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Functorial Models of Differential Linear Logic

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Abstract. There are two types of duality in Linear Logic. The first one is negation, balancing positive and negative formulas. The second one is the duality between linear and non-linear proofs, made symmetrical by Differential Linear Logic. The first duality has been here since the beginning of Linear Logic and has a significance in terms of proof-search and programming operation. However, the computational content of the second is still to understand. In this paper, we reexpress models of Differential Linear Logic in terms of models of Polarized Linear Logic, hoping to get closer to the understanding of Differential Linear Logic. It formalizes Differentiation in terms of a functor on a coslice category, within the linear–non-linear adjunction models of Linear Logic. It is also a first step in studying models mixing polarization and differentiation, concrete instances of which arise in functional analysis.

1 Introduction

Differentiation is a mathematical operation which maps a function to its closest linear approximation. Through the Curry-Howard correspondence, it has been extended to logic [ER06] and λ -terms [ER03]. Categorical presentations of differentiation originate from investigation of denotational models of these systems, resulting in a variety of interrelated categorical definitions [BCLS20]. Differentiation in such settings might be axiomatized as an external operator acting on morphisms of a category [BCS09], or as a hard-coded natural transformation refining a model of linear logic [Fio07]. However, the axiomatization of differentiation in these last models remains highly non-functorial, a shortcoming our work seeks to correct.

This novel presentation of categorical models of Differential Linear Logic is motivated by two recent research developments. The first is differentiable programming: this is a growing domain emerging from spectacular applications of machine learning, aiming to express machine learning algorithms in a functional and principled way. This led to a renewed interest for categorical axiomatizations of differentiation [Eli18].

The second is the recent trend of mixing differentiation and polarity in Linear Logic. While studying models of Differential Linear Logic using topological vector spaces [Ker18], the first-named author found that polarities and categorical models supporting these were particularly relevant. Chiralities [Mel17] are a categorical axiomatization of polarized multiplicative linear logic, discovered by Melliès after a study of game models of linear logic. In this paper, we argue that while chiralities model the interaction between positive and negative formulas in linear logic, a similar structure models the interaction between linear and non-linear proofs.

Main Idea Models of Linear Logic (LL henceforth) can be axiomatized as strong monoidal adjunctions between a category whose morphisms are ‘linear maps’ and a category whose morphisms are ‘non-linear maps’, or more accurately ‘general maps’ (smooth maps in our case). For classical LL, the situation is most naturally presented as a contravariant adjunction:

$$\begin{array}{ccc}
 \begin{array}{c} \xrightarrow{\mathcal{E}'} \\ (\mathcal{C}, \times, I) \perp (\mathcal{L}, \otimes, 1) \\ \xleftarrow{\mathcal{U}} \end{array} & \text{or} & \begin{array}{c} \xrightarrow{\mathcal{E}} \\ (\mathcal{C}, \times, I) \perp (\mathcal{L}^{op}, \mathfrak{A}, 1) \\ \xleftarrow{\mathcal{U}^\perp} \end{array}
 \end{array} \quad (1)$$

where \mathcal{U}^\perp represents the dual of the forgetful functor from linear to smooth maps, \mathcal{E} represents the contravariant hom-set functor and \mathcal{E}' its dual. Meanwhile, models of *polarized* LL, so-called chiralities, decompose *-autonomous categories into a covariant adjunction alongside a monoidal contravariant one:

$$\begin{array}{ccc}
 \begin{array}{c} \xrightarrow{(-)^\perp_P} \\ (\mathcal{P}, \otimes, 1) \perp (\mathcal{N}^{op}, \mathfrak{A}, \perp) \\ \xleftarrow{(-)^\perp_N} \end{array} & \text{and} & \begin{array}{c} \uparrow \\ \mathcal{P} \perp \mathcal{N} \\ \downarrow \end{array}
 \end{array} \quad (2)$$

Models of Differential Linear Logic (DiLL henceforth) add to the co-monad $!$ on \mathcal{L} induced by adjunction 1 in the classical setting an ad-hoc natural transformation $\bar{d} : \text{id} \Rightarrow !$ and non-functorial axioms on it.

In the present paper we take inspiration from models of polarized LL to redefine models of DiLL in terms of pairs of adjunctions. The idea is to fit models of DiLL into the pair of adjunction above, the left contravariant adjunction being a model of classical LL and the right covariant adjunction expressing the differential extension of LL. Mainly, we will be expressing differentiation as a functor

$$\mathcal{D} : (I \downarrow \mathcal{C}) \longrightarrow \mathcal{L}, \text{ or more generally } \mathcal{D} : (\mathcal{U} \downarrow \mathcal{C}) \longrightarrow \mathcal{L}.$$

Both these functors adapt to models of classical DiLL the *-autonomous setting in expressing reverse differentiation:

$${}^\perp\mathcal{D} : (I \downarrow \mathcal{C})^{op} \longrightarrow \mathcal{L} \text{ or } {}^\perp\mathcal{D} : (\mathcal{U}^\perp \downarrow \mathcal{C})^{op} \longrightarrow \mathcal{L}.$$

Contribution We formulate categorical models of DiLL with a functorial definition of differentiation (Definition 5), which we prove to be equivalent to differential storage categories under some hypotheses (Theorem 1 and Theorem 2). We get ride of the first hypotheses on \mathcal{D} by making it act as a left-adjoint in a relative reflection on the co-slice (Proposition 1), and this way give a concise definition of models of classical DiLL. We that both models of classical DiLL and chiralities are specific cases of the same categorical structure, namely pairs of well-behaving adjunctions (Definition 7 and Propositions 3 and 2). We also give an alternative definition of \mathcal{D} getting ride of a well-pointedness hypothese.

Related work In differentiable programming, differentiation is usually made functorial by making differential act on pairs [Eli18]. The only usage of slice constructions we have encountered is in reverse cartesian differential categories, where partial differential are defined on slice objects [CCG⁺]. Work on differential categories [BCS06], cartesian differential categories [BCS09] and tangent categories [CC14] have been successful in categorifying differentiation from a mathematical perspective, but place models of DiLL as the strictest in the hierarchy of categorical definitions.

Potential applications What makes this new functorial axiomatization of DiLL interesting to our eyes are its potential applications. The study of differential categories has led to the introduction of new structures including tangent categories [CC14] and calculus categories [CL17] with relevance beyond proof theory. Analogously, a functorial presentation of models of DiLL should trigger new development in Linear Logic or in type theory. One can replace differentiation DiLL by linear partial differential operators with constant coefficients [Ker18]. A first step in understanding and generalizing such operators would be to understand how they arise in categorical models, and we believe that our presentation has more chance of successfully incorporating differential operators. Another interesting novelty is the use of coslice categories, given that the dual construction is extensively used in modeling dependent type theory. We hope this work can initiate research studying differentiation from a Curry-Howard perspective with respect to dependent types.

Outline We provide background material on differential categories and chirality in Section 2. The core results of our study are presented in Section 3, where we define \mathcal{D} as a functor on the co-slice $I \downarrow \mathcal{C}$, and demonstrate that with a few assumptions, it results in a differential category on \mathcal{L} . In Section 4, we expand on this by placing \mathcal{D} within a pair of adjoint functors, generalizing chiralities. Finally, in Section 5, we discuss potential future work that can be unlocked by our study.

2 Preliminaries

2.1 Differential Categories

In this section, we give a quick introduction to the syntax and semantics of DiLL. For readers familiar with DiLL, we note that Definition 2 is the only one technically necessary to the rest of the paper; the rest of this section serves as motivation and historical introduction.

First, Linear Logic (LL) is the result of a decomposition of Intuitionistic Logic via an involutive *linear* negation. Along with multiplicative and additive versions of conjunction and disjunction, it introduces the *exponential* connective $!$ and its dual $?$. Some important intuition for working in LL coming from the Curry-Howard correspondence is that a proof of a sequent of the form $A \vdash B$ corresponds to the construction of a *linear* function from A to B , while a proof of $!A \vdash B$ constructs a *non-linear* function from A to B . When modeling LL in vector spaces, these intuitions are literally valid, in that a proof of $A \vdash B$ is interpreted as a linear map and a proof of $!A \vdash B$ is interpreted as (the transpose across the linear–non-linear adjunction of) a non-linear map.

