contraction sequences*. A *trigraph* is a graph with two types of edges: black (regular) edges and red (error) edges. The *red graph* $\mathcal{R}(H)$ of a trigraph H consists of ignoring its black edges, and considering its red edges as being normal (black) edges. We may say *red neighbor* (or *red neighborhood*) to simply mean a neighbor (or neighborhood) in the red graph. A (vertex) *contraction* consists of merging two (non-necessarily adjacent) vertices, say, u, v into a vertex w , and keeping every edge wz black if and only if uz and vz were previously black edges. The other edges incident to w become red (if not already), and the rest of the trigraph remains the same. A *contraction sequence* of an n -vertex graph G is a sequence of trigraphs $G = G_n, \dots, G_1$ such that G_i is obtained from G_{i+1} by performing one contraction. A *d -sequence* is a contraction sequence in which every vertex of every trigraph has at most d red edges incident to it. In other words, every red graph of the sequence has maximum degree at most d . The *twin-width* of G , denoted by $\text{tw}(G)$, is then the minimum integer d such that G admits a d -sequence. Figure 1 gives an example of a graph with a 2-sequence, i.e., of twin-width at most 2.

2.2 Partition sequences

Partition sequences yield an equivalent viewpoint to contraction sequences. Instead of dealing with a sequence of trigraphs $G = G_n, \dots, G_1$, we now have a sequence of partitions $\mathcal{P}_n, \dots, \mathcal{P}_1$ of $V(G)$, with $\mathcal{P}_n = \{\{v\} \mid v \in V(G)\}$ and for every $i \in [n-1]$, \mathcal{P}_i is obtained from \mathcal{P}_{i+1} by merging two parts $X, Y \in \mathcal{P}_{i+1}$ into one $(X \cup Y)$. In particular $\mathcal{P}_1 = \{V(G)\}$. Now one



■ **Figure 1** A 2-sequence witnessing that the initial graph has twin-width at most 2.

can obtain the red graph $\mathcal{R}(G_i)$ of G_i , as the graph whose vertices are the parts of \mathcal{P}_i , and whose edges link two parts $X \neq Y \in \mathcal{P}_i$ whenever there is $u, u' \in X$ and $v, v' \in Y$ such that $uv \in E(G)$ and $u'v' \notin E(G)$. We may call two such parts X, Y *inhomogeneous*. On the contrary, two parts X, Y are *homogeneous* in G when every vertex of X is adjacent to every vertex of Y , or no vertex of X is adjacent to a vertex of Y . We will also denote $\mathcal{R}(G_i)$ by $\mathcal{R}(\mathcal{P}_i)$.

2.3 Reduced parameters and functional equivalence

We detail the definition of reduced parameters mainly for the introduction, and the notions of *functional equivalence*, *component of twin-width*, and *reduced bandwidth*; the latter being conceptually close to stretch-width. The reduced parameters will not be useful, *per se*, in the rest of the paper.

As we mentioned in the introduction, there are stronger⁵ constraints that one can put on the red graphs than merely having bounded maximum degree. This leads to the *reduced parameters* as defined in [12]. If p is a graph parameter, one can define the parameter *reduced* p , denoted by p^\downarrow , of a n -vertex graph G is the minimum over every contraction sequence $G = G_n, \dots, G_1$ of $\max_{i \in [n]} p(\mathcal{R}(G_i))$. Then Δ^\downarrow is the twin-width, when Δ denotes the maximum degree. For $p = \star$, the maximum size of a connected component, p^\downarrow is the so-called *component twin-width* (see [10]).

A graph parameter p is *functionally bounded* by a graph parameter q , denoted by $p \sqsubseteq q$, if there is a function f such that for every graph G , $p(G) \leq f(q(G))$. Parameters p, q are *functionally equivalent* or *tied* if $p \sqsubseteq q$ and $q \sqsubseteq p$. We finally denote by $p \sqsubset q$ the fact that $p \sqsubseteq q$ holds but $q \sqsubseteq p$ does not. It is shown in [12] that, under some relatively mild assumptions, the strict “inclusion” $p \sqsubset q$ implies the strict “inclusion” $p^\downarrow \sqsubset q^\downarrow$.

It can be seen that $\star \sqsubset \text{bandw} \sqsubset \text{cutw} \sqsubset (\Delta + \text{tw}) \sqsubset \Delta$, where *bandw*, *cutw*, *tw* are the bandwidth, cutwidth, and treewidth, respectively. We give a definition of bandwidth here (mainly because of the apparent similarity between stretch-width and reduced bandwidth). The treewidth of a graph is defined in the next subsection, while we omit the definition of *cutwidth* as we will not need it. A *linear layout* of an n -vertex graph G is a bijective map σ from $V(G)$ to $[n]$. The *length* of an edge $e = uv \in E(G)$ under the linear layout σ is defined as $|\sigma(u) - \sigma(v)|$. The *bandwidth* $\text{bandw}(G)$ of a graph G is the minimum over every linear layout σ of the maximum length of an edge $e \in E(G)$ under σ .

By [12], $\text{ctww} = \star^\downarrow \sqsubset \text{bandw}^\downarrow \sqsubset \text{cutw}^\downarrow \sqsubset (\Delta + \text{tw})^\downarrow \sqsubset \Delta^\downarrow = \text{tw}$. As component twin-width *ctww* and clique-width *cw* are functionally equivalent [10], we get a strict ladder

⁵ Note that every graph admits a sequence where all the red graphs consist of one star together with isolated vertices (namely, any partition sequence having, at every step, only one part that is not a singleton). Stars form arguably the simplest class of unbounded degree. Thus trading the condition of maximum degree to an incomparable property on the red graphs should likely be accompanied by some extra requirement, like forcing the partitions to be reasonably “balanced.”

of classes interpolating between clique-width and twin-width. However, for every parameter p considered so far except \star (even bandwidth), the classes $\{G^{(n)} \mid n \in \mathbb{N}, G \text{ has } n \text{ vertices}\}$ and of all planar graphs have bounded p^\perp [12], where $G^{(n)}$ denotes the n -subdivision of G , that is, the graph obtained after replacing every edge of G by a $(n + 1)$ -edge path.

As this is an obstacle to exactly solving more general problems than first-order model checking, the current paper is about a new parameter stw (stretch-width) satisfying, as we will prove, $\text{cw} \sqsubset \text{stw} \sqsubset \text{bandw}^\perp$, while not containing the class of all n -subdivisions nor the one of all planar graphs.

2.4 Treewidth, separation number, and clique-width

We recall the definition of treewidth and clique-width, for completeness. We also state a useful characterization of bounded treewidth in terms of balanced separators.

A *tree-decomposition* of a graph G is a pair (T, β) where T is a tree, and β is a map from $V(T)$ to $2^{V(G)}$, such that the following three conditions are met:

- for every $v \in V(G)$, there is a $t \in V(T)$ such that $v \in \beta(t)$;
- for every $uv \in E(G)$, there is a $t \in V(T)$ such that $\{u, v\} \subseteq \beta(t)$;
- for every $v \in V(G)$, $\{t \in V(T) \mid v \in \beta(t)\}$ induces a connected graph in T (i.e., a subtree).

When dealing with treewidth in Section 7 it will more convenient to think of it in terms of the functionally equivalent *separation number*. A *separation* (A, B) of a graph G is such that $A \cup B = V(G)$ and there is no edge between $A \setminus B$ and $B \setminus A$. The *order* of the separation (A, B) is $|A \cap B|$. A separation (A, B) is *balanced* if $\max(|A \setminus B|, |B \setminus A|) \leq \frac{2}{3}|V(G)|$. The *separation number* $\text{sn}(G)$ of G is the smallest integer s such that every subgraph of G admits a balanced separation of order at most s . It is not difficult to show that for every graph G , $\text{sn}(G) \leq \text{tw}(G) + 1$. Dvořák and Norin showed the converse linear dependence:

► **Lemma 11** ([15]). *For every graph G , $\text{tw}(G) \leq 15\text{sn}(G)$.*

Note that if for some positive constant $c < 1$, every subgraph H of G has a separation (A, B) that is *c-balanced*, in the sense that $\max(|A \setminus B|, |B \setminus A|) \leq c|V(H)|$ of order at most s , then every subgraph of G has a balanced separation of order $\lceil \frac{\log c}{\log(2/3)} \rceil \cdot s$. In particular, by Lemma 11, $\text{tw}(G) = O(s)$.

We finish Section 2 with a brief definition of clique-width, just for completeness, because the introduction contains several occurrences of it. This definition can be ignored in the rest of the paper, and clique-width thought as the reduced parameter *component twin-width* (or \star^\perp). The *clique-width* $\text{cw}(G)$ of a graph G is the least number k of colors, called *labels*, needed to build G from the following operations:

- create a vertex with a label $i \in [k]$,
- make the union of two labeled graphs,
- relabel every vertex colored i with the label j , for some $i \neq j \in [k]$,
- add all edges between vertices labeled i and vertices labeled j , for some $i \neq j \in [k]$.

2.5 Outline

In Section 3, we define stretch-width and prove Theorems 1 and 2. In Section 4, we show Theorems 3 and 4. In Section 5, we prove Lemma 7. In Section 6, we establish Theorem 6. Finally in Section 7, we show Theorem 5 and draw the algorithmic consequences of Proposition 8 and Corollary 9.

3 Stretch-width

An ordered graph is a pair (G, \prec) where G is a graph and \prec a strict total order on $V(G)$. We write $u \preceq v$ whenever $u \prec v$ or $u = v$. Let (G, \prec) is an ordered graph, and $X \subseteq V(G)$. We now define some objects depending on \prec , but as the order will be clear from the context, we omit it from the corresponding notations.

The minimum and maximum of X along \prec are denoted by $\min(X)$ and $\max(X)$, respectively. The *convex closure* or *span* of X is $\text{conv}(X) := \{v \in V(G) \mid \min(X) \preceq v \preceq \max(X)\}$. Two sets $X, Y \subseteq V(G)$ are *in conflict*⁶, or X *conflicts with* Y , if $\text{conv}(X) \cap \text{conv}(Y) \neq \emptyset$. Note that this does not imply that X and Y themselves intersect, and indeed we will mostly use this notion for two disjoint sets X, Y .

Let now \mathcal{P} be a partition of $V(G)$, $\mathcal{R}(\mathcal{P})$ its red graph, and $X \in \mathcal{P}$. We say that $Y \in \mathcal{P} \setminus \{X\}$ *interferes* with X if Y conflicts with $N_{\mathcal{R}(\mathcal{P})}[X]$. Note that it may well be that Y interferes with X , but not vice versa. The *stretch* of the part $X \in \mathcal{P}$, denoted by $\text{str}(X)$, is then defined as the number of parts in \mathcal{P} interfering with X . In turn, the *stretch* of \mathcal{P} is the maximum over every part $Z \in \mathcal{P}$ of $\text{str}(Z)$. The *stretch-width* of the ordered graph (G, \prec) , denoted by $\text{stw}(G, \prec)$, is the minimum, taken among every partition sequence $\mathcal{P}_n, \dots, \mathcal{P}_1$ of G , of $\max_{i \in [n]} \text{str}(\mathcal{P}_i)$. Finally the *stretch-width* of G , denoted by $\text{stw}(G)$, is the minimum of $\text{stw}(G, \prec)$ taken among every total order \prec on $V(G)$.

Notice the similarity with reduced bandwidth. We also seek a sequence without “long” red edges. When witnessing low reduced bandwidth, one could use different (and incompatible) vertex orderings for the different red graphs. To witness low stretch-width, we need a stronger property: the existence of a “global” vertex ordering such that no red graph of the sequence has a long edge along this single order.

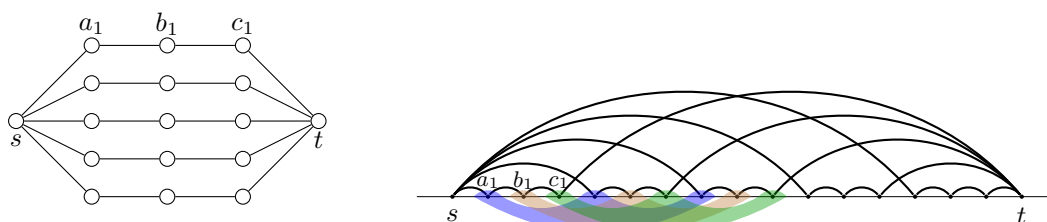
3.1 An example forcing interleaved parts

The definition of the stretch-width of an unordered graph may seem somewhat contrived. If we are to pick the order \prec , why not choosing one along which we will perform the contractions (thereby making every part an interval, and simplifying greatly the definition)? We will now see that this is in fact not always possible. There is a family of very simple graphs, with bounded treewidth, hence bounded stretch-width (see Section 3.2), but such that there is no vertex ordering \prec that simultaneously witnesses the low stretch-width, and invariably presents the two vertices to be contracted (or parts to be merged) consecutively.

