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# On the finite element approximation of the obstacle problem of a Naghdi shell

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## ABSTRACT

In this work we consider the finite element approximation of two equivalent formulations of an obstacle problem of a Naghdi shell. This second one is a new formulation of the continuous problem set on the unconstrained space of the displacement field and the rotation. Namely in order to enforce the tangency requirement on the rotation and the inequality constraint, two Lagrange multipliers are introduced. In addition to existence and uniqueness results of solutions of the continuous and the discrete problems we derive *a priori* error estimates. We further prove the convergence of the Uzawa algorithm associated with this variational inequality. Numerical tests that validate and illustrate our approach are given.

## 1. Introduction

Many phenomena are accurately and concisely described by variational inequalities instead of variational equalities. Variational inequalities can be found in mechanics (contact between deformable elastic bodies), in lubrication theory, in flows through porous media, in control theory and in financial mathematics (see [1] and the references therein).

Discretization of obstacle problems without constraint by the finite element method in its primal formulation or using Lagrange multipliers has been considered since many years. The literature on the finite element approximation of such formulations is extremely vast. Let us quote [1–6] to mention a few. However, the case of variational inequalities related to thin structures<sup>1</sup> with state constraints and in particular for thin shells seems to be not widely discussed in the academic community.

The goal of this paper is then to analyse the finite element approximation of two equivalent formulations of the obstacle problem of a Naghdi shell described in [7] (based on the Naghdi shell model introduced in [8,9]). The first one, called the reduced problem, consists in a variational inequality and a variational equality, and the second one, called the full problem, made of a variational inequality and two variational equalities. This second one is a new formulation of the continuous problem set on the unconstrained space of the displacement field and the rotation. More precisely in order to enforce the tangency requirement on the rotation (which is a state constraint) and the inequality constraint, two Lagrange multipliers are introduced. We then introduce a non conforming approximation of the reduced problem, while inspired from [1], we also consider a conforming finite element approximation of the full problem by adding the elementwise  $\mathbb{P}_3$  bubble functions to  $\mathbb{P}_1$  elements to approximate the displacement field in order to guarantee a discrete stability estimate (see [1, Theorem 3.1] and Lemma 4.10 below). In addition to existence and uniqueness results of solutions of the continuous and the discrete problems, we derive an *a priori* error estimate. We further prove the convergence

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<sup>1</sup> Thin structures are three-dimensional bodies in which one dimension/two dimensions is/are small compared to other two/one; they include beams, plates and shells, ....

of the Uzawa algorithm associated with this variational inequality. Even if its speed of convergence may be slow, we have chosen it due to its simplicity of implementation and its minimal memory requirements. Numerical tests that validate and illustrate our approach are finally given.

This paper is organized as follow: In Section 2 we recall some geometrical preliminaries of surfaces and the Naghdi shell model without contact. In Section 3 we present the contact problem described in [7], introduce its two formulations, and prove their well-posedness. Further, some comments on the regularity of the solution are made. Section 4 is devoted to the introduction of the finite element approximation of the two variational problems. For both problems, we prove the existence and uniqueness of the discrete solution and derive *a priori* error estimates assuming that the solution satisfies the  $H^2$ -regularity.

In Section 5 we discuss the Uzawa iterative method applied to the considered variational inequality. Finally numerical experiments are presented in Section 6.

Let us finish this introduction with some notation used in the paper. The usual norm and semi-norm of the Sobolev  $W^{s,p}(\omega, \mathbb{R}^\ell)$  (with  $s \geq 0$ ,  $p \in [1, \infty]$  and  $\ell \in \mathbb{N}$ ) are denoted by  $\|\cdot\|_{s,p,\omega}$  and  $|\cdot|_{s,p,\omega}$ , respectively. For  $s = 0$  (resp.  $p = 2$ ), we drop the index  $s$  (resp.  $p$ ). In the same way, we denote by  $(\cdot, \cdot)_\omega$  the  $L^2(\omega)$ -inner product. The notation  $A \lesssim B$  is used for the estimate  $A \leq C B$ , where  $C$  is a generic constant that does not depend on  $A$  and  $B$ , in particular this constant does not depend the mesh size  $h$ , but it may depend on the thickness of the shell  $\varepsilon$  which is supposed to be a strictly positive constant. The notation  $A \sim B$  means that both  $A \lesssim B$  and  $B \lesssim A$  hold.

### 2. Geometric preliminaries and notations

Let  $\omega$  be a bounded connected domain of  $\mathbb{R}^2$  with a Lipschitz boundary  $\partial\omega$ . We assume given a splitting of  $\partial\omega$  into two open subsets  $\gamma_0$  and  $\gamma_1$  such that  $\gamma_0$  is non empty,  $\partial\omega = \bar{\gamma}_0 \cup \bar{\gamma}_1$ , and  $\gamma_0 \cap \gamma_1 = \emptyset$ . We consider a shell whose midsurface is given by  $S = \varphi(\omega)$ , where  $\varphi \in W^{2,\infty}(\omega, \mathbb{R}^3)$  is a one-to-one mapping such that the two vectors

$$a_\alpha(x) = (\partial_\alpha \varphi)(x)$$

are linearly independent at each point  $x$  of  $\omega$ . Thus,

$$a_3(x) = \frac{a_1(x) \wedge a_2(x)}{|a_1(x) \wedge a_2(x)|}$$

is the unit normal vector on the midsurface at point  $\varphi(x)$ . The vectors  $a_i(x)$  define the local covariant basis at point  $\varphi(x)$ . The contravariant basis  $a^i(x)$  is defined by the relations  $a_i \cdot a^j = \delta_i^j$  where  $\delta_i^j$  is the Kronecker symbol. Note that all these vectors belong to  $W^{1,\infty}(\omega, \mathbb{R}^3)$ . Note that, Greek indices and exponents take their values in the set  $\{1, 2\}$  and Latin indices and exponents are in the set  $\{1, 2, 3\}$ . The first and second fundamental forms of the surface are given by  $a_{\alpha\beta} = a_\alpha \cdot a_\beta$  and  $b_{\alpha\beta} = a_3 \cdot \partial_\beta a_\alpha$ . The area element of the midsurface in the chart  $\varphi$  is given by  $\sqrt{a(x)}$  with  $a(x) = |a_1(x) \wedge a_2(x)|^2$ . Similarly, the length element  $\ell$  on the boundary  $\partial\omega$  is given by  $\sqrt{a_{\alpha\beta} \tau_\alpha \tau_\beta}$ , with the standard summation convention for repeated indices and exponents, the  $a^{\alpha\beta} = a^\alpha \cdot a^\beta$  being the contravariant components of the first fundamental form and  $(\tau_1, \tau_2)$  being the covariant coordinates of the unit vector tangent to  $\partial\omega$ .

In this paper, the thickness of the shell, denoted by  $\varepsilon$ , is supposed to be positive and constant. Hence the shell is given by

$$S = \{\varphi(x) + za_3(x) \mid x \in \omega, z \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})\}$$

Here,  $z$  represents the distance of a point of the shell to its midsurface. Let also  $\{e_1, e_2, e_3\}$  denote the Cartesian basis in  $\mathbb{R}^3$ . Following [7], we are interested in studying the contact of this shell with a rigid obstacle contained in the half-space  $z \cdot e_3 < 0$  and such that its boundary occupies the whole plane  $z \cdot e_3 = 0$ . So, from now on, we assume without restriction that the function  $\varphi$  satisfies  $\varphi(x) \cdot e_3 > 0$  and also  $a_3(x) \cdot e_3 \geq 0$ , for all  $x \in \bar{\omega}$ . Thus, the contact occurs on part of the lower surface of the shell, namely on the surface  $\{\varphi(x) - \frac{\varepsilon}{2}a_3, x \in \omega\}$ .

We consider the case of a homogeneous, isotropic material with Young modulus  $E > 0$  and Poisson ratio  $\nu$ ,  $0 \leq \nu < \frac{1}{2}$ . The contravariant components of the elasticity tensor  $a^{\alpha\beta\rho\sigma} \in L^\infty(\omega)$  are given by

$$a^{\alpha\beta\rho\sigma} = \frac{E}{2(1+\nu)}(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}) + \frac{E\nu}{1-\nu^2}a^{\alpha\beta}a^{\rho\sigma}. \tag{1}$$

This tensor satisfies the usual symmetry properties and is uniformly strictly positive, in the sense that there exists a positive constant  $c$  such that

$$a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta} \gamma_{\rho\sigma} \geq c \sum_{1 \leq \alpha, \beta \leq 2} |\gamma_{\alpha\beta}|^2,$$

for all  $2 \times 2$  symmetric real-valued matrices  $\gamma = (\gamma_{\alpha\beta})_{1 \leq \alpha, \beta \leq 2}$ .

In this context, the covariant components of the change of metric tensor are

$$\gamma_{\alpha\beta}(u) = \frac{1}{2}(\partial_\alpha u \cdot a_\beta + \partial_\beta u \cdot a_\alpha), \tag{2}$$

the covariant components of the change of transverse shear tensor are

$$\delta_{\alpha 3}(u, r) = \frac{1}{2}(\partial_\alpha u \cdot a_3 + r \cdot a_\alpha), \tag{3}$$

and the covariant components of the change of curvature tensor are

$$\chi_{\alpha\beta}(u, r) = \frac{1}{2}(\partial_\alpha u \cdot \partial_\beta a_3 + \partial_\beta u \cdot \partial_\alpha a_3 + \partial_\alpha r \cdot a_\beta + \partial_\beta r \cdot a_\alpha). \tag{4}$$

Following [10], when there is no obstacle, the variational problem of the Naghdi shell reads

$$\text{Find } U = (u, r) \in \mathbb{V} \quad \text{such that} \quad a(U, V) = \mathcal{L}(V), \quad \forall V \in \mathbb{V}, \tag{5}$$

where the space  $\mathbb{V}$  is given by

$$\mathbb{V} = \left\{ V = (v, s) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \mathbb{R}^3) \mid s \cdot a_3 = 0 \text{ in } \omega, \quad v|_{\gamma_0} = s|_{\gamma_0} = 0 \right\}, \tag{6}$$

the bilinear form  $a$  is defined by

$$a(U, V) := \int_\omega (\{\varepsilon a^{\alpha\beta\rho\sigma} [\gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) + \frac{\varepsilon^2}{12}\chi_{\alpha\beta}(U)\chi_{\rho\sigma}(V)] + \frac{2\varepsilon E}{1+v} a^{\alpha\beta}\delta_{\alpha 3}(U)\delta_{\beta 3}(V)\}) \sqrt{ad}x, \tag{7}$$

where  $\varepsilon$  represents the thickness of the shell that is here supposed to be a positive constant, and the linear form  $\mathcal{L}(\cdot)$  is given by

$$\mathcal{L}(V) := \int_\omega f \cdot v \sqrt{ad}x + \int_{\gamma_1} (N \cdot v + M \cdot s)\ell \, d\tau. \tag{8}$$

The data  $f \in L^2(\omega)^3$ ,  $N \in L^2(\gamma_1)^3$  and  $M \in L^2(\gamma_1)^3$  represent a given resultant force density, an applied traction density and an applied moment density, respectively.

### 3. The contact problem

First, let us now introduce the following functional spaces:

$$\mathbb{M} := H^1_{\gamma_0}(\omega) = \{ \mu \in H^1(\omega); \quad \mu = 0 \text{ on } \gamma_0 \}. \tag{9}$$

$$\mathbb{X} := H^1_{\gamma_0}(\omega, \mathbb{R}^3) \times H^1_{\gamma_0}(\omega, \mathbb{R}^3). \tag{10}$$

The space  $\mathbb{X}$  is endowed with the following natural Hilbert norm

$$\|V\|_{\mathbb{X}} = \left( \|v\|_{1,\omega}^2 + \|s\|_{1,\omega}^2 \right)^{\frac{1}{2}}, \quad \forall V = (v, s) \in \mathbb{X}. \tag{11}$$

Obviously the forms  $a(\cdot, \cdot)$  and  $\mathcal{L}$  defined in (7) and (8), respectively, are also defined (and continuous) on  $\mathbb{X} \times \mathbb{X}$  and  $\mathbb{X}$ . The space  $\mathbb{V}$  is then the kernel of the bilinear form  $b(\cdot, \cdot)$  given by

$$\forall (V, \chi) \in \mathbb{X} \times \mathbb{M}, \quad b(V, \chi) = \int_\omega \partial_\alpha (s \cdot a_3) \partial_\alpha \chi \, dx. \tag{12}$$

Following [7] the obstacle problem of Naghdi's shell reads:

**Problem 1.** Find  $(U, \psi)$  in  $\mathbb{N}_\Phi \times \mathbb{M}$  such that

$$\begin{aligned} \forall V \in \mathbb{N}_\Phi, \quad a(U, V - U) + b(V - U, \psi) &\geq \mathcal{L}(V - U), \\ \forall \chi \in \mathbb{M}, \quad b(U, \chi) &= 0, \end{aligned} \tag{13}$$

where  $\mathbb{N}_\Phi$  is a closed convex set of  $\mathbb{X}$  defined by

$$\mathbb{N}_\Phi := \left\{ V = (v, s) \in \mathbb{X}; \quad \left( v - \frac{\varepsilon}{2}s \right) \cdot e_3 \geq \Phi \text{ a.e. in } \omega \right\}. \tag{14}$$

The function  $\Phi$  belongs to the space  $W^{1,\infty}(\omega)$  and is given by

$$\Phi(x) = \left( \frac{\varepsilon}{2} a_3(x) - \varphi(x) \right) \cdot e_3. \tag{15}$$

The contact model is meaningless without the following condition,

$$\Phi(x) \leq 0, \quad \forall x \in \omega, \tag{16}$$

which follows from the positivity of  $\varphi \cdot e_3$  and the non-negativeness of  $a_3 \cdot e_3$  when the thickness  $\varepsilon$  is not too large and the shell is not flat. Indeed from the definition of  $\Phi$ , (16) holds if

$$\frac{\varepsilon}{2} \max_{x \in \bar{\omega}} a_3(x) \cdot e_3 \leq \min_{x \in \bar{\omega}} \varphi(x) \cdot e_3. \tag{17}$$

Consequently if the shell is flat, namely if  $a_3 \cdot e_3 = 0$  on  $\omega$ , then (17) holds (and hence (16)), for all  $\varepsilon > 0$ . On the contrary, if  $\max_{x \in \bar{\omega}} a_3(x) \cdot e_3 > 0$ , then (17) holds if and only if

$$\varepsilon \leq \varepsilon_0 := 2 \frac{\min_{x \in \bar{\omega}} \varphi(x) \cdot e_3}{\max_{x \in \bar{\omega}} a_3(x) \cdot e_3}.$$

Therefore (16) holds under this last constraint on  $\varepsilon$ .

Since  $r \cdot a_3 = 0$  a.e. in  $\omega$ , it is straightforward to check that Problem 1 is equivalent to the following problem:

**Problem 2.**

$$\begin{cases} \text{Find } (U, \psi) \in \mathbb{N}_\phi \times \mathbb{M} \text{ such that} \\ \forall V \in \mathbb{N}_\phi, \mathbf{a}_\rho(U, V - U) + \mathbf{b}(V - U, \psi) \geq \mathcal{L}(V - U), \\ \forall \chi \in \mathbb{M}, \mathbf{b}(U, \chi) = 0, \end{cases} \tag{18}$$

where for any real parameter  $\rho > 0$ , we set

$$\mathbf{a}_\rho(U, V) = \mathbf{a}(U, V) + \rho \int_\omega \partial_\alpha(r \cdot a_3) \partial_\alpha(s \cdot a_3) dx, \quad \forall U = (u, r), V = (v, s) \in \mathbb{X}.$$

**Remark 3.1.** Note that the bilinear form  $\mathbf{a}(\cdot, \cdot)$  is not  $\mathbb{X}$ -elliptic (see [10, Lemma 3.3]). Replacing the bilinear form  $\mathbf{a}(\cdot, \cdot), (\cdot, \cdot)$  by  $\mathbf{a}_\rho(\cdot, \cdot), (\cdot, \cdot)$  allows us to recover the ellipticity over the space  $\mathbb{X}$ , where as soon as  $\rho > 0$ .

**3.1. A compact formulation**

It is clear that neither Problem 1 nor Problem 2 is in the “standard” form of variational inequalities, i.e., a single variational inequality. In this subsection following [1], we first rewrite Problem 2 in a compact form involving a single variational inequality set in a closed convex set.

Let us consider the Hilbert space:

$$\mathcal{H} = \mathbb{X} \times \mathbb{M} \tag{19}$$

and the closed convex set

$$\mathbb{K} = \mathbb{N}_\phi \times \mathbb{M}.$$

We define the bilinear form  $\mathcal{A}_\rho : \mathbb{K} \times \mathbb{K} \rightarrow \mathbb{R}$  through

$$\mathcal{A}_\rho((W, \mu); (V, \chi)) := \mathbf{a}_\rho(W, V) + \mathbf{b}(V, \mu) + \mathbf{b}(W, \chi),$$

then, Problem 2 can be written in a compact way as follows:

**Problem 3.** Find  $(U, \psi)$  in  $\mathbb{K}$  such that

$$\mathcal{A}_\rho((U, \psi); (V - U, \chi)) \geq \mathcal{L}(V - U), \quad \forall (V, \chi) \in \mathbb{K}. \tag{20}$$

**Theorem 3.2.** For any data  $(f, N, M) \in L^2(\omega)^3 \times L^2(\gamma_1)^3 \times L^2(\gamma_1)^3$ , Problem 3 admits a unique solution.

