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# Twin-width can be exponential in treewidth

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## Abstract

For any small positive real  $\varepsilon$  and integer  $t > \frac{1}{\varepsilon}$ , we build a graph with a vertex deletion set of size  $t$  to a tree, and twin-width greater than  $2^{(1-\varepsilon)t}$ . In particular, this shows that the twin-width is sometimes exponential in the treewidth, in the so-called oriented twin-width and grid number, and that adding an apex may multiply the twin-width by at least  $2 - \varepsilon$ . Except for the one in oriented twin-width, these lower bounds are essentially tight.

## 1 Introduction

Twin-width is a graph parameter introduced by Bonnet, Kim, Thomassé, and Watrigant [12]. It is defined by means of trigraphs. A *trigraph* is a graph with some edges colored black, and some colored red. A (vertex) *contraction* consists of merging two (non-necessarily adjacent) vertices, say,  $u, v$  into a vertex  $w$ , and keeping every edge  $wz$  black if and only if  $uz$  and  $vz$  were previously black edges. The other edges incident to  $w$  become red (if not already), and the rest of the trigraph remains the same. A *contraction sequence* of an  $n$ -vertex graph  $G$  is a sequence of trigraphs  $G = G_n, \dots, G_1 = K_1$  such that  $G_i$  is obtained from  $G_{i+1}$  by performing one contraction. A *d-sequence* is a contraction sequence in which every vertex of every trigraph has at most  $d$  red edges incident to it. The *twin-width* of  $G$ , denoted by  $\text{tw}(G)$ , is then the minimum integer  $d$  such that  $G$  admits a  $d$ -sequence. Figure 1 gives an example of a graph with a 2-sequence, i.e., of twin-width at most 2. Twin-width can be naturally

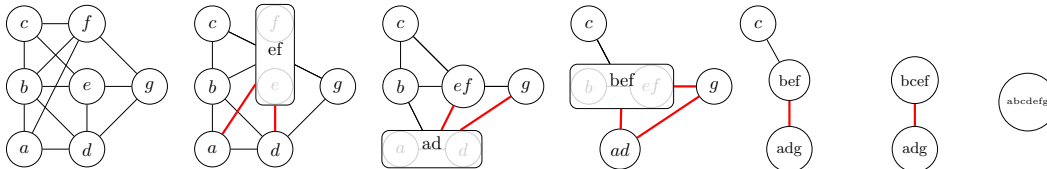


Figure 1 A 2-sequence witnessing that the initial graph has twin-width at most 2.

extended to matrices (unordered [12] or ordered [9]) over a finite alphabet, and hence to any binary structures. Classes of binary structures with bounded twin-width include graphs with bounded treewidth, bounded clique-width,  $K_t$ -minor free graphs, posets with antichains of bounded size, strict subclasses of permutation graphs, map graphs, bounded-degree string graphs [12], segment graphs with no  $K_{t,t}$  subgraph, visibility graphs of 1.5D terrains without large half-graphs, visibility graphs of simple polygons without large independent sets [6], as well as  $\Omega(\log n)$ -subdivisions of  $n$ -vertex graphs, classes with bounded queue number or bounded stack number, and some classes of cubic expanders [7].

Despite their apparent generality, classes of bounded twin-width are small [7],  $\chi$ -bounded [8], even quasi-polynomially  $\chi$ -bounded [17], preserved (albeit with a higher upper bound) by first-order transductions [12], and by the usual graph products when one graph has bounded degree [16, 7], have VC density 1 [11, 19], admit, when  $O(1)$ -sequences are given, a fixed-parameter tractable first-order model checking [12], an (almost) single-exponential

parameterized algorithm for various problems that are W[1]-hard in general [8], as well as a parameterized fully-polynomial linear algorithm for counting triangles [15], an (almost) linear representation [18], a stronger regularity lemma [19], etc.

In all these applications, the upper bound on twin-width, although somewhat hidden in the previous paragraph, plays a role. There is then an incentive to obtain as low as possible upper bounds on particular classes of bounded twin-width. To give one concrete algorithmic example, an independent set of size  $k$  can be found in time  $O(k^2 d^{2k} n)$  in an  $n$ -vertex graph given with a  $d$ -sequence [8]. This is relatively practical for moderate values of  $k$ , with the guarantee that  $d$  is below 10, but not when  $d$  is merely upperbounded by  $10^{10}$ . Another motivating example: triangle-free graphs of twin-width at most  $d$  are  $d + 2$ -colorable [8], a stronger fact in the former case than in the latter.

In that line of work, Balabán and Hliněný show that posets of width  $k$  (i.e., with antichains of size at most  $k$ ) have twin-width at most  $9k$  [2]. Unit interval graphs have twin-width at most 2 [8], and proper  $k$ -mixed-thin graphs (a recently proposed generalization of unit interval graphs) have twin-width  $O(k)$  [3]. Every graph obtained by subdividing at least  $2 \log n$  (throughout the paper, all logs are in base 2) times each edge of an  $n$ -vertex graph has twin-width at most 4 [4]. Schidler and Szeider report the (exact) twin-width of a collection of graphs [20], obtained via SAT encodings. Jacob and Pilipczuk [14] give the current best upper bound of 183 on the twin-width of planar graphs, while graphs with genus  $g$  have twin-width  $O(g)$  [13]. Most relevant to our paper, for every graph  $G$ ,  $\text{tw}(G) \leq 3 \cdot 2^{\text{tw}(G)-1}$  [14], where  $\text{tw}(G)$  denotes the treewidth of  $G$ .

