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Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry

Jean-Yves Welschinger*

10th February 2005

Abstract:
We first build the moduli spaces of real rational pseudo-holomorphic curves in a given real symplectic 4-manifold. Then, following the approach of Gromov and Witten [3, 19, 11], we define invariants under deformation of real symplectic 4-manifolds. These invariants provide lower bounds for the number of real rational J-holomorphic curves which realize a given homology class and pass through a given real configuration of points.

Introduction

Let (X, ω, c_X) be a *real symplectic* 4*-manifold*, that is a triple made of a compact differentiable 4-manifold X, a symplectic form ω on X and an involution c_X on X such that $c_X^*\omega = -\omega$. The fixed point set of c_X is called the *real part* of X and is denoted by $\mathbb{R}X$. It is either empty or a smooth lagrangian submanifold of (X, ω). The connected components of $\mathbb{R}X$ are labelled $(\mathbb{R}X)_1, \ldots, (\mathbb{R}X)_N$. Assume that the first Chern class $c_1(X)$ of the symplectic 4-manifold (X, ω) is not a torsion element and let $d \in H_2(X; \mathbb{Z})$ be a homology class satisfying $c_1(X)d > 0$ and $(c_X)_*d = -d$. Let $x \subset X$ be a *real configuration* of points, that is a subset invariant under c_X, made of $c_1(X)d - 1$ distinct points. Denote by r_i the number of such points which are in the real component $(\mathbb{R}X)_i$, $i \in \{1, \ldots, N\}$, and by $r = (r_1, \ldots, r_N)$. This N-tuple r encodes the equivariant isotopy class of the real configuration of points x in X. Let $\mathcal{J}_\omega$ be the space of almost complex structures of X, tamed by ω, and which are of class C^l where $l \gg 1$ is a fixed integer, large enough. This space is a contractible separable Banach manifold. Denote by $\mathbb{R}\mathcal{J}_\omega \subset \mathcal{J}_\omega$ the subspace consisting of those $J \in \mathcal{J}_\omega$ for which c_X is J-antiholomorphic. It is a contractible separable Banach submanifold of $\mathcal{J}_\omega$ (see Proposition 1.1). If $J \in \mathbb{R}\mathcal{J}_\omega$ is generic enough, then there are only finitely many rational J-holomorphic curves in X which pass through x and realize the homology class d (see Theorem 1.10). These curves are all irreducible and have only transversal double points as singularities. The total number of their double points is $\delta = \frac{1}{2}(d^2 - c_1(X)d + 2)$. Let C be such a curve which is assumed to be real. We define the *mass* of the curve C to be the number of its real isolated double points (see §2.1 for a definition). For every integer m ranging from 0 to δ, denote by $n_d(m)$ the total number of those real rational J-holomorphic curves in X which pass through

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x, realize the homology class d and have mass m. Define then:

$$\chi_r^d(x,J) = \sum_{m=0}^{\delta} (-1)^m n_d(m).$$

The main result of this paper is the following (see Theorem 2.1):

Theorem 0.1 *The integer $\chi_r^d(x,J)$ neither depends on the choice of J nor on the choice of x.*

From Theorem 0.1, this integer can be denoted without ambiguity by χ_r^d and when $\sum_{i=1}^N r_i$ does not have the same parity as $c_1(X)d-1$, we set χ_r^d to be 0. We then denote by $\chi^d(T)$ the generating function $\sum_{|r|=0}^{c_1(X)d-1} \chi_r^d T^r \in \mathbb{Z}[T_1,\dots,T_N]$, where $T^r = T_1^{r_1}\dots T_N^{r_N}$ and $|r| = r_1+\dots+r_N$. This polynomial function is of the same parity as $c_1(X)d-1$ and each of its monomials actually only depends on one indeterminate. It follows from Theorem 0.1 that the function $\chi : d \in H_2(X;\mathbb{Z}) \mapsto \chi^d(T) \in \mathbb{Z}[T_1,\dots,T_N]$ only depends on the real symplectic 4-manifold (X,ω,c_X) and is invariant under deformation. This means that if ω_t is a continuous family of symplectic forms for which $c_X^*\omega_t = -\omega_t$, then this function is the same for all triples (X,ω_t,c_X). This invariant is proved to be non-trivial for degree less or equal than five in the complex projective plane, see Proposition 3.7. As an application of this invariant, we obtain the following lower bounds in real enumerative geometry (see Corollary 2.2).

Corollary 0.2 *The integer $|\chi_r^d|$ provides a lower bound for the total number of real rational J-holomorphic curves of X which pass through x and realize the homology class d, independently of the choice of a generic $J \in \mathbb{R}\mathcal{J}_\omega$.*

Now, for convenience, let us assume that $\mathbb{R}X$ is connected. Let $y = (y_1,\dots,y_{c_1(X)d-2})$ be a real configuration of $c_1(X)d-2$ distinct points of X, and s be the number of those which are real. We assume $y_{c_1(X)d-2}$ to be real, so that $s \neq 0$. If $J \in \mathbb{R}\mathcal{J}_\omega$ is generic enough, then there are only finitely many rational J-holomorphic curves in X which pass through y, realize the homology class d and have a double point at $y_{c_1(X)d-2}$. These curves are all irreducible and have only transversal double points as singularities. For every integer m ranging from 0 to δ, denote by $\hat{n}_d^+(m)$ (resp. $\hat{n}_d^-(m)$) the total number of these curves which are real, of mass m and with a non-isolated (resp. isolated) real double point at $y_{c_1(X)d-2}$ (see §3.1). Define then:

$$\theta_s^d(y,J) = \sum_{m=0}^{\delta} (-1)^m (\hat{n}_d^+(m) - \hat{n}_d^-(m)).$$

Theorem 0.3 *The integer $\theta_s^d(y,J)$ neither depends on the choice of J nor on the choice of y.*

Once more, the integer $\theta_s^d(y,J)$ can then be denoted without ambiguity by θ_s^d, and we set $\theta_s^d = 0$ when s does not have the same parity as $c_1(X)d$. This invariant makes it possible to give relations between the coefficients of the polynomial χ^d, namely (see Theorem 3.2):

Theorem 0.4 *Let $d \in H_2(X;\mathbb{Z})$ and r be an integer between 0 and $c_1(X)d-3$. Then $\chi_{r+2}^d = \chi_r^d + 2\theta_{r+1}^d$.*

The text is organized as follows. The first paragraph is devoted to the construction of the moduli space $\mathcal{M}^d(x)$ (resp. $\mathbb{R}\mathcal{M}^d(x)$) of rational pseudo-holomorphic curves (resp. real rational pseudo-holomorphic curves) which pass through the given real configuration of points x and realize the homology class d. The space $\mathbb{R}\mathcal{M}^d(x)$ appears to be the fixed point set of a $\mathbb{Z}/2\mathbb{Z}$-action on $\mathcal{M}^d(x)$ induced by c_X. The main result of this paragraph is the theorem of regular values (see Theorem 1.10) which states that the set of regular values of the Fredholm projection $\pi : \mathcal{M}^d(x) \to \mathcal{J}_\omega$ intersects $\mathbb{R}\mathcal{J}_\omega$ in a dense set of the second category of $\mathbb{R}\mathcal{J}_\omega$. The second paragraph is devoted to the definition of the invariant χ and the proof of Theorem 0.1. It involves in particular several genericity arguments which are given in §2.2. Few computations of this invariant and applications to real enumerative geometry are given in §2.1. Finally, the last paragraph is devoted to the definition of the invariant θ, the statements and proofs of Theorems 0.3 and 0.4 and the proof of the non-triviality of χ^d for $d = 4, 5$ in the complex projective plane. With the exception of this non-triviality, all these results have been announced in [17].

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1 Moduli space of real rational pseudo-holomorphic curves

In this paragraph, we carry out over the reals the construction of the moduli space $\mathcal{M}^d(x)$ of rational pseudo-holomorphic curves which realize the homology class d and pass through the configuration of points x, following [9] and [12]. We prove that this moduli space comes equipped with a $\mathbb{Z}/2\mathbb{Z}$-action induced by the real structure c_X. The fixed point set of this action is the moduli space of real rational pseudo-holomorphic curves which realize the homology class d and pass through x.

1.1 The manifold $\mathbb{R}\mathcal{J}_\omega$

Let $\mathcal{J}_\omega$ be the space of almost complex structures of X tamed by ω and which are of class C^l where $l \gg 1$ is a fixed integer, large enough. This space is a contractible separable Banach manifold. The real structure c_X of (X, ω) induces a $\mathbb{Z}/2\mathbb{Z}$-action on $\mathcal{J}_\omega$ given by $\overline{c_X}^* : J \in \mathcal{J}_\omega \mapsto \overline{c_X}^*(J) = -dc_X \circ J \circ dc_X$. Denote by $\mathbb{R}\mathcal{J}_\omega$ the fixed point set of this action. It consists of those $J \in \mathbb{R}\mathcal{J}_\omega$ for which c_X is J-antiholomorphic. Let $J_0 \in \mathbb{R}\mathcal{J}_\omega$, the involution $\overline{c_X}^*$ induces an involution $d_{J_0}\overline{c_X}^*$ on the tangent space $T_{J_0}\mathcal{J}_\omega = \Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)$, where $\Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)$ denotes the Banach space of sections of class C^l of the vector bundle $\Lambda^{0,1}X \otimes_{\mathbb{C}} TX$ over X. Denote by $\Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{+1}$ (resp. $\Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{-1}$) the eigenspace of this involution associated to the eigenvalue $+1$ (resp. -1), so that $T_{J_0}\mathcal{J}_\omega = \Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{+1} \oplus \Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{-1}$.

Proposition 1.1 *The fixed point set $\mathbb{R}\mathcal{J}_\omega$ of $\overline{c_X}^*$ is a separable Banach submanifold of $\mathcal{J}_\omega$ which is non-empty and contractible. For every $J_0 \in \mathbb{R}\mathcal{J}_\omega$, the tangent space $T_{J_0}\mathbb{R}\mathcal{J}_\omega$ is isomorphic to $\Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{+1}$.*

Note that in particular, the decomposition $T_{J_0}\mathcal{J}_\omega = \Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{+1} \oplus \Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{-1}$ is a direct sum of locally trivial Banach sub-bundles.

Proof:

Let us first prove that $\mathbb{R}\mathcal{J}_\omega$ is non-empty. Let g_X be a riemannian metric on X invariant under c_X. Let $A \in \Gamma^\infty(X, End_{\mathbb{R}}(TX))$ be such that for every $x \in X$, $u, v \in T_xX$, $\omega_x(u, v) = g_x(Au, v)$. Denote by $A = QJ_0$ the polar decomposition of A, where J_0 is orthogonal for g_X. Then $J_0 \in \mathcal{J}_\omega$, $\overline{c_X}^*(A) = A$ and since $Q = \sqrt{-A^2}$, $c_X^*(Q) = Q$. We deduce that $\overline{c_X}^*(J_0) = J_0$ and thus $J_0 \in \mathbb{R}\mathcal{J}_\omega \neq \emptyset$. Now, the standard transform of Sévennec (see [1], p. 42) provides a $\mathbb{Z}/2\mathbb{Z}$-equivariant diffeomorphism between $\mathcal{J}_\omega$ and

$$\mathcal{W} = \{W \in \Gamma^l(X, End_{\mathbb{R}}(TX)) \,|\, J_0W = -WJ_0 \text{ and } 1 - W^tW \gg 0\},$$

where $\mathcal{W}$ is equipped with the involution c_X^*. Since the fixed point set $\mathbb{R}\mathcal{W} = \mathcal{W} \cap \{W \in \Gamma^l(X, End_{\mathbb{R}}(TX)) \,|\, c_X^*(W) = W\}$ is a contractible separable Banach submanifold of $\mathcal{W}$, the same holds for $\mathbb{R}\mathcal{J}_\omega \subset \mathcal{J}_\omega$. The tangent space of $\mathbb{R}\mathcal{J}_\omega$ is then $\Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{+1}$. $\square$