For every positive integer k , let H_k be the (series-parallel) graph obtained by adding k internally vertex-disjoint 4-edge paths s, a_i, b_i, c_i, t (for $i \in [k]$) between two fixed vertices s and t ; see left of Figure 2. The graph H_k has $3k + 2$ vertices and treewidth at most 2. We will see in the next section that the much more general classes of bounded clique-width have bounded stretch-width. But let us give a direct argument for H_k , to get familiar with this new width parameter.

We choose the order $s \prec a_1 \prec b_1 \prec c_1 \prec a_2 \prec b_2 \prec c_2 \prec \dots \prec a_k \prec b_k \prec c_k \prec t$; see right of Figure 2. For the partition sequence, we will maintain three parts A, B, C such that all the other parts are singletons. Initially, we have $A = \{a_1\}$, $B = \{b_1\}$, and $C = \{c_1\}$. Then, for i going from 2 to k , we merge A with $\{a_i\}$, then B with $\{b_i\}$, and C with $\{c_i\}$. In the figure, the parts A, B, C are represented after the iteration $i = 3$. When there are only five parts left (namely $\{s\}, A, B, C, \{t\}$), we finish the sequence in any fashion.

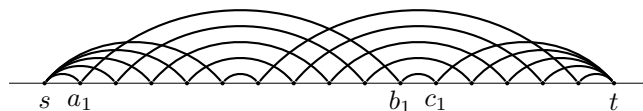
⁶ In a similar context in [9], the verb *overlap* was also used. In this paper, we will reserve *overlap* for



■ **Figure 2** Left: The graph H_5 . Right: The same graph drawn along an order able (together with an appropriate sequence) to witness low stretch-width. The parts A, B, C maintained by the partition sequence are depicted in blue, brown, green, after the iteration $i = 3$.

Importantly, the vertices within A, B , or C have the same neighborhood in $\{s, t\}$. Thus the long black edges incident to s and t never become long red edges. Note that in the midst of the i -th iteration (also at its start and end), every part P of the current partition \mathcal{P} has a closed red neighborhood contained in $Z := A \cup B \cup C \cup \{a_i, b_i, c_i\}$. Set Z is an interval along \prec , and intersects at most six parts of \mathcal{P} among $A, B, C, \{a_i\}, \{b_i\}, \{c_i\}$. Hence at most these six parts can conflict with Z , and thus interfere with P . This implies that $\text{stw}(H_k, \prec)$, hence $\text{stw}(H_k)$, is bounded by a constant (with greater care, one can show the upper bound of 3).

Now a vertex ordering such that the sequence can be done without conflicting parts forces the a_i 's, the b_i 's, and the c_i 's to be essentially consecutive. Any such attempt, like the one depicted in Figure 3 with $s \prec a_1 \prec \dots \prec a_k \prec b_k \prec \dots \prec b_1 \prec c_1 \prec \dots \prec c_k \prec t$, creates a structure which, as we will show in Section 5, entails large stretch-width: a large “biclique” of overlapping edges spanning vertices of bounded degree.



■ **Figure 3** A failed attempt at contracting along the chosen order. Observe that the first contraction involving some b_i (in the “middle” of the order) necessarily creates a long red edge.

3.2 Graphs of bounded clique-width have bounded stretch-width

Clique-width and component twin-width are *functionally equivalent*, that is, a graph class has bounded clique-width if and only if it has bounded component twin-width [11, 10]. More quantitatively, it can be observed that, for every graph G , $\text{cw}(G) \leq \text{ctww}(G) + 1 \leq 2 \cdot \text{cw}(G)$; see [2]. We show that the stretch-width of a graph is at most its component twin-width.

► **Theorem 1.** *For every graph G , $\text{stw}(G) \leq \text{ctww}(G) - 1$. Hence, $\text{stw}(G) \leq 2(\text{cw}(G) - 1)$.*

Proof. Let $\mathcal{P}_n, \dots, \mathcal{P}_1$ be a partition sequence of a graph G such that every red graph $\mathcal{R}(\mathcal{P}_i)$ has all its connected components of size at most $t := \text{ctww}(G)$. We define a total order \prec on $V(G)$ as the last order of a sequence of partial orders \prec_i on \mathcal{P}_i for i going from 1 to n . That is, \prec_n is a total order, and we imply set $\prec := \prec_n$. We maintain the following invariants:

intersecting intervals (actually edges) that are not nested, notion which we will later use.

- Parts $X, Y \in \mathcal{P}_i$ are comparable with respect to \prec_i if and only if they are in distinct connected components of $\mathcal{R}(\mathcal{P}_i)$, and
- \prec_i is a total order on the connected components of $\mathcal{R}(\mathcal{P}_i)$, that is, for every distinct connected components C, C' of $\mathcal{R}(\mathcal{P}_i)$, if $X \prec_i Y$ for some $X \in C$ and $Y \in C'$, then $X \prec_i Y$ holds for every $X \in C$ and $Y \in C'$.

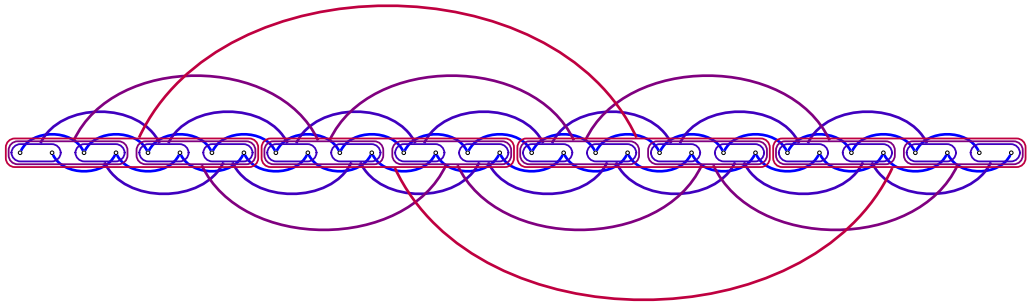
We define \prec_1 as the empty relation, which satisfies the invariant since \mathcal{P}_n has a single part, $V(G)$. We define \prec_{i+1} from \prec_i in the following way. Let X, Y be the two parts of \mathcal{P}_{i+1} being merged to $Z = X \cup Y \in \mathcal{P}_i$. Let C be the connected component of Z in $\mathcal{R}(\mathcal{P}_i)$. Let C_1, \dots, C_h be the connected components of $C \setminus \{Z\} \cup \{X, Y\}$ in $\mathcal{R}(\mathcal{P}_{i+1})$. We then set $P \prec_{i+1} Q$ whenever either $P \prec_i Q$ or $P \in C_a$ and $Q \in C_b$ for some pair $a < b \in [h]$. One can check that \prec_{i+1} is a partial order satisfying the invariants.

As the connected components of $\mathcal{R}(\mathcal{P}_n)$ are singletons, \prec_n is a total order. We now use \prec as a witness of low stretch-width for the same partition sequence $\mathcal{P}_n, \dots, \mathcal{P}_1$. Let X be any part of any partition \mathcal{P}_i , and let C be the connected component of $\mathcal{R}(\mathcal{P}_i)$ containing X . By the second invariant, no part of C can cross a part of $\mathcal{P}_i \setminus C$. Also, by definition, X may only be inhomogeneous to parts of C . Therefore, the only parts that can interfere with X are in C ; thus $\text{str}(X) \leq |C| - 1 \leq t - 1$. Finally, $\text{stw}(G) \leq \text{stw}(G, \prec) \leq t - 1 = \text{ctww}(G) - 1$. ◀

3.3 Separating construction

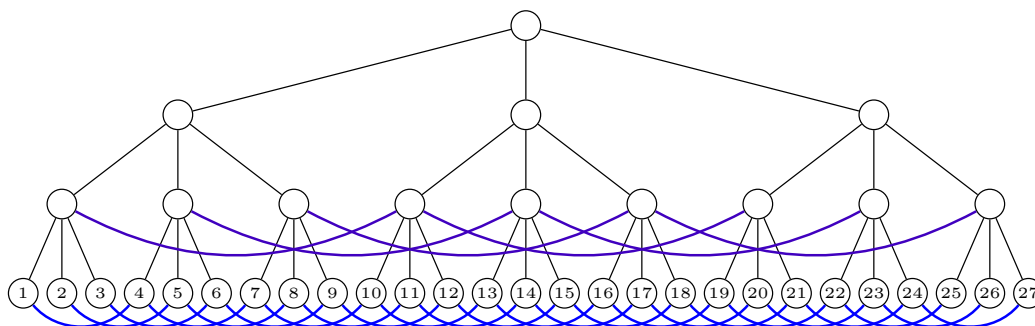
We present a graph family with bounded stretch-width but unbounded clique-width. Let $b \geq 2$ and $h \geq 1$ be two integers. We build a graph $A_b(h)$ on vertex set $[b^h]$.

Informally, $A_2(h)$ is built by first adding the paths $1, 3, 5, \dots$ on “odd” vertices, and $2, 4, 6, \dots$ on “even” vertices. Then identifying the pairs $2i - 1, 2i$, renaming them i , and adding the “odd” and “even” paths (that is, the edge between the first and the third vertices at this step corresponds to the complete adjacency between $\{1, 2\}$ and $\{5, 6\}$, in terms of original vertices). And iterating this process until it runs out of vertices; see Figure 4.



■ **Figure 4** The graph $A_2(5)$, the edge colors correspond to a step in the recursive construction.

We give a general definition, but will only use $A_3(h)$. Let $T_b(h)$ the complete b -ary tree of depth h . Think of the vertices of $A_b(h)$ as the leaves of $T_b(h)$, named from left to right $1, 2, \dots, b^h$. The level of a node of $T_b(h)$ is h minus the depth of the node. For example the level of a leaf is 0. We call $N_l(i)$ for i -th internal node (from left to right) on the l -th level. At every level l of $T_b(h)$, we add b node-disjoint paths P_1, \dots, P_b such that for each i of $[0, b - 1]$, $P_i = N_l(i), N_l(i + b), N_l(i + 2b), \dots$; see Figure 5. When two nodes at level l are linked by an edge of such a path, we say that they are l -linked.



■ **Figure 5** Tree representation of $A_3(4)$. The vertex set of $A_3(4)$ is made by the leaves, and a colored edge uv (one not part of the tree) is a biclique between the two disjoint sets of leaves of the two subtrees rooted at u and v .

From the tree $T_b(h)$, we build $A_b(h)$ as follows: the vertices of $A_b(h)$ are the leaves of $T_b(h)$ and two vertices u and v are adjacent in $A_b(h)$ if and only if there exists a level l , an ancestor $N_l(i)$ of u and an ancestor $N_l(j)$ of v such that $N_l(i)$ and $N_l(j)$ are l -linked, that is, $|j - i| = b$.

For a given node $N_l(i)$ of $T_b(h)$, we denote by $V(N_l(i))$ the set of leaves (hence, vertices of $A_b(h)$) that are descendant of $N_l(i)$. Observe that $A_b(h)[V(N_l(i))]$ is isomorphic to $A_b(l)$. If S is a set of vertices of $A_b(h)$ then the *level* of S is defined as the smallest level l for which there exists at most two consecutive nodes $N_l(i), N_l(i + 1)$ on the l -th level of $T_b(h)$ such that every element of S is a descendant of $N_l(i)$ or $N_l(i + 1)$.

We can directly establish the following bound on the stretch-width of $A_3(h)$:

► **Lemma 12.** *For every integer h , the stretch-width of $G(A_3(h))$ is at most 9.*

Proof. Let h be a positive integer. Let \prec be the left-right order on the leaves of $T_b(h)$, and for each l in $[h]$, let \mathcal{P}_l be the partition $\{V(N_l(1)), V(N_l(2)), \dots\}$.