Note that, the bilinear form  $\mathcal{A}(\cdot, \cdot)$  is not coercive on the whole space  $\mathbb{X} \times \mathbb{M}$ , hence the existence and uniqueness result for Problem 2 does not directly follow from Stampacchia’s theorem. However, we give here a proof using a perturbation technique (see [11]). Indeed, we consider a perturbed bilinear form  $\mathcal{A}^p$  depending on small positive parameter  $p$  defined as follows:

$$\mathcal{A}^p((W, \mu); (V, \chi)) := \mathbf{a}_\rho(W, V) + \mathbf{b}(V, \mu) + \mathbf{b}(W, \chi) + p(W, V)_\mathbb{X} + p(\mu, \chi)_\mathbb{M},$$

where  $(\cdot, \cdot)_\mathbb{X}$  and  $(\cdot, \cdot)_\mathbb{M}$  respectively denote the inner product in  $\mathbb{X}$  and in  $\mathbb{M}$ .

We then consider the following perturbed problem:

**Problem 4.** Find  $(U_p, \psi_p)$  in  $\mathbb{K}$  such that

$$\mathcal{A}^p((U_p, \psi_p); (V - U_p, \chi)) \geq \mathcal{L}(V - U_p), \quad \forall (V, \chi) \in \mathbb{K}. \tag{21}$$

Since the bilinear form  $\mathcal{A}^p$  is coercive on the space  $\mathcal{H}$ , Stampacchia’s theorem ensures that Problem 4 has a unique solution. We need to justify the link between the solution of Problem 4 and Problem 3.

**Lemma 3.3.** The bilinear form  $\mathbf{b}(\cdot, \cdot)$  satisfies the following inf-sup condition

$$\exists c_* > 0 \text{ such that } \sup_{W \in \mathbb{W}(\omega)} \frac{\mathbf{b}(W, \chi)}{\|W\|_\mathbb{X}} \geq c_* \|\chi\|_\mathbb{M}, \quad \forall \chi \in \mathbb{M}, \tag{22}$$

where  $\mathbb{W}$  is the closed subspace of  $\mathbb{X}$  and included into  $\mathbb{N}_\phi$  defined by

$$\mathbb{W} := \{(v, s) \in \mathbb{X}; (v - \frac{\epsilon}{2}s) \cdot e_3 = 0\}.$$

**Proof.** If we let  $\chi \in \mathbb{M}$ , then  $\vec{W} = (\frac{2}{\epsilon} \chi e_3, \chi e_3)$  belongs to  $\mathbb{W}$  and satisfies

$$\|\vec{W}\|_\mathbb{X} \sim \|\chi\|_\mathbb{M}.$$

Consequently one has

$$\sup_{W \in \mathbb{W}} \frac{\mathbf{b}(W, \chi)}{\|W\|_\mathbb{X}} \geq \frac{\mathbf{b}(\vec{W}, \chi)}{\|\vec{W}\|_\mathbb{X}} \sim \frac{|\chi|_{1,\omega}^2}{\|\chi\|_\mathbb{M}} \gtrsim \|\chi\|_\mathbb{M}. \quad \square$$

**Proof of Theorem 3.2.** The solution  $(U_p, \psi_p)$  of Problem 4 satisfies

$$\begin{cases} \forall V \in \mathbb{N}_\Phi(\omega), \quad \mathbf{a}_\rho(U_p, V - U_p) + \mathbf{b}(V - U_p, \psi_p) + p(U_p, V - U_p) \geq \mathcal{L}(V - U), \\ \forall \chi \in \mathbb{M}(\omega), \quad \mathbf{b}(U_p, \chi) - p(\psi_p, \chi)_{\mathbb{M}} = 0. \end{cases} \tag{23}$$

Since  $(\mathbf{0}_\mathbb{X}, \mathbf{0}_\mathbb{M}) \in \mathbb{K}$ , we then have the following bound

$$p\|U_p\|_\mathbb{X} + p\|\psi_p\|_\mathbb{M} \leq \|\mathcal{L}\|_{\mathbb{X}'}. \tag{24}$$

We now introduce the functional  $G : \mathbb{M} \rightarrow \mathbb{R}$  defined by

$$G(\chi) = p(\psi_p, \chi), \quad \forall \chi \in \mathbb{M}.$$

It is clear that  $G \in \mathbb{M}'$  and by (24) we have the following bound

$$\|G\|_{\mathbb{M}'} \lesssim \|\mathcal{L}\|_{\mathbb{X}'}. \tag{25}$$

By Lemma 3.3 (see [12] Lemma 4.1) there exists a unique solution  $Q_p \in \mathbb{W}$  of

$$\mathbf{b}(Q_p, \chi) = G(\chi), \quad \forall \chi \in \mathbb{M},$$

such that

$$\|Q_p\|_\mathbb{X} \lesssim \|G\|_{\mathbb{M}'}. \tag{26}$$

Then  $U_p - Q_p \in \mathbb{N}_\Phi$  and by the second line in (23) we have

$$\mathbf{b}(U_p - Q_p, \chi) = 0 \quad \forall \chi \in \mathbb{M}$$

whence,

$$U_p - Q_p \in \mathbb{N}_\Phi \cap \mathbb{V}.$$

Taking  $V = Q_p$  in the first line in (23) we get

$$\mathbf{a}_\rho(U_p, Q_p - U_p) + \mathbf{b}(Q_p - U_p, \psi_p) + p(U_p, Q_p - U_p) \geq \mathcal{L}(Q_p - U_p),$$

and subtracting  $\mathbf{a}_\rho(Q_p, Q_p - U_p)$  from both sides we obtain

$$\mathbf{a}_\rho(U_p - Q_p, Q_p - U_p) + p(U_p, Q_p - U_p) \geq \mathcal{L}(Q_p - U_p) - \mathbf{a}_\rho(Q_p, Q_p - U_p).$$

Then using (24), (25), (26) and the Cauchy–Schwarz inequality we get

$$\begin{aligned} \mathbf{a}_\rho(U_p - Q_p, U_p - Q_p) &\leq \mathcal{L}(U_p - Q_p) - \mathbf{a}_\rho(Q_p, U_p - Q_p) + p(U_p, Q_p - U_p) \\ &\lesssim \|\mathcal{L}\|_{\mathbb{X}'} \|U_p - Q_p\|_\mathbb{X}. \end{aligned}$$

Using the fact that  $U_p - Q_p \in \mathbb{V}$  and the coercivity of the bilinear form  $\mathbf{a}(\cdot, \cdot)$  on the space  $\mathbb{V}$ , we get

$$\|U_p - Q_p\|_\mathbb{X} \lesssim \|\mathcal{L}\|_{\mathbb{X}'}. \tag{27}$$

By the triangle inequality, (26) and (27) imply that

$$\|U_p\|_\mathbb{X} \lesssim \|\mathcal{L}\|_{\mathbb{X}'}. \tag{28}$$

Note that the hidden constant in this last estimate is independent of  $p$ .

Now we need to bound  $\|\psi_p\|_\mathbb{M}$ . Indeed, since

$$\forall V \in \mathbb{N}_\Phi, \quad \mathbf{a}_\rho(U_p, V - U_p) + \mathbf{b}(V - U_p, \psi_p) + p(U_p, V - U_p) \geq \mathcal{L}(V - U),$$

then (since  $\mathbb{W}$  is a closed subspace contained in  $\mathbb{N}_\Phi$ )

$$\forall V \in \mathbb{W}, \quad \mathbf{a}_\rho(U_p, V) + \mathbf{b}(V, \psi_p) + p(U_p, V) = \mathcal{L}(V)$$

which amounts to write

$$\mathbf{b}(V, \psi_p) = \mathcal{L}(V) - \mathbf{a}_\rho(U_p, V) - p(U_p, V), \quad \forall V \in \mathbb{W}.$$

Again the inf-sup inequality (22), the Cauchy–Schwarz inequality and (28) imply that

$$\|\psi_p\|_\mathbb{M} \lesssim \|\mathcal{L}\|_{\mathbb{X}'}. \tag{29}$$

From the estimate (29) and the second line in (23) we deduce that

$$\lim_{p \rightarrow 0} \mathbf{b}(U_p, \chi) = 0, \quad \forall \chi \in \mathbb{M} \quad \text{and} \quad \lim_{p \rightarrow 0} \mathbf{b}(U_p, \psi_p) = 0. \tag{30}$$

Collecting (28) and (29) we deduce that the sequence  $((U_p, \psi_p))_p$  is uniformly bounded in the Hilbert space  $\mathbb{X} \times \mathbb{M}$ . Then there exists  $(U^*, \psi^*) \in \mathbb{K}$  (recalling that  $(U_p, \psi_p)$  belongs to  $\mathbb{K}$  for all  $p$ ) such that

$$(U_p, \psi_p) \rightharpoonup (U^*, \psi^*) \in \mathbb{X} \times \mathbb{M} \text{ weakly as } p \rightarrow 0.$$

For arbitrary  $V \in \mathbb{N}_\phi$  we have,

$$\mathcal{L}(V - U_p) \leq \mathcal{A}^p((U_p, \psi_p); (V - U_p, \chi))$$

Since

$$\begin{aligned} \mathcal{A}^p((U_p, \psi_p); (V - U_p, \chi)) &= \mathcal{A}p((U_p, \psi_p); (V - U_p, \chi)) + p(U_p, V - U_p) + p(\psi_p, \chi) \\ &= \mathbf{a}_p(U_p, V - U_p) + b(U_p, \chi) + b(V - U_p, \psi_p) + p(U_p, V - U_p) + p(\psi_p, \chi) \\ &= \mathbf{a}_p(U_p, V) + b(V, \psi_p) + p(U_p, V) - \mathbf{a}(U_p, U_p) - p(U_p, U_p) + p(\psi_p, \chi) + b(U_p, \chi) - b(U_p, \psi_p) \\ &= \mathbf{a}_p(U_p, V) + b(V, \psi_p) - \mathbf{a}(U_p, U_p) + p(U_p, V) - p(U_p, U_p) + b(U_p, \chi) - b(U_p, \psi_p) + p(\psi_p, \chi), \end{aligned}$$

we then have,

$$\mathcal{L}(V - U_p) \leq \mathbf{a}_p(U_p, V) + b(V, \psi_p) - \mathbf{a}(U_p, U_p) + p(U_p, V) - p(U_p, U_p) + b(U_p, \chi) - b(U_p, \psi_p) + p(\psi_p, \chi),$$

and by letting  $p \rightarrow 0$  (using (30)) we get

$$\mathbf{a}_p(U^*, V) + b(V, \psi^*) - \lim_{p \rightarrow 0} \mathbf{a}_p(U_p, U_p) \geq \mathcal{L}(V - U^*).$$

Since,

$$\lim_{p \rightarrow 0} \mathbf{a}_p(U_p - U^*, U_p - U^*) \geq 0$$

then

$$\lim_{p \rightarrow 0} \mathbf{a}_p(U_p, U_p) \geq \mathbf{a}(U^*, U^*).$$

Those last inequalities and (30) allow us to write

$$\begin{aligned} \forall V \in \mathbb{N}_\phi, \quad \mathbf{a}_p(U^*, V - U^*) + b(V, \psi^*) &\geq \mathcal{L}(V - U^*), \\ \forall \chi \in \mathbb{M}, \quad b(U^*, \chi) &= 0. \end{aligned}$$

Hence  $(U^*, \psi^*)$  is a solution of Problem 3.

Let us now show the uniqueness. Let  $(U_1, \psi_1) \in \mathbb{K}$  and  $(U_2, \psi_2) \in \mathbb{K}$  be two solutions of Problem 3. Then

$$\begin{aligned} \mathbf{a}_p(U_1, U_2 - U_1) + b(U_2 - U_1, \psi_1) &\geq \mathcal{L}(U_2 - U_1), \\ b(U_1, \chi) &= 0 \quad \forall \chi \in \mathbb{M}. \end{aligned}$$

and

$$\begin{aligned} \mathbf{a}_p(U_2, U_1 - U_2) + b(U_1 - U_2, \psi_2) &\geq \mathcal{L}(U_1 - U_2), \\ b(U_2, \chi) &= 0 \quad \forall \chi \in \mathbb{M}. \end{aligned}$$

As  $U_1$  and  $U_2$  belong to  $\mathbb{V}$ , it follows that

$$\begin{cases} \mathbf{a}(U_1, U_2 - U_1) \geq \mathcal{L}(U_2 - U_1), \\ \mathbf{a}(U_2, U_1 - U_2) \geq \mathcal{L}(U_1 - U_2). \end{cases} \tag{31}$$

Hence

$$\mathbf{a}(U_1 - U_2, U_1 - U_2) \leq 0,$$

which implies that  $U_1 = U_2$  since  $\mathbf{a}(\cdot, \cdot)$  is coercive on  $\mathbb{V}$ .

The uniqueness of  $\psi$  follows from the inf-sup condition (22). Indeed, since,

$$\mathbf{a}(U, V - U) + b(V - U, \psi) \geq \mathcal{L}(V - U), \quad \forall V \in \mathbb{N}_\phi,$$

then

$$\mathbf{a}(U, W) + b(W, \psi) = \mathcal{L}(W), \quad \forall W \in \mathbb{W}. \tag{32}$$

If  $(U, \psi_1) \in \mathbb{K}$  and  $(U, \psi_2) \in \mathbb{K}$  are two solutions of Problem 3, the inf-sup condition (22) and (32) imply that

$$\|\psi_1 - \psi_2\|_{\mathbb{M}} \leq \sup_{W \in \mathbb{W}(\omega)} \frac{b(W, \psi_1 - \psi_2)}{\|W\|_{\mathbb{X}}} = 0,$$

which leads to  $\psi_1 = \psi_2$ . ■

Let us now denote by  $c^\#$  and  $c_\#$  the coercivity and the continuity constants of the form  $a_\rho(\cdot, \cdot)$  on  $\mathbb{X}$ , respectively. Our error analysis is built upon the following stability result.

**Lemma 3.4.** For any  $(W, \xi) \in \mathbb{X} \times \mathbb{M}$  there exists  $V \in \mathbb{X}$  such that:

$$A_\rho((W, \xi); (V, -\xi)) \geq C_1 (\|W\|_{\mathbb{X}} + \|\xi\|_{\mathbb{M}})^2, \tag{33}$$

$$\|V\|_{\mathbb{X}} + \|\xi\|_{\mathbb{M}} \leq C_2 \|W\|_{\mathbb{X}} + \|\xi\|_{\mathbb{M}}. \tag{34}$$

where  $C_1$  and  $C_2$  are two positive constants depending only on the constants  $c^\#, c_\#$  and  $c_*$ .

**Proof.** Let  $(W, \xi) \in \mathbb{X} \times \mathbb{M}$  and let  $Q \in \mathbb{X}$  be the unique solution of the following problem:

$$\begin{cases} \text{Find } Q \in \mathbb{X} \text{ such that} \\ a_\rho(Q, Z) + (Q, Z)_{\mathbb{X}} = b(Z, \xi), \quad \forall Z \in \mathbb{X}. \end{cases} \tag{35}$$

Then by taking  $Z = Q$  in (35) we get

$$\|Q\|_{\mathbb{X}}^2 \leq b(Q, \xi).$$

Cauchy–Schwarz’s inequality yields

$$\|Q\|_{\mathbb{X}}^2 \lesssim \|Q\|_{\mathbb{X}} \|\xi\|_{\mathbb{M}}.$$

which by simplification leads to

$$\|Q\|_{\mathbb{X}} \lesssim \|\xi\|_{\mathbb{M}}.$$

Furthermore by the inf-sup condition we have

$$\|\xi\|_{\mathbb{M}} \lesssim \sup_{Z \in \mathbb{X}} \frac{b(Z, \xi)}{\|Z\|_{\mathbb{X}}}.$$

Using (35) and Cauchy–Schwarz’s inequality, we obtain

$$\|\xi\|_{\mathbb{M}} \lesssim \|Q\|_{\mathbb{X}}.$$

Thus  $\|Q\|_{\mathbb{X}} \sim \|\xi\|_{\mathbb{M}}$ .