DiLL adds to LL new rules on the connector $!$, expressing the fact that one is able to transform a non-linear proof into a linear one. While LL features a **dereliction** rule (d), DiLL introduces a **codereliction** rule (\bar{d}) which acts as differentiation at 0 in the logic:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta} \text{d} \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, !A} \bar{\text{d}}$$

This points to the duality between linear and non-linear maps in DiLL. We recall also the exponential rules of LL called respectively **weakening** (w), **contraction** (c) and **promotion** (p). DiLL adds to these dual rules of **coweakening** (\bar{w}) and **cocontraction** (\bar{c}).

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta} \text{w} \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta} \text{c} \qquad \frac{! \Gamma \vdash A}{! \Gamma \vdash !A} \text{p}$$

$$\frac{\vdash}{\vdash !A} \bar{\text{w}} \qquad \frac{\vdash \Gamma, !A \quad \vdash \Delta, !A}{\vdash \Gamma, \Delta, !A} \bar{\text{c}}$$

Since the focus of this paper is on categorical models rather than the proof theory of these logics, we will not give a full presentation of the syntax of DiLL; we omit the cut-elimination rules and the corresponding commutative diagrams between their interpretation as these are direct translations of differential calculus rules. We refer the reader to the survey of Ehrhard [Ehr17] for a complete introduction. Let us now turn to the categorical models.

Definition 1. A *Seely category* [See89] [Bie] is a monoidal closed category $(\mathcal{L}, \otimes, 1)$ with products and a comonad $(!, d, \nu)$ such that $!$ is a monoidal functor $(\mathcal{L}, \times, \top) \longrightarrow (\mathcal{L}, \otimes, 1)$ (where \top denotes a terminal object) so we have ‘Seely isomorphisms’:

$$!A \otimes !B \xrightarrow{\cong} !(A \times B) \quad \text{and} \quad 1 \xrightarrow{\cong} !\top.$$

Following the LL convention, we will denote by $\ell : A \multimap B$ linear maps and $f : A \longrightarrow B$ for non-linear or smooth maps in the co-Kleisli category $\mathcal{L}_!$. To model classical LL, one requires that LL be a $*$ -autonomous category, with an involutive duality $(\cdot)^\perp \simeq \mathcal{L}(\cdot, 1)$.

The image of the diagonal map and terminal map for the product in \mathcal{L} under $!$ give each object of the form $!A$ a canonical algebra structure:

$$\text{c}_A : !A \multimap !A \otimes !A \qquad \text{w}_A : !A \multimap 1$$

That is, Seely categories are categorical models of intuitionistic Linear Logic [Mel08], with the above maps interpreting contraction and weakening, while dereliction is interpreted by the counit of the comonad, as suggested by the notation, and promotion is interpreted by (co)Kleisli extension. Moreover, we recover an instance of the linear–non-linear adjunction discussed in the Introduction by constructing the coKleisli category for the comonad $!$.

The usual definition of categorical models of DiLL refines the axiomatization of Seely categories. For instance, to interpret \bar{w} and \bar{c} in this case one only needs to replace products with biproducts. This is a natural choice: to reflect the addition of rules to a logic, one adds more constructors to the categories providing the logic’s semantics. A whole hierarchy of axiomatization of ‘differential categories’ coexist, and we’ll make use of the strongest notion.

Definition 2. A *differential category*, called a ‘differential storage category’ in [BCLS20], is a Seely category $(\mathcal{L}, \otimes, 1)$ in which:

- products coincide with coproducts, resulting in a biproduct structure¹ which we now denote by $(\diamond, 0)$, and
- the comonad $(!, d, \nu)$ is equipped with a natural transformation $\bar{d} : \text{id} \Rightarrow !$ satisfying:
 - Invariance of linear maps under differentiation: $\bar{d}; d = \text{id}$, and
 - The chain rule:

$$\text{id}_{!A} \otimes \bar{d}_A; \bar{c}_A; \nu_A = c_A \otimes \bar{d}_A; \text{id}_{!A} \otimes \bar{c}_A; \nu_A \otimes \bar{d}_{!A}; \bar{c}_{!A}. \quad (3)$$

While $\bar{d}; \nu_A$ would express the chain rule at 0, Equation 3 translates the differentiation of composed function at a point a : $D_a(g \circ f) = D_{f(a)}g \circ D_a f$. The codiagonal and initial map for the coproduct in \mathcal{L} endow $!A$ with coalgebra structure dualizing the algebra structure constructed earlier:

$$\bar{c}_A : !A \otimes !A \longrightarrow !A \quad \bar{w}_A : 1 \longrightarrow !A$$

Along with the transformation \bar{d} , these complement the components of LL interpreted in a general Seely category to produce an interpretation of DiLL. In fact, the differential categories axiomatized above are *denotational* models of intuitionistic DiLL, and of classical DiLL when we impose the extra condition of *-autonomy. The exploration of less strict axiomatization have been fruitful in exploring various mathematical examples, in functional analysis and differential geometry. However, the definition above crucially does not reflect the symmetry of exponential rules in DiLL, and does not express differentiation using the traditional objects of computer science. Notably, the chain rule is at first sight quite intricate. In this paper, we attempt to simplify this.

Rather than merely introducing new operations such as \bar{d} , our approach requires us to introduce new *categories* on top of the LL setup, even beyond the expansion of Definition 1 to an adjunction with the coKleisli category of $!$ mentioned above. Nonetheless, the extension of LL to DiLL described in Definition 2 can be recovered from our construction.

Notation When $f : A \longrightarrow B$ is a differentiable functions between two vector spaces, we denote by $D_a f : A \dashv\!\!-\!\!-\! B$ the linear map corresponding to the differential of f at a point $a \in A$.

Example 1 (The distribution model). The category of convenient spaces [BET12] provides a good example of a differential category. Formulas are interpreted by real topological vector spaces, endowed with a bornology, that is a well-behaving collection of bounded sets; we shall not attempt to give detailed definitions of these structures here. The exponential $!A$ is interpreted as the completion of the vector space generated by the dirac distributions. Diracs are defined as $\delta_x : f \in \mathcal{C}(E, \mathbb{R}) \mapsto f(x)$ and are formally objects of the dual of the space of smooth functions $\mathcal{C}(E, \mathbb{R})'$, otherwise known as the space of distributions of compact support over E [Sch66].

$$\llbracket !A \rrbracket := \overline{\langle \delta_x \rangle_{x \in \llbracket A \rrbracket}}$$

¹ Recall that the existence of biproducts (products coinciding with coproducts) and a zero object endows each homset of a category with the structure of a commutative monoid.

As the dirac distributions form a linearly independent family of distributions, to define linear morphisms on $!E$ it is enough to specify their value on diracs and then extend linearly and continuously to the completion. As such, the components of DiLL are interpreted as follows:

$$\begin{aligned}
d : \delta_x \in !E &\mapsto x \in E & \bar{d} : x \in E &\mapsto D_0(\cdot)(v) \in !E \\
c : \delta_x \in !E &\mapsto \delta_x \otimes \delta_x \in !E \otimes !E & \bar{c} : \delta_x \otimes \delta_y \in !E \otimes !E &\mapsto \delta_{x+y} \in !E \\
w : \delta_x \in !E &\mapsto 1 \in \mathbb{R} & \bar{w} : 1 \in \mathbb{R} &\mapsto \delta_0 \in !E \\
\nu : \delta_x \in !E &\mapsto \delta_{\delta_x} \in !!E
\end{aligned}$$

As $D_0(\cdot)(v) = \lim_{t \rightarrow 0} \frac{\delta_{tv} - \delta_0}{t}$, \bar{d} is well defined in $!E$.

Convenient spaces do *not* form a $*$ -autonomous category, and constructing $*$ -autonomous smooth models of DiLL with a non-trivial duality is tricky, often requiring compromise on the interpretation of the other components of DiLL, such as [DK20] which features an ad-hoc construction to allow differentiation. One can work around this issue by constructing polarized model of DiLL.

2.2 Chiralities

This section gives an introduction to chiralities, and can be skipped if one only intends to read Section 3. Chiralities, as defined by Mellès, are models of polarized multiplicative Linear Logic. Polarization offers both a syntactic refinement to Linear Logic, which makes sense in term of proof-search, and a way to relax the $*$ -autonomous requirement on categorical models of LL and DiLL. Instead of looking for an involutive equivalence on a category, one looks for a dual equivalence between two categories.