1.2 Space of pseudo-holomorphic maps $\mathcal{P}^*(x)$

Let $x = (x_1, \ldots, x_m)$ be an ordered set of distinct points of X, globally invariant under c_X. Such a set is called a *real configuration* of points. Let τ be the order two permutation of $\{1, \ldots, m\}$ induced by c_X. Let S be an oriented 2-sphere, $\mathcal{J}_S$ be the space of complex structures of class C^l of S compatible with its orientation and $z = (z_1, \ldots, z_m)$ be m distinct

points on S. Finally, let $d \in H_2(X;\mathbb{Z})$ be a fixed homology class such that $c_1(X)d > 0$ and $(c_X)_* d = -d$. Denote then by

$$\mathcal{S}^d(x) = \{u \in L^{k,p}(S,X) \,|\, u_*[S] = d \text{ and } u(z) = x\},$$

where $1 \ll k \ll l$ and $p > 2$ (see [9], §3.1). This is a Banach manifold whose tangent space at $u \in \mathcal{S}^d(x)$ is the space $T_u\mathcal{S}^d(x) = \{v \in L^{k,p}(S,E_u) \,|\, v(z) = 0\}$, where E_u is the pull-back bundle u^*TX. Let $\mathcal{E}$ (resp. $\mathcal{E}'$) be the Banach bundle over $\mathcal{S}^d(x) \times \mathcal{J}_S \times \mathcal{J}_\omega$ whose fibre over $(u, J_S, J) \in \mathcal{S}^d(x) \times \mathcal{J}_S \times \mathcal{J}_\omega$ is the space $T_u\mathcal{S}^d(x)$ (resp. the space $L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} E_u)$). Let $\sigma_{\overline{\partial}}$ be the section of $\mathcal{E}'$ defined by $\sigma_{\overline{\partial}}(u, J_S, J) = du + J \circ du \circ J_S$, and $\mathcal{P}^d(x)$ be the set of its zeros. This is the space of pseudo-holomorphic maps from S to X which pass through x. Fix some Levi-Civita connection ∇ on TX associated to some riemannian metric g_X invariant under c_X. All the induced connections on the bundles associated to TX will also be denoted by ∇, for convenience. The linearization of $\sigma_{\overline{\partial}}$ at $(u, J_S, J) \in \mathcal{P}^d(x)$ is defined by (see [9], §3.1 or [7], formula 1.2.3)

$$\nabla\sigma_{\overline{\partial}}(v, \dot{J}_S, \dot{J}) = Dv + J \circ du \circ \dot{J}_S + \dot{J} \circ du \circ J_S,$$

where D is the Gromov operator defined by

$$v \in \mathcal{E}|_{(u,J_S,J)} \mapsto D(v) = \nabla v + J \circ \nabla v \circ J_S + \nabla_v J \circ du \circ J_S \in \mathcal{E}'|_{(u,J_S,J)}.$$

Finally, denote by $\mathcal{P}^*(x)$ the subspace of $\mathcal{P}^d(x)$ consisting of non-multiple maps, that is the space of triples (u, J_S, J) such that u cannot be written $u' \circ \phi$ where $\phi : S \to S'$ is a non-trivial ramified covering and $u' : S' \to X$ is a pseudo-holomorphic map.

Proposition 1.2 *The space $\mathcal{P}^*(x)$ is a separable Banach manifold of class C^{l-k} whose tangent space at $(u, J_S, J) \in \mathcal{P}^*(x)$ is the space $T_{(u,J_S,J)}\mathcal{P}^*(x) = \{(v, \dot{J}_S, \dot{J}) \in T_{(u,J_S,J)}(\mathcal{S}^d(x) \times \mathcal{J}_S \times \mathcal{J}_\omega)) \,|\, \nabla_{(v,\dot{J}_S,\dot{J})}\sigma_{\overline{\partial}} = 0\}$.* $\square$

This proposition follows from the fact that at $(u, J_S, J) \in \mathcal{P}^*(x)$, the operator $\nabla\sigma_{\overline{\partial}} : T_{(u,J_S,J)}(\mathcal{S}^d(x) \times \mathcal{J}_S \times \mathcal{J}_\omega)) \to \mathcal{E}'|_{(u,J_S,J)}$ is surjective and has a right inverse (see [9], Proposition 3.2.1).

1.3 Gauge action of $\mathcal{D}iff(S,z)$

Denote by $\mathcal{D}iff(S,z)$ the group of diffeomorphisms of class C^{l+1} of S, which either preserve the orientation and fix z, or reverse the orientation and induce the permutation on z associated to τ. Let $\mathcal{D}iff^+(S,z)$ (resp. $\mathcal{D}iff^-(S,z)$) be the subgroup of $\mathcal{D}iff(S,z)$ of orientation preserving diffeomorphisms (resp. its complement in $\mathcal{D}iff(S,z)$) and s_* be the morphism $\mathcal{D}iff(S,z) \to \mathbb{Z}/2\mathbb{Z}$ of kernel $\mathcal{D}iff^+(S,z)$. The group $\mathcal{D}iff(S,z)$ acts on $\mathcal{S}^d(x) \times \mathcal{J}_S \times \mathcal{J}_\omega$ by

$$\phi.(u, J_S, J) = \begin{cases} (u \circ \phi^{-1}, (\phi^{-1})^* J_S, J) & \text{if} \quad s_*(\phi) = +1, \\ (c_X \circ u \circ \phi^{-1}, (\phi^{-1})^* J_S, \overline{c_X}^*(J)) & \text{if} \quad s_*(\phi) = -1, \end{cases}$$

where $\phi \in \mathcal{D}iff(S,z)$, $(u, J_S, J) \in \mathcal{S}^d(x) \times \mathcal{J}_S \times \mathcal{J}_\omega$ and $(\phi^{-1})^* J_S = s_*(\phi) d\phi \circ J_S \circ d\phi^{-1}$. The action of $\mathcal{D}iff(S,z)$ lifts to the following actions on the bundles $\mathcal{E}$ and $\mathcal{E}'$:

$$(\phi, v) \in \mathcal{D}iff(S,z) \times \mathcal{E}_{(u,J_S,J)} \mapsto \begin{cases} v \circ \phi^{-1} \in \mathcal{E}_{\phi.(u,J_S,J)} & \text{if} \quad s_*(\phi) = +1, \\ dc_X \circ v \circ \phi^{-1} \in \mathcal{E}_{\phi.(u,J_S,J)} & \text{if} \quad s_*(\phi) = -1, \end{cases}$$

and

$$(\phi,\alpha)\in\mathcal{D}iff(S,z)\times\mathcal{E}'_{(u,J_S,J)}\mapsto\begin{cases}\alpha\circ d\phi^{-1}\in\mathcal{E}'_{\phi.(u,J_S,J)} & \text{if}\quad s_*(\phi)=+1,\\ dc_X\circ\alpha\circ d\phi^{-1}\in\mathcal{E}'_{\phi.(u,J_S,J)} & \text{if}\quad s_*(\phi)=-1.\end{cases}$$

The section $\sigma_{\overline{\partial}}$ is obviously $\mathcal{D}iff(S,z)$-equivariant for these actions. As a consequence, the manifold $\mathcal{P}^*(x)$ is invariant under the action of $\mathcal{D}iff(S,z)$.

Lemma 1.3 *With the exception of the identity, only the order two elements of $\mathcal{D}iff^-(S,z)$ may have non-empty fixed point set in $\mathcal{P}^*(x)$. In particular, two such involutions have disjoint fixed point sets.* $\square$

Denote by $\mathbb{R}\mathcal{P}^*(x)$ the disjoint union of the fixed point sets of non-trivial elements of $\mathcal{D}iff(S,z)$. From Lemma 1.3 we know that each element of $\mathbb{R}\mathcal{P}^*(x)$ determines uniquely an order two element of $\mathcal{D}iff^-(S,z)$, these involutions induce bundle homomorphisms on $\mathcal{E}|_{\mathbb{R}\mathcal{P}^*(x)}$ and $\mathcal{E}'|_{\mathbb{R}\mathcal{P}^*(x)}$. We denote by $\mathcal{E}_{+1}$, $\mathcal{E}'_{+1}$ (resp. $\mathcal{E}_{-1}$, $\mathcal{E}'_{-1}$) the eigenspaces associated to the eigenvalue $+1$ (resp. -1) of these homomorphisms, so that $\mathcal{E}|_{\mathbb{R}\mathcal{P}^*(x)}=\mathcal{E}_{+1}\oplus\mathcal{E}_{-1}$ and $\mathcal{E}'|_{\mathbb{R}\mathcal{P}^*(x)}=\mathcal{E}'_{+1}\oplus\mathcal{E}'_{-1}$.

Proposition 1.4 *Let c_S be an element of order two of $\mathcal{D}iff^-(S,z)$ and $\mathbb{R}\mathcal{P}^*_{c_S}(x)$ be the fixed point set of c_S in $\mathcal{P}^*(x)$. This space $\mathbb{R}\mathcal{P}^*_{c_S}(x)$ is a Banach submanifold of $\mathcal{P}^*(x)$ of class C^{l-k} whose tangent space at $(u,J_S,J)\in\mathbb{R}\mathcal{P}^*(x)$ is the space $T_{(u,J_S,J)}\mathbb{R}\mathcal{P}^*_{c_S}(x)=\{(v,\dot{J}_S,\dot{J})\in\mathcal{E}_{+1}\times T_{J_S}\mathbb{R}\mathcal{J}_S\times T_J\mathbb{R}\mathcal{J}_\omega\,|\,\nabla_{(v,\dot{J}_S,\dot{J})}\sigma_{\overline{\partial}}=0\}$, where $\mathbb{R}\mathcal{J}_S$ is the fixed point set of c_S in $\mathcal{J}_S$.*

Proof:
The action of c_S on the product $\mathcal{S}^d(x)\times\mathcal{J}_S\times\mathcal{J}_\omega$ is antiholomorphic for the almost-complex structure defined by:

$$(v,\dot{J}_S,\dot{J})\in T_{(u,J_S,J)}(\mathcal{S}^d(x)\times\mathcal{J}_S\times\mathcal{J}_\omega)\mapsto(Jv,J_S\dot{J}_S,J\dot{J})\in T_{(u,J_S,J)}(\mathcal{S}^d(x)\times\mathcal{J}_S\times\mathcal{J}_\omega)$$

The fixed point set of this action is a Banach manifold denoted by $\mathbb{R}\mathcal{S}^d(x)\times\mathbb{R}\mathcal{J}_S\times\mathbb{R}\mathcal{J}_\omega$. The restriction of $\sigma_{\overline{\partial}}$ to $\mathbb{R}\mathcal{S}^d(x)\times\mathbb{R}\mathcal{J}_S\times\mathbb{R}\mathcal{J}_\omega$ takes values in the sub-bundle $\mathcal{E}'_{+1}$ of $\mathcal{E}'$ associated to the eigenvalue $+1$ of c_S, since $\sigma_{\overline{\partial}}$ is $\mathcal{D}iff(S,z)$-equivariant. It suffices then to prove that this restriction vanishes transversely along $\mathbb{R}\mathcal{P}^*(x)\cap(\mathbb{R}\mathcal{S}^d(x)\times\mathbb{R}\mathcal{J}_S\times\mathbb{R}\mathcal{J}_\omega)$, meaning that at $(u,J_S,J)\in\mathbb{R}\mathcal{P}^*(x)\cap(\mathbb{R}\mathcal{S}^d(x)\times\mathbb{R}\mathcal{J}_S\times\mathbb{R}\mathcal{J}_\omega)$, the operator $\nabla\sigma_{\overline{\partial}}:T_{(u,J_S,J)}(\mathbb{R}\mathcal{S}^d(x)\times\mathbb{R}\mathcal{J}_S\times\mathbb{R}\mathcal{J}_\omega)\to\mathcal{E}'_{+1}|_{(u,J_S,J)}$ is surjective and has a right inverse. This follows from the corresponding properties of $\nabla\sigma_{\overline{\partial}}:T_{(u,J_S,J)}(\mathcal{S}^d(x)\times\mathcal{J}_S\times\mathcal{J}_\omega)\to\mathcal{E}'|_{(u,J_S,J)}$ and the $\mathcal{D}iff(S,z)$-equivariance of $\sigma_{\overline{\partial}}$. $\square$

1.4 The Gromov operators D and $D_{\mathbb{R}}$

Remember that the $\mathbb{C}$-linear part of the Gromov operator D defined in §1.2 is some $\overline{\partial}$-operator which will be denoted by $\overline{\partial}$, and that its $\mathbb{C}$-antilinear part is some order 0 operator denoted by R. The latter is given by the formula $R_{(u,J_S,J)}(v)=N_J(v,du)$ where $v\in\mathcal{E}$ and N_J is the Nijenhuis tensor of J. In particular, $R\circ du=0$ (see [7], Lemma 1.3.1). Let $(u,J_S,J)\in\mathcal{P}^*(x)$, the operator $\overline{\partial}$ associated to D induces a holomorphic structure on the bundle $E_u=u^*TX$ for which the morphism $du:TS\to E_u$ is an injective homomorphism of analytic sheaves (see [7], Lemma 1.3.1). Denote by $E_{u,-z}=E_u\otimes\mathcal{O}(-z)$, $TS_{-z}=TS\otimes\mathcal{O}(-z)$ and $\mathcal{N}_{u,-z}$ the quotient sheaf $E_{u,-z}/du(TS_{-z})$. This sheaf splits