We now take  $V = W + \delta Q$ , where  $0 < \delta < \frac{c^\#}{c_\#^2}$ , and get

$$\begin{aligned} A_\rho((W, \xi); (V, -\xi)) &= A_\rho((W, \xi); (W + \delta Q, -\xi)) \\ &= a_\rho(W, W + \delta Q) + b(W + \delta Q, \xi) + b(W, -\xi) \\ &= a_\rho(W, W) + \delta a_\rho(W, Q) + b(W, \xi) + \delta b(Q, \xi) - b(W, \xi) \\ &= a_\rho(W, W) + \delta a_\rho(W, Q) + \delta b(Q, \xi) \\ &\geq c^\# \|W\|_{\mathbb{X}}^2 - \frac{\delta c_\#^2}{2} \|W\|_{\mathbb{X}}^2 - \frac{\delta}{2} \|Q\|_{\mathbb{X}}^2 + \delta \|Q\|_{\mathbb{X}}^2 \\ &\gtrsim \|W\|_{\mathbb{X}}^2 + \|\xi\|_{\mathbb{M}}^2. \quad \blacksquare \end{aligned}$$

Let us now introduce the following cone

$$A = \{ \mu \in \mathbb{M}' ; \forall \sigma \in \mathbb{M}, \sigma \geq 0, \langle \sigma, \mu \rangle \geq 0 \},$$

and the following bilinear form  $c : \mathbb{X} \times \mathbb{M}' \rightarrow \mathbb{R}$  defined by

$$c(V, \mu) = \left\langle \left( v - \frac{\varepsilon}{2} s \right) \cdot e_3, \mu \right\rangle, \forall (V, \mu) \in \mathbb{X} \times \mathbb{M}'. \tag{36}$$

**Lemma 3.5.** There exists a positive constant  $C_c$  such that

$$\inf_{\chi \in \mathbb{M}'} \sup_{v \in H^1_{\gamma_0}(\omega, \mathbb{R}^3)} \frac{c((v, 0), \mu)}{\|\mu\|_{\mathbb{M}'} \| (v, 0) \|_{\mathbb{X}}} \geq C_c. \tag{37}$$

**Proof.** See the proof of [7, Lemma 4.4].

Obviously the previous result implies that

$$\inf_{\chi \in \mathbb{M}'} \sup_{V \in \mathbb{X}} \frac{c(V, \mu)}{\|\mu\|_{\mathbb{M}'} \|V\|_{\mathbb{X}}} \geq C_c. \quad \blacksquare \tag{38}$$



**Proposition 3.6.** Let  $(U, \psi)$  be the solution of [Problem 1](#). Then there exists a unique  $\lambda \in \Lambda$  such that

$$c(V, \lambda) = a_\rho(U, V) + b(V, \psi) - \mathcal{L}(V), \quad \forall V \in \mathbb{X}. \tag{39}$$

Moreover, the multiplier  $\lambda$  satisfies the following bound

$$\|\lambda\|_{\mathbb{M}'} \leq C \|\mathcal{L}\|_{\mathbb{X}'}$$

**Proof.** See [\[7\]](#). ■

Now we consider the “full” problem

**Problem 5.** Find  $(U, \psi, \lambda) \in \mathbb{X} \times \mathbb{M} \times \Lambda$  such that:

$$\begin{cases} \forall V \in \mathbb{X}, & a_\rho(U, V) + b(V, \psi) - c(V, \lambda) = \mathcal{L}(V), \\ \forall \chi \in \mathbb{M}, & b(U, \chi) = 0, \\ \forall \mu \in \Lambda, & c(U, \mu - \lambda) \geq \langle \Phi, \mu - \lambda \rangle. \end{cases} \tag{40}$$

Note that the right-hand side of the third line of this problem makes sense only if the function  $\Phi$  belongs to  $H^1_{\gamma_0}(\omega)$ .

**Proposition 3.7.** The full [Problem 5](#) and the reduced [Problem 1](#) are equivalent, in the following sense: If  $(U, \psi, \lambda)$  is a solution of full problem, then  $(U, \psi)$  is a solution of the reduced. If  $(U, \psi)$  is a solution of the reduced problem then there exists a unique  $\lambda \in \Lambda$  such that  $(U, \psi, \lambda)$  is a solution of the full problem.

**Proof.** The proof can be done as the one of [\[7, Proposition 4.2\]](#). ■

We introduce the following forms

$$\begin{aligned} \mathcal{B}(U, \psi, \lambda; V, \chi, \mu) &:= a_\rho(U, V) + b(V, \psi) + b(U, \chi) - c(V, \lambda) + c(U, \mu), \\ \mathcal{L}(V, \chi, \mu) &:= \mathcal{L}(V) + \langle \Phi, \mu \rangle. \end{aligned}$$

Then [Problem 5](#) can now be written in the following compact form:

**Problem 6.** Find  $(U, \psi, \lambda) \in \mathbb{X} \times \mathbb{M} \times \Lambda$  such that:

$$\mathcal{B}(U, \psi, \lambda; V, \chi, \mu - \lambda) \geq \mathcal{L}(V, \chi, \mu - \lambda), \quad \forall (V, \chi, \mu) \in \mathbb{X} \times \mathbb{M} \times \Lambda.$$

**Theorem 3.8 (Continuous stability).** For any  $(V, \chi, \mu) \in \mathbb{X} \times \mathbb{M} \times \mathbb{M}'$  there exists  $W \in \mathbb{X}$  such that

$$\mathcal{B}(V, \chi, \mu; W, -\chi, \mu) \gtrsim (\|W\|_{\mathbb{X}} + \|\chi\|_{\mathbb{M}} + \|\mu\|_{\mathbb{M}'})^2, \tag{41}$$

$$\|W\|_{\mathbb{X}} \lesssim \|V\|_{\mathbb{X}}. \tag{42}$$

To prove [Theorem 3.8](#), we need the following lemma.

**Lemma 3.9.** There exists a constant  $\beta^\# > 0$  such that:

$$\inf_{(\chi, \mu) \in \mathbb{M} \times \mathbb{M}'} \sup_{V=(v,s) \in \mathbb{X}} \frac{c(V, \mu) - b(V, \chi)}{\|(\chi, \mu)\|_{\mathbb{M} \times \mathbb{M}'} \|V\|_{\mathbb{X}}} \geq \beta^\#. \tag{43}$$

**Proof.** Let  $(\chi, \mu)$  be in  $\mathbb{M} \times \mathbb{M}'$ , then there exists  $\sigma$  in  $H^1_{\gamma_0}(\omega)$  such that

$$\forall \tau \in H^1_{\gamma_0}(\omega), \int_{\omega} (\mathbf{grad} \sigma) \cdot (\mathbf{grad} \tau) dx = \langle \tau, \mu \rangle. \tag{44}$$

We directly deduce that

$$\|\sigma\|_{1,\omega} \lesssim \|\mu\|_{\mathbb{M}'}$$

Furthermore

$$\|\mu\|_{\mathbb{M}'} = \sup_{\tau \in H^1_{\gamma_0}(\omega)} \frac{\langle \tau, \mu \rangle}{\|\tau\|_{1,\omega}} = \sup_{\tau \in H^1_{\gamma_0}(\omega)} \frac{\int_{\omega} (\mathbf{grad} \sigma) \cdot (\mathbf{grad} \tau) dx}{\|\tau\|_{1,\omega}} \lesssim \|\sigma\|_{1,\omega}. \tag{45}$$

Hence  $\|\sigma\|_{1,\omega} \sim \|\mu\|_{\mathbb{M}'}$ . In [\(44\)](#), take  $\tilde{V} = (\tilde{v}, \tilde{s})$  with

$$\tilde{v} = (-\sigma + \frac{\epsilon}{2} \chi a_3 \cdot e_3) e_3 \quad \text{and} \quad \tilde{s} = \chi a_3,$$

then we have

$$b(\tilde{V}, \chi) = \int_{\omega} \partial_\alpha (\tilde{s} \cdot a_3) \partial_\alpha \chi dx = |\chi|_{1,\omega}^2,$$

$$c(\tilde{V}, \mu) = -\langle \sigma, \mu \rangle = -|\chi|_{1,\omega}^2,$$

$$\|\tilde{V}\|_{\mathbb{X}} \lesssim \|\sigma\|_{1,\omega} + \|\chi\|_{1,\omega} \lesssim \|\mu\|_{M'} + \|\chi\|_{1,\omega}.$$

These properties directly yield

$$\sup_{V \in \mathbb{X}} \frac{b(V, \chi) - c(V, \mu)}{\|V\|_{\mathbb{X}}} \geq \frac{b(\tilde{V}, \chi) - c(\tilde{V}, \mu)}{\|\tilde{V}\|_{\mathbb{X}}} = \frac{\|\chi\|_{1,\omega}^2 + \|\sigma\|_{1,\omega}^2}{\|\tilde{V}\|_{\mathbb{X}}} \\ \gtrsim (\|\chi\|_{1,\omega}^2 + \|\mu\|_{M'}^2)^{1/2}. \quad \blacksquare$$

**Proof of Theorem 3.8.** Let  $(V, \chi, \mu) \in \mathbb{X} \times \mathbb{M} \times M'$ . We consider the following variational problem:

$$\begin{cases} \text{Find } Q \text{ in } \mathbb{X} \text{ such that} \\ \mathbf{a}_\rho(Q, Z) + (Q, Z)_{\mathbb{X}} = \mathbf{b}(Z, \chi) - c(\mu, Z), \quad \forall Z \in \mathbb{X}. \end{cases} \quad (46)$$

Since, the bilinear form  $\mathbf{a}_\rho(\cdot, \cdot) + (\cdot, \cdot)_{\mathbb{X}}$  is  $\mathbb{X}$ -elliptic, problem (46) has a unique solution  $Q \in \mathbb{X}$ .

Moreover, since  $c(\cdot, \cdot) - \mathbf{b}(\cdot, \cdot)$  satisfies the inf-sup condition, the Cauchy-Schwarz inequality yields

$$\|\chi\|_{\mathbb{M}} + \|\mu\|_{M'} \lesssim \sup_{Z \in \mathbb{X}} \frac{c(\mu, Z) - \mathbf{b}(\chi, Z)}{\|Z\|_{\mathbb{X}}} = \sup_{Z \in \mathbb{X}} \frac{\mathbf{a}(Q, Z) + (Q, Z)}{\|Z\|_{\mathbb{X}}} \lesssim \|Q\|_{\mathbb{X}}.$$

Take  $W = V + \delta Q$  where  $\delta$  is a positive constant to be determined later, then we have

$$\begin{aligned} \mathcal{B}(V, \chi, \mu; W, -\chi, \mu) &= \mathbf{a}_\rho(V, V + \delta Q) + \mathbf{b}(V + \delta Q, \chi) + \mathbf{b}(V, -\chi) - c(V + \delta Q, \mu) + c(\mu, V) \\ &= \mathbf{a}_\rho(V, V) + \delta \mathbf{a}_\rho(V, Q) + \delta \mathbf{b}(Q, \chi) - \delta c(\mu, Q) \\ &\geq \mathbf{a}_\rho(V, V) - \delta c^\# \|V\|_{\mathbb{X}} \|Q\|_{\mathbb{X}} + \delta \|Q\|_{\mathbb{X}}^2 \\ &\geq (c_\# - \frac{\delta (c^\#)^2}{2}) \|V\|_{\mathbb{X}}^2 + \frac{\delta}{2} \|Q\|_{\mathbb{X}}^2 \end{aligned}$$

Then it suffices to take  $0 < \delta < \frac{c_\#^2}{2(c^\#)^2}$  to obtain

$$\mathcal{B}(V, \chi, \mu; W, -\chi, \mu) \gtrsim (\|V\|_{\mathbb{X}}^2 + \|\chi\|_1^2 + \|\mu\|_{-1}^2). \quad \square$$

### 3.2. Regularity of the solution

The regularity of the solution of any PDE problem plays an important role in its error analysis by a FEM. Accordingly, the a priori error analysis carried out in Section 4.1 requires additional regularity on the solution of the continuous problem. Let us first introduce the following quantities:

$$n^{\sigma\sigma}(u) = a^{\sigma\sigma\alpha\beta} \gamma_{\alpha\beta}(u), \quad (47)$$

$$q^\beta(u, r) = \frac{\varepsilon E}{1 + \nu} a^{\alpha\beta} \delta_{\alpha 3}(u, r), \quad (48)$$

$$m^{\alpha\beta}(u, r) = \frac{\varepsilon^3}{12} a^{\alpha\beta\rho\sigma} \chi_{\alpha\beta}(u, r), \quad (49)$$

where,  $\gamma_{\alpha\beta}$ ,  $\delta_{\alpha 3}$  and  $\chi_{\alpha\beta}$  are given by (2), (3) and (4) respectively. Then our contact problem takes the following complementarity formulation

$$\begin{cases} -\partial_\rho((n^{\sigma\sigma}(u)a_\sigma + m^{\sigma\sigma}(u,r)\partial_\sigma a_3 + q^\rho(u,r)a_3)\sqrt{a}) - \lambda e_3 = f\sqrt{a} & \text{in } \omega, \\ -\partial_\rho(m^{\rho\sigma}(u,r)a_\sigma\sqrt{a}) + q^\beta(u,r)a_\beta\sqrt{a} + \frac{\varepsilon}{2}\lambda e_3 = 0 & \text{in } \omega, \\ r \cdot a_3 = 0 & \text{in } \omega, \\ \left( u - \frac{\varepsilon}{2}r \right) \cdot e_3 \geq \Phi, \quad \lambda \geq 0, \quad \lambda \left( \left( u - \frac{\varepsilon}{2}r \right) \cdot e_3 - \Phi \right) = 0 & \text{in } \omega, \\ u = r = 0 & \text{on } \partial\omega, \end{cases} \quad (50)$$

with coefficients which are in  $L^\infty(\omega)$ , and the function  $\Phi$  belongs only to  $W^{1,\infty}(\omega)$  when the chart  $\varphi \in W^{2,\infty}(\omega, \mathbb{R}^3)$ . Therefore, the translations (or finite difference quotients) method of Nirenberg [13] cannot be applied here. The coefficients of the system satisfy the ellipticity condition and in the non contact set the system is a ‘‘standard’’ second order elliptic system. But the famous De Giorgi’s counter-example (see [14, p. 205]) indicates that the regularity problem for systems of equations (or vectorial case) cannot be treated as the case of a single elliptic equation (or scalar case), so the Stampacchia-Brezis [15] technique cannot be used here to obtain the  $H^2(\omega)$  regularity. However, if we assume that the chart  $\varphi$  is more regular, namely  $\varphi \in C^3(\omega, \mathbb{R}^3)$ , then the formulation (expressions of the tensors (2), (3) and (4)) used in this paper coincides with the classical formulation of thin shell theory (see [10]). For sufficiently smooth surfaces, recent papers (see [16–18]) improved interior regularity (i.e.  $H^2_{\text{loc}}(\omega)$ ) of the solution of elastic Koiter’s or shallow shells in the presence of obstacles, by using the Nirenberg method. The main difficulty to adapt this approach to our Nagdi’s model is the construction of admissible displacement field in term of the finite difference quotient satisfying the inequality constraint.

#### 4. Finite element discretizations

Let  $\mathcal{T}_h$  be a regular affine triangulation which cover the domain  $\omega$ . Here,  $h$  is the mesh-size, or more precisely,  $h = \max_{T \in \mathcal{T}_h} \text{diam } T$ . For a positive integer  $k$ ,  $\mathbb{P}_k(T)$  stands for the set of functions on  $T$  which are the restrictions to  $T$  of polynomials of degree less than or equal to  $k$ . For  $T \in \mathcal{T}_h$ ,  $b_T$  denotes the bubble function defined by  $b_T = \frac{\lambda_1 \lambda_2 \lambda_3}{27}$ , where  $\lambda_i, i = 1, 2, 3$  are the barycentric coordinates of  $T$ . Note that  $b_T \in H_0^1(T) \cap \mathbb{P}_3(T)$  has a maximum value of one. We further define

$$B_3(T) = \{v \in H_0^1(T); v = b_T w, \quad w \in \mathbb{P}_0(T)\}. \tag{51}$$

Let us define the finite dimensional spaces:

$$\mathbb{M}_h := \{\chi_h \in H_{\gamma_0}^1(\omega) \mid \chi_h|_T \in \mathbb{P}_1(T) \oplus B_3(T), \forall T \in \mathcal{T}_h\}.$$

$$\mathbb{Q}_h := \{\mu_h \in L^2(\omega) \mid \mu_h|_T \in \mathbb{P}_0(T), \forall T \in \mathcal{T}_h\}.$$

$$\mathbb{X}_h := (\mathbb{M}_h)^3 \times (\mathbb{M}_h)^3.$$

$$\mathbb{W}_h := \left\{ (v_h, s_h) \in \mathbb{X}_h; \left( v_h - \frac{\varepsilon}{2} s_h \right) \cdot e_3 = 0 \right\}$$

Then, we introduce the discrete convex cone

$$\mathbb{N}_h = \left\{ (v_h, s_h) \in \mathbb{X}_h; \left( v_h - \frac{\varepsilon}{2} s_h \right) \cdot e_3 \geq \Phi_h \right\}, \tag{52}$$

where  $\Phi_h := I_h \Phi$ ,  $I_h$  being the standard Lagrange interpolant operator, namely  $(I_h \Phi)_T \in \mathbb{P}_1(T)$  and  $(I_h \Phi)_T(x) = \Phi(x)$  for all vertices  $x$  of  $T$ . Clearly we have  $\mathbb{X}_h \subset \mathbb{X}$ ,  $\mathbb{M}_h \subset \mathbb{M}$  and  $\mathbb{W}_h \subset \mathbb{N}_h$  but  $\mathbb{N}_h$  is not necessarily contained in  $\mathbb{N}_\Phi$ .