Definition 3 ([Mel17]). A mixed **chirality** consists of two symmetric monoidal categories $(\mathcal{P}, \otimes, 1)$ and $(\mathcal{N}, \wp, \perp)$, and two adjunctions, one being a contravariant strong monoidal adjunction and the other an ordinary covariant adjunction:

$$\begin{array}{ccc}
& \xrightarrow{(-)^{\perp P}} & \\
(\mathcal{P}, \otimes, 1) & \perp & (\mathcal{N}^{op}, \wp, \perp) \\
& \xleftarrow{(-)^{\perp N}} &
\end{array}
\quad
\begin{array}{ccc}
& \uparrow & \\
\mathcal{P} & \perp & \mathcal{N} \\
& \downarrow &
\end{array}
\quad (4)$$

along with a suitably natural family of bijections relaxing monoidal closedness:

$$\chi_{p,n,m} : \mathcal{N}(\uparrow p, n \wp m) \simeq \mathcal{N}(\uparrow(p \otimes n^{\perp N}), m). \quad (5)$$

These are required to make the following associativity diagram commute:

$$\begin{array}{ccccc}
\mathcal{N}(\uparrow((m \wp n) \wp a), b) & \xrightarrow{\quad x \quad} & \mathcal{N}(\uparrow a, (m \wp n)^* \otimes b) & & \\
\downarrow \text{assoc} & & \uparrow \text{assoc., monoidality of negation} & & \\
\mathcal{N}(\uparrow(m \wp (n \wp a)), b) & \xrightarrow{\quad x \quad} & \mathcal{N}(\uparrow(n \wp a), m^* \otimes b) & \xrightarrow{\quad x \quad} & \mathcal{N}(\uparrow a, n^* \otimes (m^* \otimes b))
\end{array}$$

A **dialogue chirality** is a mixed chirality in which the monoidal adjunction is an equivalence. A **negative chirality** is a mixed chirality in which the two adjunctions are reflective.

Chiralities naturally arise in models based on game semantics, where positive and negative formulas are interpreted by players and opponent moves. Surprisingly, they also provide the right setting for expressing topological dualities in vector spaces [Ker19]. Chiralities are at the heart of a model of polarized DiLL resembling the convenient one described in Example 1, with a refinement on formulas allowing for involutive negations. This makes them sensible candidates of foundation for new models of DiLL.

Example 2. Metrizable complete spaces are an important class of topological vector spaces, comprising for example all spaces of smooth, h-differentiable or holomorphic functions $\mathcal{C}(\mathbb{R}^n, \mathbb{R})$. These spaces of functions are in particular Fréchet spaces, which is to say metrizable complete spaces. However, they are not stable by duality: their duals are naturally called dual Fréchet spaces or DF-spaces. When we add the criterion that topologies must be *nuclear* [Gro66], one obtains NF-spaces and NDF spaces, respectively. These assemble into a chirality which captures the duality between spaces of functions and spaces of distributions [Ker18].

$$\begin{array}{ccc}
 & \xrightarrow{(-)'\beta} & \\
 (\text{NDF}, \otimes_\pi, \mathbb{R}) & \perp & (\text{NF}^{op}, \tilde{\otimes}_\pi, \mathbb{R}) \\
 & \xleftarrow{(-)'\beta} &
 \end{array}
 \quad
 \begin{array}{ccc}
 & \xrightarrow{\tilde{(-)}} & \\
 \text{NDF} & \perp & \text{NF} \\
 & \xleftarrow{\iota} &
 \end{array}
 \quad (6)$$

in which $\tilde{(-)}$ denotes the completion of a locally convex topological vector space, $\tilde{\otimes}_\pi$ denotes the completion of the projective tensor product.

Other examples of chiralities arise naturally in functional analysis [Ker19], and models of polarized DiLL over chiralities were explored in [BK20].

3 Functorial axiomatization

This section represents the heart of this paper: here we will start from a linear–non-linear adjunction and show how defining a functor from a coslice of the non-linear category \mathcal{C} to the linear category \mathcal{L} makes \mathcal{L} into a differential category.

Definition 4. *Let I be an object of a category \mathcal{C} ; unless stated otherwise, we take it to be a terminal object. We recall that the coslice category under I , denoted $(I \downarrow \mathcal{C})$, has as objects arrows $a : I \rightarrow A$ in \mathcal{C} and as morphisms $f : (a : I \rightarrow A) \rightarrow (b : I \rightarrow B)$ those morphisms $f : A \rightarrow B$ of \mathcal{C} such that $a; f = b$.*

Notation If $f : A \rightarrow B$ is an arrow of \mathcal{C} , and $a : I \rightarrow A$ an object in $(I \downarrow \mathcal{C})$ we denote by $(a|f)$ the morphism in $(I \downarrow \mathcal{C})$ from a to $a; f$ induced by f . The coslice category comes equipped with a projection functor $\Pi : (I \downarrow \mathcal{C}) \rightarrow \mathcal{C}$ which retains only the codomain of the objects and the underlying \mathcal{C} -morphisms.

Example 3. Let \mathcal{C} be the category whose objects are real vector spaces and whose morphisms are differentiable functions, and let I be the vector space $\{0\}$. Then $(I \downarrow \mathcal{C})$ is the category of pairs (V, v) , where v is an element of V .

Differentiation defines a functor from $(I \downarrow \mathcal{C})$ to the category of vector spaces and linear maps. Denote by \mathcal{D} the operator mapping a morphism $a : I \rightarrow A$ to A and the arrow $(a|f)$ to the linear map $D_a f : A \rightarrow B$. The chain rule exactly translates the functoriality of \mathcal{D} . Indeed, morphisms $(a|f)$ and $(b|g)$ compose if and only if $b = f(a)$ and then we have

$$\mathcal{D}(a|f; g) = D_a(g \circ f) = D_{f(a)}g \circ D_a f = \mathcal{D}(a|f); D(a; f|g).$$

The example above contains the essential idea of the present section. As we shall shortly see, there is no need for an explicit co-dereliction operator once we equip our category with a differential operation, since the codereliction can be recovered from \mathcal{D} as differentiation at 0.

In order to present this differential operator, we will need some further constructions. Suppose that \mathcal{L} is a category with biproducts $(\diamond, 0)$ and that we are given a functor $\mathcal{U} : \mathcal{L} \rightarrow \mathcal{C}$ (we choose the notation for consistency) sending 0 to I . Then we have an induced functor $(0 \downarrow \mathcal{L}) \rightarrow (I \downarrow \mathcal{C})$ which we shall denote $(0 \downarrow \mathcal{U})$, which simply applies \mathcal{U} to both objects and morphisms and thus completes the square:

$$\begin{array}{ccc} (I \downarrow \mathcal{C}) & \xleftarrow{(0 \downarrow \mathcal{U})} & (0 \downarrow \mathcal{L}) \\ \downarrow \Pi & & \downarrow \Pi \\ \mathcal{C} & \xleftarrow{\mathcal{U}} & \mathcal{L} \end{array} \quad (7)$$

Note that since 0 is initial as well as terminal, the projection Π to \mathcal{L} is an equivalence of categories.

Notation Supposing that \mathcal{U} preserves products, for each object $A \in \mathcal{L}$ the subcategory of $(I \downarrow \mathcal{U})$ on the objects of the form $a : I \rightarrow \mathcal{U}(A)$ inherits a monoidal structure via the codiagonal map: for a, a' , we can define:

$$a \boxplus a' : I \xrightarrow{\langle a, a' \rangle} \mathcal{U}(A) \times \mathcal{U}(A) \cong \mathcal{U}(A \diamond A) \xrightarrow{\mathcal{U}(\nabla_A)} \mathcal{U}(A). \quad (8)$$

In Example 3 we observe the special property that

$$a \boxplus \mathcal{U}(u_A) = a, \quad (9)$$

since $\mathcal{U}(\nabla)$ coincides with vector addition. We will show in Lemma 3 that this is true in a model of DiLL.

For notational convenience, when \mathcal{U} preserves binary products we shall work as if it does so strictly, in the sense that we suppress the isomorphism $\mathcal{U}(A \diamond B) \cong \mathcal{U}(A) \times \mathcal{U}(B)$. As a consequence, the projection map $\pi_1 : A \diamond B \rightarrow A$, satisfies $\mathcal{U}(\pi_1) = \pi_1 : \mathcal{U}(A) \times \mathcal{U}(B) \rightarrow \mathcal{U}(A)$, and moreover (considering the case $B = 0$) that the \mathcal{U} preserves the left and right unit maps for 0 on the nose.