under the form $\mathcal{O}(N_{u,-z}) \oplus \mathcal{N}^{sing}_{u,-z}$, where $N_{u,-z} = N_u \otimes \mathcal{O}(-z)$, N_u being the normal bundle of $u(S)$ in X, and $\mathcal{N}^{sing}_{u,-z} = \oplus \mathbb{C}^{n_i}_{a_i}$. In the latter, the sum is taken over all the critical points a_i of du and $\mathbb{C}^{n_i}_{a_i}$ denotes the skyscraper sheaf of fibre $\mathbb{C}^{n_i}$ and support a_i, n_i being the vanishing order of du at a_i. The operator D induces on the quotient an operator $D^N : L^{k,p}(S, N_{u,-z}) \to L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} N_u)$ as well as an operator $\overline{D} : L^{k,p}(S, E_{u,-z})/du(L^{k,p}(S, TS_{-z})) \to L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} E_u)/du(L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} TS))$ (see [7], formula 1.5.1). Note that by definition, $L^{k,p}(S, E_{u,-z})$ (resp. $L^{k,p}(S, N_{u,-z})$) denotes the space of sections of $L^{k,p}(S, E_u)$ (resp. $L^{k,p}(S, N_u)$) which vanish at z. It is a Banach subspace of complex codimension $2\#(z)$ (resp. $\#(z)$) of $L^{k,p}(S, E_u)$ (resp. $L^{k,p}(S, N_u)$). Restricting to this subspace has thus the effect to decrease the real index of D (resp. D^N or $\overline{D}$) by $4\#(z)$ (resp. $2\#(z)$). Denote by $H^0_D(S, E_{u,-z})$ (resp. $H^0_D(S, N_{u,-z})$) the kernel of the operator D (resp. D^N), and by $H^1_D(S, E_{u,-z})$ (resp. $H^1_D(S, N_{u,-z})$) the cokernel of this operator. Similarly, denote by $H^0_D(S, \mathcal{N}_{u,-z})$ the kernel of the operator $\overline{D}$, which is isomorphic to $H^0_D(S, N_{u,-z}) \oplus H^0(S, \mathcal{N}^{sing}_{u,-z})$, whereas its cokernel is isomorphic to $H^1_D(S, N_{u,-z})$ (see [7], Lemma 1.5.1). Note that since the operators D, D^N and $\overline{D}$ are elliptic, all these spaces are finite dimensional and do not depend on the choices of k, p. They satisfy the following long exact sequence (see [12], Corollary 1.5.4):

$$0 \to H^0(S, TS_{-z}) \to H^0_D(S, E_{u,-z}) \to H^0_D(S, N_{u,-z}) \oplus H^0(S, \mathcal{N}^{sing}_{u,-z}) \to$$

$$\to H^1(S, TS_{-z}) \to H^1_D(S, E_{u,-z}) \to H^1_D(S, N_{u,-z}) \to 0. \tag{1}$$

Remember finally that the formal adjoint of the operator D^N is given by some operator $D^* : L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} N_u)^* = L^{1-k,p'}(S, K_S \otimes_{\mathbb{C}} N^*_u) \to L^{k,p}(S, N_{u,-z})^* = L^{-k,p'}(S, N^*_u)/L^{k,p}(S, N_{u,-z})^{\perp}$, where $K_S = \Lambda^{1,0}S$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $L^{k,p}(S, N_{u,-z})^{\perp} = \{f \in L^{-k,p'}(S, N^*_u) \mid < f, \alpha >= 0 \text{ for every } \alpha \in L^{k,p}(S, N_{u,-z})\}$ is generated by the Dirac sections of N^*_u at z. Thus, the Riemann-Roch duality provides isomorphisms $H^0_D(S, N_{u,-z}) \cong H^1_{D^*}(S, K_S \otimes_{\mathbb{C}} N^*_u)$ and $H^1_D(S, N_{u,-z}) \cong H^0_{D^*}(S, K_S \otimes_{\mathbb{C}} N^*_u)$ (see Theorem C.1.10 of [9] or Lemma 1.5.1 of [12]).

Lemma 1.5 *The operators $D : \mathcal{E} \to \mathcal{E}', D^N$ and $\overline{D}$ are $\mathcal{D}iff(S, z)$-equivariant.* $\square$

Thus, over $\mathbb{R}\mathcal{P}^*(x)$, the operator D restricts to some operator $\mathcal{E}_{+1} \to \mathcal{E}'_{+1}$ which will be denoted by $D_{\mathbb{R}}$. Similarly, let $(u, J_S, J) \in \mathbb{R}\mathcal{P}^*(x)$ and c_S be the associated order two element of $\mathcal{D}iff^-(S, z)$. Denote by $L^{k,p}(S, N_{u,-z})_{\pm 1}$ (resp. $L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} N_u)_{\pm 1}$) the eigenspace associated to the eigenvalue ± 1 of the action of c_S on $L^{k,p}(S, N_{u,-z})$ (resp. $L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} N_u)$). Similarly, denote by $L^{k,p}(S, E_{u,-z})_{\pm 1}, L^{k,p}(S, TS_{-z})_{\pm 1}$ (resp. $L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} E_u)_{\pm 1}$, $L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} TS)_{\pm 1}$) the eigenspaces associated to the eigenvalue ± 1 of the action of c_S on $L^{k,p}(S, E_{u,-z})$, $L^{k,p}(S, TS_{-z})$ (resp. $L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} E_u)$, $L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} TS)$). Denote then by $D^N_{\mathbb{R}}$ (resp. $\overline{D}_{\mathbb{R}}$) the operator $L^{k,p}(S, N_{u,-z})_{+1} \to L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} N_u)_{+1}$ (resp. $L^{k,p}(S, E_{u,-z})_{\pm 1}/du(L^{k,p}(S, TS_{-z})_{\pm 1}) \to L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} E_u)_{\pm 1}/du(L^{k-1,p}(S, \Lambda^{0,1}S \otimes_{\mathbb{C}} TS)_{\pm 1})$) induced by D^N (resp. $\overline{D}$).

Lemma 1.6 *The operators $D_{\mathbb{R}}$ and $\overline{D}_{\mathbb{R}}$ are Fredholm, of indices* $\mathrm{ind}(D_{\mathbb{R}}) = \frac{1}{2}\mathrm{ind}(D) = c_1(X)d + 2 - 2m$ *and* $\mathrm{ind}(\overline{D}_{\mathbb{R}}) = \frac{1}{2}\mathrm{ind}(\overline{D}) = c_1(X)d - 1 - m$.

Proof:

Fix some element c_S of order two of $\mathcal{D}iff^-(S,z)$. Remember that the decomposition of D into $\mathbb{C}$-linear and antilinear parts writes $\overline{\partial}+R$, where $\overline{\partial}$ and R are equivariant under the action of c_S (in fact under the whole $\mathcal{D}iff(S,z)$). The operator $\overline{\partial}$ restricts then to some operator $\mathcal{E}_{+1}\to\mathcal{E}'_{+1}$ which remains Fredholm, of kernel (resp. cokernel) the eigenspace $\ker_{+1}(\overline{\partial})$ (resp. $\operatorname{coker}_{+1}(\overline{\partial})$) associated to the eigenvalue $+1$ of the action of c_S on $\ker(\overline{\partial})$ (resp. $\operatorname{coker}(\overline{\partial})$). Since R is of order 0, it follows that $D_{\mathbb{R}}$ is Fredholm of index $\operatorname{ind}(D_{\mathbb{R}})=\operatorname{ind}(\overline{\partial}|_{\mathcal{E}_{+1}})=\frac{1}{2}\operatorname{ind}(D)=c_1(X)d+2-2m$. The last equality coming from Riemann-Roch theorem and the equality before from the fact that $\overline{\partial}$ is $\mathbb{C}$-linear. The same arguments applied to $\overline{D}$ give the result for $\overline{D}_{\mathbb{R}}$. $\square$

Denote by $H^0_D(S,E_{u,-z})_{+1}$ (resp. $H^0_D(S,N_{u,-z})_{+1}$) the kernel of the operator $D_{\mathbb{R}}$ (resp. $D^N_{\mathbb{R}}$), and by $H^1_D(S,E_{u,-z})_{+1}$ (resp. $H^1_D(S,N_{u,-z})_{+1}$) the cokernel of this operator. Denote also by $H^0_D(S,\mathcal{N}_{u,-z})_{+1}=H^0_D(S,N_{u,-z})_{+1}\oplus H^0(S,\mathcal{N}^{sing}_{u,-z})_{+1}$ the kernel of the operator $\overline{D}_{\mathbb{R}}$. Its cokernel is isomorphic to $H^1_D(S,N_{u,-z})_{+1}$. These spaces satisfy the following long exact sequence:

$$0\to H^0(S,TS_{-z})_{+1}\to H^0_D(S,E_{u,-z})_{+1}\to H^0_D(S,N_{u,-z})_{+1}\oplus H^0(S,\mathcal{N}^{sing}_{u,-z})_{+1}\to$$

$$\to H^1(S,TS_{-z})_{+1}\to H^1_D(S,E_{u,-z})_{+1}\to H^1_D(S,N_{u,-z})_{+1}\to 0. \tag{2}$$

Note that $H^0_D(S,E_{u,-z})_{+1}$ (resp. $H^0_D(S,N_{u,-z})_{+1}$) coincides with the eigenspace associated to the eigenvalue $+1$ of the action of c_S on $H^0_D(S,E_{u,-z})$ (resp. $H^0_D(S,N_{u,-z})$). Denote by $H^0_D(S,E_{u,-z})_{-1}$ (resp. $H^0_D(S,N_{u,-z})_{-1}$) the eigenspace associated to the eigenvalue -1.

Lemma 1.7 *Riemann-Roch duality provides isomorphisms:*
$H^0_D(S,N_{u,-z})_{+1}\cong H^1_{D^*}(S,K_S\otimes_{\mathbb{C}}N^*_u)_{-1}$ *and*
$H^1_D(S,N_{u,-z})_{+1}\cong H^0_{D^*}(S,K_S\otimes_{\mathbb{C}}N^*_u)_{-1}$.

Proof:
The duality between the spaces $L^{k-1,p}(S,\Lambda^{0,1}S\otimes_{\mathbb{C}}N_u)$ and $L^{1-k,p'}(S,K_S\otimes_{\mathbb{C}}N^*_u)$ writes $(\psi^*,\alpha)\in L^{1-k,p'}(S,K_S\otimes_{\mathbb{C}}N^*_u)\times L^{k-1,p}(S,\Lambda^{0,1}S\otimes_{\mathbb{C}}N_u)\mapsto\Re e\int_S<\psi^*,\alpha>$. Now, fix some element c_S of order two of $\mathcal{D}iff^-(S,z)$. We have

$$\begin{aligned}((dc^*_X)^t\circ\psi^*\circ dc_S,dc_X\circ\alpha\circ dc_S) &= \Re e\int_S<(dc^*_X)^t\circ\psi^*\circ dc_S,dc_X\circ\alpha\circ dc_S>\\ &= \Re e\int_S<\psi^*,\alpha>\circ dc_S\\ &= -\Re e\int_S<\psi^*,\alpha>,\end{aligned}$$

since c_S reverses the orientation of S. It follows that the spaces $H^1_D(S,N_{u,-z})_{\pm1}$ and $H^0_{D^*}(S,K_S\otimes_{\mathbb{C}}N^*_u)_{\pm1}$ are orthogonal to each other and that the spaces $H^1_D(S,N_{u,-z})_{\pm1}$ and $H^0_{D^*}(S,K_S\otimes_{\mathbb{C}}N^*_u)_{\mp1}$ are dual to each other. The same holds for $H^0_D(S,N_{u,-z})_{\pm1}$ and $H^1_{D^*}(S,K_S\otimes_{\mathbb{C}}N^*_u)_{\mp1}$. $\square$

1.5 Moduli space of rational pseudo-holomorphic curves

1.5.1 The manifolds $\mathcal{M}^d(x)$, $\mathbb{R}\mathcal{M}^d(x)$ and the projections π, $\pi_{\mathbb{R}}$

Denote by $\mathcal{M}^d(x)$ the quotient of $\mathcal{P}^*(x)$ by the action of $\mathcal{D}iff^+(S,z)$. The projection $\pi : (u, J_S, J) \in \mathcal{P}^*(x) \mapsto J \in \mathcal{J}_\omega$ induces on the quotient a projection $\mathcal{M}^d(x) \to \mathcal{J}_\omega$ still denoted by π. Remember the following proposition (see [9], Proposition 3.2.1 and Theorem 3.1.5, or [7], Theorem 2).

Proposition 1.8 *1) The space $\mathcal{M}^d(x)$ is a separable Banach manifold of class C^{l-k}.*

2) The projection $\pi : \mathcal{M}^d(x) \to \mathcal{J}_\omega$ is Fredholm of index $\mathrm{ind}_{\mathbb{R}}(\pi) = 2(c_1(X)d - 1 - m)$. Moreover, at $[u, J_S, J] \in \mathcal{M}^d(x)$, the kernel of π is isomorphic to $H^0_D(S, \mathcal{N}_{u,-z})$ and its cokernel to $H^1_D(S, N_{u,-z})$. $\square$

The manifold $\mathcal{M}^d(x)$ is equipped with an action of the group $\mathcal{D}iff(S,z)/\mathcal{D}iff^+(S,z) \cong \mathbb{Z}/2\mathbb{Z}$. Let us denote by $\mathbb{R}\mathcal{M}^d(x)$ the fixed point set of this action. The restriction of π to $\mathbb{R}\mathcal{M}^d(x)$ takes value in $\mathbb{R}\mathcal{J}_\omega$. Denote by $\pi_{\mathbb{R}}$ the induced projection $\mathbb{R}\mathcal{M}^d(x) \to \mathbb{R}\mathcal{J}_\omega$.