Conversely, one may ask the following.

► **Question 1.** *What is the largest twin-width a graph of treewidth  $k$  can have?*

A lower bound of  $\Omega(k)$  comes from the existence of  $n$ -vertex graphs with twin-width  $\Omega(n)$  (since the treewidth is trivially upperbounded by  $n - 1$ ). This is almost surely the case of graphs drawn from  $G(n, 1/2)$ . Alternatively, the  $n$ -vertex Paley graph (for a prime  $n$  such that  $n \equiv 1 \pmod{4}$ ) has precisely twin-width  $(n - 1)/2$  [1]. Another example to derive the linear lower bound is the power set graph [14]. Improving on this lower bound is not obvious, and  $\Theta(k)$  is indeed the answer to Question 1 within the class of planar graphs [14], or when replacing 'treewidth' by 'cliquewidth' or 'pathwidth.'

When switching 'twin-width' and 'treewidth' in Question 1, the gap is basically as large as possible: There are  $n$ -vertex graphs with treewidth  $\Omega(n)$  and twin-width at most 6, in the iterated 2-lifts of  $K_4$  [7, 5].

An important characterization of bounded twin-width is via the absence of complex divisions of an adjacency matrix. A matrix has a  $k$ -mixed minor if its row (resp. column) set can be partitioned into  $k$  sets of consecutive rows (resp. columns), such that each of the  $k^2$  cells defined by this  $k$ -division contains at least two distinct rows and at least two distinct columns. The *mixed number of a matrix*  $M$  is the largest integer  $k$  such that  $M$  admits a  $k$ -mixed minor. The *mixed number of a graph*  $G$ , denoted by  $\text{mxn}(G)$ , is the minimum, taken among all the adjacency matrices  $M$  of  $G$ , of the mixed number of  $M$ . The following was shown.

► **Theorem 1** ([12]). *For every graph  $G$ ,  $(\text{mxn}(G) - 1)/2 \leq \text{tw}(G) \leq 2^{2^{O(\text{mxn}(G))}}$ .*

In sparse graphs (here, excluding a fixed  $K_{t,t}$  as a subgraph), the previous theorem is both simpler to formulate and has a better dependency. A matrix has a  $k$ -grid minor if it has a  $k$ -division with at least one 1-entry in each of its  $k^2$  cells. The *grid number* of a matrix and of a graph  $G$ , denoted by  $\text{gn}(G)$ , are defined analogously to the previous paragraph. We

only state the inequality that is useful to bound the twin-width of a sparse class, but is valid in general.

► **Theorem 2** (follows from [12]). *For every graph  $G$ ,  $\text{tww}(G) \leq 2^{O(\text{gn}(G))}$ .*

Theorems 1 and 2 allow to bound the twin-width of a class  $\mathcal{C}$  by exhibiting, for every  $G \in \mathcal{C}$ , an adjacency matrix of  $G$  without large mixed or grid minor. Therefore one merely has to order  $V(G)$  (the vertex set of  $G$ ) in an appropriate way. The double (resp. simple) exponential dependency in mixed number (resp. grid number) implies relatively weak twin-width upper bounds. For several classes whose twin-width was originally upperbounded via Theorem 1, better bounds were later given by avoiding this theorem (see [7, 2, 14, 13, 4]). Still for some geometric graph classes, bypassing Theorem 1 seems complicated (see [6]). And in general (since this theorem is at the basis of several other applications, see for instance [7, 8, 9]) it would help to have an improved upper bound of  $\text{tww}(G)$ ; in particular a negative answer to the following question.

► **Question 2.** *Is twin-width sometimes exponential in mixed and grid number?*

A variant of twin-width, called *oriented twin-width*, adds an orientation to the red edges (see [10]). The red edge (arc) is oriented away from the contracted vertex. The *oriented twin-width*  $d$  of a graph  $G$ , denoted by  $\text{otww}(G)$ , is then defined similarly as twin-width by tolerating more than  $d$  red arcs incident to a vertex, as long as at most  $d$  of them are out-going. Rather surprisingly twin-width and oriented twin-width are tied.

► **Theorem 3** ([10]). *For every graph  $G$ ,  $\text{otww}(G) \leq \text{tww}(G) \leq 2^{2^{O(\text{otww}(G))}}$ .*

Classic results show that planar graphs have oriented twin-width at most 9 [10]. Thus it would be appreciable to lower the dependency of  $\text{tww}(G)$  in  $\text{otww}(G)$ .

► **Question 3.** *Is twin-width sometimes exponential in oriented twin-width?*

An elementary argument shows that when adding an apex (i.e., an additional vertex with an arbitrary neighborhood) to a graph  $G$ , the twin-width of the obtained graph is at most  $2 \cdot \text{tww}(G) + 1$ . Again it is not clear whether this increase could be made smaller.