Observe that $V(N_l(i))$ is not homogeneous to $V(N_l(j))$ if and only if $|j - i| = 1$. As for any $i \neq j$ $\text{conv}(V(N_l(i)) \cap \text{conv}(V(N_l(j)))$ is empty, the stretch of \mathcal{P}_l along \prec is at most 2 for every l . In addition, observe that one can go from \mathcal{P}_l to \mathcal{P}_{l+1} by merging only consecutive parts. As each part of \mathcal{P}_{l+1} is composed of exactly 3 consecutive parts of \mathcal{P}_l , the stretch of the partitions between \mathcal{P}_l and \mathcal{P}_{l+1} is at most $3 \cdot 3 = 9$. ◀

► **Lemma 13.** *If v is a vertex of $A_3(h)$ then the degree d of v verifies $\frac{3^{h-1}-1}{2} \leq d \leq 3^{h-1} - 1$.*

Proof. Let v a vertex of $A_3(h)$. For a vertex w to be adjacent to v , it is necessary that there exists a level l and two l -linked nodes $N_l(i), N_l(j)$ that are ancestors of v and w , respectively. Observe that such a level is unique. Hence, the ancestor of v on the l -th level for $l < h - 1$ yields either 3^l or $2 \cdot 3^l$ neighbors of v , depending on whether it is incident to one or two l -linked nodes in $A_b(h)$. In addition, there is no l -linked nodes for $l \geq h - 1$, thus ancestors at level h and $h - 1$ do not contribute to any edge of $A_b(h)$.

As $\sum_{i \leq h-2} 3^i = \frac{3^{h-1}-1}{2}$, the degree of v is lowerbounded by $\frac{3^{h-1}-1}{2}$ and upperbounded by $3^{h-1} - 1$. ◀

► **Lemma 14.** *For every level l , for every two integers $i \neq j$, the number of neighbors of a vertex $v \in V(N_l(i))$ that are in $V(N_l(j))$ is:*

1. in the interval $[3^{l-1}, \frac{3^l-1}{2}]$ when $|j - i| = 1$, and
2. equal to 3^l or 0 when $|j - i| > 1$.

Proof. Let v a vertex of $A_3(h)$ such that v is in $V(N_i(l))$ for a level l . Let j be such that $|j - i| = 1$. Then if $N_{l-1}(i')$ is the ancestor of v at the $(l - 1)$ -st level, there exists a child $N_{l-1}(j')$ of $V(N_l(j))$ which is $(l - 1)$ -linked to a_v . Thus the number of neighbors of v that are in $V(N_l(j))$ is at least 3^{l-1} . Observe that every ancestor of v at level $k < l$ is k -linked to at most one descendant of $N_l(j)$. In addition, for every integer $k \geq l$, the ancestor at level k of $N_l(i)$ and the one at level k of $N_l(j)$ are not k -linked. Hence the maximum number of neighbors of v that are in $V(N_l(j))$ is upperbounded by $\sum_{k < l} 3^k = \frac{3^l - 1}{2}$.

Assume now that $|j - i|$ is at least 2. Then, none of the ancestors of v at level at most $k < l$ is k -linked to a descendant of $N_l(j)$. Thus, either an ancestor of v at level $k \geq l$ is k -linked to a (non-strict) ancestor of $N_j(l)$, in which case v is fully adjacent to $V(N_l(j))$, or none of the ancestors of v is linked to an ancestor of $N_l(j)$, in which case v has no neighbors in $V(N_l(j))$. ◀

Remember that the level of a subset S of the vertices of $A_3(h)$ is the smallest level l for which there is at most two consecutive nodes $N_l(i), N_l(i + 1)$ at level l such that every vertex of l is either a descendant of $N_l(i)$ or of $N_l(i + 1)$.

We will now prove that for any partition sequence $\mathcal{P}_1, \mathcal{P}_2, \dots$ of $A_3(h)$, the size of a largest red component among every partition is lowerbounded by a function of h . To do so, we show that as long as there is no large part in the sequence, any part of the sequence has a small level, and thus is “localized” in the graph. We then prove that, given a large part S in one of the partitions, this part needs to have a red neighbor the size of which is at least linear in the size of S . We finally build a large red component based on these two facts. Informally, we take the largest part of a suitable partition. This part is localized, say on the left side, and we use the second fact to build a smaller red neighbor to the right of this part, and continue this process until we reach a part of constant size.

► **Lemma 15.** *Let $\mathcal{P}_1, \mathcal{P}_2, \dots$ be a partition sequence of $A_3(h)$ with maximum red degree at most d . Let, for any integer t , $s(t)$ be the smallest integer such that one of the parts of $\mathcal{P}_{s(t)}$ has size at least t . Then for every positive integer t , for every part $Q \in \mathcal{P}_{s(t)}$, the level of Q is at most $\log_3(t) + \log_3(d) + 5$.*

Proof. Each part of $\mathcal{P}_{s(t)}$ contains at most $2t$ vertices. Let Q be a part of $\mathcal{P}_{s(t)}$ and l be its level. As $l - 1$ is strictly smaller than the level of Q , there are two integers i, j with $i + 1 < j$ such that $Q \cap V(N_{l-1}(i))$ and $Q \cap V(N_{l-1}(j))$ are both non-empty. Going one level further down, the intermediate set $V(N_{l-1}(i + 1))$ splits into three parts. Thus, there are two integers i', j' such that $i' + 3 < j'$ and Q has a vertex u in $V(N_{l-2}(i'))$, and a vertex v in $V(N_{l-2}(j'))$.

By Lemma 14, as $|j' - i'| > 1$, u has either 3^{l-2} or 0 neighbors in $V(N_{l-2}(j'))$, while by Lemma 13, v has between $\frac{3^{l-3}-1}{2}$ and $3^{l-3} - 1$ neighbors in $V(N_{l-2}(j'))$. Thus the symmetric difference of the neighborhoods of u and v , say A , contains at least $\frac{3^{l-3}-1}{2}$ vertices. The maximum red degree of Q is at most d , and for each part $T \neq Q$ of $\mathcal{P}_{s(t)}$ containing a vertex of A , Q has a red edge toward T . Thus A is contained in at most $d + 1$ parts of $\mathcal{P}_{s(t)}$. Hence $|A| \leq 2t \cdot (d + 1)$. Since $|A| \geq \frac{3^{l-3}-1}{2}$, we get $\frac{3^{l-3}-1}{2} \leq 2t \cdot (d + 1)$, thus $l - 3 \leq \log_3(4t \cdot (d + 1))$. ◀

► **Lemma 16.** *Let \mathcal{P} any partition of $A_3(h)$ within a partition sequence of maximum red degree at most d . Let S a part in \mathcal{P} , at level l , such that there is at least 3^l vertices in $A_3(h)$ strictly to the right of S . Then, there is a part T of \mathcal{P} in the red neighborhood of S with $|T| \geq 3^{l-1}/d$, and such that T has a vertex strictly to the right of S . Furthermore, if S is*

contained in $V(N_l(i)) \cup V(N_l(i+1))$ (resp. only $V(N_l(i))$), then T contains a vertex v in $V(N_l(i+2))$ (resp. $V(N_l(i+1))$).

Proof. Let l be the level of S . There are two integers $i+1 < j$ such that $S \cap V(N_{l-1}(i))$ contains a vertex u , and $S \cap V(N_{l-1}(j))$ contains a vertex v . As there is at least 3^l vertices strictly to the left of S , $N_{l-1}(j+1)$ is well defined. By Lemma 14, the number of neighbors of u within $V(N_{l-1}(j+1))$ is either 0 or 3^{l-1} , and the number of neighbors of v within $V(N_{l-1}(j+1))$ is between 3^{l-1} and $\frac{3^l-1}{2}$. Thus the symmetric difference of neighborhoods of u and v inside $V(N_{l-1}(j+1))$, say A , contains at least 3^{l-1} vertices. As each part of \mathcal{P} containing a vertex of A is inhomogeneous to S , there is a part $T \in \mathcal{P}$ containing at least $3^{l-1}/d$ vertices of A . ◀

► **Theorem 17.** *The component twin-width of $A_3(h)$ is $\Omega(h/\log h)$.*

Proof. Consider a partition sequence $\mathcal{P}_1, \mathcal{P}_2, \dots$ of $A_3(h)$ of maximum red degree at most d . Let $t = 3^{h/2}$, and let s be the smallest integer such that \mathcal{P}_s contains a part of size at least t . We will show that the red component of \mathcal{P}_s containing its largest part is large.

Consider P the only part of \mathcal{P}_s of size at least t . Let $l = h/2 + \log_3(d) + 5$. By Lemma 15, the level of P is at most l . Thus, up to symmetry, there is at least $3^h/3$ vertices strictly to the right of P . We consider the sequence of parts $(P_k)_k$ such that $P_0 = P$, and P_{k+1} is the part obtained by the application of Lemma 16 on P_k , as long as the conditions of Lemma 16 are satisfied. We state the three following facts on the sequence $(P_k)_k$.

1. P_k is different from P_i for every $i < k$.
2. $|P_k| \geq 3^{\text{level}(P_{k-1})-1}/d \geq |P_{k-1}|/(3d)$.
3. There is a finite sequence of positive integers $(i_k)_k$ such that for any m , P_m is contained in $V(N_l(i_m)) \cup V(N_l(i_m+1))$, and $i_m \leq i_{m+1} \leq i_m + 2$.

Indeed for every k , P_{k+1} always has a vertex strictly to the right of P_k , thus Item 1 holds. Item 2 directly follows from the bound on the size of the part of Lemma 16. The lower bound of Item 3 comes from the fact that P_m has a vertex strictly to the right of P_{m-1} , and the upper bound from the fact that P_m contains a vertex in $V(N_l(i_{m-1}+1))$.

Consider the largest integer n for which P_n is defined. Integer n satisfies $3^{\text{level}(P_n)-1} > d$, or there is less than $3^{\text{level}(P_n)}$ (thus, less than 3^l) vertices strictly to the right of P_n . As the level of P_n is at least $\log_3 |P_n|$, if the former condition is satisfied, $|P_n| \leq 3d$. But Item 2 ensures that $|P_n|$ is at least $P_0/(3d)^n$, thus $n \geq \log_{3d} t$. If the latter condition is satisfied, then i_n is at least $(1/3^{h/2}) \cdot (3^h/3) - 1 = 3^{h/2-1} - 1$, and thus Item 3 ensures that n is at least $\frac{3^{h/2-1}-1}{2}$.

Hence, as Item 1 ensures that the red component containing P is of size at least n , this component is of size at least $\log_{3d} t$. Let x be the component twin-width of $A_3(h)$. Then in a sequence witnessing that fact, the size of a largest red component is at most x , thus the maximum red degree is at most x (even $x-1$). Therefore $\log_{3d} t \leq x$, and so $\log t \leq x \log(3x)$. Thus $3x \geq \frac{h}{2} / \log \frac{h}{2}$, and $x = \Omega(h/\log h)$. ◀

Theorem 2 is a consequence of Lemma 12 and Theorem 17.

4 Matrix characterization

Let us first reinterpret the definition of stretch-width on symmetric (ordered) matrices. A symmetric *partition* of a symmetric matrix M is a pair $(\mathcal{R}, \mathcal{C})$ such that \mathcal{R} is a partition of the row set of M , $\text{rows}(M)$, \mathcal{C} is a partition of the column set, $\text{columns}(M)$, and \mathcal{C} is symmetric to \mathcal{R} , i.e., two rows r_i and r_j are in the same part if and only if the symmetric

columns c_i and c_j are in the same part. Hence each row part corresponds to a (unique) symmetric column part.

A (symmetric) *division* of a symmetric matrix M is a (symmetric) partition of M every row (resp. column) part of which is on consecutive rows (resp. columns). Given a row part $R \in \mathcal{R}$, and a column part $C \in \mathcal{C}$, the *zone* $R \cap C$ of M is the submatrix of M with row set R and column set C . A zone $R \cap C$ is *diagonal* if R and C are symmetric parts. A zone is *non-constant* if it contains two distinct entries. A symmetric *partition sequence* of an $n \times n$ 0, 1-matrix M is a sequence $(\mathcal{R}_n, \mathcal{C}_n), \dots, (\mathcal{R}_1, \mathcal{C}_1)$ where $(\mathcal{R}_n, \mathcal{C}_n)$ is the *finest partition* (with n row parts and n column parts), $(\mathcal{R}_1, \mathcal{C}_1)$ is the coarsest partition (with one row part and one column part), and for every $i \in [2, n]$, $(\mathcal{R}_{i-1}, \mathcal{C}_{i-1})$ is obtained from $(\mathcal{R}_i, \mathcal{C}_i)$ by merging together two row parts, and the symmetric two column parts.