We first consider the discrete version of [Problem 2](#), namely

**Problem 7.** Find  $(U_h, \psi_h) \in \mathbb{N}_h \times \mathbb{M}_h$  such that:

$$\begin{cases} \forall V_h \in \mathbb{N}_h, & \mathbf{a}_\rho(U_h, V_h - U_h) + \mathbf{b}(V_h - U_h, \psi_h) \geq \mathcal{L}(V_h - U_h), \\ \forall \chi_h \in \mathbb{M}_h, & \mathbf{b}(U_h, \chi_h) = 0. \end{cases} \tag{53}$$

As in the continuous case, we may rewrite [Problem 7](#) in the following compact form

**Problem 8.** Find  $(U_h, \psi_h) \in \mathbb{N}_h \times \mathbb{M}_h$  such that:

$$\mathbf{A}_\rho((U_h, \psi_h); (V_h - U_h, \chi_h)) \geq \mathcal{L}(V_h - U_h), \quad \forall (V_h, \chi_h) \in \mathbb{N}_h \times \mathbb{M}_h. \tag{54}$$

**Lemma 4.1.** *If the mesh size  $h$  is sufficiently small then, there exists a positive constant  $C_b$  such that*

$$\inf_{\chi_h \in \mathbb{M}_h} \sup_{V_h \in \mathbb{W}_h} \frac{\mathbf{b}(V_h, \chi_h)}{\|\chi_h\|_{\mathbb{M}} \|V_h\|_{\mathbb{X}}} \geq C_b. \tag{55}$$

**Proof.** Recalling that  $I_h$  is the standard Lagrange interpolation operator, for an arbitrary  $\chi_h \in \mathbb{M}_h$ , we take

$$V_h = (v_h, s_h) = \left( \frac{2}{\varepsilon} I_h(\chi_h a_3), I_h(\chi_h a_3) \right),$$

then clearly,  $V_h \in \mathbb{W}_h$  and

$$\mathbf{b}(V_h, \chi_h) \gtrsim \|\chi_h\|_{\mathbb{M}}$$

since the inverse estimate  $\|\nabla v_h\|_{\infty, \omega} \lesssim h^{-1} \|v_h\|_{\infty, \omega}$  is valid for all  $v_h \in \mathbb{M}_h$ , see [[19](#), Lemma 3.3] and [[10](#), Lemma 5.6]. ■

**Theorem 4.2.** *If the mesh size  $h$  is sufficiently small, then [Problem 7](#) admits a unique solution.*

**Proof.** Since  $\mathbb{W}_h$  is a closed subspace of  $\mathbb{N}_h$  (see for instance [[11](#)]), the proof can be done by using the same perturbation technique as for the continuous problem. ■

Now we introduce the closed convex cone

$$A_h = \{ \mu_h \in \mathbb{Q}_h; \mu_h \geq 0 \}, \tag{56}$$

that is clearly a subspace of  $A$ . We then consider the ‘‘full’’ discrete problem (compare with [Problem 5](#))

**Problem 9.** Find  $(U_h, \psi_h, \lambda_h)$  in  $\mathbb{X}_h(\omega) \times \mathbb{M}_h(\omega) \times A_h$  such that:

$$\begin{cases} \forall V_h \in \mathbb{X}_h, & \mathbf{a}_\rho(U_h, V_h) + \mathbf{b}(V_h, \psi_h) - c(V_h, \lambda_h) = \mathcal{L}(V_h), \\ \forall \chi_h \in \mathbb{M}_h, & \mathbf{b}(U_h, \chi_h) = 0, \\ \forall \mu_h \in A_h, & c(U_h, \mu_h - \lambda_h) \geq \langle I_h \Phi, \mu_h - \lambda_h \rangle. \end{cases} \tag{57}$$

Let us also introduce the following  $h$ -dependent norm:

$$\|\chi_h\|_h^2 = \sum_{T \in \mathcal{T}_h} h_T^2 \|\chi_h\|_T^2, \forall \chi_h \in \mathbb{Q}_h. \tag{58}$$

**Lemma 4.3.** *There exist two positive constants  $C_1$  and  $C_2$  (independent of  $h$ ) such that*

$$\forall \chi_h \in \mathbb{Q}_h, \sup_{V_h \in \mathbb{X}_h \cap \ker b} \frac{c(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}_h}} \geq C_1 \|\chi_h\|_{M'} - C_2 \|\chi_h\|_h. \tag{59}$$

**Proof.** Let  $\chi_h \in \mathbb{Q}_h \subset M'$ , then the inf-sup condition (37) implies that there exists  $v \in H_{y_0}^1(\omega, \mathbb{R}^3)$  and  $C_1 > 0$  such that

$$c((v, 0), \chi_h) \geq C_1 \|(v, 0)\|_{\mathbb{X}} \|\chi_h\|_{M'}.$$

Let  $V_h$  be the Clément interpolant of  $(v, 0)$  (hence  $V_h$  is in the form  $(v_h, 0)$  and belongs to  $\ker b$ ), then we have

$$\begin{aligned} c(V_h, \chi_h) &= c(V_h - V, \chi_h) + c(V, \chi_h) \\ &= \sum_{T \in \mathcal{T}_h} ((v_h - v) \cdot e_3, \chi_h)_T + c(V, \chi_h) \\ &\geq \sum_{T \in \mathcal{T}_h} ((v_h - v) \cdot e_3, \chi_h)_T + C_1 \|V\|_{\mathbb{X}} \|\chi_h\|_{M'} \\ &\geq - \sum_{T \in \mathcal{T}_h} \|(v_h - v) \cdot e_3\|_T \|\chi_h\|_T + C_1 \|V\|_{\mathbb{X}} \|\chi_h\|_{M'} \\ &= - \sum_{T \in \mathcal{T}_h} h_T^{-1} \|(v_h - v) \cdot e_3\|_T h_T \|\chi_h\|_T + C_1 \|V\|_{\mathbb{X}} \|\chi_h\|_{M'}. \end{aligned} \tag{60}$$

From the properties of the Clément interpolant and the fact that  $\mathcal{T}_h$  is quasi-uniform, we have

$$T \left( \sum_{T \in \mathcal{T}_h} h_T^{-2} \|(v_h - v) \cdot e_3\|_T^2 \right)^{1/2} \leq C_2 \|V\|_{\mathbb{X}} \quad \text{and} \quad \|V_h\|_{\mathbb{X}} \lesssim \|V\|_{\mathbb{X}},$$

consequently

$$\sum_{T \in \mathcal{T}_h} h_T^{-1} \|(v_h - v) \cdot e_3\|_T h_T \|\chi_h\|_T \leq C_2 \|\chi_h\|_h \|V\|_{\mathbb{X}},$$

which together with (60) shows that

$$c(V_h, \chi_h) \geq C_1 \|V\|_{\mathbb{X}} \|\chi_h\|_{M'} - C_2 \|V\|_{\mathbb{X}} \|\chi_h\|_h. \tag{61}$$

Now if  $C_1 \|\chi_h\|_{M'} - C_2 \|\chi_h\|_h \geq 0$ , then (61) implies that

$$C_1 \|\chi_h\|_{M'} - C_2 \|\chi_h\|_h \leq \frac{c(V_h, \chi_h)}{\|V\|_{\mathbb{X}}} \lesssim \frac{c(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}},$$

since  $c(V_h, \chi_h) \geq 0$  because the left-hand side of this estimate is positive.

Otherwise, if  $C_1 \|\chi_h\|_{M'} - C_2 \|\chi_h\|_h \leq 0$ , then clearly

$$\sup_{V_h \in \mathbb{X}_h} \frac{c(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}} \gtrsim C_1 \|\chi_h\|_{M'} - C_2 \|\chi_h\|_h,$$

since, for  $W_h = (0, 0, \chi_h b_T, 0, 0, 0)$ , for some interior triangle  $T$ , we have  $c(W_h, \chi_h) \geq 0$ . ■

**Lemma 4.4.** *We have the following inf-sup condition for the mesh-dependent norm (58), namely there exists a positive constant  $C_3$  (independent of  $h$ ) such that*

$$\forall \chi_h \in \mathbb{Q}_h, \sup_{V_h \in \mathbb{X}_h \cap \ker b} \frac{c(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}} \geq C_3 \|\chi_h\|_h. \tag{62}$$

**Proof.** Let  $\chi_h \in \mathbb{Q}_h$ , we define  $V_h \in \mathbb{X}_h$  as follow:

$$V_h = (0, 0, 0, \sigma_h, 0, 0), \quad \text{with } (\sigma_h)|_T = h_T^2 \chi_h b_T, \forall T \in \mathcal{T}_h.$$

Then clearly  $V_h \in \mathbb{X}_h \cap \ker b$ , and we have

$$c(V_h, \chi_h) = \sum_{T \in \mathcal{T}_h} \left( (v_h - \frac{\varepsilon}{2} s_h) \cdot e_3, \chi_h \right) = \sum_{T \in \mathcal{T}_h} \int_T h_T^2 \chi_h^2 b_T \gtrsim \|\chi_h\|_h^2$$

and

$$\|V_h\|_{\mathbb{X}}^2 = \|\sigma_h\|_{1,T}^2 \lesssim \sum_{T \in \mathcal{T}_h} h_T^{-2} \|\sigma_h\|_{0,T}^2 \leq \sum_{T \in \mathcal{T}_h} h_T^2 \|\chi_h\|_{0,T}^2 = \|\chi_h\|_h^2$$

We then conclude that

$$c(V_h, \chi_h) \gtrsim \|V_h\|_{\mathbb{X}} \|\chi_h\|_h. \quad \square$$

**Lemma 4.5.** *It holds*

$$\sup_{V_h \in \mathbb{X} \cap \ker b} \frac{c(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}} \gtrsim \|\chi_h\|_{\mathbb{M}'}. \tag{63}$$

**Proof.** For  $\delta \in (0, 1)$ , owing to (59) and (62), we have

$$\begin{aligned} \sup_{V_h \in \mathbb{X} \cap \ker b} \frac{c(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}} &= \delta \sup_{V_h \in \mathbb{X} \cap \ker b} \frac{c(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}} + (1 - \delta) \sup_{V_h \in \mathbb{X} \cap \ker b} \frac{c(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}} \\ &\geq \delta (C_1 \|\chi_h\|_{\mathbb{M}'} - C_2 \|\chi_h\|_h) + (1 - \delta) C_3 \|\chi_h\|_h \\ &\geq \delta C_1 \|\chi_h\|_{\mathbb{M}'} + ((1 - \delta) C_3 - \delta C_2) \|\chi_h\|_h. \end{aligned}$$

Choosing  $\delta$  such that  $(1 - \delta) C_3 - \delta C_2 = 0$  or equivalently  $\delta = \frac{C_3}{C_2 + C_3}$  that is indeed in  $(0, 1)$ , we obtain (63). ■

**Proposition 4.6.** *The full problem (57) and the reduced problem (53) are equivalent, in the following sense: If  $(U_h, \psi_h, \lambda_h)$  is a solution of full problem, then  $(U_h, \psi_h)$  is a solution of the reduced. If  $(U_h, \psi_h)$  is a solution of the reduced problem then there exists  $\lambda_h \in \Lambda_h$  such that  $(U_h, \psi_h, \lambda_h)$  is a solution of the full problem.*

**Proof.** Suppose that  $(U_h, \psi_h, \lambda_h)$  satisfies (57), then from the first equation of (57) we have

$$a_\rho(U_h, V_h) + b(V_h, \psi_h) = \mathcal{L}(V_h) + c(V_h, \lambda_h), \quad \forall V_h \in \mathbb{X}_h. \tag{64}$$

Taking  $\mu_h = 0$  and  $\mu_h = 2\lambda_h$  in the third line of (57) gives

$$c(U_h, \lambda_h) = \langle I_h \Phi, \lambda_h \rangle, \tag{65}$$

and since  $I_h \Phi \leq 0$  in  $\omega$ , for any  $V_h \in \mathbb{N}_h$  we have

$$c(V_h, \lambda_h) \geq \langle I_h \Phi, \lambda_h \rangle. \tag{66}$$

So, combining (65) and (66) we get

$$c(V_h - U_h, \lambda_h) \geq 0, \quad \forall V_h \in \mathbb{N}_h.$$

Hence (60), and the second line in (57) amount to write

$$\begin{aligned} a_\rho(U_h, V_h - U_h) + b(V_h - U_h, \psi_h) &\geq \mathcal{L}(V_h - U_h), & \forall V_h \in \mathbb{N}_h, \\ b(U_h, \chi_h) &= 0. & \forall \chi_h \in \mathbb{M}_h. \end{aligned}$$

Conversely, if  $(U_h, \psi_h)$  is a solution of Problem 7 we want to prove that there exists  $\lambda_h \in \Lambda_h$  such that  $(U_h, \psi_h, \lambda_h)$  is a solution of Problem 9.

Let us first recall that the first line of (53) with  $V_h = 0$  and  $V_h = 2U_h$  yields

$$a_\rho(U_h, U_h) + b(U_h, \psi_h) = \mathcal{L}(U_h). \tag{67}$$

Since the bilinear form  $c(\cdot, \cdot)$  satisfies the inf-sup condition (see Lemma 4.5), then there exists a unique  $\lambda_h \in \mathbb{Q}_h$  such that

$$c(V_h, \lambda_h) = a_\rho(U_h, V_h) + b(V_h, \psi_h) - \mathcal{L}(V_h), \quad \forall V_h \in \mathbb{X}_h. \tag{68}$$

Now we need to prove that  $\lambda_h \in \Lambda_h$ .

First let  $T \in \mathcal{T}_h$  be an arbitrary triangle and let  $V_h \in \mathbb{X}_h$  be chosen such that

$$V_h = (u_h + (0, 0, b_T), r_h),$$

where  $U_h = (u_h, r_h)$ . Then since  $b_T \in H_0^1(T)$ ,  $b_T \geq 0$ , it is clear that  $V_h \in \mathbb{N}_h$  and

$$c(V_h - U_h, \lambda_h) = \int_T \lambda_h b_T = a_\rho(U_h, V_h - U_h) + b(V_h - U_h, \psi_h) - \mathcal{L}(V_h - U_h) \geq 0$$

Hence  $\lambda_h \geq 0$  in  $\omega$ , which means that  $\lambda_h \in \Lambda_h$ .

It remains to prove the last property of Problem 9. First as  $U_h$  belongs to  $\mathbb{N}_h$ , it holds

$$\left(u_h - \frac{\varepsilon}{2} r_h\right) \cdot e_3 \geq I_h \Phi, \tag{69}$$

which directly implies that

$$c(U_h, \mu_h) \geq \langle I_h \Phi, \mu_h \rangle \quad \forall \mu_h \in \Lambda_h. \tag{70}$$

The last inequality of (57) then holds if we show that

$$\left\langle \left( u_h - \frac{\varepsilon}{2} r_h \right) \cdot e_3 - I_h \Phi, \lambda_h \right\rangle = 0, \tag{71}$$

Since  $U_h \in \mathbb{N}_h$  and  $\lambda_h \in \Lambda_h$  then we have

$$\left\langle \left( u_h - \frac{\varepsilon}{2} r_h \right) \cdot e_3 - I_h \Phi, \lambda_h \right\rangle \geq 0. \tag{72}$$

On the other hand, by (67) and (68), we have

$$c(U_h, \lambda_h) = 0,$$

while the fact that  $\lambda_h \in \Lambda_h$  and that  $I_h \Phi \leq 0$  lead to

$$\langle I_h \Phi, \lambda_h \rangle \geq 0.$$

This directly implies

$$\left\langle \left( u_h - \frac{\varepsilon}{2} r_h \right) \cdot e_3 - I_h \Phi, \lambda_h \right\rangle \leq 0. \tag{73}$$

Hence, (72) and (73) imply that (71) holds. ■

#### 4.1. A priori error analysis of the reduced problem

The purpose of this subsection is the a priori error analysis of Problem 2. We assume that the mesh size  $h$  is sufficiently small so that Lemma 4.1 holds.