► **Question 4.** *Does twin-width sometimes essentially double when an apex is added?*

Note that Question 1 is asked by Jacob and Pilipczuk [14], and Question 3 is posed by Bonnet et al. [10], and is closely related to Question 2.

### Our contribution.

With a single construction, we answer all these questions. The answer to Questions 2, 3, and 4 is affirmative, while the answer to Question 1 is  $2^{\Theta(k)}$ , which confirms the intuition of the authors of [14]. More precisely, we show the following.

► **Theorem 4.** *For every real  $0 < \varepsilon \leq 1/2$  and integer  $t > 1/\varepsilon$ , there is a graph  $G_{t,\varepsilon}$  with a feedback vertex set of size  $t$  and such that  $\text{tww}(G_{t,\varepsilon}) > 2^{(1-\varepsilon)t}$ .*

The graph  $G_{t,\varepsilon}$  has in particular treewidth at most  $t + 1$ , grid number at most  $t + 2$ , and oriented twin-width at most  $t + 1$ . Thus

- $\text{tww}(G_{t,\varepsilon}) > 2^{(1-\varepsilon)(\text{tw}(G_{t,\varepsilon})-1)}$ ,
- $\text{tww}(G_{t,\varepsilon}) > 2^{(1-\varepsilon)(\text{gn}(G_{t,\varepsilon})-2)}$ , and
- $\text{tww}(G_{t,\varepsilon}) > 2^{(1-\varepsilon)(\text{otww}(G_{t,\varepsilon})-1)}$ .

Hence Theorem 4 has the following consequences.

► **Corollary 5.** *For every small  $\varepsilon > 0$ , there is a family  $\mathcal{F}$  of graphs with unbounded twin-width such that for every  $G \in \mathcal{F}$ :  $\text{tw}(G) > 2^{(1-\varepsilon)(\text{tw}(G)-1)}$ .*

Up to multiplicative factors, it matches the known upper bound [14, 12], and essentially settles Question 1. The following answers Question 2.

► **Corollary 6.** *For every small  $\varepsilon > 0$ , there is a family  $\mathcal{F}$  of graphs with unbounded twin-width such that for every  $G \in \mathcal{F}$ :  $\text{tw}(G) > 2^{(1-\varepsilon)(\text{gn}(G)-2)}$ .*

The following answers Question 3.

► **Corollary 7.** *For every small  $\varepsilon > 0$ , there is a family  $\mathcal{F}$  of graphs with unbounded twin-width such that for every  $G \in \mathcal{F}$ :  $\text{tw}(G) > 2^{(1-\varepsilon)(\text{otw}(G)-1)}$ .*

The following answers Question 4.

► **Corollary 8.** *For every small  $\varepsilon > 0$ , there is a family  $\mathcal{F}$  of graphs with unbounded twin-width such that for every  $G \in \mathcal{F}$ :  $\text{tw}(G) > (2 - \varepsilon)\text{tw}(G - \{v\})$ , where  $v$  is a single vertex of  $G$ .*

We leave as an open question if the twin-width upper bound in oriented twin-width and mixed number can be made single-exponential.

## 2 Preliminaries

For  $i$  and  $j$  two integers, we denote by  $[i, j]$  the set of integers that are at least  $i$  and at most  $j$ . For every integer  $i$ ,  $[i]$  is a shorthand for  $[1, i]$ . We use the standard graph-theoretic notations:  $V(G)$  denotes the vertex set of a graph  $G$ ,  $E(G)$  denotes its edge set,  $G[S]$  denotes the subgraph of  $G$  induced by  $S$ , etc.

We give an alternative approach to contraction sequences. The *twin-width* of a graph, introduced in [12], can be defined in the following way (complementary to the one given in introduction). A *partition sequence* of an  $n$ -vertex graph  $G$ , is a sequence  $\mathcal{P}_n, \dots, \mathcal{P}_1$  of partitions of its vertex set  $V(G)$ , such that  $\mathcal{P}_n$  is the set of singletons  $\{\{v\} : v \in V(G)\}$ ,  $\mathcal{P}_1$  is the singleton set  $\{V(G)\}$ , and for every  $2 \leq i \leq n$ ,  $\mathcal{P}_{i-1}$  is obtained from  $\mathcal{P}_i$  by merging two of its parts into one. Two parts  $P, P'$  of a same partition  $\mathcal{P}$  of  $V(G)$  are said *homogeneous* if either every pair of vertices  $u \in P, v \in P'$  are non-adjacent, or every pair of vertices  $u \in P, v \in P'$  are adjacent. Two non-homogeneous parts are also said *red-adjacent*. The *red degree* of a part  $P \in \mathcal{P}$  is the number of other parts of  $\mathcal{P}$  which are red-adjacent to  $P$ . Finally the twin-width of  $G$ , denoted by  $\text{tw}(G)$ , is the least integer  $d$  such that there is a partition sequence  $\mathcal{P}_n, \dots, \mathcal{P}_1$  of  $G$  with every part of every  $\mathcal{P}_i$  ( $1 \leq i \leq n$ ) having red degree at most  $d$ .