So far, we were following the definitions of [11, 8] (in the symmetric case). Instead of defining the *error value* which leads to the twin-width of a matrix, we introduce the *stretch value*. The *stretch value* of a row part R of a matrix partition $(\mathcal{R}, \mathcal{C})$ is the number of column parts conflicting with the union of columns parts C such that C is the symmetric of R , or $R \cap C$ is non-constant. The *stretch value* of a column part is defined symmetrically. The *stretch value* of a partition $(\mathcal{R}, \mathcal{C})$ is the maximum stretch value of a part of $(\mathcal{R}, \mathcal{C})$. Finally, the *stretch-width* of a symmetric 0, 1-matrix M is the minimum among every symmetric partition sequence \mathcal{S} of M of the maximum stretch value among partitions of \mathcal{S} . Observe that for any ordered graph (G, \prec) , the stretch-width of (G, \prec) is equal to the stretch-width of its adjacency matrix.

R_1	0	0	0	0	1	0	0	1	1	0	1	0	1	0
	0	0	1	0	1	1	1	0	1	0	0	0	0	0
R_2	0	1	0	1	0	1	0	0	1	1	1	0	1	1
	0	0	1	0	1	1	0	0	1	0	1	0	1	0
	1	1	0	1	0	1	1	1	0	1	1	1	1	1
R_3	0	1	1	1	1	0	1	0	1	0	1	0	0	1
	0	1	0	0	1	1	0	0	1	0	0	1	0	1
	1	0	0	0	1	0	0	0	0	1	0	0	0	0
R_4	1	1	1	0	0	1	1	0	0	0	0	1	0	0
	0	0	1	0	1	0	0	1	0	0	0	0	0	1
	1	0	1	1	1	1	0	0	0	0	0	1	1	1
	0	0	0	0	1	0	1	0	1	0	1	0	1	0
	1	0	1	1	1	0	0	0	0	0	1	1	0	0
	0	0	1	0	1	1	1	0	0	1	1	0	0	0
	C_3													

■ **Figure 6** A symmetric division of a symmetric 0, 1-matrix. R_3 and C_3 are symmetric parts. The zone $R_2 \cap C_3$ is shaded. Column part C_3 is not 3-wide since the deletion of R_2, R_3, R_4 leaves only two distinct column vectors in C_3 , namely $(0, 1, 0, 1)$ and $(1, 0, 0, 0)$. It is however 2-wide since $R_1 \cap C_3$ has two distinct column vectors (and R_1 is too far from the diagonal zone to be removed).

The following is the counterpart of the so-called *rich divisions* [9] tailored for stretch-width. If R is a set of rows, and C is a set of columns of a matrix M , we denote by $R \setminus C$ the zone $R \cap (\text{columns}(M) \setminus C)$, that is the submatrix formed by R deprived of the columns of C (and symmetrically for $C \setminus R$). In a division $(\mathcal{R} = (R_1, \dots, R_n), \mathcal{C} = (C_1, \dots, C_m))$, a row part R_i is k -wide if for every k consecutive columns parts C_j, \dots, C_{j+k-1} containing the symmetric

of R_i , $R_i \setminus \bigcup_{j \leq h \leq j+k-1} C_h$ contains at least k distinct rows. The k -wideness of column parts is defined symmetrically; see Figure 6. A division $(\mathcal{R}, \mathcal{C})$ is k -wide if all its row and column parts are k -wide. The division is k -diagonal if none of the row and column parts is k -wide.

Given a set of rows (or columns) X of a matrix M , we keep the notation $\text{conv}(X)$ for the set of rows (or columns) of M with indices between the minimum and the maximum indices of X .

► **Theorem 18.** *For every symmetric 0,1-matrix M and natural k , if $\text{stw}(M) \leq k$, then M has no $9k$ -wide division.*

Proof. Let $\mathcal{D} = (\mathcal{R}, \mathcal{C})$ be a symmetric division of M . Let $\mathcal{P} = (\mathcal{R}'_n, \mathcal{C}'_n), \dots, (\mathcal{R}'_1, \mathcal{C}'_1)$ be a symmetric partition sequence of M with stretch value at most k . Let s be the largest integer (that is, first time within the partition sequence) for which there is a row part $R' \in \mathcal{R}'_s$ such that $\text{conv}(R')$ contains a row part $R \in \mathcal{R}$ of the division \mathcal{D} . Observe that, by symmetry of \mathcal{D} and \mathcal{P} , it happens at the same time in columns. We will prove that R is not $9k$ -wide.

Set $\mathcal{S} := \{T \in \mathcal{R}'_s \mid \text{conv}(T) \cap R \neq \emptyset\}$. Note that \mathcal{S} is the set of row parts of \mathcal{R}'_s conflicting with R , and that $R' \in \mathcal{S}$. As $\text{conv}(R) \subset \text{conv}(R')$, every part in \mathcal{S} conflicts with R' . Thus, it should hold that $|\mathcal{S}| \leq k$ because $(\mathcal{R}'_s, \mathcal{C}'_s)$ has stretch value at most k .

For each T in \mathcal{S} , we define $C_T := \{c \in \text{columns}(M) \mid c \in C, C \in \mathcal{C}'_s, \text{ and } C \cap T \text{ is non-constant or } C \text{ is the symmetric of } T\}$. For every $T \in \mathcal{S}$, C_T conflicts with at most k parts of \mathcal{C}'_s , as the stretch value of T is at most k . Let C' be the symmetric of R' . As T conflicts with R' , the symmetric of T conflicts with C' . Since the symmetric of T is contained in C_T , C_T conflicts with C' , thus $\text{conv}(C') \cap \text{conv}(C_T) \neq \emptyset$.

As for every $T \in \mathcal{S}$, $\text{conv}(C') \cap \text{conv}(C_T) \neq \emptyset$, there is $T_1, T_2 \in \mathcal{S}$ such that

$$\text{conv}(C_{T_1}) \cup \text{conv}(C_{T_2}) \cup \text{conv}(C') = \text{conv}\left(\bigcup_{T \in \mathcal{S}} C_T\right).$$

Take indeed $T_1 \in \mathcal{S}$ such that C_{T_1} contains the column of minimum index in $\bigcup_{T \in \mathcal{S}} C_T$, and $T_2 \in \mathcal{S}$ such that C_{T_2} realizes the maximum index. As C_{T_1} and C_{T_2} conflict with C' , the parts in conflict with C_{T_1} , with C' , and with C_{T_2} are consecutive, so their union contains at most $3k$ parts of \mathcal{C}'_s . Thus $\bigcup_{T \in \mathcal{S}} C_T$ conflicts with at most $3k$ parts of \mathcal{C}'_s .

Observe that, except for C' , every part in \mathcal{C}'_s is covered by the union of two consecutive parts of \mathcal{C} . Part C' is itself covered by the union of three consecutive parts of \mathcal{C} . Thus, overall, each part of \mathcal{C}'_s is covered by the union of at most three consecutive parts of \mathcal{C} . Hence, as we showed that $\bigcup_{T \in \mathcal{S}} C_T$ conflicts with at most $3k$ consecutive parts of \mathcal{C}'_s , it is contained in $9k$ consecutive parts of \mathcal{C} , say C_j, \dots, C_{j+9k-1} . Thus for any $T \in \mathcal{S}$, $T \setminus \bigcup_{j \leq h \leq j+9k-1} C_h$ is constant. Therefore $R \setminus \bigcup_{j \leq h \leq j+9k-1} C_h$ contains at most k distinct rows, as $|\mathcal{S}| \leq k$. ◀

► **Theorem 19.** *For every symmetric 0,1-matrix M and natural k , if M does not have a k -wide division, then M admits a sequence of symmetric $2(k+1)$ -diagonal divisions.*

Proof. Let M be a symmetric $n \times n$ matrix that does not admit a k -wide division. The finest division $(\mathcal{R}_n, \mathcal{C}_n)$ of M is 2-diagonal, hence, $2(k+1)$ -diagonal. Now, we greedily merge some consecutive parts, to form a division sequence in which every division is $2(k+1)$ -diagonal. Assume, for the sake of contradiction, that after a partial sequence of symmetric $2(k+1)$ -diagonal divisions $(\mathcal{R}_n, \mathcal{C}_n), \dots, (\mathcal{R}_s, \mathcal{C}_s)$ of M , no division $(\mathcal{R}, \mathcal{C})$ obtained by merging two symmetric pairs of consecutive parts of $(\mathcal{R}_s, \mathcal{C}_s)$ is $2(k+1)$ -diagonal.

Say, $\mathcal{R}_s = (R_1, \dots, R_s)$ and $\mathcal{C}_s = (C_1, \dots, C_s)$. By assumption, for each $i \in [s-1]$, the division $((R_1, \dots, R_i \uplus R_{i+1}, \dots, R_s), (C_1, \dots, C_i \uplus C_{i+1}, \dots, C_s))$ is not $2(k+1)$ -diagonal. As $(\mathcal{R}_s, \mathcal{C}_s)$ is $2(k+1)$ -diagonal, $R_i \uplus R_{i+1}$ has to be the (only) $2(k+1)$ -wide row part in

$(R_1, \dots, R_i \uplus R_{i+1}, \dots, R_s)$. Indeed, for any other row part R_j (with $j \in [i-1] \cup [i+2, s]$), R_j was also a part of \mathcal{R}_s , while each part of $(C_1, \dots, C_i \uplus C_{i+1}, \dots, C_s)$ contains a part of \mathcal{C}_s . Thus, as R_j is not $2(k+1)$ -wide in $(\mathcal{R}_s, \mathcal{C}_s)$, it is not $2(k+1)$ -wide in $((R_1, \dots, R_i \uplus R_{i+1}, \dots, R_s), (C_1, \dots, C_i \uplus C_{i+1}, \dots, C_s))$.

Hence, in the division $((R_1 \uplus R_2, R_3 \uplus R_4, \dots, R_{s-1} \uplus R_s), (C_1 \uplus C_2, C_3 \uplus C_4, \dots, C_{s-1} \uplus C_s))$ (resp. $((R_1 \uplus R_2, R_3 \uplus R_4, \dots, R_{s-2} \uplus R_{s-1} \uplus R_s), (C_1 \uplus C_2, C_3 \uplus C_4, \dots, C_{s-2} \uplus C_{s-1} \uplus C_s))$ when s is odd), each row part is at least $k+1$ -wide (resp. k -wide). Indeed, k consecutive parts of $(C_1 \uplus C_2, C_3 \uplus C_4, \dots, C_{s-1} \uplus C_s)$ (resp. $(C_1 \uplus C_2, C_3 \uplus C_4, \dots, C_{s-1} \uplus C_{s-1} \uplus C_s)$) are always covered by $2(k+1)$ -consecutive parts of \mathcal{C}_s . Thus M admits a k -wide division, a contradiction. \blacktriangleleft

► **Observation 20.** *If M is a matrix on which $\mathcal{D} = (\mathcal{R} = (R_1, \dots, R_p), \mathcal{C} = (C_1, \dots, C_p))$ is a symmetric k -diagonal division, then for every $i \in [p]$, $R_i \setminus \bigcup_{i-k+1 \leq h \leq i+k-1} C_h$ (for the sake of legibility, an out-of-range value h indexes an empty C_h) has less than k distinct rows.*

Proof. By definition, for each $i \in [p]$, there are k consecutive column parts C_j, \dots, C_{j+k-1} such that $R_i \setminus \bigcup_{j \leq h \leq j+k-1} C_h$ contains less than k different rows, with $j \leq i \leq j+k-1$. Hence, $\{C_{i-k+1}, \dots, C_{i+k-1}\}$ contains $\{C_j, \dots, C_{j+k-1}\}$, and thus $R_i \setminus \bigcup_{i-k+1 \leq j \leq i+k-1} C_h$ also contains less than k distinct rows. \blacktriangleleft

► **Theorem 21.** *If a symmetric $0, 1$ -matrix M admits a sequence of symmetric k -diagonal divisions, then $\text{stw}(M) \leq 4k^3$.*

Proof. Let M be an $n \times n$ $0, 1$ -matrix, and $(\mathcal{R}_n, \mathcal{C}_n), \dots, (\mathcal{R}_1, \mathcal{C}_1)$, a sequence of symmetric k -diagonal divisions. For any $s \in [2, n]$, let $\mathcal{R}_s = (R_1, \dots, R_s)$ and $\mathcal{C}_s = (C_1, \dots, C_s)$. By Observation 20, for any $i \in [s]$, $R_i \setminus \bigcup_{i-k+1 \leq h \leq i+k-1} C_h$ contains less than k distinct rows. Let $(\mathcal{R}'_s, \mathcal{C}'_s)$ be the symmetric partition of M such that for each $P \in \mathcal{R}'_s$, there is $R_i \in \mathcal{R}_s$ with $P \subseteq R_i$, and P is a maximal subset of equal rows of $R_i \setminus \bigcup_{i-k+1 \leq h \leq i+k-1} C_h$.