Proposition 1.9 *The projection $\pi_{\mathbb{R}} : \mathbb{R}\mathcal{M}^d(x) \to \mathbb{R}\mathcal{J}_\omega$ is Fredholm of index $\mathrm{ind}_{\mathbb{R}}(\pi_{\mathbb{R}}) = c_1(X)d - 1 - m$. Moreover, at $[u, J_S, J] \in \mathbb{R}\mathcal{M}^d(x)$, the kernel of $\pi_{\mathbb{R}}$ is isomorphic to $H^0_D(S, \mathcal{N}_{u,-z})_{+1}$ and its cokernel to $H^1_D(S, N_{u,-z})_{+1}$.*

Proof:

The projection π is $\mathbb{Z}/2\mathbb{Z}$-equivariant. Let $[u, J_S, J] \in \mathbb{R}\mathcal{M}^d(x)$ and $(u, J_S, J) \in \mathbb{R}\mathcal{P}^*(x)$ be a lift of this element. Denote by c_S the associated element of order two of $\mathcal{D}iff^-(S,z)$. Then $Im(d_{[u,J_S,J]}\pi_{\mathbb{R}}) = Im(d_{[u,J_S,J]}\pi) \cap \Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{+1}$, this image is thus closed in $\Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{+1} = T_J(\mathbb{R}\mathcal{J}_\omega)$. From Proposition 1.8, we know that the cokernel of $d_{[u,J_S,J]}\pi_{\mathbb{R}}$ is finite dimensional and isomorphic to $H^1_D(S, N_{u,-z})_{+1}$. Similarly, note that all the order two elements of $\mathcal{D}iff^-(S,z)$ which have non-empty fixed point sets in S are conjugated to each other by some element of $\mathcal{D}iff^+(S,z)$. This follows from the similar result in $\mathcal{D}iff(S)$ which can be proved as follows. Every order two element of $\mathcal{D}iff^-(S)$ can be made J_S-antiholomorphic for some almost complex structure on S. Then, using Riemann uniformization Theorem, it is conjugated to an antiholomorphic involution of $\mathbb{C}P^1$ having non-empty fixed point set. The latter are well known to be conjugated to each other. Thus, $\ker(d_{[u,J_S,J]}\pi_{\mathbb{R}}) = \ker\left(d_{(u,J_S,J)}\pi \cap T_{(u,J_S,J)}(\mathcal{S}^d(x) \times \mathcal{J}_S)_{+1}\right)/\mathcal{D}iff^+(S,z)_{c_S}$, where $T_{(u,J_S,J)}(\mathcal{S}^d(x) \times \mathcal{J}_S)_{+1}$ is the eigenspace associated to the eigenvalue $+1$ of the action of c_S on $T_{(u,J_S,J)}(\mathcal{S}^d(x) \times \mathcal{J}_S)$ and $\mathcal{D}iff^+(S,z)_{c_S}$ is the subgroup of diffeomorphisms of S which preserve the orientation and commute with c_S. The latter acts by reparameterization. From Proposition 1.8, we deduce now that $\ker(d_{[u,J_S,J]}\pi_{\mathbb{R}})$ is finite dimensional and isomorphic to $H^0_D(S, \mathcal{N}_{u,-z})_{+1}$. The index formula follows from the long exact sequence (2), from Lemma 1.6 and from the Riemann-Roch formula applied to the bundle TS_{-z}, since the operator $\overline{\partial}$ on this bundle is $\mathbb{C}$-linear. $\square$

1.5.2 The theorem of regular values

The following theorem is the main result of this first paragraph.

Theorem 1.10 *The set of regular values of the projection $\pi : \mathcal{M}^d(x) \to \mathcal{J}_\omega$ intersects $\mathbb{R}\mathcal{J}_\omega$ in a dense subset of the second category of $\mathbb{R}\mathcal{J}_\omega$.*

Proposition 1.11 *The submanifold $\mathbb{R}\mathcal{J}_\omega$ of $\mathcal{J}_\omega$ is transversal to the restriction of π to $\mathcal{M}^d(x) \setminus \mathbb{R}\mathcal{M}^d(x)$.*

Proof:

Let $J \in \mathbb{R}\mathcal{J}_\omega$ and $[u, J_S, J] \in \mathcal{M}^d(x) \setminus \mathbb{R}\mathcal{M}^d(x)$. Fix some element $(u, J_S, J) \in \mathcal{P}^*(x)$ lifting $[u, J_S, J]$. Then from Proposition 1.8, $\mathrm{coker}(d_{[u,J_S,J]}\pi)$ is isomorphic to $H^1_D(S, N_{u,-z})$. Let $0 \neq \psi \in H^0_{D^*}(S, K_S \otimes_{\mathbb{C}} N_u^*) \cong H^1_D(S, N_{u,-z})^*$, it suffices to prove that there exists $\dot{J} \in \Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{+1} = T_J(\mathbb{R}\mathcal{J}_\omega)$ such that $< \psi, \dot{J} \circ du \circ J_S > \neq 0$. But $D^*\psi = 0$, which means that $(\overline{\partial} - R^*)(\psi)$ is a linear combination of Dirac sections of N_u^* at z, thus ψ vanishes only at a finite number of points outside a neighbourhood of z (see [4]). Since u is neither real, nor multiple, there exists an open subset U of S, disjoint from $z \subset S$, such that $u|_U$ is an embedding, $u(U) \cap u(S \setminus U) = \emptyset$, $c_X(u(U)) \cap u(S) = \emptyset$ and such that ψ does not vanish on U. Then, there exists a section α of $\Lambda^{0,1}S \otimes_{\mathbb{C}} E_u$ with support in U such that $< \psi, \alpha > \neq 0$. Let $\dot{J} \in \Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)$ be a section with support in a neighbourhood of $u(U)$ such that $\dot{J} \circ du \circ J_S = \alpha$. The section $\dot{J}_{\mathbb{R}} = \dot{J} + \overline{c_X}^*(\dot{J})$ then belongs to $\Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{+1}$ and also satisfies $\dot{J}_{\mathbb{R}} \circ du \circ J_S = \alpha$, hence the result. $\square$

Proof of Theorem 1.10:

From Proposition 1.11 and the theorem of Sard-Smale (see [15]), there exists a dense set of the second category of $\mathbb{R}\mathcal{J}_\omega$, denoted by $\mathcal{U}_1$, such that every point of $\pi^{-1}(\mathcal{U}_1) \setminus \mathbb{R}\mathcal{M}^d(x)$ is regular for π. Similarly, from Proposition 1.9 and the theorem of Sard-Smale, the set of regular values of $\pi_{\mathbb{R}}$ is a dense subset of the second category of $\mathbb{R}\mathcal{J}_\omega$ denoted by $\mathcal{U}_2$. Then $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ is suitable. Indeed, let $J \in \mathcal{U}$ and $[u, J_S, J] \in \pi_{\mathbb{R}}^{-1}(J)$. Choose some element $(u, J_S, J) \in \mathbb{R}\mathcal{P}^*(x)$ lifting it and denote by c_S the associated order two element of $\mathcal{D}iff^-(S, z)$. By hypothesis, $H^1_D(S, N_{u,-z})_{+1} = 0$. It suffices thus to prove that $H^1_D(S, N_{u,-z})_{-1} = 0$. If this would not be the case, since S is rational, we would have $H^0_D(S, N_{u,-z}) = 0$ (see [4], Theorem 1′). Since u is real and $H^0(S, \mathcal{N}^{sing}_{u,-z})$ is carried by the cuspidal points of u, we see that $\dim H^0(S, \mathcal{N}^{sing}_{u,-z})_{+1} = \dim H^0(S, \mathcal{N}^{sing}_{u,-z})_{-1} = \frac{1}{2} \dim H^0(S, \mathcal{N}^{sing}_{u,-z})$. From this we would obtain

$$\mathrm{ind}(\pi) = 2 \dim H^0(S, \mathcal{N}^{sing}_{u,-z})_{+1} - \dim H^1_D(S, N_{u,-z})_{-1} < 2\,\mathrm{ind}(\pi_{\mathbb{R}}),$$

which contradicts Proposition 1.9. $\square$

2 The invariant χ of real symplectic 4-manifolds

2.1 Statements of the results

Let (X, ω, c_X) be a real symplectic 4-manifold and $\mathcal{J}_\omega$ be the space of almost complex structures of X tamed by ω and of class C^l. Label the connected components of the real part $\mathbb{R}X$ by $(\mathbb{R}X)_1, \ldots, (\mathbb{R}X)_N$. Let C be a real irreducible rational pseudo-holomorphic curve of X having only transversal double points as singularities, and $d \in H_2(X; \mathbb{Z})$ be its homology class. The total number of double points of C is given by adjunction formula and is equal to $\delta = \frac{1}{2}(d^2 - c_1(X)d + 2)$. The real double points of C are of two different natures. They are either the local intersection of two real branches, or the local intersection of two complex conjugated branches. In the first case they are called *non-isolated* and in the second case they

are called *isolated*.

Non-isolated real double point Isolated real double point

We define the *mass* of the curve C to be the number of its isolated real double points, it is denoted by $m(C)$. This integer satisfies the upper and lower bounds $0 \leq m(C) \leq \delta$. Now, let $x \subset X$ be a real configuration of $c_1(X)d - 1$ distinct points, r_i be the number of such points which are in $(\mathbb{R}X)_i$, $i \in \{1, \ldots, N\}$, and $r = (r_1, \ldots, r_N)$. Let $J \in \mathbb{R}\mathcal{J}_\omega$ be generic enough, then there are only finitely many rational J-holomorphic curves in X which pass through x and realize the homology class d. Moreover, these curves are all irreducible and have only transversal double points as singularities. For every integer m ranging from 0 to δ, denote by $n_d(m)$ the number of these curves which are real and of mass m. Then define:

$$\chi_r^d(x, J) = \sum_{m=0}^{\delta} (-1)^m n_d(m).$$

The main result of this paper is the following:

Theorem 2.1 *Let (X, ω, c_X) be a real symplectic 4-manifold and $d \in H_2(X; \mathbb{Z})$ be such that $c_1(X)d > 0$. The connected components of the real part $\mathbb{R}X$ are labelled $(\mathbb{R}X)_1, \ldots, (\mathbb{R}X)_N$. Let $x \subset X$ be a real configuration of $c_1(X)d - 1$ distinct points, r_i be the cardinality of $x \cap (\mathbb{R}X)_i$, and $r = (r_1, \ldots, r_N)$. Finally, let $J \in \mathbb{R}\mathcal{J}_\omega$ be an almost complex structure generic enough, so that the integer $\chi_r^d(x, J)$ is well defined. Then, this integer $\chi_r^d(x, J)$ neither depends on the choice of J nor on the choice of x.*

From Theorem 2.1, this integer can be denoted without ambiguity by χ_r^d and when $\sum_{i=1}^N r_i$ does not have the same parity as $c_1(X)d - 1$, we set χ_r^d to be 0. We then denote by $\chi^d(T)$ the generating function $\sum_{|r|=0}^{c_1(X)d-1} \chi_r^d T^r \in \mathbb{Z}[T_1, \ldots, T_N]$, where $T^r = T_1^{r_1} \ldots T_N^{r_N}$ and $|r| = r_1 + \cdots + r_N$. This polynomial function is of the same parity as $c_1(X)d - 1$ and each of its monomials actually only depends on one indeterminate. The latter follows from the fact that the real part $\mathbb{R}S$ of a real rational pseudo-holomorphic curve S is always connected and thus cannot pass through points in different connected components of $\mathbb{R}X$. It follows from Theorem 2.1 that the function $\chi : d \in H_2(X; \mathbb{Z}) \mapsto \chi^d(T) \in \mathbb{Z}[T_1, \ldots, T_N]$ only depends on the real symplectic 4-manifold (X, ω, c_X) and is invariant under deformation of this real symplectic 4-manifold. This means that if ω_t is a continuous family of symplectic forms for which $c_X^* \omega_t = -\omega_t$, then this function is the same for all triples (X, ω_t, c_X). As an application of this invariant, we obtain the following lower bounds in real enumerative geometry.

Corollary 2.2 *Under the hypothesis of Theorem 2.1, the integer $|\chi_r^d|$ provides a lower bound for the total number of real rational J-holomorphic curves of X which pass through x and realize the homology class d, independently of the choice of a generic $J \in \mathbb{R}\mathcal{J}_\omega$.* $\square$

Note that this number of real curves is always bounded from above by the total number N_d of rational J-holomorphic curves of X which pass through x and realize the homology class d. The latter does not depend on the choice of J, it is a Gromov-Witten invariant of the

symplectic 4-manifold (X, ω) and was computed by Kontsevich in [8]. A typical problem of real enumerative geometry is, in this context, to know whether there exists a generic real almost-complex structure J such that all these rational J-holomorphic curves are real. The following corollary provides a criterium for the existence of such a structure.