The definition of the previous paragraph is equivalent to the one given in introduction, via contraction sequences. Indeed the trigraph  $G_i$  is obtained from partition  $\mathcal{P}_i$ , by having one vertex per part of  $\mathcal{P}_i$ , a black edge between any fully adjacent pair of parts, and a red edge between red-adjacent parts. A *partial contraction sequence* is a sequence of trigraphs  $G_n, \dots, G_i$ , for some  $i \in [n]$ . A (full) *contraction sequence* is one such that  $i = 1$ . We naturally consider the trigraph  $G_j$  to come *after* (resp. *before*)  $G_{j'}$  if  $j < j'$  (resp.  $j > j'$ ). Thus when we write *the first trigraph of the sequence  $\mathcal{S}$  to satisfy  $X$*  (or *the first time a trigraph of  $\mathcal{S}$  satisfies  $X$* ) we mean the trigraph  $G_j$  with largest index  $j$  among those satisfying  $X$ . The same goes for partition sequences.

If  $u$  is a vertex of a trigraph  $H$ , then  $u(G)$  denotes the set of vertices of  $G$  eventually contracted into  $u$  in  $H$ . We denote by  $\mathcal{P}_G(H)$  (and  $\mathcal{P}(H)$  when  $G$  is clear from the context) the partition  $\{u(G) : u \in V(H)\}$  of  $V(G)$ . We may refer to a *part* of  $H$  as any set in  $\{u(G) : u \in V(H)\}$ . We may also refer to a *part* of a contraction/partition sequence as any part of one its trigraphs/partitions. A contraction *involves* a vertex  $v$  if it produces a new part (of size at least 2) containing  $v$ . In general, we use trigraphs and partitioned graphs somewhat interchangeably, when one notion appears more convenient than the other.

### 3 Proof of Theorem 4

We fix once and for all,  $0 < \varepsilon \leq 1/2$ , a possibly arbitrarily small positive real. We build for every integer  $t > 1/\varepsilon$ , a graph  $G_{t,\varepsilon}$ , that we shorten to  $G_t$ . We set

$$f(t) = \left\lceil 2 + C_t 2^{(1-\varepsilon)t(2+C_t(2^{(1-\varepsilon)t}+1))} \right\rceil$$

where  $C_t = 2^{(1-\varepsilon)t}/\varepsilon$ .

**Construction of  $G_t$ .** Let  $T$  be the full  $2^t$ -ary tree of depth  $f(t)$ , i.e., with root-to-leaf paths on  $f(t)$  edges. Let  $X$  be a set of  $t$  vertices, that we may identify to  $[t]$ . The vertex set of  $G_t$  is  $X \uplus V(T)$ . The edges of  $G_t$  are such that  $G[X]$  is an independent set, and  $G[V(T)] = T$ . The edges between  $V(T)$  and  $X$  are such that

- the root of  $T$  has no neighbor in  $X$ , and
  - the  $2^t$  children (in  $T$ ) of every internal node of  $T$  each have a distinct neighborhood in  $X$ .
- Note that this defines a single graph up to isomorphism. By a slight abuse of language, we may utilize the usual vocabulary on trees directly on  $G_t$ . By *root*, *internal node*, *child*, *parent*, *leaf* of  $G_t$ , we mean the equivalent in  $T$ .

We start with this straightforward observation.

► **Lemma 9.**  $G_t$  has treewidth at most  $t + 1$ .

**Proof.** The set  $X$  is a feedback vertex set of  $G_t$  of size  $t$ , thus  $\text{tw}(G_t) \leq \text{fvs}(G_t) + 1 \leq t + 1$ . ◀

The following is the core lemma, which occupies us for the remainder of the section.

► **Lemma 10.**  $G_t$  has twin-width greater than  $2^{(1-\varepsilon)t}$ .

**Proof.** We assume, by way of contradiction, that  $G_t$  admits a  $d$ -sequence with  $d \leq 2^{(1-\varepsilon)t}$ . We consider the partial  $d$ -sequence  $\mathcal{S}$ , starting at  $G_t$ , and ending right before the first contraction involving a child of the root. We first show that no vertex of  $X$  can be involved in a contraction of  $\mathcal{S}$ . Note that it implies, in particular, that the root cannot be involved in a contraction of  $\mathcal{S}$ .

▷ **Claim 11.** No part of  $\mathcal{S}$  contains more than one vertex of  $X$ .

**PROOF OF THE CLAIM:** Observe that, for every  $i \neq j \in [t]$ , there are  $2^{t-1}$  sets of  $2^t$  containing exactly one of  $i, j$ :  $2^{t-2}$  only contain  $i$ , and  $2^{t-2}$  only contain  $j$ . Recall now that by assumption, in every trigraph of  $\mathcal{S}$ , every child of the root is alone in its part. Thus a part  $P$  of  $\mathcal{S}$  such that  $|P \cap X| \geq 2$  would have red degree at least  $2^{t-1} > 2^{(1-\varepsilon)t} \geq d$ . ◊

▷ **Claim 12.** No part of  $\mathcal{S}$  intersects both  $X$  and  $V(T)$ .