As each $R \in \mathcal{R}_s$ is split into at most k parts in \mathcal{R}'_s , the stretch value of $(\mathcal{R}'_s, \mathcal{C}'_s)$ is at most $(2k-1)k$. By construction, $(\mathcal{R}'_{s-1}, \mathcal{C}'_{s-1})$ is a coarsening of $(\mathcal{R}'_s, \mathcal{C}'_s)$. Indeed, every $R'_i \in \mathcal{R}'_s$ is an equivalence class of $R_i \setminus \bigcup_{i-k+1 \leq h \leq i+k-1} C_h$. Thus it is contained in an equivalence class of the row part of $(\mathcal{R}_{s-1}, \mathcal{C}_{s-1})$ containing R'_i .

To go from $(\mathcal{R}'_s, \mathcal{C}'_s)$ to $(\mathcal{R}'_{s-1}, \mathcal{C}'_{s-1})$, we perform any symmetric partition sequence. As any part in the latter partition contains at most $2k$ parts in the former one (all the possible parts partitioning two row parts R_i and R_{i+1} that are merged), the stretch value in between these two partitions is bounded by $2k^2 \cdot 2k$. Hence $\text{stw}(M) \leq 4k^3$. \blacktriangleleft

► **Theorem 22.** *If a symmetric $0, 1$ -matrix M does not admit a k -wide division, then $\text{stw}(M) \leq 32(k+1)^3$.*

Proof. Indeed, by Theorem 19, M admits a sequence of $2(k+1)$ -diagonal divisions. Applying Theorem 21 on this sequence yields a witness of stretch-width $4 \cdot (2(k+1))^3 = 32(k+1)^3$. \blacktriangleleft

► **Theorem 4.** *Given an integer k and a symmetric $n \times n$ $0, 1$ -matrix M , there is an $n^{O(1)}$ -time algorithm that outputs a symmetric partition sequence witnessing that $\text{stw}(M) = O(k^3)$ or correctly reports that $\text{stw}(M) > k$.*

Proof. Given any $n \times n$ $0, 1$ -matrix M , division $(\mathcal{R} = (R_1, \dots, R_p), \mathcal{C} = (C_1, \dots, C_p))$ of M , part $R \in \mathcal{R}$, and integer q , one can decide in polynomial time if the row part R_i is q -wide in $(\mathcal{R}, \mathcal{C})$. Indeed, it suffices to check for every $i-q+1 \leq j \leq i$, if $R_i \setminus \bigcup_{j \leq h \leq j+q-1} C_h$ contains at least q different rows, which can be done in $n^{O(1)}$. Thus, in time $n^{O(1)}$, one can check if

the division $(\mathcal{R}, \mathcal{C})$ is q -diagonal, and for every $i \in [p - 1]$ if there is a q -diagonal division of the form $((R_1, \dots, R_i \uplus R_{i+1}, \dots, R_p), (C_1, \dots, C_i \uplus C_{i+1}, \dots, C_p))$.

Let M an $n \times n$ symmetric 0, 1-matrix, and k an integer. We start from the finest division $(\mathcal{R}_n, \mathcal{C}_n)$. If at some point we have a division $(\mathcal{R}_s = (R_1, \dots, R_s), \mathcal{C}_s = (C_1, \dots, C_s))$ such that none of the divisions $(R_1, \dots, R_i \uplus R_{i+1}, \dots, R_s), (C_1, \dots, C_i \uplus C_{i+1}, \dots, C_s)$ are $2(9k + 1)$ -diagonal, then by Theorem 19, the division $((R_1 \uplus R_2, \dots, R_{s-1} \uplus R_s), (C_1 \uplus C_2, \dots, C_{s-1} \uplus C_s))$ is $9k$ -wide. Thus by Theorem 19, $\text{stw}(M) > k$.

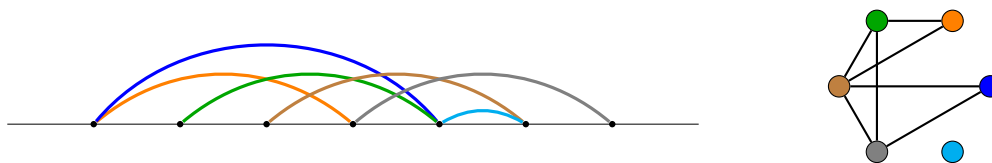
Otherwise, in time $n^{O(1)}$, we get a sequence of $2(9k + 1)$ -diagonal divisions $(\mathcal{R}_n, \mathcal{C}_n), \dots, (R_1, \mathcal{C}_1)$. At this point, we can find a sequence witnessing that $\text{stw}(M) = O(k^3)$, by the proof of Theorem 21. We build the symmetric partitions $(\mathcal{R}'_n, \mathcal{C}'_n), \dots, (\mathcal{R}'_1, \mathcal{C}'_1)$ of M , where for each $P \in \mathcal{R}'_s$, there is $R_i \in \mathcal{R}_s$ with $P \subseteq R_i$, and P is a maximal subset of equal rows of $R_i \setminus \bigcup_{i-k+1 \leq h \leq i+k-1} C_h$. The partition \mathcal{R}'_s (and its symmetric \mathcal{C}'_s) can be found in polynomial time: For each $R_i \in \mathcal{R}_s$, one can sort the rows of $R_i \setminus \bigcup_{i-k+1 \leq h \leq i+k-1} C_h$ by lexicographic order, and obtain the desired equivalence classes of equal rows.

Theorem 21 ensures that any sequence going from $(\mathcal{R}'_s, \mathcal{C}'_s)$ to $(\mathcal{R}'_{s-1}, \mathcal{C}'_{s-1})$ maintains a stretch value of $O(k^3)$. ◀

5 Overlap graph

Consider an ordered graph (G, \prec) , and think of \prec as a *left-to-right* order (with the smallest vertex being the leftmost one). For any edge $e \in E(G)$, we denote by $L(e)$ (resp. $R(e)$) the left (resp. right) endpoint of e . Given two edges $e, f \in E(G)$, we say that e is *left of* f if $L(e) \preceq L(f)$, and e is *strictly left of* f if $L(e) \prec L(f)$. By extension, we say that $X \subset E(G)$ is *left of* (resp. *strictly left of*) $Y \subset E(G)$ if for every $e \in X$ and $f \in Y$, $L(e) \preceq L(f)$ (resp. $L(e) \prec L(f)$). If u, v are vertices of (G, \prec) , we denote by $[u, v]$ the set of vertices that are, in \prec , at least u and at most v . We also denote by $[\leftarrow, u]$ (resp. $[u, \rightarrow]$) the set of vertices that are at most u (resp. at least u).

We say that two edges e, f are *crossing* if $L(e) \prec L(f) \prec R(e) \prec R(f)$ (or symmetrically) and we denote $e \times f$ this relation. Observe that two edges sharing an endpoint are not crossing. The relation \times is symmetric and anti-reflexive, hence defines an undirected graph on $E(G)$. We denote by $\text{Ov}(G, \prec)$ the graph $(E(G), \times)$. $\text{Ov}(G, \prec)$ is called the *overlap graph* of (G, \prec) ; see Figure 7.



■ **Figure 7** An ordered graph (left) and its overlap graph (right).

We relate the structure of $\text{Ov}_{\prec}(G)$ and the stretch-width of G among bounded-degree graphs, by proving the following theorem:

► **Theorem 23.** *A class \mathcal{C} of ordered graphs of bounded degree has bounded stretch-width if and only if $\{\text{Ov}(G, \prec) \mid G \in \mathcal{C}\}$ does not admit $K_{t,t}$ subgraph, for some integer t .*

The next two lemmas prove the forward implication, by considering a special point in the partition sequence. The last lemma of this section proves the backward implication, using the matrix characterization of Section 4. We say that a $K_{t,t}$ subgraph of $\text{Ov}(G, \prec)$ is *clean* if the sides of the $K_{t,t}$ are $X, Y \subset E(G)$ such that X is strictly left of Y .

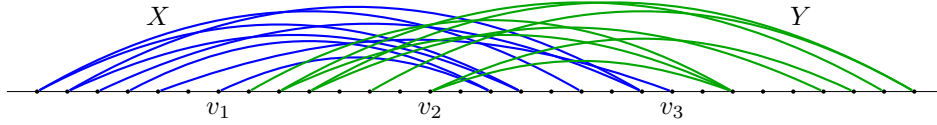
► **Lemma 24.** *For every ordered graph (G, \prec) , if $\text{Ov}(G, \prec)$ contains a $K_{t,t}$ as a subgraph, then $\text{Ov}(G, \prec)$ contains a clean $K_{\lfloor t/2 \rfloor, \lfloor t/2 \rfloor}$ subgraph.*

Proof. Assuming that $\text{Ov}(G, \prec)$ has a $K_{t,t}$ subgraph, there is two disjoint sets $X, Y \subset E(G)$ each of size t such that for every $x \in X$ and $y \in Y$, $x \times y$. Let $L(x_1) \prec L(x_2) \prec \dots \prec L(x_t)$ be the elements of X , and $L(y_1) \prec L(y_2) \prec \dots \prec L(y_t)$, the elements of Y . As $x_{\lfloor t/2 \rfloor}$ and $y_{\lfloor t/2 \rfloor}$ are crossing, either $L(x_{\lfloor t/2 \rfloor}) \prec L(y_{\lfloor t/2 \rfloor})$ or $L(y_{\lfloor t/2 \rfloor}) \prec L(x_{\lfloor t/2 \rfloor})$. The sides of the clean $K_{\lfloor t/2 \rfloor, \lfloor t/2 \rfloor}$ are $\{x_1, \dots, x_{\lfloor t/2 \rfloor}\}$ and $\{y_{\lfloor t/2 \rfloor}, \dots, y_t\}$ in the former case, and $\{y_1, \dots, y_{\lfloor t/2 \rfloor}\}$ and $\{x_{\lfloor t/2 \rfloor}, \dots, x_t\}$ in the latter. ◀

► **Lemma 25.** *For any ordered graph (G, \prec) , if $\Delta(G) \leq d$ and $\text{stw}(G, \prec) \leq t$, then $\text{Ov}(G, \prec)$ does not contain $K_{N,N}$ with $N = 4td^2$ as a subgraph.*

Proof. We will prove the contrapositive. Let (G, \prec) be an ordered graph of maximum degree at most d . We suppose that $\text{Ov}(G, \prec)$ contains a $K_{N,N}$ as a subgraph, with $N = 4td^2$. By Lemma 24, there are two sets $X, Y \subset E(G)$ forming a clean $K_{N/2, N/2}$ of $\text{Ov}(G, \prec)$.

Let v_1 (resp. v_2) be the rightmost vertex among left endpoints of edges in X (resp. Y), and let v_3 be the rightmost vertex among (right) endpoints of edges in X ; see Figure 8. Observe that for any edge $e \in X$, $L(e) \prec v_1$, and $v_2 \prec R(e) \prec v_3$. The relation $v_2 \prec R(e)$ holds because every edge of X crosses every edge of Y . In addition, for any edge $f \in Y$, $v_1 \prec L(f) \prec v_2$, and $v_3 \prec R(f)$.



■ **Figure 8** The edge subsets X (blue) and Y (green) forming a clean biclique $K_{N/2, N/2}$ of $\text{Ov}(G, \prec)$ (here, with $N = 16$), and the vertices $v_1, v_2, v_3 \in V(G)$. The vertices non-incident to a blue or green edge are in $V(G) \setminus V(H)$.