**Lemma 4.7.** For any  $(W_h, \xi_h) \in \mathbb{X}_h \times \mathbb{M}_h$  there exists  $V_h \in \mathbb{X}_h$  such that  $V_h - W_h \in \mathbb{W}_h$  and satisfying

$$\mathcal{A}_\rho((W_h, \xi_h); (V_h, -\xi_h)) \gtrsim (\|W_h\|_{\mathbb{X}} + \|\xi_h\|_{\mathbb{M}})^2, \tag{74}$$

$$\|V_h\|_{\mathbb{X}} + \|\chi_h\|_{\mathbb{M}} \lesssim \|W_h\|_{\mathbb{X}} + \|\xi_h\|_{\mathbb{M}}. \tag{75}$$

**Proof.** The proof can be done by the same way as in Lemma 3.4 for the continuous problem. Let  $(W_h, \xi_h) \in \mathbb{X}_h \times \mathbb{M}_h$  and let  $Q_h \in \mathbb{W}_h$  be the unique solution of

$$\begin{cases} \text{Find } Q_h \in \mathbb{W}_h \text{ such that} \\ \mathbf{a}_\rho(Q_h, Z_h) + (Q_h, Z_h)_{\mathbb{X}} = \mathbf{b}(Z_h, \xi_h), \quad \forall Z_h \in \mathbb{W}_h. \end{cases} \tag{76}$$

By taking  $Z_h = Q_h$  in (76) we get

$$\|Q_h\|_{\mathbb{X}}^2 \leq \mathbf{b}(Q_h, \xi_h)$$

which by Cauchy–Schwarz’s inequality leads to

$$\|Q_h\|_{\mathbb{X}} \lesssim \|\xi_h\|_{\mathbb{M}}.$$

Furthermore Lemma 4.1 yields

$$C_b \|\xi_h\|_{\mathbb{M}} \leq \sup_{V_h \in \mathbb{W}_h} \frac{\mathbf{b}(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}},$$

and by (76) and Cauchy–Schwarz’s inequality, we find

$$\|\xi_h\|_{\mathbb{M}} \lesssim \|Q_h\|_{\mathbb{X}}.$$

This means that

$$\|Q_h\|_{\mathbb{X}} \sim \|\xi_h\|_{\mathbb{M}}.$$

We now take  $V_h = W_h + \delta Q_h$  where  $0 < \delta < \frac{c^\#}{c_2^\#}$ , and get

$$\begin{aligned} \mathcal{A}_\rho((W_h, \xi_h); (V_h, -\xi_h)) &= \mathcal{A}_\rho((W_h, \xi_h); (W_h + \delta Q_h, -\xi_h)) \\ &= \mathbf{a}_\rho(W_h, W_h + \delta Q_h) + \mathbf{b}(W_h + \delta Q_h, \xi_h) + \mathbf{b}(W_h, -\xi_h) \\ &= \mathbf{a}_\rho(W_h, W_h) + \delta \mathbf{a}_\rho(W_h, Q_h) + \mathbf{b}(W_h, \xi_h) + \delta \mathbf{b}(Q_h, \xi_h) - \mathbf{b}(W_h, \xi_h) \\ &= \mathbf{a}_\rho(W_h, W_h) + \delta \mathbf{a}_\rho(W_h, Q_h) + \delta \mathbf{b}(Q_h, \xi_h) \end{aligned}$$

$$\begin{aligned} &\geq c^\# \|W_h\|_{\mathbb{X}}^2 - \frac{\delta c^\#}{2} \|W_h\|_{\mathbb{X}}^2 - \frac{\delta}{2} \|Q_h\|_{\mathbb{X}}^2 + \delta \|Q_h\|_{\mathbb{X}}^2 \\ &\geq \|W_h\|_{\mathbb{X}}^2 + \|\xi_h\|_{\mathbb{M}}^2. \end{aligned}$$

Let us finally notice that  $V_h - W_h = \delta Q_h$ , which indeed belongs to  $\mathbb{W}_h$ . ■

**Theorem 4.8.** *Let  $(U, \psi)$  and  $(U_h, \psi_h)$  be the solution of Problem 2 and Problem 7 respectively. Then*

$$\|U - U_h\|_{\mathbb{X}} + \|\psi - \psi_h\|_{\mathbb{M}} \lesssim \inf_{V_h \in \mathbb{N}_h} (\|U - V_h\|_{\mathbb{X}} + \sqrt{c(V_h - U, \lambda) - \langle I_h \Phi - \Phi, \lambda \rangle}) + \inf_{\chi_h \in \mathbb{M}_h} \|\psi - \chi_h\|_{\mathbb{M}}$$

**Proof.** Let  $(V_h, \chi_h, \mu_h) \in \mathbb{N}_h \times \mathbb{M}_h \times \Lambda_h$  and let  $Q_h \in \mathbb{W}_h$  be the solution of

$$\mathbf{a}_\rho(Q_h, Z_h) + (Q_h, Z_h)_{\mathbb{X}_h} = \mathbf{b}(\chi_h, Z_h) - c(\mu_h, Z_h), \quad \forall Z_h \in \mathbb{W}_h$$

For an arbitrary  $\chi_h \in \mathbb{M}_h$ , we apply Lemma 4.7 to the pair  $(V_h - U_h, \chi_h - \psi_h)$ , hence there exists  $W_h \in \mathbb{X}_h$  such that  $D_h := W_h - (V_h - U_h) \in \mathbb{W}_h$  and satisfying

$$\begin{aligned} (\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}})^2 &\lesssim \mathcal{A}_\rho((V_h - U_h, \chi_h - \psi_h); (W_h, \psi_h - \chi_h)) \\ &= \mathcal{A}_\rho((V_h - U, \chi_h - \psi); (W_h, \psi_h - \chi_h)) + \mathcal{A}_\rho((U, \psi); (W_h, \psi_h - \chi_h)) \\ &\quad - \mathcal{A}_\rho((U_h, \psi_h); (W_h, \psi_h - \chi_h)) \end{aligned} \tag{77}$$

Taking as test function  $V_h$  in the first line of Problem 7 the function  $U_h + W_h = D_h + V_h$  that belongs to  $\mathbb{N}_h$  because  $D_h$  is in  $\mathbb{W}_h$  and  $V_h$  is in  $\mathbb{N}_h$ , we find that

$$\mathbf{a}_\rho(U_h, W_h) + \mathbf{b}(W_h, \psi_h) \geq \mathcal{L}(W_h).$$

Since we also have

$$\mathbf{b}(U_h, \psi_h - \chi_h) = 0,$$

we obtain

$$\mathbf{a}_\rho(U_h, W_h) + \mathbf{b}(W_h, \psi_h) + \mathbf{b}(U_h, \psi_h - \chi_h) \geq \mathcal{L}(W_h),$$

or equivalently,

$$\mathcal{A}_\rho((U_h, \psi_h); (W_h, \psi_h - \chi_h)) \geq \mathcal{L}(W_h), \tag{78}$$

recalling that

$$\mathcal{A}_\rho((U_h, \psi_h); (W_h, \psi_h - \chi_h)) = \mathbf{a}_\rho(U_h, W_h) + \mathbf{b}(W_h, \psi_h) + \mathbf{b}(U_h, \psi_h - \chi_h).$$

Then (77) and (78) amount to write

$$\begin{aligned} (\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}})^2 &\lesssim \mathcal{A}_\rho((V_h - U, \chi_h - \psi); (W_h, \psi_h - \chi_h)) + \mathcal{A}_\rho((U, \psi); (W_h, \psi_h - \chi_h)) \\ &\quad - \mathcal{L}(W_h). \end{aligned}$$

But by Proposition 3.7, we may write

$$\mathcal{A}_\rho((U, \psi); (W_h, \psi_h - \chi_h)) - \mathcal{L}(W_h) = \mathbf{c}(W_h, \lambda).$$

Then we get

$$(\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}})^2 \lesssim \mathcal{A}_\rho((V_h - U, \chi_h - \psi); (W_h, \psi_h - \chi_h)) + \mathbf{c}(W_h, \lambda). \tag{79}$$

As  $W_h - (V_h - U_h) \in \mathbb{W}_h$ , we directly have

$$\mathbf{c}(W_h, \lambda) = \mathbf{c}(V_h - U_h, \lambda) = \mathbf{c}(V_h - U, \lambda) + \mathbf{c}(U - U_h, \lambda).$$

Now we recall that

$$\mathbf{c}(U, \lambda) = \langle \Phi, \lambda \rangle \quad \text{and} \quad \mathbf{c}(U_h, \lambda) \geq \langle I_h \Phi, \lambda \rangle,$$

whence,

$$\mathbf{c}(U - U_h, \lambda) + \langle I_h \Phi - \Phi, \lambda \rangle \leq 0.$$

All together we then have

$$\mathbf{c}(W_h, \lambda) \leq \mathbf{c}(V_h - U, \lambda) - \langle I_h \Phi - \Phi, \lambda \rangle.$$

This estimate in (79) implies

$$(\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}})^2 \lesssim \mathcal{A}_\rho((V_h - U, \chi_h - \psi); (W_h, -\xi_h)) + \mathbf{c}(V_h - U, \lambda) - \langle I_h \Phi - \Phi, \lambda \rangle.$$

The continuity of the bilinear form  $\mathcal{A}_\rho$  gives

$$(\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}})^2 \lesssim \|V_h - U, \chi_h - \psi\|_{\mathcal{H}} \|(W_h, -\xi_h)\|_{\mathcal{H}} + c(V_h - U, \lambda) - \langle \mathcal{I}_h \Phi - \Phi, \lambda \rangle.$$

Note that

$$c(V_h - U, \lambda) - \langle \mathcal{I}_h \Phi - \Phi, \lambda \rangle \geq 0, \quad \forall V_h \in \mathbb{N}_h.$$

Applying Young's inequality to the first term on the right-hand side and completing the square gives

$$\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}} \lesssim \|V_h - U\|_{\mathbb{X}} + \|\chi_h - \psi\|_{\mathbb{M}} + \sqrt{c(V_h - U, \lambda) - \langle \mathcal{I}_h \Phi - \Phi, \lambda \rangle}$$

Finally, the desired estimate follows from the last inequality and the triangle inequality.  $\blacksquare$

The next a priori error estimate is a direct consequence of the Lagrange interpolant properties, in particular we may notice that if  $U$  belongs to  $\mathbb{N}$ , then  $\mathcal{I}_h U$  belongs to  $\mathbb{N}_h$ .

**Corollary 4.9.** Assume that the solution  $(U, \psi)$  of Problem 1 belongs to  $(H^2(\omega, \mathbb{R}^3))^2 \times H^2(\omega)$  and the function  $\Phi$  belongs to  $H^2(\omega)$ . Let  $(U_h, \psi_h)$  be the solution of Problem 7. Then

$$\|U - U_h\|_{\mathbb{X}} + \|\psi - \psi_h\|_{\mathbb{M}} \lesssim \sqrt{h} [|U|_{2,\omega} + |\psi|_{2,\omega} + |\Phi|_{2,\omega}].$$

#### 4.2. A priori error analysis of the full problem

In this section we derive a priori error analysis for Problem 5. We recall that it consists in finding  $(U, \psi, \lambda) \in \mathbb{X} \times \mathbb{M} \times \Lambda$  such that

$$\begin{cases} \forall V \in \mathbb{X}, \mathbf{a}_\rho(U, V) + \mathbf{b}(V, \psi) - c(V, \lambda) = \mathcal{L}(V), \\ \forall \chi \in \mathbb{M}, \mathbf{b}(U, \chi) = 0, \\ \forall \mu \in \Lambda, c(U, \mu - \lambda) \geq \langle \Phi, \mu - \lambda \rangle. \end{cases} \tag{80}$$

while its discrete approximation (Problem 9) consists in finding  $(U_h, \psi_h, \lambda_h) \in \mathbb{X}_h \times \mathbb{M}_h \times \Lambda_h$  such that

$$\begin{cases} \forall V_h \in \mathbb{X}_h, \mathbf{a}_\rho(U_h, V_h) + \mathbf{b}(V_h, \psi_h) - c(V_h, \lambda_h) = \mathcal{L}(V_h), \\ \forall \chi_h \in \mathbb{M}_h, \mathbf{b}(U_h, \chi_h) = 0 \\ \forall \mu_h \in \Lambda_h, c(U_h, \mu_h - \lambda_h) \geq \langle \Phi_h, \mu_h - \lambda_h \rangle. \end{cases} \tag{81}$$

First, we observe that (81) can now be written in a compact way as follows:

**Problem 10.** Find  $(U_h, \psi_h, \lambda_h) \in \mathbb{X}_h \times \mathbb{M}_h \times \Lambda_h$  such that:

$$\mathcal{B}(U_h, \psi_h, \lambda_h; V_h, \chi_h, \mu_h - \lambda_h) \geq \mathcal{L}_h(V_h, \mu_h - \lambda_h), \quad \forall (V_h, \chi_h, \mu_h) \in \mathbb{X}_h \times \mathbb{M}_h \times \Lambda_h,$$

where

$$\begin{aligned} \mathcal{B}(U_h, \psi_h, \lambda_h; V_h, \chi_h, \mu_h) &:= \mathbf{a}_\rho(U_h, V_h) + \mathbf{b}(V_h, \psi_h) + \mathbf{b}(U_h, \chi_h) - c(V_h, \lambda_h) + c(U_h, \mu_h) \\ \mathcal{L}_h(V_h, \chi_h, \mu_h) &:= \mathcal{L}(V) + \langle \Phi_h, \mu_h \rangle \end{aligned}$$

**Lemma 4.10.** There exists a constant  $\tilde{\beta}^\# > 0$  such that:

$$\inf_{(\chi_h, \mu_h) \in \mathbb{M}_h \times \mathbb{Q}_h} \sup_{Z_h = (\varepsilon_h, t_h) \in \mathbb{X}_h} \frac{c(Z_h, \mu_h) - \mathbf{b}(Z_h, \chi_h)}{\|(\chi_h, \mu_h)\|_{\mathbb{M}_h \times \mathbb{M}'} \|Z_h\|_{\mathbb{X}}} \geq \tilde{\beta}^\#. \tag{82}$$

**Proof.** Let us fix  $(\chi_h, \mu_h) \in \mathbb{M}_h \times \mathbb{Q}_h$  such that  $(\chi_h, \mu_h) \neq (0, 0)$ . First note that by Lemma 4.5 there exists  $V_h \in \mathbb{X}_h \cap \ker \mathbf{b}$  and  $\|V_h\|_{\mathbb{X}} = 1$  such that

$$\|\mu_h\|_{\mathbb{M}'} \lesssim c(V_h, \mu_h) = c(V_h, \mu_h) - \mathbf{b}(V_h, \chi_h),$$

while by Lemma 4.1 there exists  $W_h \in \mathbb{W}_h = \ker c$  with  $\|W_h\|_{\mathbb{X}} = 1$  such that:

$$\|\chi_h\|_{\mathbb{M}} \lesssim -\mathbf{b}(W_h, \chi_h) = c(W_h, \mu_h) - \mathbf{b}(W_h, \chi_h).$$

Now we may notice that  $\|V_h + W_h\|_{\mathbb{X}}$  is positive, indeed if  $V_h + W_h = 0$ , we deduce that  $W_h = -V_h$  and therefore  $U_h$  and  $W_h$  belong to  $\ker \mathbf{b} \cap \ker c$ , and by the previous estimates we would have  $\chi_h = \mu_h = 0$ , which contradicts our assumption. Then by the triangular inequality, one has  $0 < \|V_h + W_h\|_{\mathbb{X}} \leq 2$ , and consequently  $1 \leq \frac{2}{\|V_h + W_h\|_{\mathbb{X}}}$ . Using all these estimates we get

$$\begin{aligned} \|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'} &\lesssim c(V_h + W_h, \mu_h) - \mathbf{b}(V_h + W_h, \chi_h) \\ &\lesssim \frac{c(V_h + W_h, \mu_h) - \mathbf{b}(V_h + W_h, \chi_h)}{\|V_h + W_h\|_{\mathbb{X}}}. \end{aligned}$$



This obviously implies that

$$\|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'} \lesssim \sup_{Z_h \in \mathbb{X}_h} \frac{c(Z_h, \mu_h) - b(Z_h, \chi_h)}{\|Z_h\|_{\mathbb{X}}}. \quad \blacksquare \tag{83}$$

**Lemma 4.11.** For any  $(W_h, \chi_h, \mu_h) \in \mathbb{X}_h \times \mathbb{M}_h \times \mathbb{Q}_h$  there exists  $Y_h \in \mathbb{X}_h$  such that:

$$\mathcal{B}(W_h, \chi_h, \mu_h; Y_h, -\chi_h, \mu_h) \gtrsim (\|W_h\|_{\mathbb{X}} + \|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'})^2, \tag{84}$$

$$\|Y_h\|_{\mathbb{X}} + \|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'} \lesssim \|W_h\|_{\mathbb{X}} + \|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'}. \tag{85}$$

**Proof.** The proof follows the lines of the one of [Theorem 3.8](#) using the previous Lemma. We give it for completeness. Let  $(W_h, \chi_h, \mu_h) \in \mathbb{X}_h \times \mathbb{M}_h \times \mathbb{Q}_h$ . We consider the following variational problem:

$$\begin{cases} \text{Find } Q_h \text{ in } \mathbb{X}_h \text{ such that} \\ \alpha_\rho(Q_h, Z_h) + (Q_h, Z_h)_{\mathbb{X}} = b(Z_h, \chi_h) - c(\mu_h, Z_h), \quad \forall Z_h \in \mathbb{X}_h \end{cases} \tag{86}$$

Since, the bilinear form  $\alpha_\rho(\cdot, \cdot) + (\cdot, \cdot)_{\mathbb{X}}$  is  $\mathbb{X}$ -elliptic and  $\mathbb{X}_h \subset \mathbb{X}$ , problem (86) has a unique solution  $Q_h \in \mathbb{X}_h$ .