Definition 5. A *functorial model of intuitionistic* DiLL consists of the following:

- A monoidal closed category $(\mathcal{L}, \otimes, 1)$, admitting a biproduct $(\diamond, 0)$.
- A cartesian monoidal category (\mathcal{C}, \times, I) .
- A **linear-non-linear adjunction** between \mathcal{L} and \mathcal{C} , which is a strong monoidal adjunction:

$$\begin{array}{ccc} & \xrightarrow{\varepsilon'} & \\ (\mathcal{C}, \times, I) & \perp & (\mathcal{L}, \otimes, 1); \\ & \xleftarrow{\mathcal{U}} & \end{array} \quad (10)$$

– A functor $\mathcal{D} : (I \downarrow \mathcal{C}) \longrightarrow \mathcal{L}$ such that

$$\mathcal{D} \circ (0 \downarrow \mathcal{U}) = \Pi. \quad (11)$$

Note that Equation 11 implies in particular that for any object A of \mathcal{L} we have $\mathcal{D}(\mathcal{U}(u_A)|\mathcal{U}(A)) = A$, and that for any arrow $\ell : A \multimap B$ in \mathcal{L} and any object $a : I \longrightarrow \mathcal{U}(A)$ in $(\mathcal{U} \downarrow \mathcal{C})$ we have: $\mathcal{D}(a|\mathcal{U}(\ell)) = \ell$. It also implies that \mathcal{D} preserves products of objects in $\mathcal{U}(\mathcal{L})$. Most importantly, adjunction 10 also enforces $I \simeq \mathcal{U}(0)$.

Notation We denote the endofunctor on \mathcal{L} from the contravariant adjunction 10 by $! := \mathcal{E}' \circ \mathcal{U}$, the endofunctor on \mathcal{C} by $\square := \mathcal{U} \circ \mathcal{E}'$, the unit by $\eta : \text{id}_{\mathcal{C}} \Rightarrow \square$ and the counit by $d : ! \multimap \text{id}_{\mathcal{L}}$. We moreover write $\nu = \mathcal{E}' \eta_{\mathcal{U}}$ for the comultiplication of the induced comonad $(!, d, \nu)$ on \mathcal{L} .

Following linear logic conventions, we denote an arrow $f \in \mathcal{C}(A, B)$ by $f : A \longrightarrow B$ and an arrow $\ell \in \mathcal{L}(A, B)$ by $\ell : A \multimap B$. In both \mathcal{C} and \mathcal{L} we denote the composite of f and g by $f; g$. We denote by id_A the identity morphism in $A \in \mathcal{C}$ or $A \in \mathcal{L}$, the category being clear from the context.

The chain rule arises from the functoriality of \mathcal{D} :

Lemma 1. *Let $a : I \longrightarrow \mathcal{U}(A)$, $b : I \longrightarrow B$, $f : B \longrightarrow C$, $g : C \longrightarrow D$ be morphisms in \mathcal{C} . Then we have:*

$$\mathcal{D}(\text{id}_I | a) = u_A \text{ and } \mathcal{D}(a|f; g) = \mathcal{D}(a|f); \mathcal{D}(a; f|g)$$

Proof. The former is from the fact that u_A is the unique morphism with the required domain and codomain, the latter is by functoriality of \mathcal{D} . \square

We will use techniques similar to the one used in [BCLS20]: with additive structure, the invariance of linear maps under differentiation imply the product and constant rule of differentiation. We will then use the additive bialgebra modality structure $(!, \nu, d, c, w, \bar{c}, \bar{w})$ [BCLS20, Def 3.4] recalled in Section 2. Comparing with differential categories, the only thing that \mathcal{L} is missing now is a co-derelection $\bar{d} : \text{id} \Rightarrow !$ satisfying the required identities.

Definition 6. *For A an object of \mathcal{L} , we define:*

$$\bar{d}_A := \mathcal{D}(\mathcal{U}(u_A)|\eta_{\mathcal{U}(A)}) : A \multimap !A.$$

Lemma 2. *The transformation $\bar{d} : \text{id}_{\mathcal{L}} \Rightarrow !$ is natural.*

Proof. Observe that we have a functor, which by abuse of notation we denote $u : \mathcal{L} \longrightarrow (I \downarrow \mathcal{C})$, sending A to $\mathcal{U}(u_A) : I \cong \mathcal{U}(0) \longrightarrow \mathcal{U}(A)$. Now, $\eta_{\mathcal{U}(-)}$ lifts to a natural transformation $\mathcal{U} \Rightarrow \mathcal{U} \circ !$. Indeed, by uniqueness of morphisms from 0 in \mathcal{L} , all of the required diagrams commute. \bar{d} is the whiskering of this natural transformation by \mathcal{D} , and so is a natural transformation, as required. \square

Thanks to equation 11, for all morphisms $\ell : !A \longrightarrow B$ in \mathcal{L} we have:

$$\mathcal{D}(\mathcal{U}(u_A)|\ell^\#) = \bar{d}; \ell, \quad (12)$$

where $\ell^\# : \mathcal{U}(A) \longrightarrow \mathcal{U}(B)$ is the transpose of ℓ across the adjunction. In particular, the linear rule $\bar{d}_A; d_A = id_A$ holds for any object A of \mathcal{L} , since the counit morphism d_A transposes to the identity in (12). As a consequence, by making use of proposition [BCLS20, 6.2], we have that the product rule is satisfied:

$$\bar{d}; c = \bar{d} \otimes w + w \otimes \bar{d}. \quad (13)$$

The structure on $!A$ allows us to show an auxiliary lemma, namely that $\mathcal{U}(u_A)$ is the unit for the sum operation defined in (8), that will be useful below when proving Lemma 4.

Lemma 3. *For every pointed object $a : I \longrightarrow \mathcal{U}(A)$, we have*

$$a = a \boxplus \mathcal{U}(u_A)$$

Proof. Consider $a \boxplus \mathcal{U}(u_A) : I \cong (I \times I) \longrightarrow \mathcal{U}(A \diamond A) \longrightarrow \mathcal{U}(A)$. The transpose of this morphism is $\mathcal{E}'(a \times \mathcal{U}(u_A)); \mathcal{E}'(\mathcal{U}(\nabla_A)); d_A = \mathcal{E}'(a) \otimes \bar{w}_A; \bar{c}_A; d_A$ by monoidality of \mathcal{E}' . The bialgebra laws over $!A$, as described for example in [BCLS20, Section 7] give us $id_A \otimes \bar{w}_A; \bar{c}_A = id_A$ and as such $\mathcal{E}'(a) \otimes \bar{w}_A; \bar{c}_A; d_A = \mathcal{E}'(a); d_A$, the transpose of a , so $a = a \boxplus \mathcal{U}(u_A)$ as required.

The following is a partial reinterpretation of Equation 13 with the functor \mathcal{D} .

Lemma 4. *The functor \mathcal{D} can internalize translations, meaning for any object $a : I \longrightarrow \mathcal{U}(A)$ and morphism $(a|g)$ in $(I \downarrow \mathcal{C})$, we have:*

$$\mathcal{D}(a|g) = \mathcal{D}(\mathcal{U}(u_A)|\mathcal{U}(i_A); (id_{\mathcal{U}(A)} \times a); \mathcal{U}(\nabla); g),$$

where i_A is the isomorphism $A \longrightarrow A \diamond 0$ inverse to the projection π_1 .