We consider H , the subgraph of (G, \prec) induced by the endpoints of edges in $X \cup Y$, and let h be $|V(H)|$. We will show that the stretch-width of H (hence that of G) is at least t . In the remaining of the proof, the intervals of vertices and the lengths of edges are all with respect to H . The intervals $[\leftarrow, v_1[$, $[v_1, v_2[$, $[v_2, v_3[$ and $[v_3, \rightarrow]$ are all of size at least $N/(2d)$. Indeed, observe that every vertex of H has at most d incident edges. Thus, the $N/2$ left (resp. right) endpoints of edges in X (resp. Y) make for at least $\frac{1}{d}N/2$ distinct vertices.

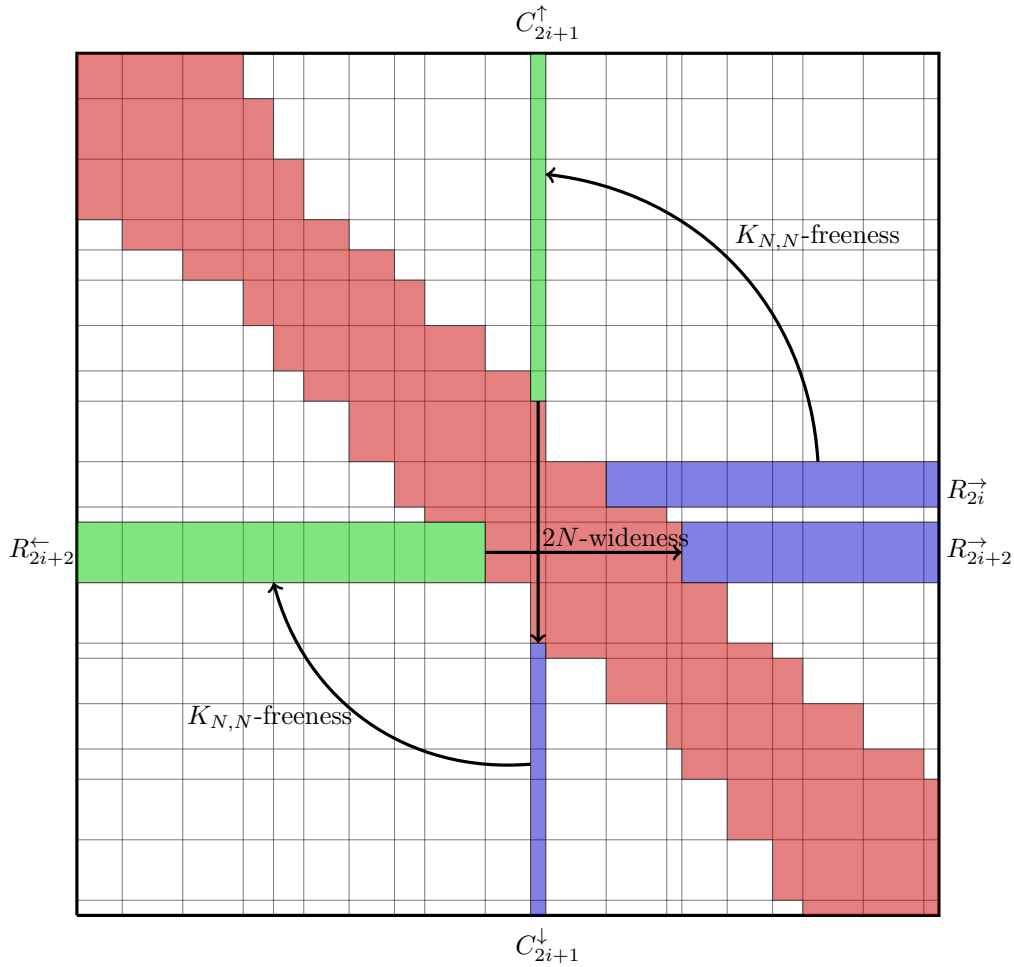
Therefore, every edge in $X \cup Y$ has length at least $N/(2d)$. Let $\mathcal{P}_h, \dots, \mathcal{P}_1$ be any partition sequence of H , and let $i \in [h]$ be the maximum integer such that $d + 1$ vertices are contained in a single part $P \in \mathcal{P}_i$. As $\Delta(H) \leq \Delta(G) \leq d$, every part $Q \in \mathcal{P}_i$ adjacent to P is such that P and Q are inhomogeneous. Let $e = uv$ be an edge of $X \cup Y$ with at least one endpoint in P , say u , and let P' (possibly equal to P) be the part of \mathcal{P}_i containing the other endpoint, v . The span of $N_{\mathcal{R}(\mathcal{P}_i)}(P)$ contains the interval $I = [u, v]$ (or $I = [v, u]$ if $v \prec u$). As I has length at least $N/(2d)$ and every part of \mathcal{P}_i has size at most $2d$, the number of parts of \mathcal{P}_i conflicting with I (hence, in particular with $N_{\mathcal{R}(\mathcal{P}_i)}(P)$) is at least $N/(4d^2)$. Thus, $\text{stw}(G, \prec) \geq \text{stw}(H, \prec) \geq \frac{N}{4d^2} = t$. ◀

► **Lemma 7.** *For every ordered graph (G, \prec) and positive integer N , if $\text{Ov}(G, \prec)$ does not contain $K_{N,N}$ as a subgraph, then $\text{stw}(G, \prec) \leq 32(2N + 1)^3$.*

Proof. Let (G, \prec) be an ordered graph such that $\text{Ov}(G, \prec)$ does not contain $K_{N,N}$ as a subgraph, and let M be the adjacency matrix of (G, \prec) . We prove that $\text{stw}(M) \leq 32(2N + 1)^3$.

Suppose, for the sake of contradiction, that $\text{stw}(M) > 32(2N + 1)^3$. By Theorem 22, there is a $2N$ -wide division $(\mathcal{R} = \{R_1, \dots, R_k\}, \mathcal{C} = \{C_1, \dots, C_k\})$ of M . In particular, for any row R_i , $R_i \setminus C_{i-N+1}, \dots, C_{i+N-1}$ contains more than $2N$ different rows. Let D be the union of the zones $R_i \cap C_j$ such that $|i - j| < N$, that is, the $2N - 1$ “longest” diagonals of zones of the division $(\mathcal{R}, \mathcal{C})$. As, for every $i \in [k]$, the number of distinct rows in $R_i \setminus D$ (resp. distinct columns in $C_i \setminus D$) is at least $2N$, $R_i \setminus D$ (resp. $C_i \setminus D$) contains at least $2N$ 1-entries.

To simplify the coming notations, let denote by $\|M'\|$ the number of 1-entries of any submatrix M' of M . For example, $\|R_i \setminus D\| \geq 2N$. Observe that R_i (resp. C_j) is split by D in at most two sets R_i^\leftarrow and R_i^\rightarrow (resp. C_j^\uparrow and C_j^\downarrow), namely, $R_i^\leftarrow = \bigcup_{j \leq i-N} R_i \cap C_j$ and $R_i^\rightarrow = \bigcup_{j \geq i+N} R_i \cap C_j$; see Figure 9.



■ **Figure 9** Visual depiction of the proof of Lemma 7. The red zones represent D , the blue zones contain more than N 1-entries, and the green zones have fewer than N 1-entries. The invariant leading to a contradiction (that R_{2i}^\rightarrow has more than N 1-entries) propagates by invoking twice $K_{N,N}$ -freeness followed by $2N$ -wideness.

Observe that for every i, j such that $i + 1 \leq j < i + N$, each 1-entry of R_i^{\rightarrow} (resp. C_i^{\downarrow}) and 1-entry of C_j^{\uparrow} (resp. R_j^{\rightarrow}) correspond to crossing edges in (G, \prec) . As $\text{Ov}(G, \prec)$ does not contain any $K_{N,N}$ subgraph we have, for every i, j such that $i + 1 \leq j < i + N$:

1. $\min(\|R_i^{\rightarrow}\|, \|C_j^{\uparrow}\|) < N$, and
2. $\min(\|C_i^{\downarrow}\|, \|R_j^{\leftarrow}\|) < N$.

Indeed, if the first item does not hold, N 1-entries in R_i^{\rightarrow} and N 1-entries in C_j^{\uparrow} form the two sides of a $K_{N,N}$.

We finally prove by induction on i that, while $2i \leq k$, the property $\|R_{2i}^{\rightarrow}\| > N$, henceforth called (\mathcal{Q}_i) , holds. Note that R_0^{\leftarrow} is empty. Thus $\|R_0^{\rightarrow}\| \geq 2N > N$, hence (\mathcal{Q}_0) holds. Now assume that (\mathcal{Q}_i) holds. By the first item, we have $\|C_{2i+1}^{\uparrow}\| < N$. Thus $\|C_{2i+1}^{\downarrow}\| > N$, since

$$C_{2i+1} \setminus D = C_{2i+1}^{\downarrow} \cup C_{2i+1}^{\uparrow} \text{ and } \|C_{2i+1} \setminus D\| \geq 2N.$$

Symmetrically, by the second item, $\|R_{2i+2}^{\leftarrow}\| < N$, and hence $\|R_{2i+2}^{\rightarrow}\| > N$. Thus (\mathcal{Q}_{i+1}) holds. As R_{k-N+1}^{\rightarrow} is empty, (\mathcal{Q}_i) can no longer be true when $2i \geq k - N + 1$, a contradiction. Therefore $\text{stw}(M) \leq 32(2N)^3$. \blacktriangleleft

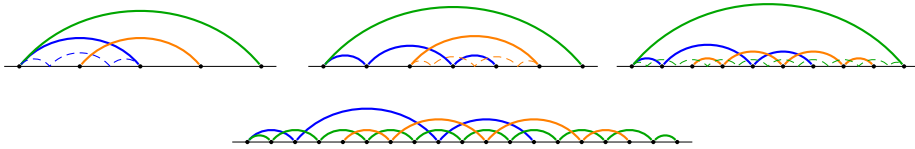
6 Subdivisions

When subdividing the edges of an ordered graph, there is a simple way of updating its vertex ordering without creating larger bicliques in its overlap graph.

► **Lemma 26.** *Let (G, \prec) be an ordered graph, and H be obtained by subdividing an edge of G . There is an order \prec' such that, for every integer t , if $\text{Ov}(G, \prec)$ has no $K_{t,t}$ subgraph, then $\text{Ov}(H, \prec')$ has no $K_{t,t}$ subgraph.*

Proof. Let $e = uv$ be the edge of G subdivided to form H , and let $w \in V(H)$ be the new vertex resulting from this subdivision. The total order \prec' is obtained from \prec , by adding w next to u , say, just to its right. This way $\text{Ov}(H, \prec')$ is simply $\text{Ov}(G, \prec)$ plus an isolated vertex. Indeed the edge $uw \in E(H)$ is an isolated vertex in $\text{Ov}(H, \prec')$, since u and w are consecutive along \prec' , whereas $wv \in E(H)$ crosses the same edges as uv was crossing. \blacktriangleleft

We now define a long subdivision process that is actually “erasing” large bicliques in the overlap graph of a bounded-degree graph. Let uv be an edge of an ordered graph (G, \prec) , with h vertices between u and v , say, $u \prec u_1 \prec u_2 \prec \dots \prec u_h \prec v$. We describe an $h + 1$ -subdivision of uv in (G, \prec) that we call *flattening of uv* . We delete uv , and create $h + 1$ new vertices w_1, \dots, w_{h+1} such that $u \prec w_1 \prec u_1 \prec w_2 \prec u_2 \prec \dots \prec w_h \prec u_h \prec w_{h+1} \prec v$. We then create the edges $uw_1, w_i w_{i+1}$ for every $i \in [h]$, and $w_{h+1}v$. We may say that these *edges stem* from uv . An iterated subdivision of (G, \prec) chooses a total order on the edges of G , and iteratively flattens the edges of G in this order (note that the created edges are *not* flattened themselves); see Figure 10.



■ **Figure 10** An iterated subdivision. Created edges have the color of the edge they stem from.

► **Lemma 27.** *Any iterated subdivision (G', \prec') of an ordered graph (G, \prec) of maximum degree d , is such that $\text{Ov}(G', \prec')$ has no $K_{2d+2, 2d+2}$ subgraph.*

Proof. Assume for the sake of contradiction that $\text{Ov}(G', \prec')$ has a $K_{2d+2, 2d+2}$ subgraph. Then by Lemma 24, $\text{Ov}(G', \prec')$ has a clean $K_{d+1, d+1}$ subgraph. Let X, Y be the two sides of this clean biclique, where X is left of Y . As every vertex of G' (like G) is incident to at most d edges, there is $\{x_1, x_2\} \subseteq X$ and $\{y_1, y_2\} \subseteq Y$ such that $L(x_1) \preceq L(x_2) \prec L(y_1) \prec L(y_2) \prec R(x_i) \prec R(x_{3-i}) \prec R(y_j) \preceq R(y_{3-j})$ with $i, j \in [2]$.