Corollary 2.3 *Under the hypothesis of Theorem 2.1, assume that $\chi_r^d \geq 0$ (resp. $\chi_r^d \leq 0$). Assume that there exists a generic $J \in \mathbb{R}\mathcal{J}_\omega$ such that X has $\frac{1}{2}(N_d - |\chi_r^d|)$ real J-holomorphic curves of odd (resp. even) mass passing through x in the homology class d. Then, all the rational J-holomorphic curves of X which pass through x and realize the homology class d are real.* $\square$

Example:

Let (X, ω, c_X) be the complex projective plane equipped with its standard symplectic form and real structure. We denote the homology classes of the complex curves of $\mathbb{C}P^2$ by integers. Then $\chi^1(T) = 1 + T^2$, $\chi^2(T) = T + T^3 + T^5$ and $\chi^3(T) = \sum_{r=0}^4 2rT^{2r}$. The latter can be obtained using the observation of V. Kharlamov (see [2], Proposition 4.7.3 or [16], Theorem 3.6). Indeed, the Euler characteristic of the real part of the projective plane blown up at the nine base points of a pencil of elliptic curves, $(r+1)$ of which are real, is $-r$. This real pencil of cubics provides this surface with an elliptic fibration over $\mathbb{R}P^1$. The generic fibres of this fibration are real parts of smooth cubics, so their Euler characteristic vanishes. The singular fibres of this fibration are the real parts of the rational curves passing through the configuration of points. Their Euler characteristic is thus $+1$ or -1 depending on whether they have an isolated or non-isolated real double point. The relation $\chi_{2r}^3 = 2r$ follows. The non-triviality of the polynomials $\chi^4(T)$ and $\chi^5(T)$ is proved in §3.4.2.

The following question arise from Corollary 2.2. Are the lower bounds given by this corollary sharp? In the above example, for the degree 3 and $r = 8$, the lower bound is sharp from [2], Proposition 4.7.3. Also, is it possible to define a similar invariant using higher genus curves in real symplectic 4-manifolds, or in real symplectic manifolds of higher dimensions? Note that the straightforward generalization of the integer $\chi_r^d(x, J)$ using higher genus curves, even taking into account the coherent orientation of the complex moduli space $\mathcal{M}_g^d(x, J)$, certainly does depend on x, J. This can be noticed for plane projective curves of degree $d \geq 4$ with one nodal point, using the same trick as for the degree 3 curves, see the above example. Indeed, let x be a generic real configuration of $\frac{d(d+3)}{2} - 1$ distinct points, r of which being real. Then the linear pencil of curves of degree d passing through x has $\frac{d(d-3)}{2} + 1$ other fixed points. Denote by s the number of those additional points which are real. The Euler characteristic of $\mathbb{R}P^2$ blown up at these $r + s$ real points is $1 - r - s$. We deduce as before that $\chi_r^d(x, J) = r + s - 1$. Now, using linear pencils close to ones generated by lines arrangements, one realizes that s does depend in general on the choice of x.

Remark 2.4 Some lower bounds for the invariant $\chi_{c_1(X)d-1}^d$ have been obtained in [5] when X is a toric Del Pezzo surface. In particular, it is proven there that $\chi_{3d-1}^d \geq \frac{d!}{2}$ for the projective plane. Also, I recently defined similar invariants in complex dimension 3, providing thus a partial answer to the above question, see [18].

2.2 Genericity arguments

From now on, the real symplectic 4-manifold (X, ω, c_X) is **fixed**, so that it will not be mentioned in general in the following statements.

Denote by B^2 (resp. $\overline{B}^2$) the open (resp. closed) unit disk of $\mathbb{C}$ and by j_{st} the standard complex structure on this disk. Similarly, denote by B^4 the open unit ball of $\mathbb{C}^2$ and by J_{st} (resp. conj) the standard complex (resp. real) structure on this ball.

Lemma 2.5 *Let J be an almost complex structure of class C^l on B^4 tamed by the standard symplectic form and satisfying $\overline{\text{conj}}^*(J) = J$. Let $u_0 : \overline{B}^2 \to B^4$ be a real J-holomorphic map having an isolated singularity of order μ at $0 = u_0(0)$. Then, as soon as J is close enough to $J_{st} = J(0)$ in C^1 norm, for every $v \in \mathbb{R}^2$ and every integer $\nu \leq 2\mu+1$, there exist $\epsilon > 0$ and a family of real maps $w_\lambda \in L^{k,p}(\overline{B}^2, \mathbb{C}^2)$, $\lambda \in]-\epsilon, \epsilon[$, such that $w_0 = 0$, $\dot{w}_0 = \frac{d}{d\lambda}|_{\lambda=0}(w_\lambda)(0) = 0$ and for every $\lambda \in]-\epsilon, \epsilon[$, the map $u_\lambda(t) = u_0(t) + t^\nu(\lambda v + w_\lambda(t))$ is J-holomorphic and real.*

This is a real version of a weaker form of Lemma 3.1.1 of [12]. For the reader's convenience, we reproduce a sketch of proof here.

Sketch of proof:
One can write the equation $\sigma_{\overline{\partial}}(u_\lambda, j_{st}, J) = 0$ in the form

$$(x + yj_{st})^{-\nu}\sigma_{\overline{\partial}}(u_0(t) + (x + yj_{st})^\nu(\lambda v + w_\lambda(t)), j_{st}, J) = 0,$$

where by definition $x + yj_{st} = t$. The linearization of this equation writes

$$(x + yj_{st})^{-\nu}(\overline{\partial} + R)|_{(u_\lambda, j_{st}, J)}((x + yj_{st})^\nu(v + \dot{w}_\lambda\,(t))) = 0,$$

which takes the form

$$(\overline{\partial} + R)^{(\nu)}|_{(u_\lambda, j_{st}, J)}\ \dot{w}_\lambda\,(t) = -(\overline{\partial} + R)^{(\nu)}|_{(u_\lambda, j_{st}, J)}(v)$$

for some generalized $\overline{\partial}$-type operator $D^{(\nu)} = (\overline{\partial}+R)^{(\nu)} = (x+yj_{st})^{-\nu}(\overline{\partial}+R)|_{(u_\lambda, j_{st}, J)}(x+yj_{st})^\nu$. To solve this equation, it is thus sufficient to find a right inverse $T^{(\nu)}$ of this operator $D^{(\nu)}$ such that $T^{(\nu)}(\alpha)(0) = 0$ for every $\alpha \in L^{k-1,p}(\overline{B}^2, \Lambda^{0,1}\overline{B}^2 \otimes \mathbb{C}^2)$. As soon as J is close enough to $J_{st} = J(0)$ in C^1 norm, the existence of such a right inverse follows from the existence of a right inverse for the standard $\overline{\partial}$-operator, see Lemma 1.2.2 of [7] for instance. Moreover, the operators $\overline{\partial}^\nu, R^\nu, D^\nu$ and T^ν are $\mathbb{Z}/2\mathbb{Z}$-equivariants, hence the result. $\square$

Denote by $\mathbb{R}\mathcal{J}_{\overline{B}^2}$ the space of complex structures of $\overline{B}^2$ compatible with the complex conjugation conj.

Lemma 2.6 *Let $\eta > 0$ and $\mathbb{R}\mathcal{P}' = \{(u, J_{\overline{B}^2}, J) \in L^{k,p}(\overline{B}^2, B^4) \times \mathbb{R}\mathcal{J}_{\overline{B}^2} \times \mathbb{R}\mathcal{J}_\omega \,|\, du + J \circ du \circ J_{\overline{B}^2} = 0,\ c_X \circ u = u \circ \text{conj}\,,\ ||J - J_{st}||_{C^1} < \eta$ and $u(\overline{B}^2)$ has a unique singularity of multiplicity two outside $u(\partial\overline{B}^2)\}$. Let $\mathbb{R}\mathcal{P}'_s$ be the subspace of $\mathbb{R}\mathcal{P}'$ consisting of maps having a unique cuspidal point. Moreover, let $(u_\lambda, J_B^\lambda, J_\lambda)_{\lambda \in]-1,1[}$ be a path of $\mathbb{R}\mathcal{P}'$ such that $(u_0, J_B^0, J_0) \in \mathbb{R}\mathcal{P}'_s$ and du_0 has a real ordinary cusp at the point $0 \in \overline{B}^2$. Then, as soon as η is small enough,*

1) The space $\mathbb{R}\mathcal{P}'$ is a separable Banach manifold of class C^{l-k} and $\mathbb{R}\mathcal{P}'_s$ is a separable Banach submanifold of $\mathbb{R}\mathcal{P}'$ of codimension one.

2) The path $(u_\lambda, J_B^\lambda, J_\lambda)_{\lambda\in]-1,1[}$ is transversal to $\mathbb{R}\mathcal{P}'_s$ at $\lambda = 0$ if and only if $\nabla\ \dot{u}_0\ (T_0B^2)$ is not the tangent line of $u_0(B^2)$ at the cusp $u_0(0)$. Under this condition, there exists $\epsilon > 0$ such that for every $\lambda \in]-\epsilon, 0[$ (resp. $\lambda \in]0, \epsilon[$), the curve $u_\lambda(\overline{B}^2)$ has a non-isolated (resp. isolated) real double point in a neighbourhood of the cusp, or vice-versa.

Proof:

1) Consider the map

$$\Phi : \begin{array}{ccc} L^{k,p}(\overline{B}^2, B^4) \times \mathbb{R}\mathcal{J}_{\overline{B}^2} \times \mathbb{R}\mathcal{J}_\omega & \to & L^{k-1,p}(\overline{B}^2, \Lambda^{0,1}\overline{B}^2 \otimes_{\mathbb{C}} E_u) \times \mathbb{R}\mathcal{J}_{\overline{B}^2} \times \mathbb{R}\mathcal{J}_\omega \\ (u, J_{\overline{B}^2}, J) & \mapsto & (du + J \circ du \circ J_{\overline{B}^2}, J_{\overline{B}^2}, J). \end{array}$$

As in §1.2, the linearization of Φ writes

$$d|_{(u,J_{\overline{B}^2},J)}\Phi : \begin{array}{ccc} L^{k,p}(\overline{B}^2, E_u) \times T\mathbb{R}\mathcal{J}_{\overline{B}^2} \times T\mathbb{R}\mathcal{J}_\omega & \to & L^{k-1,p}(\overline{B}^2, \Lambda^{0,1}\overline{B}^2 \otimes_{\mathbb{C}} E_u) \times T\mathbb{R}\mathcal{J}_{\overline{B}^2} \times T\mathbb{R}\mathcal{J}_\omega \\ (v, \dot{J}_{\overline{B}^2}, \dot{J}) & \mapsto & (Dv + \dot{J} \circ du \circ J_{\overline{B}^2} + J \circ du \circ \dot{J}_{\overline{B}^2}, \dot{J}_{\overline{B}^2}, \dot{J}). \end{array}$$

Remember that as soon as η is close enough to zero, the operator D has a continuous right inverse $T : L^{k-1,p}(\overline{B}^2, \Lambda^{0,1}\overline{B}^2 \otimes_{\mathbb{C}} E_u) \to L^{k,p}(\overline{B}^2, E_u)$, since the operator $\overline{\partial}_{st}$ has one, see, for example, Lemma 1.2.2 of [7]. It follows that the operator $d|_{(u,J_{\overline{B}^2},J)}\Phi$ between separable Banach spaces is surjective as soon as η is close enough to zero, and its kernel has a closed complement. From the implicit function theorem (see Theorem A.3.3 of [9] for example) follows then that $\mathbb{R}\mathcal{P}'$ is a separable Banach manifold of class C^{l-k}. Now, let E (resp. F) be the vector bundle over $\mathbb{R}\mathcal{P}'\times]-1,1[$ whose fibre over $((u, J_{\overline{B}^2}, J), t)$ is the vector space $T_{u(t)}(\mathbb{R}^2 \cap B^4)$ (resp. $T^*_t]-1,1[$). Denote by Γ the section of the bundle $F \otimes E$ defined by $\Gamma((u, J_{\overline{B}^2}, J), t) = d_t u$. Then Γ vanishes transversely over $\mathbb{R}\mathcal{P}'_s$. Indeed, let $((u'_0, J'_{\overline{B}^2}, J'), t) \in \mathbb{R}\mathcal{P}'\times]-1,1[$ be such that $d_t u'_0 = 0$, and let v be an element of $T_{u'_0(t)}(\mathbb{R}^2 \cap B^4)$. We can apply Lemma 2.5 with $\nu = 1$. Let $(u'_\lambda, J'_{\overline{B}^2}, J')$ be the path of $\mathbb{R}\mathcal{P}'$ given by this lemma. We have $\Gamma((u'_\lambda, J'_{\overline{B}^2}, J'), t) = d_t u'_\lambda$, thus

$$\nabla_{((\dot{u}'_0,0,0),\frac{\partial}{\partial t})}\Gamma = \nabla\ \dot{u}'_0\ |_t + \nabla_{\frac{\partial}{\partial t}}(du'_0) = v \otimes \frac{d}{dt} + \nabla_{\frac{\partial}{\partial t}}(du'_0).$$

Choosing t constant in $]-1,1[$, we deduce the surjectivity of $\nabla\Gamma$ since the vector v has been chosen arbitrarily in $T_{u'_0(t)}(\mathbb{R}^2 \cap B^4)$. Thus, $\Gamma^{-1}(0)$ is a codimension two submanifold of class C^{l-k} of $\mathbb{R}\mathcal{P}'\times]-1,1[$. Now the projection $\mathbb{R}\mathcal{P}'\times]-1,1[\to \mathbb{R}\mathcal{P}'$ restricted to $\Gamma^{-1}(0)$ is an immersion. Moreover, if $(u', J'_{\overline{B}^2}, J') \in \mathbb{R}\mathcal{P}'_s$ and $((u', J'_{\overline{B}^2}, J'), t) \in \mathbb{R}\mathcal{P}'\times]-1,1[$ lifts this element, then there exists a neighbourhood V of $(u', J'_{\overline{B}^2}, J')$ in $\mathbb{R}\mathcal{P}'$ such that the projection $\Gamma^{-1}(0) \cap (V\times]-1,1[) \to \mathbb{R}\mathcal{P}'_s \cap V$ is one to one, since by hypothesis the cusp of u' is unique. Finally, it is proper, since the condition to have a cuspidal point is a closed condition in $\mathbb{R}\mathcal{P}'$. The first part of the lemma follows.