Moreover, since  $c(\cdot, \cdot) - b(\cdot, \cdot)$  satisfies the inf-sup condition (see [Lemma 4.10](#)) and the Cauchy–Schwarz inequality yields:

$$\|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'} \lesssim \sup_{Z_h \in \mathbb{X}_h} \frac{c(\mu_h, Z_h) - b(\chi_h, Z_h)}{\|Z_h\|_{\mathbb{X}}} = \sup_{Z_h \in \mathbb{X}_h} \frac{\alpha(Q_h, Z_h) + (Q_h, Z_h)_{\mathbb{X}}}{\|Z_h\|_{\mathbb{X}}} \lesssim \|Q_h\|_{\mathbb{X}}.$$

Take  $Y_h = W_h + \delta Q_h$  where  $\delta$  is a positive constant to be determined later, then we have:

$$\begin{aligned} \mathcal{B}(W_h, \chi_h, \mu_h; Y_h, -\chi_h, \mu_h) &= \alpha_\rho(W_h, W_h + \delta Q_h) + b(W_h + \delta Q_h, \chi_h) + b(W_h, -\chi_h) - c(W_h + \delta Q_h, \mu_h) \\ &\quad + c(\mu_h, W_h) \\ &= \alpha_\rho(W_h, W_h) + \delta \alpha_\rho(W_h, Q_h) + \delta b(Q_h, \chi_h) - \delta c(\mu_h, Q_h) \\ &\geq \alpha_\rho(W_h, W_h) - \delta c^\# \|W_h\|_{\mathbb{X}} \|Q_h\|_{\mathbb{X}} + \delta \|Q_h\|_{\mathbb{X}}^2 \\ &\geq (c_\# - \frac{\delta (c^\#)^2}{2}) \|W_h\|_{\mathbb{X}}^2 + \frac{\delta}{2} \|Q_h\|_{\mathbb{X}}^2. \end{aligned}$$

Then it suffice to take  $0 < \delta < \frac{c_\#^2}{2(c^\#)^2}$  to obtain

$$\mathcal{B}(W_h, \chi_h, \mu_h; Y_h, -\chi_h, \mu_h) \gtrsim (\|W_h\|_{\mathbb{X}}^2 + \|\chi_h\|_{\mathbb{M}}^2 + \|\mu_h\|_{\mathbb{M}'}^2). \quad \blacksquare \tag{87}$$

**Theorem 4.12.** Let  $(U, \psi, \lambda)$  and  $(U_h, \psi_h, \lambda_h)$  be the solution of [Problem 5](#) and [Problem 9](#) respectively. Then

$$\begin{aligned} \|U - U_h\|_{\mathbb{X}} + \|\psi - \psi_h\|_{\mathbb{M}} + \|\lambda - \lambda_h\|_{\mathbb{M}'} &\lesssim \inf_{V_h \in \mathbb{N}_h} \|U - V_h\|_{\mathbb{X}} + \inf_{\chi_h \in \mathbb{M}_h} \|\psi - \chi_h\|_{\mathbb{M}} \\ &\quad + \inf_{\mu_h \in \Lambda_h} (\|\mu_h - \lambda\|_{\mathbb{M}'} + \sqrt{c(U, \mu_h - \lambda) - \langle \Phi, \mu_h - \lambda \rangle}) \\ &\quad + \|\Phi - \Phi_h\|_{\mathbb{M}}. \end{aligned}$$

**Proof.** Let  $V_h \in \mathbb{N}_h$  and let  $Q_h \in \mathbb{W}_h$  be the solution of

$$\alpha_\rho(Q_h, Z_h) + (Q_h, Z_h)_{\mathbb{X}_h} = b(\chi_h - \psi_h, Z_h) - c(\mu_h - \lambda_h, Z_h), \quad \forall Z_h \in \mathbb{W}_h.$$

Using [Lemma 4.11](#) with  $W_h = V_h - U_h, \chi_h = \chi_h - \psi_h$  and  $\mu_h = \mu_h - \lambda_h$ , there exists  $Y_h \in \mathbb{X}_h$  satisfying (84) and (85), namely

$$\|Y_h\|_{\mathbb{X}} \lesssim \|V_h - U_h\|_{\mathbb{X}}. \tag{88}$$

as well as

$$\begin{aligned} (\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}} + \|\mu_h - \lambda_h\|)^2 &\lesssim \mathcal{B}(V_h - U_h, \chi_h - \psi_h, \mu_h - \lambda_h; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h) \\ &= \mathcal{B}(V_h - U_h, \chi_h - \psi_h, \mu_h - \lambda_h; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h) \\ &\quad + \mathcal{B}(U, \psi, \lambda; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h) \\ &\quad - \mathcal{B}(U_h, \psi_h, \lambda_h; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h). \end{aligned}$$

By the definition of [Problem 10](#)

$$-\mathcal{B}(U_h, \psi_h, \lambda_h; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h) \leq -\mathcal{L}_h(Y_h, \mu_h - \lambda_h).$$

For the second term we have, since  $\mathbb{X}_h \subset \mathbb{X}$  then the definition of the bilinear form  $c(\cdot, \cdot)$  implies that:

$$\begin{aligned} \mathcal{B}(U, \psi, \lambda; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h) &= a_\rho(U, Y_h) + \mathbf{b}(Y_h, \psi) + \underbrace{\mathbf{b}(U, \psi_h - \chi_h)}_{=0} - c(Y_h, \lambda) + c(U, \mu_h - \lambda_h) \\ &= \mathcal{L}(Y_h) + c(U, \mu_h - \lambda_h), \end{aligned}$$

we then get

$$\begin{aligned} (\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}} + \|\mu_h - \lambda_h\|)^2 &\lesssim \mathcal{B}(V_h - U, \chi_h - \psi, \mu_h - \lambda; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h) \\ &\quad + c(U, \mu_h - \lambda_h) + \mathcal{L}(Y_h) - \mathcal{L}_h(Y_h, \mu_h - \lambda_h) \\ &= \mathcal{B}(V_h - U, \chi_h - \psi, \mu_h - \lambda; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h) \\ &\quad + c(U, \mu_h - \lambda_h) - \langle \Phi_h, \mu_h - \lambda_h \rangle. \end{aligned}$$

Since  $\Lambda_h \subset \Lambda$  then,

$$c(U, \lambda_h) \geq \langle \Phi, \lambda_h \rangle \tag{89}$$

On the other hand, we have,

$$\begin{aligned} c(U, \mu_h - \lambda_h) - \langle \Phi_h, \mu_h - \lambda_h \rangle &= c(U, \mu_h - \lambda) + c(U, \lambda - \lambda_h) - \langle \Phi_h - \Phi, \mu_h - \lambda_h \rangle - \langle \Phi, \mu_h - \lambda \rangle \\ &\quad - \langle \Phi, \lambda - \lambda_h \rangle \\ &\leq c(U, \mu_h - \lambda) - \langle \Phi_h - \Phi, \mu_h - \lambda_h \rangle - \langle \Phi, \mu_h - \lambda \rangle \end{aligned}$$

whence we obtain

$$\begin{aligned} (\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}} + \|\mu_h - \lambda_h\|)^2 &\lesssim \mathcal{B}(V_h - U, \chi_h - \psi, \mu_h - \lambda; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h) \\ &\quad + c(U, \mu_h - \lambda) - \langle \Phi, \mu_h - \lambda \rangle - \langle \Phi_h - \Phi, \mu_h - \lambda_h \rangle \end{aligned}$$

Applying Young’s inequality to the first term on the right-hand side, using the estimate (88), and completing the square gives

$$\begin{aligned} \|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}} + \|\mu_h - \lambda_h\|_{\mathbb{M}'} &\lesssim \|V_h - U\|_{\mathbb{X}} + \|\chi_h - \psi\|_{\mathbb{M}} \\ &\quad + \|\mu_h - \lambda\|_{\mathbb{M}'} + \sqrt{c(U, \mu_h - \lambda) - \langle \Phi, \mu_h - \lambda \rangle} + \|\Phi - \Phi_h\|_{\mathbb{M}} \end{aligned}$$

Finally, the desired estimate follows from this last inequality and the triangle inequality. ■

**Corollary 4.13.** Assume that the solution  $(U, \psi, \lambda)$  of Problem 5 belongs to  $(H^2(\omega, \mathbb{R}^3))^2 \times H^2(\omega) \times L^2(\omega)$  and the function  $\Phi$  belongs to  $H^2(\omega) \cap H^1_{\gamma_0}(\omega)$ . Let  $(U_h, \psi_h, \lambda_h)$  be the solution of Problem 9. Then

$$\|U - U_h\|_{\mathbb{X}} + \|\psi - \psi_h\|_{\mathbb{M}} + \|\lambda - \lambda_h\|_{\mathbb{M}'} \lesssim \sqrt{h} [ \|U\|_{2,\omega} + \|\psi\|_{2,\omega} + \|\Phi\|_{2,\omega} + \|\lambda\|_{\omega} ].$$

**Proof.** The proof is based on the a priori error estimate shown in Theorem 4.12. The estimate for the terms,  $\|V_h - U\|_{\mathbb{X}}$ ,  $\|\chi_h - \psi\|_{\mathbb{M}}$ ,  $\|\mu_h - \lambda\|_{\mathbb{M}'}$  and  $\|\Phi - \Phi_h\|_{\mathbb{M}}$  can be easily obtained by standard interpolation procedure. To show the result, we need to prove the estimate for the term  $\sqrt{c(U, \mu_h - \lambda) - \langle \Phi, \mu_h - \lambda \rangle}$ .

As  $U \in H^1_{\gamma_0}(\omega; \mathbb{R}^3)^2$  and  $\Phi \in H^1_{\gamma_0}(\omega)$ , one has

$$|c(U, \mu_h - \lambda) - \langle \Phi, \mu_h - \lambda \rangle| \lesssim (\|U\|_{1,\omega} + \|\Phi\|_{1,\omega}) \|\mu_h - \lambda\|_{\mathbb{M}'}. \tag{90}$$

Now we take  $\mu_h$  as the weighted Clément type interpolation operator of  $\lambda$  [20], namely  $\mu_h = \mathcal{Q}_h \lambda$ , defined by

$$\mathcal{Q}_h \lambda = \sum_{x \in \mathcal{N}_h} \pi_x(\lambda) \lambda_x,$$

where for any  $\varphi \in L^1(\omega)$ , one sets

$$\pi_x(\varphi) = \begin{cases} \frac{\int_{\omega_x} \varphi \lambda_x}{\int_{\omega_x} \lambda_x} & \text{if } x \notin \bar{\gamma}_0, \\ 0 & \text{if } x \in \bar{\gamma}_0. \end{cases}$$

Hence as

$$\|\lambda - \mathcal{Q}_h \lambda\|_{\mathbb{M}'} = \sup_{\varphi \in H^1_{\gamma_0}(\omega), \varphi \neq 0} \frac{\int_{\omega} (\lambda - \mathcal{Q}_h \lambda) \varphi}{\|\varphi\|_{1,\omega}},$$

and as we directly check that

$$\int_{\omega} (\lambda - \mathcal{Q}_h \lambda) \varphi = \int_{\omega} \lambda (\varphi - \mathcal{Q}_h \varphi),$$

we obtain

$$\|\lambda - Q_h \lambda\|_{M'} = \sup_{\varphi \in H^1_{\gamma_0}(\omega), \varphi \neq 0} \frac{\int_{\omega} \lambda(\varphi - Q_h \varphi)}{\|\varphi\|_{1,\omega}}.$$

By Cauchy-Schwarz's inequality and Lemma 6.2 of [20], we get

$$\|\lambda - Q_h \lambda\|_{M'} \lesssim h \|\lambda\|_{\omega}.$$

Inserting this estimate in (90), we obtain

$$|c(U, \mu_h - \lambda)| \lesssim h(\|U\|_{1,\omega} + \|\Phi\|_{1,\omega}) \|\lambda\|_{\omega} \lesssim h(\|U\|_{1,\omega}^2 + \|\Phi\|_{1,\omega}^2 + \|\lambda\|_{\omega}^2). \quad \square$$

### 5. Iterative solution: Uzawa-type stationary methods

In this section we analyse the Uzawa method for solving Problem 9. The most attractive character of the Uzawa method is its simplicity of implementation and its minimal memory requirements. Even if its speed of convergence may be slow, we have chosen it due to its simplicity of implementation and its minimal memory requirement. A primal–dual active set method [1,21] may certainly be used, this will be investigated in the future. In each Uzawa iteration, an elliptic solver for computing the inverse of a large sparse matrix is needed. The convergence of the Uzawa method for saddle point systems was discussed by several authors (see for instance [22,23]). Here we prove the convergence of the Uzawa algorithm for the case of variational inequality, namely Problem 9.

It is important to observe that Problem 9 in its matrix form can be regarded as a  $2 \times 2$  block matrix in two different ways, according to the used partitioning strategy. The first way consists of finding  $(U_h^{k+1}, \psi_h^{k+1})$ , for a given  $\lambda_h^k \in \Lambda_h$ . It highlights the fact that Problem 9 can in principle be treated as a standard saddle point problem to compute  $U_h^{k+1}, \psi_h^{k+1}$  followed by a projection procedure that computes  $\lambda_h^{k+1}$ . The second way consists of finding  $U_h^{k+1}$ , for a given  $(\psi_h^k, \lambda_h^k) \in \mathbb{M}_h \times \Lambda_h$ . More precisely, assume that we consider iterative methods for solving large, sparse linear systems of equations of the form

$$\mathcal{A}x = b, \quad \text{with } \mathcal{A} \equiv \begin{bmatrix} A & B^T & C^T \\ B & 0 & 0 \\ C & 0 & 0 \end{bmatrix}, \tag{91}$$

where  $A \in \mathbb{R}^{n \times n}$  is symmetric positive definite,  $B \in \mathbb{R}^{m \times n}$  and  $C \in \mathbb{R}^{p \times n}$ . We recall that iterative methods are based on the splitting of  $\mathcal{A}$ , i.e.,

$$\mathcal{A} = \mathcal{M} - \mathcal{N}, \quad \text{with invertible matrix } \mathcal{M}$$

and the iterative scheme

$$x_{k+1} = \mathcal{M}^{-1} \mathcal{N} x_k + \mathcal{M}^{-1} b, \quad k = 0, 1, 2, \dots,$$

For our considered case, the matrix  $\mathcal{A}$  can be regarded as a  $2 \times 2$  block matrix in two different ways, according to which of the following partitioning strategies is used:

$$\mathcal{A} \equiv \left[ \begin{array}{c|cc} A & B^T & C^T \\ \hline B & 0 & 0 \\ C & 0 & 0 \end{array} \right] \quad \text{or} \quad \mathcal{A} \equiv \left[ \begin{array}{cc|c} A & B^T & C^T \\ \hline B & 0 & 0 \\ C & 0 & 0 \end{array} \right]. \tag{92}$$

Then two Uzawa-like iterative methods can be considered for solving (91) according to the splittings

$$\mathcal{A} = \mathcal{M}_1 - \mathcal{N}_1 \quad \text{or} \quad \mathcal{A} = \mathcal{M}_2 - \mathcal{N}_2,$$

with

$$\mathcal{M}_1 = \begin{bmatrix} A & 0 & 0 \\ B & \frac{-1}{\alpha} I & 0 \\ C & 0 & \frac{-1}{\beta} I \end{bmatrix}, \quad \mathcal{N}_1 = \begin{bmatrix} 0 & -B^T & -C^T \\ 0 & \frac{-1}{\alpha} I & 0 \\ 0 & 0 & \frac{-1}{\beta} I \end{bmatrix},$$

$$\mathcal{M}_2 = \begin{bmatrix} A & B^T & 0 \\ B & 0 & 0 \\ C & 0 & \frac{-1}{\alpha} I \end{bmatrix}, \quad \mathcal{N}_2 = \begin{bmatrix} 0 & 0 & -C^T \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\alpha} I \end{bmatrix}.$$

where  $\alpha$  and  $\beta$  are two given nonzero parameters (see [24,25]).

Then if the splitting  $\mathcal{A} = \mathcal{M}_1 - \mathcal{N}_1$  is used, we need to initialize our iterative method with a given  $(\psi_h^0, \lambda_h^0) \in \mathbb{M}_h \times \Lambda_h$  and two parameters  $\alpha$  and  $\beta$ . While for the second choice we need only one parameter and an initial guess  $\lambda_h^0 \in \Lambda_h$ .

In this paper we will make use only the second type with only one parameter  $\alpha$ , but we are especially interested in studying the convergence of the considered method when the third line of the system (91) is replaced by an inequality.

The Uzawa algorithm of [Problem 9](#) in variational form reads,

$$\left\{ \begin{aligned} \mathbf{a}_\rho(U_h^{k+1}, V_h) + \mathbf{b}(V_h, \psi_h^{k+1}) &= \mathcal{L}(V_h) + \mathbf{c}(V_h, \lambda_h^k), \quad \forall V_h \in \mathbb{X}_h, \\ \mathbf{b}(U_h^{k+1}, \chi_h) &= 0, \quad \forall \chi_h \in \mathbb{M}_h, \\ (\tilde{\lambda}_h^{k+1}, \tilde{\mu}_h) &= (\lambda_h^k, \tilde{\mu}_h) + \alpha \mathbf{c}(U_h^{k+1}, \tilde{\mu}_h) - \alpha(\Phi_h, \tilde{\mu}_h), \quad \forall \tilde{\mu}_h \in \mathbb{M}_h, \\ \lambda_h^{k+1} &= \mathcal{P}_{\Lambda_h}(\tilde{\lambda}_h^{k+1}). \end{aligned} \right. \tag{93}$$

For  $k \geq 1$ , we introduce the following notation

$$E_h^k = U_h - U_h^k, \quad \psi_h^k = \psi_h - \psi_h^k, \quad \text{and} \quad e_h^k = \lambda_h - \lambda_h^k,$$

then we can easily write

$$\begin{aligned} \mathbf{a}_\rho(E_h^{k+1}, V_h) + \mathbf{b}(V_h, E_h^{k+1}) &= \mathbf{c}(V_h, e_h^k), \quad \forall V_h \in \mathbb{X}_h \\ \mathbf{b}(E_h^{k+1}, \chi_h) &= 0, \quad \forall \chi_h \in \mathbb{M}_h \end{aligned}$$

then using the coercivity of  $\mathbf{a}_\rho(\cdot, \cdot)$ , the continuity of  $\mathbf{c}(\cdot, \cdot)$  and the inf-sup condition of  $\mathbf{b}(\cdot, \cdot)$ , we get,

$$\|E_h^{k+1}\|_{\mathbb{X}} + \|E_h^{k+1}\|_{\mathbb{M}} \lesssim \|e_h^k\|$$

and therefore, the convergence to zero of the sequence  $\|e_h^k\|$  will imply immediately the convergence of  $\|E_h^{k+1}\|_{\mathbb{X}}$  and  $\|E_h^{k+1}\|_{\mathbb{M}}$ .