As x_1 and x_2 cross y_1 and y_2 , there is no $i, j \in [2]$ such that x_i and y_j stem from the same edge of G . We can thus assume without loss of generality that the last edge among x_1, x_2, y_1, y_2 to be created is in X (since the argument is symmetric if this happens in Y), i.e., x_i for some $i \in [2]$. When x_i is created, the vertices $L(y_1)$ and $L(y_2)$ already exist and form a non-trivial interval since $L(y_1) \prec L(y_2)$. This contradicts the construction of the iterated subdivision, since x_i jumps over $[L(y_1), L(y_2)]$, when it should have at least created an intermediate vertex in $[L(y_1), L(y_2)]$. ◀

We are now equipped to show the main result of this section, that exponentially-long subdivisions of bounded-degree graphs have bounded stretch-width.

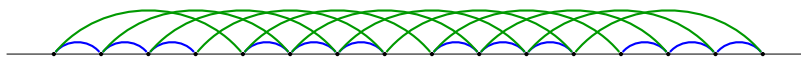
► **Theorem 6.** *Every $(\geq n2^m)$ -subdivision of every n -vertex m -edge graph G of maximum degree d has stretch-width at most $32(4d + 5)^3$.*

Proof. Let G be any graph of \mathcal{C} with n vertices and m edges, and let G'' be any $(\geq n2^m)$ -subdivision of G . Choose an arbitrary order \prec of $V(G)$. Let (G', \prec') be the iterated subdivision of (G, \prec) , performed with an arbitrary order on the edges G . By Lemma 27, $\text{Ov}(G', \prec')$ has no $K_{2d+2, 2d+2}$ subgraph. Every edge of G is subdivided at most $n2^m$ times by the process of iterated subdivision. By Lemma 26, the edges of G' can be further subdivided to obtain G'' such that $\text{Ov}(G'', \prec'')$ has no $K_{2d+2, 2d+2}$ subgraph, for some vertex ordering \prec'' . Therefore, by Lemma 7, $\text{stw}(G'', \prec'') \leq 32(4d + 5)^3$, and in particular, $\text{stw}(G'') \leq 32(4d + 5)^3$. ◀

► **Corollary 28.** *There are graph classes with bounded stretch-width and maximum degree, and yet unbounded treewidth.*

Proof. Consider the family $\Gamma_1, \Gamma_2, \dots$, where Γ_k is the $k^2 2^{2k(k-1)}$ -subdivision of the $k \times k$ -grid, for every positive integer k . The graphs from this family have degree at most 4, and stretch-width at most 296352, but unbounded treewidth since $\text{tw}(\Gamma_k) = k$. ◀

The above argument gives an example of n -vertex graphs with bounded degree and stretch-width, and treewidth $\Omega(\sqrt{\log n})$. We can do better by picking the vertex ordering \prec , and the order on the edges (for the iterated subdivision) more carefully. We simply order



■ **Figure 11** The 4×4 grid ordered row by row, with the horizontal edges in blue, and vertical edges in green.

the grid row by row, and from left to right within each row; see Figure 11. We perform the iterated subdivision of this ordered grid, with the following edge ordering. First we flatten every horizontal edge (in blue), in any order. When this is done, the total number of vertices has less than doubled. Then we flatten every vertical edge (in green) from left to right. It can

be observed that, starting from the $k \times k$ grid, we now obtain an iterated subdivision with less than 2^{ck} vertices, for some constant c . Thus, there are n -vertex graphs with bounded degree and stretch-width, and treewidth $\Omega(\log n)$.

7 Classes with bounded $\Delta + \text{stw}$ have logarithmic treewidth

For any edge e of an ordered graph (G, \prec) , we denote $e^i =]L(e), R(e)[$, the *interior* of e , and $e^o := [\leftarrow, L(e)[\cup]R(e), \rightarrow]$, the *exterior* of e ; note that $L(e)$ and $R(e)$ are neither part of e^i nor of e^o . The *length* of e according to \prec is $\ell(e, \prec) = R(e) - L(e)$. When F is a set of edges we define $\ell(F, \prec)$ to be the maximum length of an edge of F .

We say that a set C of vertices is a *c-balanced separator* of G when there is a c -balanced separation (A, B) of G such that $C = A \cap B$. In an ordered graph (G, \prec) , a set of vertices C is called *left/right c-balanced separator* if there is a c -balanced separation (A, B) , with $C = A \cap B$, A contains the leftmost $c \cdot n$ vertices, and B contains the rightmost $c \cdot n$ vertices. Given a subset U of vertices of (G, \prec) , we define the ordered induced subgraph $(G, \prec)[U]$ as the subgraph obtained from G by removing the vertices not in U , and restraining \prec to U . When \prec is clear from the context, we use the notation $G[U]$.

We start with an observation on paths going from the interior to the exterior of an edge.

► **Observation 29.** *For every ordered graph G and for every edge $e \in E(G)$, any path P starting in e^i and ending in e^o contains an endpoint of e , or an edge crossing e .*

Proof. Let $e = uv$ be an edge of G , and $P = v_1, \dots, v_k$, a path with $v_1 \in e^i$ and $v_k \in e^o$. Let j be the smallest index such that $v_j \notin e^i$. As $V(G) = \{u, v\} \cup e^i \cup e^o$, we either have $v_j \in \{u, v\}$, or $v_j \in e^o$ and $v_{j-1} \in e^i$, hence $v_{j-1}v_j$ is an edge of P crossing e . ◀

To simplify the notations, if the vertices of (G, \prec) are $v_1 \prec \dots \prec v_n$, we will write $G\langle i, j \rangle$ instead of $G[[v_i, v_j]]$ and $\langle i, j \rangle$ instead of $[v_i, v_j]$.

We say that a set S of edges of (G, \prec) is a *rainbow* if for every pair e, f of S , $e^i \subset f^i$ or $f^i \subset e^i$. Notice that a rainbow induces an independent set in $\text{Ov}(G, \prec)$. When S contains t edges we say that S is a *t-rainbow*, or a rainbow of order t . The following is an application of Dilworth's theorem on permutation graphs (or the Erdős-Szekeres theorem); see for instance [28, Lemma 2.1].

► **Lemma 30** ([28]). *Let (G, \prec) be an ordered graph, such that $\text{Ov}(G, \prec)$ does not contain a clique on t vertices. Then, for every vertex v of $V(G)$, for every set F of edges from $[\leftarrow, v[$ to $]v, \rightarrow]$ we have $|F| \leq kt$ where k is the maximum order of a rainbow in F .*

For any rainbow S , we set $S^i := \bigcup_{e \in S} e^i$, that is, S^i is the interior of the edge of S with maximal interior. If S is the empty set, by convention, S^i is also empty. A *rainbow over v* is a rainbow S contained in the set of edges from $[\leftarrow, v[$ to $]v, \rightarrow]$. A *maximum rainbow over v* is a rainbow of maximum cardinality among the rainbows over v .

In the next lemma, we find, by induction on the minimum length of a maximum rainbow over some vertex, a small (but not always balanced) left/right separator.

► **Lemma 31.** *Let (G, \prec) be an ordered graph such that $\text{Ov}(G, \prec)$ does not contain a $K_{t,t}$ subgraph. Then, if S is a maximum rainbow over $v \in V(G)$, there is a vertex $x \in S^i \cup \{v\}$ and a set U that separates $[\leftarrow, x[$ from $]x, \rightarrow]$ with $|U| = O(t^2 \log \ell(S, \prec))$.*

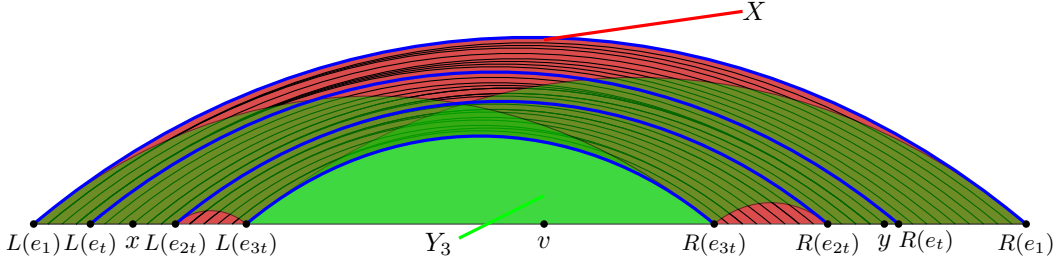
Proof. First, note that we write $S^i \cup \{v\}$ instead of simply S^i in the event that $S = \emptyset$ (in which case v is not contained in S^i). We define $h(l) = 2^{\lceil \log(l+1) \rceil + 1}$ and $g(t) = 6t^2 + 3t$. Thus $g(t) = O(t^2)$ and $h(l) = O(l)$.

We prove by induction on l that for every ordered graph (G, \prec) for which $\text{Ov}(G, \prec)$ has no $K_{t,t}$ subgraph, and for every vertex $v \in V(G)$, if S is a maximum rainbow over v such that $\ell(S, \prec) \leq l$, then there exists a vertex $x \in S^i \cup \{v\}$ and a set U that separates $[\leftarrow, x[$ from $]x, \rightarrow]$ with $|U| \leq g(t) \log h(l)$.

If $l = 0$, there is no edge over v . Hence, S is empty and $\{v\}$ separates $[\leftarrow, v[$ from $]v, \rightarrow]$. Assume that the property holds for any $k < l$. We show the property also holds for l . Let v be a vertex of G , and S be a maximum rainbow over v such that $\ell(S, \prec) \leq l$. Let F be the set of edges from $[\leftarrow, v[$ to $]v, \rightarrow]$. We make the following case distinction: the rainbow S contains either at most $3t$ edges or more than $3t$ edges.

In the former case, Lemma 30 ensures that $|F| \leq 2t|S| \leq 6t^2$. Indeed, S is a rainbow of F of maximum cardinality, and $\text{Ov}(G, \prec)$ does not contain any $K_{t,t}$ subgraph, hence any K_{2t} . Thus, consider $L_F := \{L(f) \mid f \in F\} \cup \{v\}$. The set L_F separates $[\leftarrow, v[$ from $]v, \rightarrow]$ since any edge with exactly one endpoint in $[\leftarrow, v[$ has an endpoint in L_F . As $|L_F| \leq 6t^2 + 1 \leq g(t) \leq g(t) \log h(l)$, the property holds.

We now handle the case when $|S| > 3t$. Let $\{e_1, \dots, e_k\}$ be the edges in S with $e_{j+1}^i \subsetneq e_j^i$ for every $j \in [k-1]$ (we are sorting the edges from longest to smallest). Let X the set of edges from $[L(e_1), L(e_{3t})]$ to $[R(e_1), R(e_{3t})]$. Set X cannot contain a rainbow of size $3t+1$, otherwise S would not be maximum. As X does not contain a K_{2t} , Lemma 30 implies that $|X| \leq 6t^2$. Let Y_1 be the set of edges from e_1^o to e_t^i , Y_2 be the set of edges from e_t^o to e_{2t}^i , and Y_3 be the set of edges from e_{2t}^o to e_{3t}^i ; see Figure 12.



■ **Figure 12** The black (or blue) edges are $e_1, e_2, \dots, e_{3t} \in S$, and the edges e_1, e_t, e_{2t}, e_{3t} are in thick blue. The red arch represents the edge set X , while the green shape symbolizes the edge set Y_3 . We will now remove the left endpoints of these edges sets (as well as those of Y_1 and Y_2), define the vertices $x \in [L(e_t), L(e_{2t})]$ and $y \in [R(e_{2t}), R(e_t)]$, and apply the induction on the *shorter* maximum rainbow over x or over y .

We have $|Y_1| \leq t$, as each edge of Y_1 crosses e_1, \dots, e_t . Similarly, $|Y_2| \leq t$ and $|Y_3| \leq t$. Let $L_{X \cup Y} := \{L(f) \mid f \in X \cup Y_1 \cup Y_2 \cup Y_3\}$. Observe that $|L_{X \cup Y}| \leq 6t^2 + 3t = g(t)$.