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PROOF OF THE CLAIM: For the sake of contradiction, consider the first occurrence of a part  $P \supseteq \{x, v\}$  with  $x \in X$  and  $v \in V(T)$ . Vertex  $x$  is adjacent to half of the children of the root, whereas  $v$  is adjacent to at most one of them, or all of them (if  $v$  is itself the root). In both cases, this entails at least  $2^{t-1} - 1$  red edges for  $P$  towards children of the root. If  $v$  is not a grandchild of the root, the red degree of  $P$  is at least  $2^{t-1}$ . We thus assume that  $v$  is a grandchild of the root.

As  $t \geq 2$ , there is a  $y \in X \setminus \{x\}$ . Let  $v'$  be the child of  $v$  whose neighborhood in  $X$  is exactly  $\{y\}$ . This vertex exists since  $f(t) \geq 3$ . If  $P$  contains  $v'$ ,  $P$  is also red-adjacent to  $\{y\}$  (indeed a part, by Claim 11). If instead,  $P$  does not contain  $v'$ , then  $P$  is also red-adjacent to the part containing  $v'$ .

Thus, in any case, the red degree of  $P$  is at least  $2^{t-1} > 2^{(1-\varepsilon)t} \geq d$ .  $\diamond$

From Claims 11 and 12, we immediately obtain:

▷ **Claim 13.** Every part of  $\mathcal{S}$  intersecting  $X$  is a singleton.

Crucial to the proof, we introduce two properties  $\mathcal{P}$ , and later  $\mathcal{Q}$ , on internal nodes  $v \in V(T)$  in trigraphs  $H \in \mathcal{S}$ . Property  $\mathcal{P}$  is defined by

$$\mathcal{P}(v, H) = \text{“At least } 2^{\varepsilon t} \text{ children of } v \text{ are in the same part of } \mathcal{P}(H)\text{.”}$$

We first remark that any internal node in a non-singleton part verifies  $\mathcal{P}$ .

▷ **Claim 14.** Let  $H$  be any trigraph of  $\mathcal{S}$  and  $v$  be any internal node of  $T$  whose part in  $\mathcal{P}(H)$  is not a singleton. Then  $\mathcal{P}(v, H)$  holds.

PROOF OF THE CLAIM: Let  $P$  be the part of  $v$  (i.e., the one containing  $v$ ) in  $\mathcal{P}(H)$ , and  $u \in P \setminus \{v\}$ . At least  $2^t - 1$  children of  $v$  are not adjacent to  $u$ . Thus these  $2^t - 1$  vertices have to be in at most  $d + 1 \leq 2^{(1-\varepsilon)t} + 1$  parts. These parts are part  $P$ , plus at most  $d$  parts linked to  $P$  by a red edge. Since  $(2^{\varepsilon t} - 1)(2^{(1-\varepsilon)t} + 1) < 2^t - 1$  (recall that  $\varepsilon < 1/2$ ), one of these parts (possibly  $P$ ) contains at least  $2^{\varepsilon t}$  children of  $v$ .  $\diamond$

As the merge of a singleton part  $\{v\}$  with any other part does not change the intersections of parts with the set of children of  $v$ , we get a slightly stronger claim.

▷ **Claim 15.** Let  $v$  be an internal node of  $T$ , and  $H$  be the last trigraph of  $\mathcal{S}$  for which  $v$  is in a singleton part of  $\mathcal{P}(H)$ . Then  $\mathcal{P}(v, H)$  holds.

A *preleaf* is an internal node of  $T$  adjacent to a leaf, i.e., the parent of some leaves. We obtain the following as a direct consequence of Claim 14.

▷ **Claim 16.** In any trigraph  $H \in \mathcal{S}$ , any non-preleaf internal node  $v \in V(T)$  that verifies  $\mathcal{P}(v, H)$  has at least  $2^{\varepsilon t}$  children  $u$  verifying  $\mathcal{P}(u, H)$ .

We define the property  $\mathcal{Q}$  on internal nodes  $v$  of  $T$  and trigraphs  $H \in \mathcal{S}$  by induction:

$$\mathcal{Q}(v, H) = \begin{cases} \mathcal{P}(v, H) & \text{if } v \text{ is a preleaf, and otherwise} \\ \mathcal{Q}(u_1, H) \wedge \mathcal{Q}(u_2, H) & \text{for some pair } u_1 \neq u_2 \text{ of children of } v. \end{cases}$$

That is,  $\mathcal{Q}$  is defined as  $\mathcal{P}$  for preleaves, and otherwise,  $\mathcal{Q}$  holds when it holds for at least two of its children. Observe that  $\mathcal{P}$  and  $\mathcal{Q}$  are monotone in the following sense: If  $\mathcal{P}(v, H)$  (resp.  $\mathcal{Q}(v, H)$ ) holds, then  $\mathcal{P}(v, H')$  (resp.  $\mathcal{Q}(v, H')$ ) holds for every subsequent trigraph  $H'$  of the partial  $d$ -sequence  $\mathcal{S}$ . We may write that  $v$  *satisfies*  $\mathcal{P}$  (resp.  $\mathcal{Q}$ ) *in*  $H$  when  $\mathcal{P}(v, H)$  (resp.  $\mathcal{Q}(v, H)$ ) holds, and may add *for the first time* if no trigraph  $H' \in \mathcal{S}$  before  $H$  is such that  $\mathcal{P}(v, H')$  (resp.  $\mathcal{Q}(v, H')$ ) holds.