We end those preliminary remarks by the following observation. We recall that,

$$\lambda_h^{k+1} = \mathcal{P}_{\Lambda_h}(\tilde{\lambda}_h^{k+1}),$$

i.e,  $\lambda_h^{k+1}$  is the projection of  $\tilde{\lambda}_h^{k+1}$  on the closed convex set  $\Lambda_h$ , then,

$$(\tilde{\lambda}_h^{k+1} - \lambda_h^{k+1}, \mu_h - \lambda_h^{k+1}) \leq 0, \quad \forall \mu_h \in \Lambda_h \text{ and } \|\lambda_h^{k+1}\| \leq \|\tilde{\lambda}_h^{k+1}\|.$$

Since  $(\tilde{\lambda}_h^{k+1} - \lambda_h^{k+1}, \lambda_h) \leq 0$ , we then have

$$(\tilde{\lambda}_h^{k+1} - \lambda_h^{k+1}, \lambda_h) + \|\lambda_h^{k+1}\|^2 \leq \|\tilde{\lambda}_h^{k+1}\|^2,$$

which is equivalent to

$$(\lambda_h - \lambda_h^{k+1}, \lambda_h - \lambda_h^{k+1}) \leq (\lambda_h - \tilde{\lambda}_h^{k+1}, \lambda_h - \tilde{\lambda}_h^{k+1}).$$

This amounts to write

$$\|e_h^{k+1}\| \leq \|\tilde{e}_h^{k+1}\|, \quad \text{where } \tilde{e}_h^k = \lambda_h - \tilde{\lambda}_h^k. \tag{94}$$

Hence, the convergence to zero of the sequence  $\|\tilde{e}_h^{k+1}\|$ , will imply the convergence to zero of the sequence  $\|e_h^{k+1}\|$ .

**Theorem 5.1.** Let  $K_0 = \frac{c_\#}{\tilde{c}_c}$  where  $\tilde{c}_c$  is the inf-sup constant of [Lemma 4.4](#). Denote by  $c_{c,\#}$  the continuity constant of the bilinear form  $c$  in  $\mathbb{X} \times L^2(\omega)$ , namely the smallest positive constant such that

$$|c(V, \mu)| \leq c_{c,\#} \|U\|_{\mathbb{X}} \|\mu\|, \quad \forall V \in \mathbb{X}, \mu \in L^2(\omega). \tag{95}$$

If the parameter  $\alpha$  is chosen such that

$$0 < 1 + \alpha(\alpha c_{c,\#}^2 - 2c_\#)K_0^{-2}h^2 < 1 \tag{96}$$

then

$$\lim_{k \rightarrow +\infty} \|\tilde{e}_h^{k+1}\| = 0.$$

**Proof.** First we have

$$\mathbf{a}_\rho(E_h^{k+1}, V_h) + \mathbf{b}(V_h, E_h^{k+1}) - \mathbf{c}(V_h, e_h^k) = 0, \quad \forall V_h \in \mathbb{X}_h. \tag{97}$$

Take  $V_h = E_h^{k+1}$ , we get

$$\mathbf{a}_\rho(E_h^{k+1}, E_h^{k+1}) = \underbrace{\mathbf{c}(E_h^{k+1}, e_h^k) - \mathbf{b}(E_h^{k+1}, E_h^{k+1})}_{=0} = \mathbf{c}(E_h^{k+1}, e_h^k). \tag{98}$$

The third line of the scheme [\(81\)](#) and [\(93\)](#) amount to write

$$\begin{aligned} (\tilde{\lambda}_h^{k+1}, \tilde{\mu}_h) &= (\lambda_h^k, \tilde{\mu}_h) + \alpha \mathbf{c}(U_h^{k+1}, \tilde{\mu}_h) + \alpha(\Phi_h, \tilde{\mu}_h), \\ \mathbf{c}(U_h, \lambda_h^k - \lambda_h) &\geq \langle \Phi_h, \lambda_h^k - \lambda_h \rangle. \end{aligned}$$

For  $\tilde{\mu}_h = e_h^k$ , we get

$$\begin{aligned} (\tilde{\lambda}_h^{k+1}, e_h^k) &= (\lambda_h^k, e_h^k) + \alpha c(U^{k+1}, e_h^k) + \alpha(\Phi_h, e_h^k), \\ (\lambda_h, e_h^k) &\geq (\lambda_h, e_h^k) + \alpha c(U_h, e_h^k) + \alpha(\Phi_h, e_h^k). \end{aligned}$$

Then,

$$(\tilde{e}_h^{k+1}, e_h^k) \geq (e_h^k, e_h^k) + \alpha c(E_h^{k+1}, e_h^k)$$

or equivalently,

$$\alpha c(E_h^{k+1}, \mu_h) \leq (\tilde{e}_h^{k+1} - e_h^k, e_h^k).$$

Using (98) we find

$$a_\rho(E_h^{k+1}, E_h^{k+1}) \leq -\frac{1}{\alpha}(\tilde{e}_h^{k+1} - e_h^k, e_h^k) = -\frac{1}{2\alpha}(\|\tilde{e}_h^{k+1}\|^2 - \|e_h^k\|^2 - \|\tilde{e}_h^{k+1} - e_h^k\|^2), \tag{99}$$

where in the last equality we have used the identity

$$\begin{aligned} (b - a, b - a) - (b, b) + (a, a) &= (b, b - a) - (a, b - a) - (b, b) + (a, a) \\ &= (b, b) - (b, a) - (a, b - a) - (b, b) + (a, a) \\ &= (a, a - b) - (a, b - a) \\ &= 2(a, a - b). \end{aligned}$$

The estimate (99) implies that

$$2\alpha c^\# \|E_h^{k+1}\|_{\mathbb{X}}^2 + \|\tilde{e}_h^{k+1}\|^2 \leq \|e_h^k\|^2 + \|\tilde{e}_h^{k+1} - e_h^k\|^2, \tag{100}$$

where  $c^\#$  is the coercivity constant of  $a_\rho$ . The next step is to evaluate  $\|\tilde{e}_h^{k+1} - e_h^k\|^2$ .

Let  $\tilde{\mu}_h \in \mathbb{M}_h$  and  $\mu_h \in \Lambda_h$  be given by

$$\tilde{\mu}_h = \tilde{e}_h^{k+1} - e_h^k, \quad \mu_h = \lambda_h - (\lambda_h^k - \tilde{\lambda}_h^{k+1})$$

then,

$$\tilde{\mu}_h = \mu_h - \lambda_h.$$

We take this  $\tilde{\mu}_h$  in the third line of (93) to obtain

$$(\tilde{\lambda}_h^{k+1}, \tilde{\mu}_h) = (\lambda_h^k, \tilde{\mu}_h) + \alpha c(U_h^{k+1}, \tilde{\mu}_h) - \alpha(\Phi_h, \tilde{\mu}_h),$$

while by the third line of (57) one has

$$(\lambda_h, \mu_h - \lambda_h) \leq (\lambda_h, \mu_h - \lambda_h) + \alpha c(U_h, \mu_h - \lambda_h) - \alpha(\Phi_h, \mu_h - \lambda_h).$$

So by taking the difference

$$(\tilde{e}_h^{k+1}, \tilde{e}_h^{k+1} - e_h^k) \leq (e_h^k, \tilde{e}_h^{k+1} - e_h^k) + \alpha c(E_h^{k+1}, \tilde{e}_h^{k+1} - e_h^k) \tag{101}$$

Then the estimate (95) yields

$$\|\tilde{e}_h^{k+1} - e_h^k\|^2 \leq \alpha c(E_h^{k+1}, \tilde{e}_h^{k+1} - e_h^k) \leq \alpha c_{c,\#} \|E_h^{k+1}\|_{\mathbb{X}} \|\tilde{e}_h^{k+1} - e_h^k\|.$$

This estimate is finally equivalent to

$$\|\tilde{e}_h^{k+1} - e_h^k\|^2 \leq \alpha^2 c_{c,\#}^2 \|E_h^{k+1}\|_{\mathbb{X}}^2.$$

By (100), we deduce that,

$$(2\alpha c^\# - \alpha^2 c_{c,\#}^2) \|E_h^{k+1}\|_{\mathbb{X}}^2 + \|\tilde{e}_h^{k+1}\|^2 \leq \|e_h^k\|^2,$$

or equivalently

$$\|\tilde{e}_h^{k+1}\|^2 \leq \|e_h^k\|^2 + \alpha(\alpha c_{c,\#}^2 - 2c^\#) \|E_h^{k+1}\|_{\mathbb{X}}^2. \tag{102}$$

We therefore chose  $\alpha > 0$  small enough such that  $\alpha c_{c,\#}^2 - 2c^\# < 0$ . So that we will conclude if one can show that

$$\|e_h^k\| \leq K_0 h^{-1} \|E_h^{k+1}\|_{\mathbb{X}}. \tag{103}$$

Indeed if this estimate is valid then (102) becomes

$$\|\tilde{e}_h^{k+1}\|^2 \leq (1 + \alpha(\alpha c_{c,\#}^2 - 2c^\#) K_0^{-2} h^2) \|e_h^k\|^2,$$

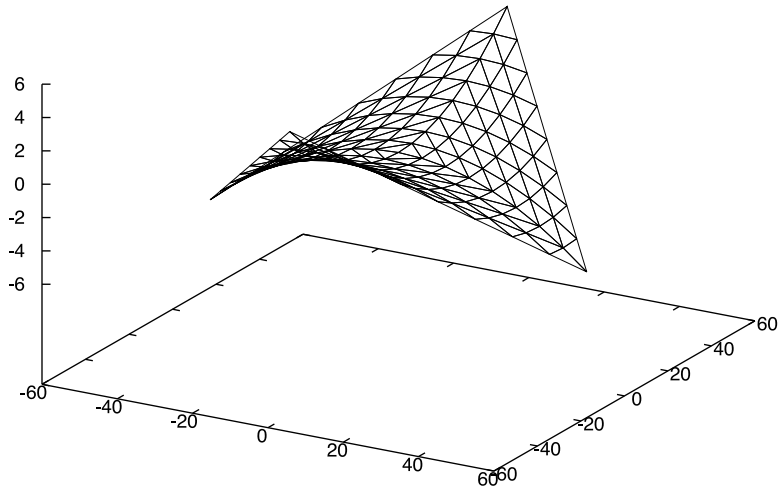


Fig. 1. The surface  $S = \varphi(\omega)$ .

and by (94)

$$\|e_h^{k+1}\|^2 \leq (1 + \alpha(ac_{c,\#}^2 - 2c^\#)K_0^{-2}h^2)\|e_h^k\|^2.$$

By iteration we finally find

$$\|e_h^k\| \leq (1 + \alpha(ac_{c,\#}^2 - 2c^\#)K_0^{-2}h^2)^{\frac{k}{2}} \|e_h^0\|,$$

and proves the convergence  $\|e_h^k\|$  to zero if  $0 < 1 + \alpha(ac_{c,\#}^2 - 2c^\#)K_0^{-2}h^2 < 1$ .

It remains to prove (103). For that purpose, we use the identity (97), which says that

$$c(V_h, e_h^k) = \mathbf{a}_\rho(\mathbf{E}_h^{k+1}, V_h) + \mathbf{b}(V_h, \mathbf{E}_h^{k+1}), \quad \forall V_h \in \mathbb{X}_h,$$

which reduces to

$$c(V_h, e_h^k) = \mathbf{a}_\rho(\mathbf{E}_h^{k+1}, V_h) \quad \forall V_h \in \mathbb{X}_h \cap \ker b,$$

Hence using Lemma 4.4, we deduce that

$$\tilde{c}_c \|e_h^k\|_h \leq c_\# \|\mathbf{E}_h^{k+1}\|_{\mathbb{X}}.$$

By the definition of the norm  $\|\cdot\|_h$ , we get (103) with  $K_0 = \frac{c_\#}{\tilde{c}_c}$ . ■

### 6. Numerical tests

In this section we first discuss the performance of the Uzawa algorithm presented in the previous section for solving the discrete problem Problem 9. We consider the hyperbolic paraboloid shell  $S = \varphi(\omega)$ , where the reference domain  $\omega$  is

$$\omega = \left\{ (x, y) \in \mathbb{R}^2, |x| + |y| < 50\sqrt{2} \right\} \tag{104}$$

and the chart is defined by (see Fig. 1)

$$\varphi(x, y) = (x, y, 1.4 + \frac{x^2 - y^2}{R^2}), \quad \text{with } R = 50\sqrt{2}. \tag{105}$$

The shell is clamped on  $\partial\omega$ , namely, we choose  $\gamma_0 = \partial\omega$  (hence  $\gamma_1$  is empty) and subjected to a uniform pressure  $f_3 = q = -0.25 \text{ kp/cm}^2$ . In other words, in (8), we choose  $f = (0, 0, q)$ . As Young’s modulus and Poisson’s ratio we take  $E = 2.85 \times 10^4 \text{ kp/cm}^2$ ,  $\nu = 0.4$  respectively, while the thickness of the shell is  $\varepsilon = 0.8 \text{ cm}$ .

Then the function  $\Phi$  defined by (15) is here given by

$$\Phi(x, y) = \frac{0.4R^2}{\sqrt{4(x^2 + y^2) + R^4}} - \frac{x^2 - y^2}{R^2} - 1.4.$$

Note that the function  $\Phi$  (see Fig. 4(a)) satisfies the condition (16) which ensures that the surface satisfies the required conditions that we discussed in the introductory section. The numerical tests that we now present have been performed on the finite element code FreeFEM++ [26]. We study the convergence of the Uzawa method with respect to the number of iterations.

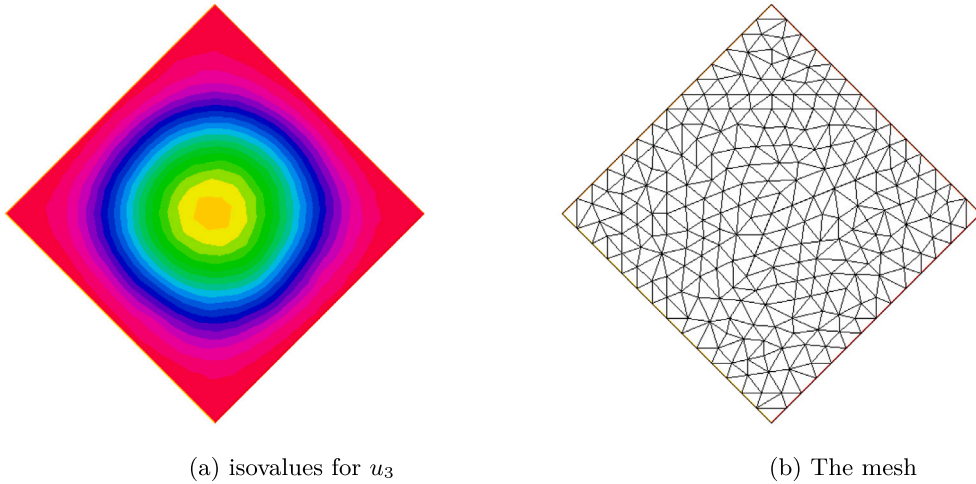


Fig. 2. The mesh and isovalues for  $u_3$ .

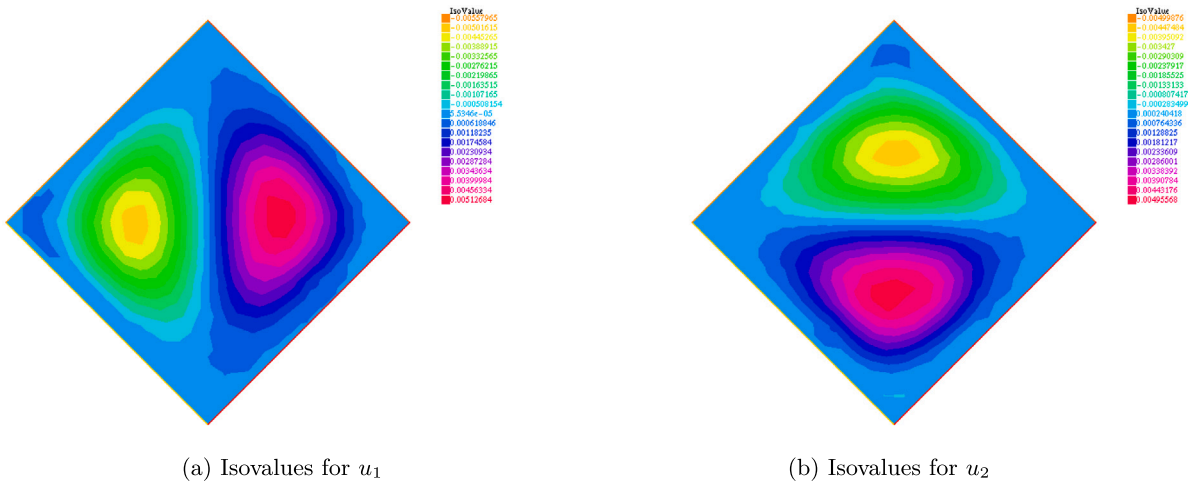


Fig. 3. Isovalues for  $u_\beta, \beta = 1, 2$ .