2) The kernel of the projection $\mathbb{R}\mathcal{P}'\times]-1,1[\to \mathbb{R}\mathcal{P}'$ is generated by vectors of the form $((0,0,0), \frac{\partial}{\partial t})$. From what we have done, $\nabla_{((0,0,0),\frac{\partial}{\partial t})}\Gamma = \nabla_{\frac{\partial}{\partial t}}(du_0)$. Since the image of $\nabla_{\frac{\partial}{\partial t}}(du_0)$ is the tangent line of the curve $u_0(]-1,1[)$ at the real cusp, and $\nabla_{((\dot{u}_0,\dot{J}_{\overline{B}^2},\dot{J}),0)}\Gamma = \nabla\ \dot{u}_0$, we deduce the transversality condition. For example, let $u^\lambda_{st} : t \in \overline{B}^2 \mapsto (\frac{1}{4}t^2, \frac{1}{8}t^3 + \frac{1}{2}\lambda t) \in B^4$, where $|\lambda|$ is small enough so that it indeed takes

values in B^4. Then, the path $(u^\lambda_{st}, j_{st}, J_{st})$ of $\mathbb{R}\mathcal{P}'$ is transversal to $\mathbb{R}\mathcal{P}'_s$ at $\lambda = 0$. Indeed, $\nabla\, \dot{u}^0_{st} = (0, \frac{1}{2}) \otimes \frac{d}{dt}$ and the tangent line of u^0_{st} at the cusp is generated by the vector $(1, 0)$, so that the transversality condition is satisfied. Note that for $\lambda < 0$ (resp. $\lambda > 0$), u^λ_{st} has a non-isolated (resp. isolated) real double point at the parameters $t = \pm 2\sqrt{-\lambda}$. Since $\mathbb{R}\mathcal{P}'_s$ is smooth at $(u^0_{st}, j_{st}, J_{st})$, we deduce that there exists some neighbourhood W of $(u^0_{st}, j_{st}, J_{st})$ in $\mathbb{R}\mathcal{P}'$ such that for any path $(u_\lambda, J^\lambda_B, J_\lambda)_{\lambda\in]-1,1[}$ of W which is transversal to $\mathbb{R}\mathcal{P}'_s$ at $\lambda = 0$, there exists $\epsilon > 0$ such that for every $\lambda \in]-\epsilon, 0[$ (resp. $\lambda \in]0, \epsilon[$), the curve $u_\lambda(\overline{B}^2)$ has a non-isolated (resp. isolated) real double point in a neighbourhood of the cusp, or vice-versa. Now, let $(u_\lambda, J^\lambda_B, J_\lambda)_{\lambda\in]-1,1[}$ be any path of $\mathbb{R}\mathcal{P}'$ transversal to $\mathbb{R}\mathcal{P}'_s$ at $\lambda = 0$. Let us prove that there exists an isotopy among paths which are transversal to $\mathbb{R}\mathcal{P}'_s$ at $\lambda = 0$ between this path and a path included in W. This isotopy will be obtained via some rescaling argument. After performing a $\mathbb{Z}/2\mathbb{Z}$-equivariant linear transformation which conjugates $J_0(0)$ to the standard complex structure of $\mathbb{C}^2$ if necessary, we can assume that $J_0(0) = J_{st}$. Then, from Corollary 1.4.3 of [7] for example, we can assume that u_0 writes $t \mapsto (\frac{1}{4}t^2, \frac{1}{8}t^3) + t^3\epsilon_0(t)$ in complex coordinates, where $\epsilon_0 : \overline{B}^2 \to B^4$ satisfies $\epsilon_0(0) = 0$. For every $\delta \in]0, 1[$, denote by $a_\delta : (z_1, z_2) \in \mathbb{C}^2 \mapsto (\delta^2 z_1, \delta^3 z_2) \in \mathbb{C}^2$ and $h_\delta : t \in \mathbb{C} \mapsto \delta t \in \mathbb{C}$. For $\delta \in]0, 1[$ and $|\lambda| \leq \epsilon_1(\delta)$ where $\epsilon_1(\delta)$ is close enough to zero, denote by $u(\lambda, \delta) = a_\delta^{-1} \circ u_\lambda \circ h_\delta$, $J_B(\lambda, \delta) = h^*_\delta J^\lambda_B$ and $J(\lambda, \delta) = a^*_\delta J_\lambda$, so that the map $u(\lambda, \delta) : (\overline{B}^2, J_B(\lambda, \delta)) \to (B^4, J(\lambda, \delta))$ is pseudo-holomorphic. This map $u(\lambda, \delta)$ indeed takes value in B^4 as soon as $\epsilon_1(\delta)$ is close enough to zero, since the map $u(0, \delta)$ itself takes value in B^4, restricting the domain of definition of $u(0, 1)$ from the beginning if necessary, since ϵ_0 is bounded. Note that $J(0, \delta)$ converges to J_{st} in C^1-norm as δ converges to zero, and $J_B(0, \delta)$ to j_{st}. Moreover, the map $u(0, \delta)$ converges to the real ordinary cusp $u^0_{st} : t \in \overline{B}^2 \mapsto (\frac{1}{4}t^2, \frac{1}{8}t^3) \in B^4$ as δ converges to zero. As a consequence, as soon as δ is small enough, the element $(u(0, \delta), J_B(0, \delta), J(0, \delta))$ belongs to W. Hence, for δ small enough and $|\lambda| \leq \epsilon_1(\delta)$ where $\epsilon_1(\delta)$ is close enough to zero, the path $(u(\lambda, \delta), J_B(\lambda, \delta), J(\lambda, \delta))_{\lambda\in]-\epsilon_1(\delta),\epsilon_1(\delta)[}$ belongs to W and is transversal to $\mathbb{R}\mathcal{P}'_s$ at $\lambda = 0$. We deduce a smooth family parameterized by $\delta \in]0, 1]$ of paths $(u(\lambda, \delta), J_B(\lambda, \delta), J(\lambda, \delta))_{\lambda\in]-\epsilon_1(\delta),\epsilon_1(\delta)[}$ of $\mathbb{R}\mathcal{P}'$, which are transversal to $\mathbb{R}\mathcal{P}'_s$ at $\lambda = 0$. The topology of the real double point of $u(\lambda, \delta)$, $\lambda > 0$, and $u(\lambda, \delta)$, $\lambda < 0$, thus does not depend on $\delta \in]0, 1]$. The result follows since it has already been checked for paths included in W. $\square$

The moduli space $\mathbb{R}\mathcal{M}^d(x)$ which has been constructed in §1.5.1 admits a natural stratification whose strata are given by equisingular rational pseudo-holomorphic curves. The remaining part of this paragraph is devoted to the computation of the dimension of each stratum. For this purpose, a stratum will be said to be of *codimension* k if it is the image of a separable Banach manifold N under a smooth Fredholm map ϕ of Fredholm index $-k$ such that all the limits of sequences $\phi(x_n)$, where x_n is a diverging sequence in N, belong to a countable union of strata of higher codimensions. In particular, the map ϕ is not required to be proper. The point for us is that a generic path $\gamma : [0, 1] \to \mathbb{R}\mathcal{M}^d(x)$ avoids every stratum of codimension greater than one, and intersects each stratum of codimension one at only finitely many points.

Proposition 2.7 *Let (X, ω, c_X) be a real symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ and x be a real configuration of distinct points of X.*

1) The subspace of $\mathbb{R}\mathcal{M}^d(x)$ consisting of curves having a real ordinary cusp (resp. a non-ordinary cusp or several cusps) is an immersed submanifold of codimension one (resp.

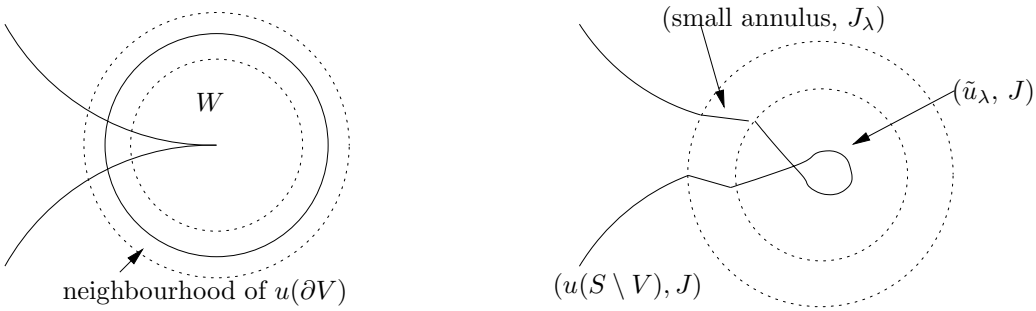
two).

2) The subspace of $\mathbb{R}\mathcal{M}^d(x)$ consisting of curves having a real tacnode or a real ordinary triple point (resp. a multiple point of higher order or several such points) is a stratum of codimension one (resp. two) transversal to the previous one.

3) The subspace of $\mathbb{R}\mathcal{M}^d(x)$ consisting of curves having a real ordinary cusp, a real tacnode or a real ordinary triple point at some point of $x \cap \mathbb{R}X$ is a stratum of codimension two.

Proof:

Let us prove the first part of the proposition. For this purpose, let $[u, J_S, J] \in \mathbb{R}\mathcal{M}^d(x)$ and $(u, J_S, J) \in \mathbb{R}\mathcal{P}^*(x)$ be a lift of this element. Denote by $c_S \in \mathcal{D}iff^-(S, z)$ the associated involution and by $\mathbb{R}S$ (resp. $\mathcal{C}$) the fixed point set of c_S in S (resp. $\mathbb{R}\mathcal{P}^*(x)$). Assume that (u, J_S, J) has an ordinary cusp at $t \in \mathbb{R}S$. Fix then some real neighbourhood V of t in S diffeomorphic to $\overline{B}^2$ and a real neighbourhood W of $u(t)$ in X diffeomorphic to B^4. As soon as these neighbourhoods have been chosen small enough, we deduce some "restriction map" $rest : \mathcal{C}^W \to \mathbb{R}\mathcal{P}'$, where $\mathcal{C}^W$ is such that $(u|_V, J_S|_B, J|_W)$ belongs to $\mathbb{R}\mathcal{P}'$ (see Lemma 2.6). This map is transversal to $\mathbb{R}\mathcal{P}'_s$. Indeed, denote by $(\tilde{u}_0, \widetilde{J}^0_S, \widetilde{J}) = rest(u, J_S, J) \in \mathbb{R}\mathcal{P}'_s$. Extend this into a path $(\tilde{u}_\lambda, \widetilde{J}^\lambda_S, \widetilde{J})_{\lambda \in]-1,1[}$ transversal to $\mathbb{R}\mathcal{P}'_s$. Using the $\mathbb{Z}/2\mathbb{Z}$-equivariant identification between W and B^4, we deduce a path $(\tilde{u}_\lambda, \widetilde{J}^\lambda_S, \widetilde{J})_{\lambda \in]-1,1[}$ of maps in W. There exists then a smooth family $(J_\lambda)_{\lambda \in]-1,1[}$ in $\mathbb{R}\mathcal{J}_\omega$ extending $J = J_0$ such that for every $\lambda \in]-1, 1[\setminus\{0\}$, J_λ differs from J only in a neighbourhood of $u(\partial V)$, and a smooth family $(u_\lambda, J^\lambda_S, J_\lambda)$ of real J_λ-holomorphic curves extending (u, J_S, J), such that for $\lambda \neq 0$, the image of u_λ is obtained topologically from $\tilde{u}_\lambda(V)$ and $u(S \setminus V)$ by gluing a small annulus embedded in a neighbourhood of $u(\partial V)$. This proves the transversality of the restriction map.