▷ Claim 17. For any trigraph  $H \in \mathcal{S}$  and internal node  $v$  of  $T$ ,  $\mathcal{P}(v, H)$  implies  $\mathcal{Q}(v, H)$ .

PROOF OF THE CLAIM: This is a tautology if  $v$  is a preleaf. The induction step is ensured by Claim 16, since  $2^{\varepsilon t} \geq 2$ .  $\diamond$

At the end of the partial  $d$ -sequence  $\mathcal{S}$ , we know, by Claim 15, that at least one child of the root satisfies  $\mathcal{P}$ , hence satisfies  $\mathcal{Q}$ , by Claim 17. Thus the first time in the partial  $d$ -sequence  $\mathcal{S}$  that  $\mathcal{Q}(v, H)$  holds, for a trigraph  $H \in \mathcal{S}$  and a child  $v$  of the root, is well-defined. We call  $F$  this trigraph, and  $v_0$  a child of the root such that  $\mathcal{Q}(v_0, F)$  holds.

We now find many nodes satisfying  $\mathcal{Q}$  in  $F$ , whose parents form a vertical path of singleton parts.

▷ Claim 18. There is a set  $Q \subset V(T)$  of at least  $f(t) - 2$  internal nodes such that

- for every  $v \in Q$ ,  $\mathcal{Q}(v, F)$  holds,
- the parent of any  $v \in Q$  is in a singleton part of  $\mathcal{P}(F)$ , and
- and *no* two distinct nodes of  $Q$  are in an ancestor-descendant relationship.

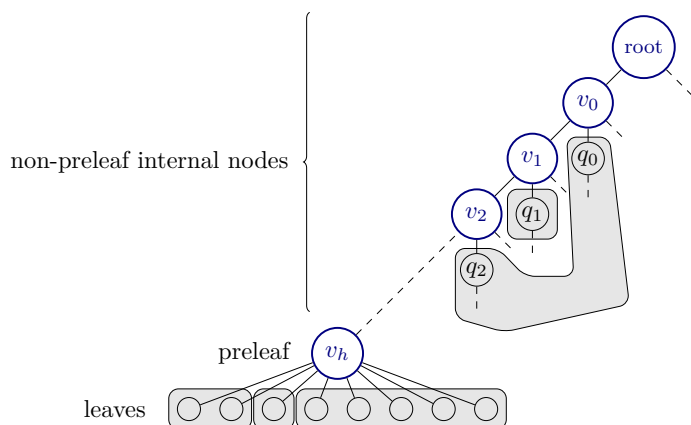
PROOF OF THE CLAIM: We construct by recurrence two sequences  $(v_i)_{i \in [f(t)-2]}$ ,  $(q_i)_{i \in [0, f(t)-3]}$  of internal nodes of  $T$  such that for all  $i \in [f(t) - 2]$ ,  $v_i$  is a child of  $v_{i-1}$ ,  $v_{i-1}$  is in a singleton part of  $\mathcal{P}(F)$ , and  $v_{i-1}$  has a child  $q_{i-1} \neq v_i$  for which  $\mathcal{Q}(q_{i-1}, F)$  holds.

Assume that the sequence is defined up to  $v_i$ , for some  $i < f(t) - 2$ . We will maintain the additional invariant that  $v_i$  satisfies  $\mathcal{Q}$  for the first time in  $F$ . This is the case for  $i = 0$ .

As  $v_i$  is not a preleaf, it satisfies  $\mathcal{Q}$  for the first time when a second child of  $v_i$  satisfies  $\mathcal{Q}$ . Let  $v_{i+1}$  be this second child, and  $q_i$  be the first child to satisfy  $\mathcal{Q}$  (breaking ties arbitrarily if both children satisfy  $\mathcal{Q}$  for the first time in  $F$ ). The vertex  $v_{i+1}$  satisfies  $\mathcal{Q}$  for the first time in  $F$ . Thus our invariant is preserved.

For every  $i \in [f(t) - 2]$ ,  $v_i$  is in a singleton part of  $\mathcal{P}(F)$ . Indeed, by Claim 15, if  $v_i$  was not in a singleton part of  $\mathcal{P}(F)$ ,  $v_i$  would satisfy  $\mathcal{P}$ , hence  $\mathcal{Q}$ , in the trigraph preceding  $F$ ; a contradiction.

The set  $Q$  can thus be defined as  $\{q_i : i \in [0, f(t) - 3]\}$ . We already checked that the first two requirements of the lemma are fulfilled. No pair in  $Q$  is in an ancestor-descendant relationship since the nodes of  $Q$  are all children of a root-to-leaf path made by the  $v_i$ s (see Figure 2).  $\diamond$



■ **Figure 2** The nodes  $(v_i)_{i \in [0, h]}$  and  $(q_i)_{i \in [0, h-1]}$  ( $h = f(t) - 2$ ) satisfy  $\mathcal{P}$  and  $\mathcal{Q}$  in  $F$ . The  $v_i$ s and the root (nodes circled in blue) are in singleton parts of  $F$ . The other represented nodes can be in larger parts (shaded areas).