Proof. Consider the diagram in $(I \downarrow \mathcal{C})$:

$$\begin{array}{ccccccc}
 & & & I & & & \\
 & & & \downarrow & & & \\
 & & & \langle \mathcal{U}(u_A), id_I \rangle & & \langle \mathcal{U}(u_A), a \rangle & \\
 & & & \downarrow & & \downarrow & \\
 \mathcal{U}(A) & \xrightarrow{\mathcal{U}(i_A)} & \mathcal{U}(A) \times I & \xrightarrow{\mathcal{U}(id_A) \times a} & \mathcal{U}(A) \times \mathcal{U}(A) & \xrightarrow{\mathcal{U}(\nabla_A)} & \mathcal{U}(A) \xrightarrow{g} B
 \end{array}$$

Which wish to show that the image under \mathcal{D} of this composite morphism is equal to $\mathcal{D}(a|g)$. By compositionality, we may without loss of generality assume $g = id_{\mathcal{U}(A)}$ and show that the remaining composite is mapped to the identity. Indeed, the image under \mathcal{D} is:

$$A \xrightarrow{i} A \times 1 \xrightarrow{id_A \times u_A} A \times A \xrightarrow{\nabla_A} A$$

We know the image of the left-hand and right-hand morphisms by the assumption that \mathcal{D} cancels \mathcal{U} , and the middle morphism by the further assumption that \mathcal{D} preserves products on images of \mathcal{U} and $\mathcal{D}(id_I | a) = u_A$ by Lemma 1. Thus by the left unit law for i and ∇ , the composite is the identity, as required. \square

A further useful identity is the following.

Lemma 5. For $a : I \longrightarrow \mathcal{U}(A)$, we have:

$$\mathcal{D}(a|\eta_{\mathcal{U}(A)}) = j_A; \bar{d}_A \otimes \mathcal{E}'(a); \bar{c}_A, \quad (14)$$

where $j_A : A \multimap A \otimes 1$ is the right unit map for \otimes .

Proof. By direct calculation:

$$\begin{aligned} \mathcal{D}(a|\eta_{\mathcal{U}(A)}) &= \mathcal{D}(\mathcal{U}(u_A)|\mathcal{U}(i_A); (id_{\mathcal{U}(A)} \times a); \mathcal{U}(\nabla); \eta_{\mathcal{U}(A)}) \text{ by Lemma 4} \\ &= \mathcal{D}(\mathcal{U}(u_A) | (!i_A; \mathcal{E}'(id_{\mathcal{U}(A)} \times a); !\nabla)^\#) \text{ by naturality of } \eta \\ &= \bar{d}_A; !i_A; \mathcal{E}'(id_{\mathcal{U}(A)} \times a); !\nabla \text{ by (12)} \\ &= \bar{d}_A; !i_A; id_{!A} \otimes \mathcal{E}'(a); \bar{c}_A \text{ by monoidality of } \mathcal{E}' \text{ and definition of } \bar{c} \\ &= j_A; \bar{d} \otimes \mathcal{E}'(a); \bar{c}_A \text{ by naturality of the right unit for } \otimes, \end{aligned}$$

noting that $!$ being strong monoidal implies $!i_A = j_A$ up to the suppressed isomorphism. \square

This identity enables us to generalize (12).

Lemma 6. For a morphism $\ell : !A \multimap B$ in \mathcal{L} and $a : I \longrightarrow \mathcal{U}(A)$,

$$\mathcal{D}(a|\ell^\#) = j_A; \bar{d}_A \otimes \mathcal{E}'(a); \bar{c}_A; \ell.$$

Proof. As for equation (12), we have:

$$\begin{aligned} \mathcal{D}(a|\ell^\#) &= \mathcal{D}(a|\eta_{\mathcal{U}(A)}; \mathcal{U}(\ell)) \\ &= \mathcal{D}(a|\eta_{\mathcal{U}(A)}); \mathcal{D}(a; \eta_{\mathcal{U}(A)}|\mathcal{U}(\ell)) \text{ thanks to Lemma 1} \\ &= \mathcal{D}(a|\eta_{\mathcal{U}(A)}); \ell \text{ by (11)} \\ &= j_A; \bar{d}_A \otimes \mathcal{E}'(a); \bar{c}_A; \ell \text{ by Lemma 5.} \square \end{aligned}$$

Lemma 7. The co-dereliction \bar{d}_A satisfies the chain rule without a context (alternate chain rule $[dC.A']$ in [BCLS20]):

$$\bar{d}_A; \nu_A = j_A; (\bar{d}_A \otimes !u_A); (\bar{d}_{!A} \otimes \nu_A); \bar{c}_{!A}$$

Proof. Consider the following diagram in $(I \downarrow \mathcal{C})$,

$$\begin{array}{ccccc} I & \xrightarrow{\mathcal{U}(u_A)} & \mathcal{U}(A) & \xrightarrow{\eta_{\mathcal{U}(A)}} & \mathcal{U}(!A) \\ \eta_I \downarrow & \searrow \text{dashed} & \downarrow \eta_{\mathcal{U}(A)} & \searrow \text{dashed} & \downarrow \mathcal{U}\nu_A \\ \mathcal{U}(1) & \xrightarrow{\mathcal{U}(!u_A)} & \mathcal{U}(!A) & \xrightarrow{\eta_{\mathcal{U}(!A)}} & \mathcal{U}(!!A) \end{array}$$

which is obtained by pasting together naturality squares for η , recalling that $\nu = \mathcal{E}'\eta_{\mathcal{U}(A)}$. The left-hand dashed arrow is $(!u_A)^\#$ and the right-hand dashed arrow is $\nu^\#$. Applying \mathcal{D} to the upper and lower paths and applying (12), we conclude:

$$\bar{d}_A; \nu_A = \bar{d}_A; \mathcal{D}\left((!u_A)^\# \mid \eta_{\mathcal{U}(!A)}\right).$$

Applying Lemma 5, we have:

$$\mathcal{D}\left((!u_A)^\# \mid \eta_{\mathcal{U}(!A)}\right) = j_{!A}; \bar{d}_{!A} \otimes \mathcal{E}'((!u_A)^\#); \bar{c}_{!A},$$

but $\mathcal{E}'(!u_A)^\#$ simplifies to $!u_A; \nu_A$. Putting this together, we conclude

$$\bar{d}_A; \nu_A = \bar{d}_A; j_{!A}; (\bar{d}_{!A} \otimes (!u_A; \nu_A)); \bar{c}_{!A},$$

which can be manipulated into to the stated expression by exploiting naturality of j and rearranging parallel tensored terms. \square

For this coderelection to fully interpret DiLL, one needs the commutation of \bar{d} with promotion within a context, or stated otherwise one need the validity of the chain rule at any point.

Lemma 8. *If \mathcal{C} is well-pointed with respect to I , then the co-derelection \bar{d}_A agrees with the generalized chain rule (Equation 3):*

$$\bar{d}_A \otimes \text{id}_{!A}; \bar{c}_A; \nu_A = \bar{d}_A \otimes c_A; \bar{c}_A \otimes \text{id}_{!A}; \bar{d}_{!A} \otimes \nu_A; \bar{c}_{!A}.$$

Proof. Using the isomorphism of the adjunction and the hypothesis that \mathcal{C} is well-pointed, for morphisms $f, g : A \rightarrow B$ we have $f = g$ if and only if for all $a : I \rightarrow \mathcal{U}(A)$, $\mathcal{E}'(a); f = \mathcal{E}'(a); g$. So we just need to prove:

$$\text{id}_A \otimes \mathcal{E}'(a); \bar{d}_A \otimes \text{id}_{!A}; \bar{c}_A; \nu_A = \text{id}_A \otimes \mathcal{E}'(a); \bar{d}_A \otimes c_A; \bar{c}_A \otimes \text{id}_{!A}; \bar{d}_{!A} \otimes \nu_A; \bar{c}_{!A}.$$

By Lemma 6,

$$\begin{aligned} \mathcal{D}(a|\nu_A^\#) &= j_A; \bar{d}_A \otimes \mathcal{E}'(a); \bar{c}_A; \nu_A. \\ &= j_A; \text{id}_A \otimes \mathcal{E}'(a); \bar{d}_A \otimes \text{id}_{!A}; \bar{c}_A; \nu_A. \end{aligned}$$

which corresponds to the left-side of our equation.