We deduce that the subspace $rest^{-1}(\mathbb{R}\mathcal{P}'_s) \subset \mathcal{C}^W$ made of curves having a real cuspidal point in a neighbourhood of $u(t)$ is an immersed codimension one submanifold of $\mathcal{C}^W$. Hence, the subspace of $\mathbb{R}\mathcal{M}^d(x)$ consisting of curves having a real ordinary cusp is an immersed submanifold of codimension one. Moreover, it follows from this proof that the condition to have two different cusps is transversal, so that the subspace of these curves is an immersed submanifold of codimension two of $\mathbb{R}\mathcal{M}^d(x)$. It remains to prove that the same holds for curves having some cuspidal point of higher order or some non-ordinary cusp. For this purpose, we can assume that the cuspidal point is unique. Denote by $\mathbb{R}\mathcal{M}^d(x)_{cusp}$ the subspace of $\mathbb{R}\mathcal{M}^d(x)$ consisting of curves having a unique cuspidal point which is thus real. Let $[u, J_S, J] \in \mathbb{R}\mathcal{M}^d(x)_{cusp}$ and $(u, J_S, J) \in \mathbb{R}\mathcal{P}^*(x)$ be a lift of this element. Denote as before by $c_S \in \mathcal{D}iff^-(S, z)$ the associated involution and by $\mathbb{R}S$ (resp. $\mathcal{C}_{cusp}$) the fixed point set of c_S in S (resp. in $\mathbb{R}\mathcal{P}^*(x)_{cusp}$). Let $t \in \mathbb{R}S$ be the point where du vanishes. Remember that the order three jet $j^3_t(u)$ of u at t is well defined (see [7], Corollary 1.4.3). It is a polynomial map from $T_t\mathbb{R}S$ to $T_{u(t)}\mathbb{R}X$ whose first order term vanishes since we restrict ourselves to $\mathcal{C}_{cusp}$.

Denote by Γ_2 the section of the space of 2-jets over $\mathcal{C}_{cusp}$ which maps (u, J_S, J) to the term of order two of $j^3_t(u)$. Writing u in a local chart in a neighbourhood of $u(t)$, we prove that this section is smooth, of class C^{l-k} (see [12], Lemma 3.2.3). Moreover, it follows as before from Lemma 2.5 with $\nu = 2$ that this section is transversal to the zero section. Indeed, from Lemma 2.5 with $\nu = 2$, it is possible to deform u in a neighbourhood of $u(t)$ into a path of J-holomorphic maps from a disc $\overline{B}^2$ to X. Then it is as before possible to glue this path to the initial map outside this neighbourhood by adding a small embedded annulus. We thus obtain a smooth family $(u_\lambda, J^\lambda_S, J_\lambda)$ of real J_λ-holomorphic curves extending $(u, J_S, J) = (u_0, J^0_S, J)$ and over which Γ_2 vanishes transversely. It follows that the subspace of $\mathbb{R}\mathcal{M}^d(x)$ consisting of curves having a cuspidal point of order ≥ 2 is an immersed submanifold of codimension three. Let us now restrict ourselves to the open set $\mathcal{V} \subset \mathcal{C}_{cusp}$ on which Γ_2 does not vanish. Let $(u, J_S, J) \in \mathcal{V}$ and $t \in \mathbb{R}S$ be the cuspidal point. The term of order two of $j^3_t(u)$ defines a line in $T_{u(t)}\mathbb{R}X$, it is the tangent line of the curve $u(S)$ at $u(t)$. Denote by $N_u(t)$ the quotient of $T_{u(t)}\mathbb{R}X$ by this line. Projecting the term of order three of $j^3_t(u)$ on $N_u(t)$, we define a section Γ_3 of the bundle of 3-jets from $T_t\mathbb{R}S$ to $N_u(t)$, bundle defined over $\mathcal{V}$. As before, this section is smooth of class C^{l-k-1} (the bundle $N_u(t)$ is only of class C^{l-k-1}), and it follows from Lemma 2.5 with $\nu = 3$ that it is transversal to the zero section. This ends the proof of the first part of Proposition 2.7.

Now let us prove the second part of Proposition 2.7. Since all the cases are proved almost along the same lines, we only give a proof in the case of the tacnode. Let $c_S \in \mathcal{D}iff^-(S, z)$ be an involution and $\mathcal{D}iff^+(S, z)_{c_S}$ be the subgroup of orientation preserving diffeomorphisms of (S, z) which commute with c_S. Denote by $\mathbb{R}S$ (resp. $\mathcal{C}$) the fixed point set of c_S in S (resp. $\mathcal{P}^*(x)$), so that $\mathcal{C}/\mathcal{D}iff^+(S, z)_{c_S}$ is a union of connected components of $\mathbb{R}\mathcal{M}^d(x)$. The latter follows from the fact that all the order two elements of $\mathcal{D}iff^-(S, z)$ which have non-empty fixed point sets in S are conjugated to each other by some element of $\mathcal{D}iff^+(S, z)$ (see the proof of Proposition 1.9). Let

$$\mathcal{C}_r = \{((u, J_S, J), t_1, t_2) \in \mathcal{C} \times \mathbb{R}S \times \mathbb{R}S \,|\, t_1 \neq t_2\,,\ u(t_1) = u(t_2) \text{ and } d_{t_1}u \neq 0 \neq d_{t_2}u\}.$$

This is a codimension two submanifold of class C^{l-k} of $\mathcal{C} \times \mathbb{R}S \times \mathbb{R}S$. Denote by N_r (resp. F_r) the vector bundle of class C^{l-k-1} over $\mathcal{C}_r$ whose fibre over $((u, J_S, J), t_1, t_2)$ is the quotient space $T_{u(t_1)}\mathbb{R}X/d_{t_1}u(T_{t_1}\mathbb{R}S)$ (resp. $T^*_{t_2}\mathbb{R}S$). Denote by Θ_r the section of the bundle $F_r \otimes N_r$ defined by $\Theta_r((u, J_S, J), t_1, t_2) = d_{t_2}u$. As before, the section Θ_r is smooth of class C^{l-k-1}, and from Lemma 2.5 with $\nu = 1$, it is transversal to the zero section. We deduce that $\Theta_r^{-1}(0)$ is a codimension one submanifold of $\mathcal{C}_r$, and a codimension three submanifold of $\mathcal{C} \times \mathbb{R}S \times \mathbb{R}S$. This manifold thus maps onto a codimension one stratum of $\mathbb{R}\mathcal{M}^d(x)$, since the projection $(\mathcal{C} \times \mathbb{R}S \times \mathbb{R}S)/\mathcal{D}iff^+(S, z)_{c_S} \to \mathbb{R}\mathcal{M}^d(x)$ is of index two. This codimension one stratum of $\mathbb{R}\mathcal{M}^d(x)$ is transversal to the one defined in the first part of this proposition. Now, let

$$\mathcal{C}_i = \{((u, J_S, J), t) \in \mathcal{C} \times (S \setminus \mathbb{R}S) \,|\, u(t) = u \circ c_S(t) \text{ and } d_t u \neq 0\}.$$

This is a codimension two submanifold of class C^{l-k} of $\mathcal{C} \times S$. Denote by N_i (resp. F_i) the vector bundle of class C^{l-k-1} over $\mathcal{C}_i$ whose fibre over $((u, J_S, J), t)$ is the quotient space $T_{u(t)}X/d_t u(T_t S)$ (resp. $T^*_{c_S(t)}S$). Denote by Θ_i the section of the bundle $F_i \otimes N_i$ defined by $\Theta_i((u, J_S, J), t) = d_{c_S(t)}u$. As before, the section Θ_i is smooth of class C^{l-k-1}, and from Lemma 2.5 with $\nu = 1$, it is transversal to the zero section. We deduce that $\Theta_i^{-1}(0)$ is a codimension one submanifold of $\mathcal{C}_i$, and a codimension three submanifold of $\mathcal{C} \times S$.

This manifold thus maps onto a codimension one stratum of $\mathbb{R}\mathcal{M}^d(x)$, since the projection $(\mathcal{C}\times S)/\mathcal{D}iff^+(S,z)_{c_S} \to \mathbb{R}\mathcal{M}^d(x)$ is of index two. This codimension one stratum of $\mathbb{R}\mathcal{M}^d(x)$ is transversal to the one defined in the first part of this proposition. The second part of the proposition follows and the proof of the third part of the proposition is left to the reader. $\square$

Denote by $\mathbb{R}\mathcal{M}^d(x)_s$ the immersed codimension one submanifold of $\mathbb{R}\mathcal{M}^d(x)$ consisting of curves having a unique cuspidal point which is a real ordinary cusp. Let $[u, J_S, J] \in \mathbb{R}\mathcal{M}^d(x)_s$, we have $\dim H^0_D(S, \mathcal{N}^{sing}_{u,-z})_{+1} = 1$. Since $\mathrm{ind}(\pi_{\mathbb{R}}) = 0$, it follows that $\dim H^1_D(S, N_{u,-z})_{+1} \geq 1$. Thus, S being rational, $\dim H^0_D(S, N_{u,-z})_{+1} = 0$ (see [4]) and $\dim H^1_D(S, N_{u,-z})_{+1} = 1$. Let ψ_u be a generator of the vector space $H^0_{D^*}(S, K_S \otimes_{\mathbb{C}} N^*_u)_{-1} = H^1_D(S, N_{u,-z})^*_{+1}$. Remember that since $\psi_u \neq 0$ and $D^*(\psi_u) = 0$, this section ψ_u vanishes at a finite number of points outside a neighbourhood of z.

Proposition 2.8 *The subspace of $\mathbb{R}\mathcal{M}^d(x)_s$ consisting of curves $[u, J_S, J]$ for which the section ψ_u vanishes at the unique real cuspidal point of u is an immersed codimension one submanifold of $\mathbb{R}\mathcal{M}^d(x)_s$.*

Proof:

The following proof is analogous to the proof of Lemma 4.4.3 of [12]. Fix some component of $\mathbb{R}\mathcal{M}^d(x)_s$ and a lift $\mathcal{C}_s$ of this component in $\mathbb{R}\mathcal{P}^*(x)$. Denote by $c_S \in \mathcal{D}iff^-(S,z)$ the associated involution, so that $\mathcal{C}_s$ is included in the fixed point set of c_S in $\mathcal{P}^*(x)$. Denote also by $\mathbb{R}S$ the fixed point set of c_S in S and by $\mathcal{D}iff^+(S,z)_{c_S}$ the subgroup of orientation preserving diffeomorphisms of (S,z) which commute with c_S. For every $(u, J_S, J) \in \mathcal{C}_s$, we denote by $t_u \in \mathbb{R}S$ the unique point at which du is not injective. Let then F be the real vector bundle of rank one over $\mathcal{C}_s$ whose fibre over (u, J_S, J) is the vector space $(K_S \otimes N^*_u)_{-1}|_{t_u}$. It is a vector bundle of class C^{l-k-1}. Let ψ be a local section of class C^{l-k-1} of the vector bundle over $\mathcal{C}_s$ whose fibre over (u, J_S, J) is the vector space $H^0_{D^*}(S, K_S \otimes_{\mathbb{C}} N^*_u)_{-1}$. We assume that ψ does not vanish and we will denote as before by ψ_u the value $\psi(u, J_S, J)$. We then deduce a section Γ_ψ of the bundle F defined by $\Gamma_\psi(u, J_S, J) = \psi_u(t_u)$. This section Γ_ψ is of class C^{l-k-1} and we have to prove that it vanishes transversely. For this purpose, we fix a riemannian metric on S invariant under c_S and the associated Levi-Civita connection. This connection as well as the connection ∇ on X induce connections on all the associated bundles, like F for instance. For convenience, all these connections will be denoted by ∇. Hence, suppose that $(u, J_S, J) \in \mathcal{C}_s$ is such that $\Gamma_\psi(u, J_S, J) = 0$, we have to prove that $\nabla|_{(u,J_S,J)}\Gamma_\psi : T_{(u,J_S,J)}\mathcal{C}_s \to F_{(u,J_S,J)}$ is surjective. Let then $j_0 \in (K_S \otimes N^*_u)_{-1}|_{t_u}$, we are searching for $(v, \dot{J}_S, \dot{J}) \in T_{(u,J_S,J)}\mathcal{C}_s$ such that $\nabla_{(v,\dot{J}_S,\dot{J})}\Gamma_\psi = j_0$. Let $(u_\lambda, J^\lambda_S, J_\lambda)_{\lambda\in]-1,1[}$ be a path in $\mathcal{C}_s$ such that $(u_0, J^0_S, J_0) = (u, J_S, J)$ and $\frac{d}{d\lambda}(u_\lambda, J^\lambda_S, J_\lambda)|_{\lambda=0} = (v, \dot{J}_S, \dot{J})$. For every $w_0 \in L^{k,p}(S, N_{u,-z})_{+1}$, we extend w_0 into a path $w_\lambda \in L^{k,p}(S, N_{u_\lambda,-z})_{+1}$ such that $\nabla_{(\dot{u}_\lambda,\dot{J}^\lambda_S,\dot{J}_\lambda)} w_\lambda = 0$. Then the relation $D^*_\lambda \psi_{u_\lambda} = 0$ becomes equivalent to

$$\forall w_0 \in L^{k,p}(S, N_{u,-z})_{+1}, < \psi_{u_\lambda}, D_\lambda w_\lambda >= 0.$$

We deduce after differentiation

$$\forall w_0 \in L^{k,p}(S, N_{u,-z})_{+1}, <\dot{\psi}_{u_\lambda}, D_\lambda w_\lambda > + < \psi_{u_\lambda}, \big(\nabla_{(v_\lambda,\dot{J}^\lambda_S,\dot{J}_\lambda)} D_\lambda\big)(w_\lambda) >= 0. \tag{3}$$