Let  $B$  the vertices  $w \in V(F)$  such that  $w(G)$  contains at least  $2^{\varepsilon t}$  children of the same node of  $T$ . Each vertex of  $B$  is red-adjacent to at least  $\log(2^{\varepsilon t}) = \varepsilon t$  (singleton) parts of  $X$ . Therefore, since the red degree of (singleton) parts of  $X$  is at most  $2^{(1-\varepsilon)t}$ :

$$|B| \leq \frac{2^{(1-\varepsilon)t}}{\varepsilon}.$$

Next we show that there is relatively large set of vertices of  $F$  each corresponding to a non-singleton part that contains an internal node of  $T$ .

▷ **Claim 19.** There is a set  $B' \subseteq V(F)$  of size at least

$$\frac{1}{(1-\varepsilon)t} \log \left( \frac{f(t)-2}{|B|} \right) - 1$$

such that for every  $b \in B'$  there is an internal node  $v$  of  $T$  with  $v \in b(G_t)$  and  $|b(G_t)| \geq 2$ .

**PROOF OF THE CLAIM:** Let  $s := \frac{1}{(1-\varepsilon)t} \log \left( \frac{f(t)-2}{|B|} \right) - 1$ . Our goal is to construct a sequence  $(b_i)_{i \in [0, s]}$  of distinct vertices of  $F$  such that for every  $i \in [s]$ ,

$$\text{part } b_i(G_t) \text{ is not a singleton and contains an internal node of } T. \quad (1)$$

We first focus on finding  $b_0$ . Note that  $b_0$  need not satisfy Invariant (1), but will be chosen to force the existence of  $b_1$  itself satisfying (1) and starting the induction.

Let  $Q := \{q_j : 0 \leq j \leq f(t) - 3\} \subset V(T)$  be as described in Claim 18. Every  $q_j \in Q$  has (at least) one descendant  $q'_j$  that is a preleaf and satisfies  $\mathcal{Q}$ , hence  $\mathcal{P}$ , in  $F$ . The  $q'_j$ s are pairwise distinct because no two nodes of  $Q$  are in an ancestor-descendant relationship. We set  $Q' := \{q'_j : 0 \leq j \leq f(t) - 3\}$ .

Now for every  $q'_j$ , at least  $2^{\varepsilon t}$  of its children are in the same part of  $\mathcal{P}(F)$ ; hence, this part corresponds to a vertex in  $B$ . By the pigeonhole principle, there is a  $b_0 \in B$  that contains at least  $2^{\varepsilon t}$  children of at least  $(f(t) - 2)/|B|$  nodes of  $Q'$ .

For each  $b_i$ , we define  $Q_i \subset Q$  as the set of vertices  $q_j$  such that

- $b_i(G_t)$  contains a (not necessarily strict) descendant  $z$  of  $q_j$ , and
- no part  $b_{i'}(G_t)$  with  $i' < i$  contains a node on the path between  $q_j$  and  $z$  in  $T$ .

Thus  $|Q_0| \geq (f(t) - 2)/|B|$ .

We now assume that  $b_i \in V(F)$ , for some  $0 \leq i < s$ , has been found with

$$|Q_i| \geq \frac{f(t) - 2}{|B| \cdot 2^{i(1-\varepsilon)t}}. \quad (2)$$

Observe that  $Q_0$  satisfies (2). We construct  $b_{i+1}, Q_{i+1}$  satisfying the invariants (1) and (2).

For each  $q_j \in Q_i$ , consider the highest descendant  $z_j$  of  $q_j$  in  $b_i(G_t)$ , and  $z'_j$  the parent of  $z_j$  in  $T$ . By construction, the part  $P_j$  of  $\mathcal{P}(F)$  containing  $z'_j$  is not a  $b_k(G_t)$  for any  $k \leq i$ . Part  $P_j$  is linked to  $b_i(G_t)$  by a red edge. Therefore there are at most  $2^{(1-\varepsilon)t}$  such parts  $P_j$ . In particular, there is a  $b_{i+1} \in V(F)$  such that  $b_{i+1}(G_t)$  contains at least

$$\frac{|Q_i|}{d} \geq \frac{f(t) - 2}{|B| \cdot 2^{i(1-\varepsilon)t}} \cdot \frac{1}{2^{(1-\varepsilon)t}} = \frac{f(t) - 2}{|B| \cdot 2^{(i+1)(1-\varepsilon)t}}$$

parents  $z'_j$  of highest descendants  $z_j$ .

Remark that  $b_{i+1}(G_t)$  has size at least two while  $(f(t) - 2)/(|B| \cdot 2^{(i+1)(1-\varepsilon)t}) > 1$ , which holds since  $i < s$ . Thus  $b_{i+1}(G_t)$  does not contain any parent  $v_j$  of a  $q_j$  (since the  $v_j$ s are in singleton parts). In particular,  $|Q_{i+1}| \geq (f(t) - 2)/(|B| \cdot 2^{(i+1)(1-\varepsilon)t})$ , and  $b_{i+1}, Q_{i+1}$  satisfy (1) and (2).