On the other hand, generalizing the proof of Lemma 7, consider the diagram,

$$\begin{array}{ccccc} I & \xrightarrow{a} & \mathcal{U}(A) & \xrightarrow{\eta_{\mathcal{U}(A)}} & \mathcal{U}(!A) \\ & \searrow & \downarrow \eta_{\mathcal{U}(A)} & \searrow & \downarrow \mathcal{U}\nu_A \\ & & \mathcal{U}(!A) & \xrightarrow{\eta_{\mathcal{U}(!A)}} & \mathcal{U}(!!A), \end{array}$$

we see that

$$\mathcal{D}(a|\nu_A^\#) = \mathcal{D}(a|\eta_{\mathcal{U}(A)}); \mathcal{D}(a; \eta_{\mathcal{U}(A)}|\eta_{\mathcal{U}(!A)}),$$

which expands via two applications of Lemma 5 to conclude:

$$\begin{aligned} \mathcal{D}(a|\nu_A^\#) &= j_A; \bar{d}_A \otimes \mathcal{E}'(a); \bar{c}_A; j_{!A}; \bar{d}_{!A} \otimes \mathcal{E}'(a; \eta_{\mathcal{U}(A)}); \bar{c}_{!A} \\ &= j_A; \bar{d}_A \otimes \mathcal{E}'(a); \bar{c}_A; j_{!A}; \bar{d}_{!A} \otimes (\mathcal{E}'(a); \nu_A); \bar{c}_{!A} \\ &= j_A; \bar{d}_A \otimes \mathcal{E}'(a) \otimes \mathcal{E}'(a); \bar{c}_A \otimes \text{id}; \bar{d}_{!A} \otimes \nu_A; \bar{c}_{!A} \\ &= j_A; \text{id}_A \otimes \mathcal{E}'(a); \bar{d}_A \otimes c_A; \bar{c}_A \otimes \text{id}; \bar{d}_{!A} \otimes \nu_A; \bar{c}_{!A} \end{aligned}$$

The third line is by associativity of \otimes and neutrality of I while the last one holds as $\mathcal{E}'(a); c_A = \mathcal{E}'(a) \otimes \mathcal{E}'(a)$. \square

We just proved the following theorem, and then prove its converse.

Theorem 1. *A functorial model of Intuitionistic DiLL in which \mathcal{L} is well-pointed results in a differential category (Definition 2).*

Theorem 2. Any differential category is a functorial model of Intuitionistic DiLL, as axiomatized by definition 5.

Proof. Consider \mathcal{L} a differential category, with notations as in Definition 2. Let us note $\mathcal{C} = \mathcal{L}_!$ the co-Kleisli, \times the product on $\mathcal{L}_!$ inherited by \diamond . Then define the functor $\mathcal{D} : 0 \downarrow \mathcal{C} \longrightarrow \mathcal{L}$ by:

$$\mathcal{D} \begin{cases} (a : !0 \longrightarrow A) \mapsto A \\ (a|f) \mapsto !a \otimes \bar{d}; \bar{c}; f \end{cases}$$

The image of \mathcal{D} on function is the exact categorical translation of $D_a f$, given the intuitions given in Example 1. Then \mathcal{D} is functorial. On one hand we have by definition and by the chain rule of Definition 2:

$$\begin{aligned} \mathcal{D}((a|f)(f(a); g)) &= \mathcal{D}((a|\nu; !f; g)) = !a \otimes \bar{d}; \bar{c}; \nu; !f; g \\ &= (!a \otimes id); id_{!A} \otimes \bar{d}; \bar{c}; \nu; !f; g \\ &= (!a \otimes id); c \otimes \bar{d}; 1 \otimes \bar{c}; \nu \otimes \bar{d}; \bar{c}; !f; g \end{aligned}$$

On the other hand we have

$$\begin{aligned} \mathcal{D}((a|f))\mathcal{D}((f(a); g)) &= !a \otimes \bar{d}; \bar{c}; f; !(a; f) \otimes \bar{d}; \bar{c}; g \\ &= !a \otimes \bar{d}; \bar{c}; !a \otimes \bar{d}; \bar{c}; !f; g \text{ by naturality} \\ &= (!a \otimes id); c \otimes \bar{d}; 1 \otimes \bar{c}; \nu \otimes \bar{d}; \bar{c}; !f; g \text{ hence the result} \end{aligned}$$

The fact that $\mathcal{D}(a|(d; \ell)) = \ell$ follows directly from $\bar{d}; d = id$, as

$$\mathcal{D}(a|(d; \ell)) = !a \otimes \bar{d}; \bar{c}; d; \ell = (!a; d) \otimes id; \bar{c}; \ell = !u_A \otimes id; \bar{c}; \ell.$$

\mathcal{D} preserves product by definition, and property 9 is straightforward. \square

4 Generalizing chiralities

In this section, we define categorical models of classical DiLL in terms of pairs of adjunctions. Definition 5 asked that \mathcal{D} should be a (Π -relative) left inverse to the functor $(0 \downarrow \mathcal{U})$. A sufficient condition for this is that \mathcal{D} might be a Π -relative reflection or coreflection. We now make this intuition more formal, by making it fit into a framework that generalizes chiralities, as introduced in Section 2.2.

Alternatively, the following definitions can be seen as a way to incorporate *-autonomy into functorial models of DiLL, and as such define models of classical DiLL. We first give a definition in the well-pointed case, generalizing it afterwards to eliminate that hypothesis.

4.1 \mathcal{D} as an adjoint functor in the well-pointed case

If \mathcal{L} is *-autonomous, then the linear–non-linear adjunction 10 at the heart of Linear Logic can be reformulated as follows:

$$\begin{array}{ccc} & \xrightarrow{\varepsilon} & \\ (\mathcal{C}, \times, I) & \perp & (\mathcal{L}^{op}, \otimes, 1) \\ & \xleftarrow{\eta} & \end{array} \quad (15)$$

where \mathcal{U}^\perp is the composition of \mathcal{U} with the interpretation of $(-)^\perp$, and \mathcal{E} is thought of as the contravariant hom-set of non-linear scalar maps $\mathcal{E}(A) = \mathcal{C}(A, \mathbb{K})$. Likewise, we can either express \mathcal{D} as a functor from $(\mathcal{U}(0) \downarrow \mathcal{C})$ to \mathcal{L} or as a backward differentiation functor ${}^\perp\mathcal{D} : (\mathcal{U}(0) \downarrow \mathcal{C})^{\text{op}} \rightarrow \mathcal{L}$, intuitively mapping $(a|f)$ to $(D_a f)'$, the dual of the linear map $(D_a f)$.

Recall the adjunctions involved in a chirality (Definition 3):

$$\begin{aligned} (-)^{\perp P} : \mathcal{P} &\rightarrow \mathcal{N}^{\text{op}} \dashv (-)^{\perp N} : \mathcal{N}^{\text{op}} \rightarrow \mathcal{P} \\ \uparrow : \mathcal{P} &\rightarrow \mathcal{N} \dashv \downarrow : \mathcal{N} \rightarrow \mathcal{P} \end{aligned}$$

Beware of the notational coincidence: the functor \downarrow has nothing to do with slice categories. If $(\mathcal{E} \dashv \mathcal{U}^\perp)$ is to be compared with $((-)^{\perp P} \dashv (-)^{\perp N})$, we would like to relate $\mathcal{D} : (\mathcal{I} \downarrow \mathcal{C}) \rightarrow \mathcal{L}$ to \uparrow by showing that \mathcal{D} acts on \mathcal{E} and \mathcal{U}^\perp analogously to how \uparrow acts on $(-)^{\perp P}$ and $(-)^{\perp L}$. We can reconcile the evident difference of typing of functors between differentiation and polarities by noticing that in an additive setting $(0^\perp \downarrow \mathcal{P}) \simeq \mathcal{P}$ and as such one can present \uparrow as a functor $(0^\perp \downarrow \mathcal{P}) \rightarrow \mathcal{N}$. To recover an adjunction involving \mathcal{D} , we need to consider a contravariant functor modelling reverse differentiation:

$${}^\perp\mathcal{D} : (\mathcal{U}(0) \downarrow \mathcal{C})^{\text{op}} \rightarrow \mathcal{L}.$$

With this goal in mind, we define the structure below and show that it generalizes chiralities (Proposition 2) and gives our model of DiLL (Proposition 1).