It is well known that (after the elimination of the unknowns  $U$  and  $\psi$ ) the Uzawa method can be seen as a fixed-parameter first-order Richardson iteration by using Schur complement (for the unknown  $\lambda$ ) (see [24]). So as stopping criterion we can stop the process when  $\|\lambda^{k+1} - \lambda^k\|_{L^\infty}$  is very small.

In our problem the contact zone is defined to be the set of points  $(x, y) \in \omega$  that satisfies  $((u - \frac{\epsilon}{2}r) \cdot e_3)(x, y) = \Phi(x, y)$ . For contact problems, the contact zone and the free boundary are a priori unknowns. However, for the considered example we have observed that, at the origin  $(0, 0)$  we have  $((u_h - \frac{\epsilon}{2}r_h) \cdot e_3)(0, 0) \approx \Phi_h(0, 0)$  when the number of iterations becomes large. Since the analytic expression of the function  $\Phi_h$  is available and by analytical computations we have  $\Phi_h(0, 0) = -1$ , we have reported the quantity  $|((u_h - \frac{\epsilon}{2}r_h) \cdot e_3)(0, 0) - \Phi_h(0, 0)|$  at different steps. The results indicate that this quantity relatively decays to zero by the same rate as  $\|\lambda^{k+1} - \lambda^k\|_{L^\infty}$  (see Fig. 6). This observation may be interpreted as follows: the type of the considered loading and the position of the function  $\Phi$  with respect to the shell imply that the origin  $(0, 0)$  belongs to the contact zone at least for the discrete problems.

In Fig. 2(a) the isovalues for  $u_3$  are plotted using the quasi uniform mesh shown in Fig. 2(b). Due to the form of the considered loading we can expect that the displacement  $u_3$  will be larger than the tangential displacement  $u_\beta, \beta = 1, 2$ . Fig. 3 shows this significant difference between  $u_3$  and  $u_\beta, \beta = 1, 2$ . Indeed the range of  $u_3$  is between 0.029 and  $-1.09$  while the values of  $u_1$  and  $u_2$  varies between  $-0.005$  and  $0.005$ .

The constraint  $(u_h - \frac{\epsilon}{2}r_h) \cdot e_3$  and the function  $\Phi_h$  are presented in Fig. 4(b), there we observe that the function  $\Phi_h$  can be considered as an obstacle for the unknown  $(u_h - \frac{\epsilon}{2}r_h) \cdot e_3$ . In Fig. 5 we plot the “contact zone”, and the free boundary after 350 iterations. It seems to be a connected and non convex subset of  $\omega$  that contains the origin.

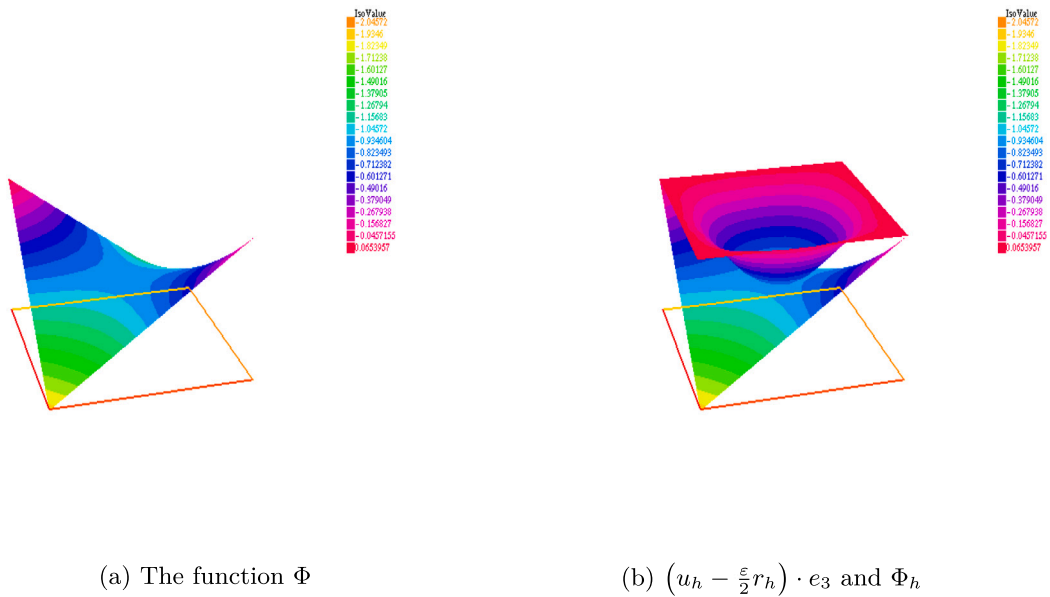


Fig. 4. The constraint  $(u_h - \frac{\epsilon}{2}r_h) \cdot e_3$  and the functions  $\Phi$  and  $\Phi_h$ .

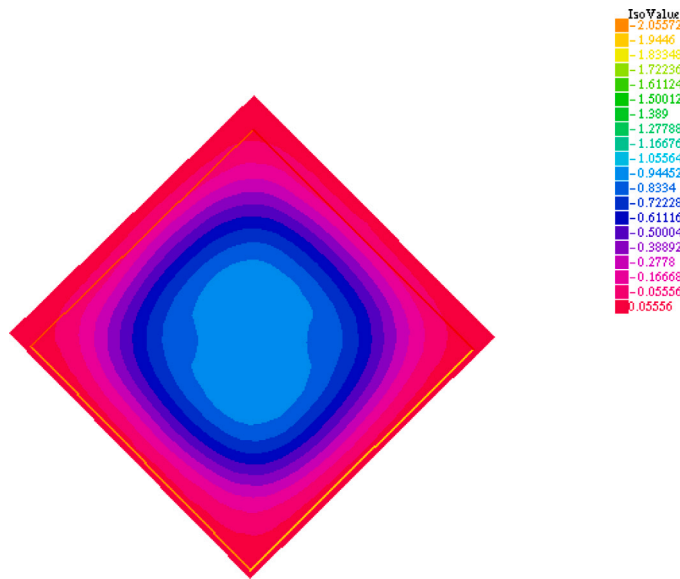


Fig. 5. The contact zone  $\{(x, y) \in \omega; ((u_h - \frac{\epsilon}{2}r_h) \cdot e_3)(x, y) = \Phi_h(x, y)\}$ .

Table 1

Convergence results for the Uzawa scheme using  $\mathbb{P}_1 \oplus B_3 - \mathbb{P}_1 \oplus B_3 - \mathbb{P}_0$ .

Iteration	100	150	200	250	300	350
$\ \lambda^{k+1} - \lambda^k\ _{L^2(\omega)}$	0.0485238	0.0327921	0.0288491	0.0220641	0.021193	0.0192298
Value of The constraint at the point (0,0)	0.0317884	0.024239	0.0207364	0.0169701	0.0156183	0.0140509

We present in Table 1 and Fig. 6, the evolution of  $\|\lambda^{k+1} - \lambda^k\|_{L^\infty(\omega)}$  and of  $|(u_h - \frac{\epsilon}{2}r_h) \cdot e_3(0,0) - \Phi_h(0,0)|$  at different iterations with the choice  $\rho = 10^3$ . Note that the number of iterations to stop the algorithm for some reasonable stopping criteria is huge.



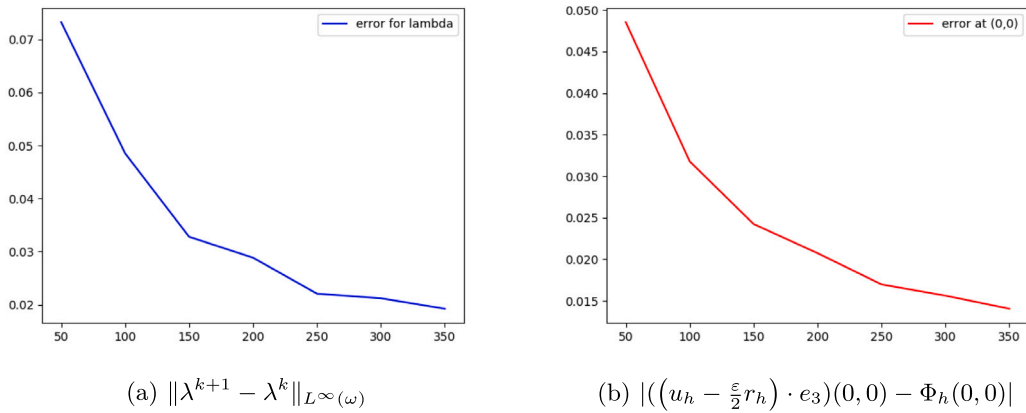


Fig. 6. The error as a function of the number of iterations.

Table 2

Convergence results for the Uzawa scheme for  $\alpha = 0.01$  and different values of  $\rho$ .

$\rho$	$10^3$	$10^5$	$10^7$	$10^9$
$\ \lambda_h^{k+1} - \lambda_h^k\ _{L^2(\omega)}$	0.0119165	0.0119167	0.0119348	0.0087216
$ ((u_h - \frac{\varepsilon}{2}r_h) \cdot e_3 - \Phi_h)(0,0) $	0.0635194	0.0636253	0.0637142	0.0621998

Table 3

Comparison of the number of iteration for  $\alpha = 0.01$  and different values of  $\rho$ .

$\rho$	1	$10^1$	$10^3$	$10^5$	$10^7$	$10^9$
Number of iterations	701	688	686	686	686	684

Table 4

Convergence results for the Uzawa scheme for  $\rho = 10^3$  and different values of  $\alpha$ .

$\alpha$	$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$
$\ \lambda_h^{k+1} - \lambda_h^k\ _{L^2(\omega)}$	0.01851140	0.01191650	0.00732018	0.00148822

Indeed, we have observed that in order to obtain  $\|\lambda^{k+1} - \lambda^k\|_{L^\infty(\omega)} < 10^{-6}$  or  $|((u_h - \frac{\varepsilon}{2}r_h) \cdot e_3)(0,0) - \Phi_h(0,0)| < 10^{-6}$  more than 100000 iterations is needed but no pathological behaviour is observed. Therefore we have preferred to present the evolution of the errors up to 350 iterations.

The convergence of the Uzawa method depends strongly on the value of the parameter  $\alpha$ . The optimal choice of this parameter depends on the eigenvalues of the system. Based on the inequality (96), it must satisfy

$$0 < \alpha < \frac{2c^\#}{c_{c,\#}^2}$$

but the coercivity and the continuity of the bilinear form depend on the parameter  $\rho$ . Indeed, for  $\rho = 0$ , we have observed that the method does not converge, choosing  $\rho > 0$  big enough gives a large range of  $\alpha$  for which the method converges. In order to show the influence of the parameter  $\rho$  on the performance of the algorithm we have made numerical experiments on a mesh consisting of 512 triangles (6119 degrees of freedom) with fixed value of  $\alpha = 0.01$  and different values of  $\rho = 1, 10^1, 10^3, 10^5, 10^7$  and  $10^9$ . First as above, we look at the convergence of the Uzawa algorithm after 350 iterations for different values of  $\rho$ . The results are listed in Table 2, where we may remark that the choice of large enough  $\rho$  does not significantly influence the error. Secondly, we have chosen the stopping criteria  $\|\lambda^{k+1} - \lambda^k\|_{L^\infty(\omega)} < 0.01$ , in that case we observe that augmenting the value of  $\rho$  reduces slightly the number of iterations, see Table 3

Finally for a fixed (large enough) value of  $\rho$ , we have performed numerical tests with different values of  $\alpha$  and look at the variation of the error  $\|\lambda^{k+1} - \lambda^k\|_{L^\infty(\omega)}$  after 350 iterations. The results are shown in Table 4, where contrary to the previous case, we observe that changing  $\alpha$  and fixing  $\rho$  large enough may affect significantly the convergence.

In the context of finite element approximation of PDEs, the rate of convergence depends strongly on the regularity of the solution of the exact solution and the degree of the used polynomials, an inverse theorem also exists (see [27]). It should be noticed that for contact problems, the limited regularity of the solution due to the unknown contact boundary limits the convergence rate. For our problem the exact solution and the a priori regularity are unknown. In order to overcome this issue, we follow the algorithm proposed in [28, Sec. 6]. Indeed, the prescribed numerical test is solved by our mixed formulation discretized using  $(\mathbb{P}_1 \oplus B_3, \mathbb{P}_1 \oplus B_3, \mathbb{P}_0)$  and the Uzawa algorithm with fixed parameters  $\rho = 10^3$  and  $\alpha = 0.01$ . The meshes are refined uniformly and the mesh size of the mesh

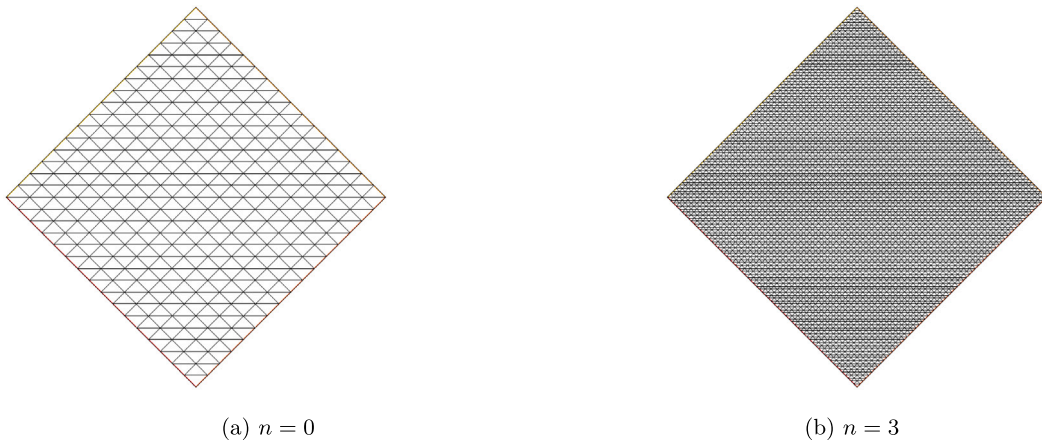


Fig. 7. The first and the fourth meshes.

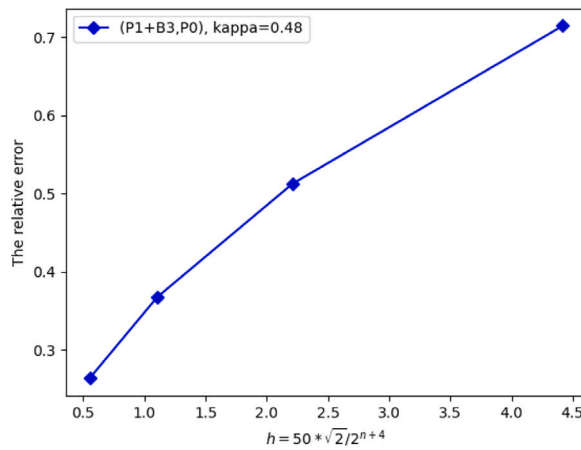


Fig. 8. The “ global” rate of the convergence as a function of the mesh size.

$\mathcal{T}_n$  after  $n$  refinements is given by  $h_{\max,n} = 50\sqrt{2}/2^{(n+4)}$ ,  $n = 0, 1, 2, 3$  (see Fig. 7). Hence if  $(U_n, \psi_n)$  is the solution of our Problem 9 for the mesh  $\mathcal{T}_n$ , then we define the numerical convergence rate by

$$\kappa_n := \log_2 \frac{\|(U_n, \psi_n) - (U_{n-1}, \psi_{n-1})\|_{\mathbb{X} \times \mathbb{M}}}{\|(U_{n+1}, \psi_{n+1}) - (U_n, \psi_n)\|_{\mathbb{X} \times \mathbb{M}}}.$$

Hence assuming that

$$\|(U_n, \psi_n) - (U, \psi)\|_{\mathbb{X} \times \mathbb{M}} \sim h_{\max,n}^\kappa,$$

we may expect that

$$\lim_{n \rightarrow \infty} \kappa_n = \kappa.$$

Since the different components have very different order of magnitude, we prefer to use the relative error instead of the absolute error.

The numerical tests from Fig. 8 show that the algorithm converges with a numerical convergence rate  $\kappa_n \approx 0.5$ , which is in good agreement with the theoretical results obtained in Proposition 4.6 and Corollary 4.9. Note that the numerical convergence rate is not the same for the two components of the error.

Let us finally mention that it would be a very interesting question to investigate whether the rate of convergence can be improved by an automatic adaptive refinement strategy using a reliable and efficient a posteriori error indicator together with high order polynomial spaces like  $\mathbb{P}_k + \mathbb{B}_{k+1}, \mathbb{P}_{k-1}$ ,  $k \geq 2$ .

## Data availability

Data will be made available on request.

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