Finally, the set  $B' := \{b_i : 1 \leq i \leq s\}$  has the required properties.  $\diamond$

We can now finish the proof of the lemma.

For every  $b_i \in B'$ , let  $u_i \in b_i(G_t)$  be an internal node of  $T$ . As  $b_i(G_t) \geq 2$ ,  $u_i$  satisfies  $\mathcal{P}$  in  $F$ . This implies that  $b_i$  or a red neighbor of  $b_i$  is in  $B$ . Therefore, the total number of red edges incident to a vertex of  $B$  is at least  $|B'| - |B|$ . Thus there is a vertex in  $B$  with red degree at least  $(|B'| - |B|)/|B|$ . This is a contradiction since

$$\begin{aligned} \frac{|B'| - |B|}{|B|} &= \frac{|B'|}{|B|} - 1 \geq \left( \frac{1}{(1-\varepsilon)t} \log \left( \frac{f(t) - 2}{|B|} \right) - 1 \right) \cdot \frac{1}{|B|} - 1 \\ &\geq \left( \frac{1}{(1-\varepsilon)t} \log \left( 2^{(1-\varepsilon)t(2+C_t \cdot (2^{(1-\varepsilon)^t+1}))} \right) - 1 \right) \cdot \frac{1}{|B|} - 1 \\ &= \left( (2 + C_t \cdot (2^{(1-\varepsilon)^t+1})) - 1 \right) \cdot \frac{1}{|B|} - 1 > 2^{(1-\varepsilon)t} + 1 - 1 = 2^{(1-\varepsilon)t} \geq d. \end{aligned}$$

since, we recall,  $f(t) = \left\lceil 2 + C_t \cdot 2^{(1-\varepsilon)t(2+C_t \cdot (2^{(1-\varepsilon)^t+1}))} \right\rceil$  and  $C_t = \frac{2^{(1-\varepsilon)t}}{\varepsilon} \geq |B|$ .  $\blacktriangleleft$

Since  $X$  is a feedback vertex set of size  $t$  of  $G_t$ , Lemma 10 implies Theorem 4, and hence Corollary 5.

As the twin-width of  $T$  is 2, adding the  $t$  apices in  $X$ , multiplies the twin-width by at least  $2^{t(1-\varepsilon-\frac{1}{t})}$ . Thus one apex in  $X$  multiplies the twin-width by at least  $2^{1-\varepsilon-\frac{1}{t}}$ , which can be made arbitrarily close to 2. This establishes Corollary 8.

## 4 Oriented twin-width and grid number

In this section, we check that  $G_t$  has oriented twin-width at most  $t + 1$ , and grid number at most  $t + 2$ .

A *(partial) oriented contraction sequence* is defined similarly as a *(partial) contraction sequence* with every red edge replaced by a red arc leaving the newly contracted vertex. Then a *(partial) oriented  $d$ -sequence* is such that all the vertices of all its *ditrigraphs* have at most  $d$  out-going red arcs. The *oriented twin-width* of a graph  $G$ , denoted by  $\text{otww}(G)$ , is the minimum integer  $d$  such that  $G$  admits an oriented  $d$ -sequence.

► **Lemma 20.** *The oriented twin-width of  $G_t$  is at most  $t + 1$ .*

**Proof.** We observe that the 2-sequence for trees [12] is an oriented 1-sequence. We contract  $T$  to a single vertex (without touching  $X$ ) in that manner. This yields a partial oriented  $t + 1$ -sequence for  $G_t$  ending on a  $t + 1$ -vertex ditrigraph, which can be contracted in any way. This contraction sequence witnesses that  $\text{otww}(G_t) \leq t + 1$ .  $\blacktriangleleft$

Thus Corollary 7 holds.

We finish by establishing Corollary 6.

► **Lemma 21.** *The grid number of  $G_t$  is at most  $t + 2$ .*

**Proof.** Recall that  $V(G_t) = X \uplus V(T)$ . Let  $\prec$  be the total order on  $V(G_t)$  that puts first all the vertices of  $X$  in any order, then from left to right, all the leaves of  $T$ , followed by the preleaves, the nodes at depth  $f(t) - 2$ , the nodes at depth  $f(t) - 3$ , and so on, up to the root. We denote by  $M$  the adjacency matrix of  $G_t$  ordered by  $\prec$ .

Let  $M_T$  be the submatrix of  $M$  obtained by deleting the  $t$  rows and  $t$  columns corresponding to  $X$ . Note that the grid number of  $M$  is at most  $\text{gn}(M_T) + t$ . We claim that there is no 3-grid minor in  $M_T$ .

Indeed, in the order  $\prec$ , above the diagonal of  $M_T$  there is no pair of 1-entries in strictly decreasing positions. Thus overall there is no triple of 1-entries in strictly decreasing positions. Thus no 3-grid minor is possible in  $M_T$ . ◀

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