Let $G' = G - L_{X \cup Y}$. In G' , for any $w \in]L(e_t), L(e_{2t})]$, the edges over w have both endpoints in $[L(e_1), L(e_{3t})]$. Indeed let f be an edge over w and assume that f has an endpoint outside of $[L(e_1), L(e_{3t})]$. By assumption, this endpoint is in $e_1^o \cup e_{3t}^i \cup [R(e_{3t}), R(e_1)]$ (the set of remaining vertices). As f is over w , the left endpoint of f is either in $[\leftarrow, e_1]$ or in $[e_1, w[$, and its right endpoint is in $[w, \rightarrow]$. Suppose the left endpoint of f is in $[\leftarrow, e_1]$. Then f is in Y_1 when its right endpoint is in e_t^i , and in X , otherwise. Assume now that the left endpoint of f is in $[\leftarrow, e_1]$, then its right endpoint is in $[L(e_{3t}), \rightarrow]$ (since by assumption, f is not in $[L(e_1), L(e_{3t})]$), and f is an edge of Y_3 when its right endpoint is in e_{3t}^i , and an edge of X when its right endpoint is in $[R(e_{3t}), \rightarrow]$.

In any case, we reach the contradiction that f is in $Y_1 \cup Y_3 \cup X$. Note that if there is no vertex of G' in $]L(e_t), L(e_{2t})]$, $L_{X \cup Y}$ separates G' : As we removed X, Y_1, Y_2 and Y_3 , there is

no edges between $[\leftarrow, L(e_t)]$ and $]L(e_{2t}), \rightarrow]$. The same holds symmetrically when there is no vertex of G' in $[R(e_{2t}), R(e_t)[$.

Thus we consider two vertices x and y respectively in (the non-empty sets) $]L(e_t), L(e_{2t})[\cap V(G')$ and $[R(e_{2t}), R(e_t)[\cap V(G')$. In G' , edges over x are contained in $]L(e_1), L(e_{3t})[$, and edges over y , in $[R(e_{3t}), R(e_1)[$. Hence no edge is both over x and y . Consider S_x and S_y two maximum rainbows over x and y , respectively. We have $\ell(S_x, \prec) + \ell(S_y, \prec) \leq \ell(S, \prec)$. Assume, without loss of generality, that $\ell(S_x, \prec) \leq \ell(S_y, \prec)$. We have $\ell(S_x, \prec) \leq \lfloor l/2 \rfloor$. By the induction hypothesis, there is a vertex x' in $S_x^i \cup \{x\}$ and a set U that separates $[\leftarrow, x'[$ from $]x', \rightarrow]$ in G' with $|U| \leq g(t) \log h(\lfloor l/2 \rfloor)$.

Thus $U \cup L_{X \cup Y}$ separates $[\leftarrow, x'[$ from $]x', \rightarrow]$ in G , and $|U \cup L_{X \cup Y}| \leq g(t) \log h(\lfloor l/2 \rfloor) + g(t) = g(t)(\lceil \log(\lfloor l/2 \rfloor) + 1 \rceil) + 2 \leq g(t)(\lceil \log(l+1) \rceil + 1) = g(t) \log h(l)$. \blacktriangleleft

Leveraging Lemma 31, we now show how to obtain a relatively small *balanced* separator.

► **Theorem 32.** *For any ordered graph (G, \prec) , if $\text{Ov}(G, \prec)$ does not contain any $K_{t,t}$ subgraph, then G contains a $1/12$ -balanced separator of order at most $\gamma t^2 \log n$, for some constant γ .*

Proof. We first examine the case when there is a vertex $v \in V(G)$ and a maximum rainbow S over x such that at least $3t$ edges of S have length in $[n/12, 11n/12]$. We denote by f_1, \dots, f_{3t} the $3t$ longest edges of S of length at most $11n/12$, still ordered from longest (f_1) to shortest (f_{3t}). As in the proof of Lemma 31, we consider X the edges from $]L(f_1), L(f_{3t})[$ to $[R(f_{3t}), R(f_1)[$, and Lemma 30 ensures that $|X| \leq 6t^2$. Let Y_1 be the set of edges from f_1^o to f_t^i , Y_2 be the set of edges from f_t^o to f_{2t}^i , and Y_3 be the set of edges from f_{2t}^o to f_{3t}^i . We again have that Y_1, Y_2 and Y_3 are of size at most t , otherwise $\text{Ov}(G, \prec)$ would contain a $K_{t,t}$ subgraph. Let $L_{X \cup Y} = \{L(z) \mid z \in X \cup Y_1 \cup Y_2 \cup Y_3\}$.

Let $G' = G - L_{X \cup Y}$. As observed in Lemma 31, the separator is only simpler when $]L(f_t), L(f_{2t})[\cap V(G')$ or $[R(f_{2t}), R(f_t)[\cap V(G')$ are empty. Hence we only deal with the case when both sets are non-empty (in the other cases, a subset of our eventual separator works). Thus, let $x \in]L(f_t), L(f_{2t})[\cap V(G')$ and $y \in [R(f_{2t}), R(f_t)[\cap V(G')$.

By construction of G' , any edge over x is contained in $]L(f_1), L(f_{3t})[$, or is going from $[\leftarrow, L(f_1)[$ to $]R(f_1), \rightarrow]$. Thus, by applying Lemma 31 on $G_x = G'[\leftarrow, R(f_1)]$ and x , we find a set U_x separating $[\leftarrow, u_x[$ from $]u_x, \rightarrow]$, where u_x is a vertex in $]L(f_1), L(f_{3t})[$. Similarly, considering $G_y = G'[L(f_1), \rightarrow]$ and y , we find a set U_y separating $[\leftarrow, u_y[$ from $]u_y, \rightarrow]$, with $u_y \in [R(f_{3t}), R(f_1)[$. The set $U_x \cup U_y$ separates $]u_x, u_y[$ from $[\leftarrow, u_x[\cup]u_y, \rightarrow]$ in G' . As the length between u_x and u_y is between $n/12$ and $11n/12$, $L_{X \cup Y} \cup U_x \cup U_y$ is a $1/12$ -balanced separator of G of size $6t^2 + 3t + 2 \cdot O(t^2 \log n) = O(t^2 \log n)$.

Now, we deal with the case when for any vertex v of G , the number of edges of length in $[n/12, 11n/12]$ in a (maximum) rainbow over v is less than $3t$. Let $v_1 \prec v_2 \prec \dots \prec v_n$ be the vertices of G , and say, $v = v_i$. In particular, the set M_v of edges over v going from $\langle n/12, i - n/12 \rangle$ to $\langle i + n/12, 11n/12 \rangle$ is of size at most $6t^2$ by Lemma 30. Set $x = v_{\lfloor n/3 \rfloor}$ and $y = v_{\lfloor 2n/3 \rfloor}$. Let $A = \{L(e) \mid e \in M_x \cup M_y\}$ and let $H = G - A$. Consider $H_x = H \langle 1, 11n/12 \rangle$, and $H_y = H \langle n/12, n \rangle$. Then the length of a maximum rainbow over x (resp. y) in H_x (resp. H_y) is at most $n/12$. Indeed, any edge over x (resp. y) of length more than $11n/12$ is not contained in H_x (resp. H_y), and any edge over x (resp. y) of length in $[n/12, 11n/12]$ has been deleted when removing A .

Hence we can apply Lemma 31 on H_x and x , and on H_y over y . This yields two sets U_x, U_y of size $O(t^2 \log n)$ and two vertices w_x, w_y such that the indices of w_x and x (resp. w_y and y) are at distance at most $n/12$, and such that U_x separates $[\leftarrow, w_x[$ from $]w_x, \rightarrow]$ in H_x , while U_y separates $[\leftarrow, w_y[$ from $]w_y, \rightarrow]$ in H_y . The set $A \cup U_x \cup U_y$ then separates $]w_x, w_y[$ from $[\leftarrow, w_x[\cup]w_y, \rightarrow]$ in G , and $|A \cup U_x \cup U_y| = O(t^2 \log n)$. Notice that $]w_x, w_y[$

contains at least $\lfloor n/3 \rfloor - 2n/12 \geq \lfloor n/6 \rfloor$ vertices because there is $\lfloor n/3 \rfloor$ vertices between x and y , and w_x is separated from x by at most $n/12$ vertices (and symmetrically for y). In addition, there is at most $n/3 + 2n/12 = n/2$ vertices in $]w_x, w_y[$. Thus $A \cup U_x \cup U_y$ is a $1/6$ -balanced separator. \blacktriangleleft

Pipelining Lemma 25 and the previous theorem, we get small balanced separators for graphs of bounded degree and stretch-width.

► **Theorem 33.** *Let G be any graph on n vertices, such that $\Delta(G) \leq d$ and $stw(G) \leq t$. Then G contains a $1/12$ -balanced separator of size at most $\gamma(4td^2)^2 \log n$, for a constant γ .*

Proof. Let G a graph on n vertices, such that $stw(G) \leq t$ and $\Delta(G) \leq d$. Let \prec be an order such that (G, \prec) has stretch-width t . By Lemma 25, $\text{Ov}(G, \prec)$ does not admit any $K_{4td^2, 4td^2}$ as a subgraph. Hence Theorem 32 ensures the existence of a $1/12$ -balanced separator of (G, \prec) , hence of G , of size $\gamma(4td^2)^2 \log n$. \blacktriangleleft

By Lemma 11 (and the remark following it) and Theorem 33, we obtain the bound on the treewidth of a graph of bounded degree and bounded stretch-width.

► **Theorem 5.** *There is a c such that for every graph G , $tw(G) \leq c\Delta(G)^4 stw(G)^2 \log |V(G)|$.*

We draw two algorithmic consequences from Theorem 5. First, Existential Counting Modal Logic (see the introduction for a description) can be model-checked in polynomial time in classes where both the maximum degree and the stretch-width is bounded.

► **Corollary 9.** *Problems definable in ECML (resp. ECML+C) can be solved in polynomial time (resp. randomized polynomial time) in bounded-degree graphs of bounded stretch-width.*

Proof. Let \mathcal{C} be a class with bounded $\Delta + stw$. By Theorem 5, there is a constant c such that every graph $G \in \mathcal{C}$ has treewidth at most $c \log n$. There is a single-exponential 2-approximation of treewidth [21], which returns here a tree-decomposition of width at most $2c \log n$ in time $2^{O(c \log n)} = n^{O(1)}$. Any problem definable in ECML (resp. ECML+C) can be solved in time $2^{O(w)}$ (resp. randomized time $2^{O(w)}$, with a Monte Carlo algorithm) where w is the width of a tree-decomposition of the input graph [23]. This gives an algorithm in (randomized) time $2^{O(2c \log n)} = n^{O(1)}$ for any problem definable in ECML(+C) on \mathcal{C} . \blacktriangleleft

The second consequence is a subexponential-time algorithm for MIS (and any problem with a similar branching rule on high-degree vertices) in graphs of bounded stretch-width.

► **Proposition 8.** *There is an algorithm that solves MAX INDEPENDENT SET in graphs of bounded stretch-width with running time $2^{\tilde{O}(n^{4/5})}$.*

Proof. Let \mathcal{C} be a class of bounded stretch-width. By Theorem 5, for every n -vertex graph $G \in \mathcal{C}$, the treewidth of G is in $O(\Delta(G)^4 \log n)$. We use a standard branching rule that Turing-reduces the problem to MIS on subinstances of small maximum degree, hence small treewidth.

While there is in the current graph a vertex v of degree at least $n^{1/5}$, we branch on two options: either (first branch) we put v in (an initially empty set) I , and remove its closed neighborhood, or (second branch) we remove v from the current graph (without adding it to I). With the former choice, the number of vertices drops by at least $n^{1/5}$, and it drops by 1 in the latter. The former outcome can only happen at most $n^{4/5}$ times, as we started with n vertices. Hence the branching tree has at most $\binom{n}{n^{4/5}} = 2^{\tilde{O}(n^{4/5})}$ leaves.

The treewidth of the graph at every leaf is in $O(n^{4/5} \log n)$ since there are no more vertices of degree at least $n^{4/5}$. As explained in the proof of Corollary 9, we can solve such instances in time $2^{O(n^{4/5} \log n)}$, and append the corresponding I to the output. The overall running time is $2^{\tilde{O}(n^{4/5})} \cdot 2^{O(n^{4/5} \log n)} = 2^{\tilde{O}(n^{4/5})}$. ◀

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