Definition 7. A generalized chirality consists of two monoidal categories (\mathcal{A}, \otimes) and (\mathcal{B}, \otimes) , where \mathcal{B} has terminal object 0 , and four functors:

- $\mathcal{X} : \mathcal{A} \rightarrow \mathcal{B}^{\text{op}}$ and $\mathcal{Y} : \mathcal{B}^{\text{op}} \rightarrow \mathcal{A}$,
- $\mathcal{F} : (\mathcal{G}(0) \downarrow \mathcal{A})^\alpha \rightarrow \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \rightarrow \mathcal{A}^\alpha$,

where α is either empty or op , so that \mathcal{F} and \mathcal{G} might be covariant or contravariant functors. We ask these four functors to form a pair of adjunctions, with the first being strong monoidal,

$$\begin{array}{ccc} (\mathcal{A}, \otimes) & \begin{array}{c} \xrightarrow{\mathcal{X}} \\ \perp \\ \xleftarrow{\mathcal{Y}} \end{array} & (\mathcal{B}^{\text{op}}, \otimes) \\ (\mathcal{G}(0) \downarrow \mathcal{A}) & \begin{array}{c} \xleftarrow{(\text{0}\downarrow\mathcal{G})} \\ \downarrow \Pi_{\mathcal{A}} \quad \searrow \mathcal{F} \quad \downarrow \Pi_{\mathcal{B}} \\ \mathcal{A}^\alpha \quad \xleftarrow{\mathcal{G}} \quad \mathcal{B} \end{array} & (0 \downarrow \mathcal{B}) \end{array} \quad (16)$$

where on the right-hand side we require that there are natural isomorphisms,

$$\mathcal{A}(\Pi_{\mathcal{A}}(-), \mathcal{G}(-)) \simeq \mathcal{B}(\mathcal{F}(-), \mathcal{Y} \circ \mathcal{X}(-)) \quad (17)$$

and

$$\mathcal{F} \circ (0 \downarrow \mathcal{G}) \simeq \Pi_{\mathcal{B}}. \quad (18)$$

Proposition 1. Consider the case when \mathcal{B} is a well-pointed $*$ -autonomous additive category, $(\mathcal{A}, \otimes, I)$ is cartesian monoidal with terminal object I , and $\alpha = \text{op}$. Then a generalized chirality such that $\mathcal{Y} = \mathcal{G}^\perp$ makes \mathcal{B} a differential category, and as such a categorical model of classical DiLL.

Proof. We will simply show that we fit in the setting of Definition 5 and let Theorem 1 conclude. We adapt the notation of Definition 5 to place ourselves in the following situation:

$$\begin{array}{ccc}
 (\mathcal{C}, \times) & \xrightarrow{\mathcal{E}} & (\mathcal{L}^{op}, \mathfrak{F}) \\
 & \perp & \\
 & \xleftarrow{U^\perp} & \\
 \end{array}
 \quad
 \begin{array}{ccc}
 (I \downarrow \mathcal{C}^{op}) & \xleftarrow{0 \downarrow U^\perp} & (0 \downarrow \mathcal{L}) \\
 \downarrow \Pi_{\mathcal{C}} & \searrow \uparrow \mathcal{D} & \downarrow \Pi_{\mathcal{L}} \\
 \mathcal{C}^{op} & \xleftarrow{U^\perp} & \mathcal{L}
 \end{array}$$

Let us reproduce equation 17:

$$\mathcal{C}^{op}(U^\perp(-), \Pi_{\mathcal{C}}(-)) \simeq \mathcal{L}(\mathcal{E} \circ U(-)^\perp, {}^\perp\mathcal{D}(-))$$

By definition and *-autonomy of \mathcal{L} we have ${}^\perp\mathcal{D} = (-)^\perp \circ \mathcal{D}$. As ${}^\perp\mathcal{D} \circ (I \downarrow U^\perp) = \Pi$ by definition (equation 18), we recover $\mathcal{D} \circ (I \downarrow U) = \Pi$. \square

Proposition 2. *A generalized chirality leads to a negative chirality under these hypotheses:*

1. $\alpha = id$, meaning \mathcal{F} and \mathcal{G} are covariant functors,
2. 0 is initial in \mathcal{B} and $\mathcal{G}(0)$ is initial in \mathcal{A} ,
3. \mathcal{X} and \mathcal{Y} form a reflection,
4. Equation 3 and the commutativity diagram of definition 3 hold.

In particular, if \mathcal{A} is additive then a generalized chirality on \mathcal{A} is precisely a chirality.

Proof. Let's adopt the notation from definition 3, with $\mathcal{A} = \mathcal{P}$ and $\mathcal{B} = \mathcal{N}$.

$$\begin{array}{ccc}
 (\mathcal{P}, \otimes) & \xrightarrow{(-)^\perp_P} & (\mathcal{N}^{op}, \mathfrak{F}) \\
 & \perp & \\
 & \xleftarrow{(-)^\perp_N} & \\
 \end{array}
 \quad
 \begin{array}{ccc}
 (I \downarrow \mathcal{P}) & \xleftarrow{(0 \downarrow)} & (0 \downarrow \mathcal{N}) \\
 \downarrow \Pi_{\mathcal{P}} & \searrow \uparrow \mathcal{Q} & \downarrow \Pi_{\mathcal{N}} \\
 \mathcal{P} & \xleftarrow{\quad} & \mathcal{N}
 \end{array}$$

As I and 0 are initial in \mathcal{P} and \mathcal{N} respectively, one has $(0 \downarrow \downarrow) \cong \downarrow$ up to the equivalences Π , so the right diagram amounts to:

$$\begin{array}{ccc}
 (I \downarrow \mathcal{P}) & \xleftarrow{(\downarrow)} & (0 \downarrow \mathcal{N}) \\
 \downarrow \cong & \searrow \uparrow \mathcal{Q} & \downarrow \cong \\
 \mathcal{P} & \xleftarrow{\quad} & \mathcal{N}
 \end{array}$$

Equation 17 combined with the fact that $(-)^\perp_N \circ (-)^\perp_P = id$ gives us that \uparrow is left adjoint to \downarrow . Equation 18 gives us that they form a reflection. \square

Proposition 3. *Any *-autonomous differential category results in a generalized chirality.*

Proof. Let us borrow the notations from Theorem 2, and define $\mathcal{B} = \mathcal{A}^{op} = \mathcal{L}$, $\mathcal{A} = \mathcal{L}$, $\mathcal{X} = (!)^\perp$. \mathcal{Y} is the contravariant forgetful functor from \mathcal{L} to \mathcal{L} defined by $\mathcal{Y}(A) = A^\perp$ and $\mathcal{Y}(\ell) = d; \ell^\perp$, and $\mathcal{G} = \mathcal{Y}^\perp$. We just need to show that, when \mathcal{L} is *-autonomous with duality $(-)^\perp$, the functor ${}^\perp\mathcal{D} : 0 \downarrow \mathcal{C} \rightarrow \mathcal{L}^{op}$ verifies indeed equation 17. This equation translates as :

$$\begin{aligned}
 \mathcal{L}_!^\perp(A, B^\perp) &\simeq \mathcal{L}({}^\perp\mathcal{D}(A), !(B^\perp)^\perp) \\
 \mathcal{L}(!B^\perp, A) &\simeq \mathcal{L}(A^\perp, !(B^\perp)^\perp)
 \end{aligned}$$

which holds as \mathcal{L} is *-autonomous. \square

4.2 Avoiding well-pointedness

Definition 5 included a well-pointedness hypotheses on the linear category \mathcal{L} exclusively for the purpose of proving Proposition 1. This is not an inherent problem, since our motivating example of the functor \mathcal{D} is most readily specified on elements of $(\mathcal{U} \downarrow \mathcal{C})$, which is to say pointed objects of \mathcal{C} . In this section, we examine how the hypothesis of well-pointedness of the linear category \mathcal{L} or \mathcal{C} may be dropped.

The simplest option is to simply drop the version of the chain rule in Definition 2. Alternatively, one can generalize by taking the domain of \mathcal{D} to be a broader category, such as:

$$\mathcal{D} : (\mathcal{U} \downarrow \mathcal{C}) \longrightarrow \mathcal{L}.$$

Objects of $(\mathcal{U} \downarrow \mathcal{C})$ are triples (E, A, f) with $E \in \mathcal{L}$, $A \in \mathcal{C}$ and $f \in \mathcal{C}(\mathcal{U}(E), A)$, and morphisms in $\mathcal{U} \downarrow \mathcal{C}$ are pairs of a morphism $\ell \in \mathcal{L}(E_1, E_2)$ and $g \in \mathcal{C}(A_1, A_2)$ making the following commute:

$$\begin{array}{ccc} \mathcal{U}(E_1) & \xrightarrow{f_1} & A_1 \\ \downarrow \mathcal{U}(\ell) & & \downarrow g \\ \mathcal{U}(E_2) & \xrightarrow{f_2} & A_2 \end{array}$$

We choose this example because it extends \mathcal{D} when \mathcal{C} and \mathcal{L} have real vectors spaces objects and respectively smooth maps and linear maps as morphisms, by taking:

$$\mathcal{D} \begin{cases} (E, A, f) \mapsto A \\ (\ell, g) \mapsto D_{f(0)}g \end{cases}$$

This example is not really more general, since it is obtained by composing our original definition with the coreflection $(\mathcal{U} \downarrow \mathcal{C}) \longrightarrow (\mathcal{I} \downarrow \mathcal{C})$. We do not have any non-well-pointed examples to hand which might provide strong intuition for what differentiation might mean outside of this example. This is why we chose to present the simpler well-pointed version, but in case it should be desired, the above extends to the definition below.

Theorem 3. *The following structure restricts to the data of a differential category:*

- A monoidal closed category $(\mathcal{L}, \otimes, 1)$, admitting a biproduct \diamond with neutral object 0, which is well-pointed.
- A cartesian monoidal category (\mathcal{C}, \times, I) .
- A **linear-non-linear adjunction** between \mathcal{L} and \mathcal{C} , which is a strong monoidal adjunction:

$$\begin{array}{ccc} & \xrightarrow{\varepsilon'} & \\ (\mathcal{C}, \times, I) & \perp & (\mathcal{L}, \otimes, 1); \\ & \xleftarrow{u} & \end{array} \quad (19)$$

- A functor $\mathcal{D} : (\mathcal{U} \downarrow \mathcal{C}) \longrightarrow \mathcal{L}$ such that

$$\mathcal{D} \circ (\mathcal{U} \downarrow \mathcal{U}) = \Pi.$$

Conversely, any differential category verifies the hypotheses of Theorem 3.

Proof. In the forward direction, a functor $\mathcal{D} : (\mathcal{U} \downarrow \mathcal{C}) \longrightarrow \mathcal{L}$ induces a functor $\mathcal{D} : (I \downarrow \mathcal{C}) \longrightarrow \mathcal{L}$ by composition with the properties required for Lemmas 1, 4, 6 and 7.

Conversely, adapting Theorem 2, consider a differential category \mathcal{L} with the notations of Definition 2. Define $\mathcal{C} = \mathcal{L}$, \mathcal{U} the forgetful functor from \mathcal{L} to \mathcal{L} and $\mathcal{D} : \mathcal{U} \downarrow \mathcal{C} \longrightarrow \mathcal{L}$ via:

$$\mathcal{D} \begin{cases} (E, A, f) \mapsto A \\ ((\ell, g) : (E_1, A_1, f_1) \longrightarrow (E_2, A_2, f_2)) \mapsto \bar{d} \otimes (!u_A; f_1); \bar{d}; g \end{cases} \quad (20)$$

One checks easily that \mathcal{D} is functorial and preserves products, and is such that $\mathcal{D}((\ell, \mathcal{U}(\ell'))) = \ell'$. \square

Beware that Lemma 1 relied on the fact that there is a unique morphism $0 \longrightarrow A$ in \mathcal{L} , which affects Lemma 6, since the identity $\mathcal{U}(\text{id}_A) \boxplus a = a$ would need to be substituted for another identity, possibly imposing the need for additive structure on morphisms in \mathcal{L} . Assuming this is possible, Lemma 8 should work likewise without the well-pointed hypothesis at the beginning by computing $\mathcal{D}(\text{id}_A, \nu_A^\#)$ instead of $\mathcal{D}(a | \nu_A^\#)$. We leave the details of these computations to future work. The definition of chiralities can also be generalized, to make \mathcal{D} a relative right-adjoint in a relative reflection, and Proposition 1 straightforwardly generalizes.

Definition 8. *In the non well-pointed setting, Definition 7 generalized to the following diagram*

$$\begin{array}{ccc} & \mathcal{X} & \\ \curvearrowright & & \curvearrowleft \\ (\mathcal{A}, \mathbb{O}) & \perp & (\mathcal{B}^{\text{op}}, \mathbb{O}) \\ \curvearrowleft & & \curvearrowright \\ & \mathcal{Y} & \end{array} \quad \begin{array}{ccc} (\mathcal{G} \downarrow \mathcal{A})^\alpha & \xleftarrow{(0 \downarrow \mathcal{G})} & (0 \downarrow \mathcal{B}) \\ \downarrow \Pi_{\mathcal{A}} & \searrow \mathcal{F} \circ \Omega & \downarrow \Pi_{\mathcal{B}} \\ \mathcal{A}^\alpha & \xleftarrow{\mathcal{G}} & \mathcal{B} \end{array} \quad (21)$$

with equations

$$\mathcal{A}(\Pi_{\mathcal{A}}(-), \mathcal{G}(-)) \simeq \mathcal{B}(\mathcal{F}(-), \mathcal{Y} \circ \mathcal{X}(-)) \quad \mathcal{F} \circ 0 \downarrow \mathcal{G} = \Pi_{\mathcal{B}}$$

Proposition 4. *A generalized chirality in the sense of Definition 8 gives a differential category when \mathcal{B} is a $*$ -autonomous additive category, $\alpha = \text{op}$ and $(\mathcal{A}, \mathbb{O}, I)$ a category with product \mathbb{O} and terminal object I*

Proposition 2 also generalizes as one would expect, as does Proposition 3; we invite the interested reader to compute the details.

5 Conclusion

In this paper, we gave a categorical definition of models of intuitionistic DiLL, by proving it equivalent to differential categories. This gives a functorial and concise axiomatization of differentiation in the setting of a linear–non-linear adjunction. We also generalized the first definition to model classical DiLL, making it fit into the framework of chiralities, that is into a pair of adjunctions. We give all definitions in the well-pointed and not necessarily well-pointed setting. There are several perspectives opened by this work, which we detail now.

Additives Models of DiLL ask for an additive category of linear maps, and generalized chiralities as defined in Definition 7 and 8 restrict to models of polarized multiplicative LL in particular when they are additive. More generally, when generalized chiralities model multiplicative and exponential connectives of DiLL, the role of additive connectives \oplus and \otimes is not well understood; neither are their categorical interpretations in models of polarized Multiplicative and Additive LL. In [BK20], the authors asked for a coproduct over \mathcal{P} and \mathcal{N} . Can this be refined to shed light on the interaction between polarity and differentiation?

Polarity and Differentiation Once this work stabilizes, we would like to express more precisely what polarized models of DiLL should be, and refine diagram 1 when \mathcal{L} is not $*$ -autonomous but decomposed in a polarized chirality as in diagram 4. Combined with a study of additive within our setting, this might ameliorate Definition 7, unifying the covariant and contravariant case.

Integration One can also notice that when \mathcal{L} is a calculus category [CL17], that is a category with both differentiation and integration, then we have an $! \otimes$ id-relative adjunction between $(I \downarrow \mathcal{C})$ and \mathcal{L} , where the fundamental theorem of calculus is expressed exactly as a relative $! \otimes$ id adjunction between \mathcal{D} and an extension of \mathcal{U} :

$$(I \downarrow \mathcal{C}) \begin{array}{c} \xrightarrow{\mathcal{D}} \\ \perp \\ \xleftarrow{0 \downarrow \mathcal{U}} \end{array} \mathcal{L} \quad (I \downarrow \mathcal{C})(u_A, (b, B)) \simeq \mathcal{L}(!A \otimes A, B)$$

Indeed, this theorem states that given the derivative of a function at every point $Df : a \mapsto v \mapsto D_a(f)(v)$, there is a unique function $f \in \mathcal{C}(A, B)$ such that $f(0) = b$. Reformulated within the setting of a linear–non-linear adjunction, this gives $\mathcal{C}((u_A, A), (b, B)) \simeq \mathcal{C}(A, \mathcal{L}(A, B)) \simeq \mathcal{L}(!A \otimes A, B)$ as \mathcal{L} is monoidal closed. It would be worthwhile to try and make calculus categories fit into the framework of generalized chiralities.

Co-digging While our setting expresses the chain rule, it’s also fun to see what happens in a dual setting with the “co-chain” rule of exponential maps [KL23]. In that situation, $!$ is a monad and an object of a slice of its Kleisli category are morphisms $a' : A \multimap \mathbb{K}$, that is elements of the dual of A . The co-chain rule is expressed in a generalized exponential map $E : \mathcal{L}^! \downarrow I \multimap \mathcal{L}$ and one might explore models of DiLL with co-digging in our framework.

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