Note that $\nabla_{(v,\dot{J}_S,\dot{J})}\Gamma_\psi = \dot{\psi}\,(t_u) + \nabla_{\dot{t}_{u_\lambda}}\psi$. It is then enough to find $(0, 0, \dot{J}) \in T_{(u,J_S,J)}\mathcal{C}_s$ and a

section $\dot{\psi}\in L^{1-k,p'}(S, K_S \otimes_{\mathbb{C}} N_u^*)_{-1}$ such that $\dot{\psi}\ (t_u) = j_0$ and

$$\forall w \in L^{k,p}(S, N_{u,-z})_{+1}, < D^*\ \dot{\psi}, w > + < \psi_u, \nabla_{(0,0,\dot{j})} D(w) >= 0. \tag{4}$$

Indeed, this relation (4) ensures that $\dot{\psi}$ equals $\dot{\psi}_{u_0}$ up to the addition of a multiple of ψ_{u_0}, and thus the evaluation $\dot{\psi}\ (t_u)$ equals $\dot{\psi}_{u_0}\ (t_u)$. Moreover, the choice $v = 0$ implies that $\dot{t}_{u_\lambda}= 0$ and thus $\nabla_{(v,\dot{j}_S,\dot{j})}\Gamma_\psi =\dot{\psi}\ (t_u)$. Let us start to construct the section $\dot{\psi}$. Let $\dot{\psi}_1\in L^{1-k,p'}(S, K_S \otimes_{\mathbb{C}} N_u^*)_{-1}$ be a local section such that $\dot{\psi}_1\ (t_u) = j_0$ and $D^*(\dot{\psi}_1) = 0$ in a neighbourhood of t_u. Such a section does exist. Indeed, extend j_0 in a neighbourhood of t_u as a holomorphic section of $(K_S \otimes N_u^*)_{-1}$ for the complex structure associated to the complex linear part $\overline{\partial}$ of D^*. It suffices to solve locally the equation $D^*(j_0 + z\phi(z)) = 0$ where the unknown ϕ is defined in the neighbourhood of t_u. This equation is equivalent to $(z^{-1}D^*z)(\phi(z)) = -z^{-1}D^*(j_0)$. Note that since j_0 is holomorphic, $D^*(j_0) = R^*(j_0)$ and since t_u is a cusp of u and $R(v) = N_J(v, du(.))$, the right hand side $z^{-1}D^*(j_0)$ of the previous equation is well defined. The operator $z^{-1}D^*z$ is equivariant under the action of c_S and has, once restricted to a neighbourhood of t_u, a right inverse T also $\mathbb{Z}/2\mathbb{Z}$-equivariant. Thus $\phi = -T \circ z^{-1}D^*(j_0)$ is a local solution and satisfies $dc_X \circ \phi \circ c_S = -\phi$, which provides the existence of $\dot{\psi}_1$. Using partition of unity, this local section is completed to a global section $\dot{\psi}\in L^{1-k,p'}(S, K_S \otimes_{\mathbb{C}} N_u^*)_{-1}$ which has support in a neighbourhood of t_u where ψ_u has no other zero than t_u. It remains to find $(0, 0, \dot{J}) \in T_{(u,J_S,J)}\mathcal{C}_s$ such that (4) is satisfied. Let us search for such a vector among those for which $\dot{J} = 0$ in a neighbourhood of the cusp $u(t_u)$ and along $u(S)$. Such a vector is then tangent to $\mathcal{C}_s$ as soon as $\overline{c_X}^*(\dot{J}) = \dot{J}$. Remember that by definition, $Dw = \nabla w + J \circ \nabla w \circ J_S + \nabla_w J \circ du \circ J_S$, thus at such a vector $(0, 0, \dot{J})$, we have $\nabla_{(0,0,\dot{j})} D(w) = \nabla_w \dot{J} \circ du \circ J_S$. Now the relation (4) rewrites:

$$\forall w \in L^{k,p}(S, N_{u,-z})_{+1}, < D^*\ \dot{\psi}, w > + < \psi_u, \nabla_w \dot{J} \circ du \circ J_S >= 0,$$

$$\text{or} \quad -D^*\ \dot{\psi}=< \psi_u, \nabla \dot{J} \circ du \circ J_S > . \tag{5}$$

Outside of a neighbourhood of t_u, this equation determines the value of the derivative of $\dot{J}$ in the normal direction of the curve $u(S)$. After integration of this condition, we construct a solution $\dot{J}$ satisfying (5) and $\overline{c_X}^*(\dot{J}) = \dot{J}$. Hence the result. □

Proposition 2.9 *Let (X, ω, c_X) be a real symplectic 4-manifold, $d_1, d_2 \in H_2(X; \mathbb{Z})$, and y_1, y_2 be two finite disjoint subsets of X invariant under c_X. Denote by*

$$\Delta = \{([u_1, J_{S_1}, J], [u_2, J_{S_2}, J]) \in \mathbb{R}\mathcal{M}^{d_1}(y_1) \times \mathbb{R}\mathcal{M}^{d_2}(y_2) \,|\, u_1(S_1) = u_2(S_2)\}.$$

Then, the projections $\pi^1_{\mathbb{R}} : \mathbb{R}\mathcal{M}^{d_1}(y_1) \to \mathbb{R}\mathcal{J}_\omega$ and $\pi^2_{\mathbb{R}} : \mathbb{R}\mathcal{M}^{d_2}(y_2) \to \mathbb{R}\mathcal{J}_\omega$ are transversal to each other outside of Δ.

Note that as soon as $d_1 \neq d_2$, the diagonal Δ is empty.

Corollary 2.10 *Let (X, ω, c_X) be a real symplectic 4-manifold, $d_1, \ldots, d_l \in H_2(X; \mathbb{Z})$ and $y_1, \ldots, y_l$ be finite disjoint subsets of X invariant under c_X. Denote by*

$$\Delta = \{([u_1, J_{S_1}, J], \ldots, [u_l, J_{S_l}, J]) \in \mathbb{R}\mathcal{M}^{d_1}(y_1) \times \cdots \times \mathbb{R}\mathcal{M}^{d_l}(y_l) \,|\, \exists i \neq j \text{ for which } u_i(S_i) = u_j(S_j)\}.$$

Then the fibre product $(\Pi_{i=1}^l \mathbb{R}\mathcal{M}^{d_i}(y_i)) \setminus \Delta$ *over* $\mathbb{R}\mathcal{J}_\omega$ *is a separable Banach manifold of class* C^{l-k}*. Moreover, the projection* $(\Pi_{i=1}^l \mathbb{R}\mathcal{M}^{d_i}(y_i)) \setminus \Delta \to \mathbb{R}\mathcal{J}_\omega$ *is Fredholm of index* $c_1(X)d - l - \#y$*, where* $d = \sum_{i=1}^l d_i$ *and* $y = \cup_{i=1}^l y_i$*.* $\square$

Proof of Proposition 2.9:

Let $[u_1, J_{S_1}, J] \in \mathbb{R}\mathcal{M}^{d_1}(y_1)$ and $[u_2, J_{S_2}, J] \in \mathbb{R}\mathcal{M}^{d_2}(y_2)$, so that $\pi^1_{\mathbb{R}}([u_1, J_{S_1}, J]) = \pi^2_{\mathbb{R}}([u_2, J_{S_2}, J]) = J \in \mathbb{R}\mathcal{J}_\omega$. Fix (u_1, J_{S_1}, J) (resp. (u_2, J_{S_2}, J)) a lift of $[u_1, J_{S_1}, J]$ (resp. $[u_2, J_{S_2}, J]$) in $\mathbb{R}\mathcal{P}^*(y_1)$ (resp. $\mathbb{R}\mathcal{P}^*(y_2)$), so that u_1 (resp. u_2) is a J-holomorphic map $(S_1, z_1) \to (X, y_1)$ (resp. $(S_2, z_2) \to (X, y_2)$). Assume that $u_1(S_1) \neq u_2(S_2)$. From Proposition 1.9, $\mathrm{coker}(\pi^i_{\mathbb{R}}|_{[u_i, J_{S_i}, J]}) \cong H^1_D(S_i, N_{u_i, -z_i})_{+1}$. Let $0 \neq \psi_1 \in H^0_{D^*}(S_1, K_{S_1} \otimes_{\mathbb{C}} N^*_{u_1})_{-1} \cong H^1_D(S_1, N_{u_1, -z_1})^*_{+1}$ (see Lemma 1.7). It suffices to prove the existence of $\dot{J} \in \Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{+1}$ such that $< \psi_1, \dot{J} \circ du_1 \circ J_{S_1} > \neq 0$ and $< \psi_2, \dot{J} \circ du_2 \circ J_{S_2} > = 0$ for every $\psi_2 \in H^0_{D^*}(S_2, K_{S_2} \otimes_{\mathbb{C}} N^*_{u_2})_{-1}$. Since $D^*(\psi_1) = 0$ and $\psi_1 \neq 0$, the section ψ_1 vanishes only at a finite number of points outside a neighbourhood of z_1 (see [4]). Since u_1 is not multiple, there exists an open set $U \subset S_1 \setminus z_1$ such that $u_1|_U$ is an embedding, $u_1(U) \cap u_1(S_1 \setminus U) = \emptyset$, $c_X(u_1(U)) \cap u_1(U) = \emptyset$ and such that ψ_1 does not vanish on U. Moreover, since $u_1(S_1) \neq u_2(S_2)$, the intersection $u_1(S_1) \cap u_2(S_2)$ consists only of a finite number of points and thus the open set U can be chosen such that $u_1(U) \cap u_2(S_2) = \emptyset$.

Denote by c_{S_1} the order two element of $\mathcal{D}iff^-(S_1, z_1)$ whose fixed point set in $\mathbb{R}\mathcal{P}^*(y_1)$ contains (u_1, J_{S_1}, J), it is in fact the J_{S_1}-antiholomorphic involution of S_1 induced by u_1 and c_X. Let then α_U be a section of $\Lambda^{0,1}S_1 \otimes E_{u_1}$ with support in U such that $< \psi_1, \alpha_U > \neq 0$. There exists $\dot{J} \in \Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)$, with support in a neighbourhood of $u_1(U)$ in X, such that $\dot{J} \circ du_1 \circ J_{S_1} = \alpha_U$. Denote by $\dot{J}_{\mathbb{R}} = \dot{J} - dc_X \circ \dot{J} \circ dc_X \in \Gamma^l(X, \Lambda^{0,1}X \otimes_{\mathbb{C}} TX)_{+1}$. We have

$$\begin{aligned}
< \psi_1, \dot{J}_{\mathbb{R}} \circ du_1 \circ J_{S_1} > &= < \psi_1, \dot{J} \circ du_1 \circ J_{S_1} > + < \psi_1, \overline{c_X}^*(\dot{J}) \circ du_1 \circ J_{S_1} > \\
&= < \psi_1, \alpha_U > + < \psi_1, dc_X \circ \dot{J} \circ du_1 \circ J_{S_1} \circ dc_{S_1} > \\
&= < \psi_1, \alpha_U > - < (dc_X)^t \circ \psi_1 \circ dc_{S_1}, \alpha_U > \\
&\quad \text{(changing of variables, since } c_{S_1} \text{ reverses the orientation of } S_1) \\
&= 2 < \psi_1, \alpha_U > \neq 0.
\end{aligned}$$

Since the support of $\dot{J}$ is disjoint from $u_2(S_2)$, we have $\dot{J}_{\mathbb{R}} \circ du_2 \circ J_{S_2} = 0$ and thus $< \psi_2, \dot{J}_{\mathbb{R}} \circ du_2 \circ J_{S_2} > = 0$ for every $\psi_2 \in H^0_{D^*}(S_2, K_{S_2} \otimes_{\mathbb{C}} N^*_{u_2})_{-1}$. $\square$

Under the hypothesis of Proposition 2.9, we will denote by $\mathbb{R}\mathcal{M}^{d_1, d_2}(y_1, y_2)$ the fibre product $(\mathbb{R}\mathcal{M}^{d_1}(y_1) \times_{\mathbb{R}\mathcal{J}_\omega} \mathbb{R}\mathcal{M}^{d_2}(y_2)) \setminus \Delta$.

Proposition 2.11 *The subspace of* $\mathbb{R}\mathcal{M}^{d_1, d_2}(y_1, y_2)$ *consisting of couples* $([u_1, J_{S_1}, J], [u_2, J_{S_2}, J]))$ *for which the union* $u_1(S_1) \cup u_2(S_2)$ *does not have only transversal double points as singularities, or has a double point in* $y_1 \cup y_2$*, is a stratum of codimension one.*

Proof:

This subspace consists of couples for which u_1 or u_2 is not an immersion; or u_1 and u_2 are immersions, but $u_1(S_1)$ or $u_2(S_2)$ has some multiple points or tacnode; or $u_1(S_1)$ and $u_2(S_2)$ have only transversal double points as singularities, but the intersection $u_1(S_1) \cap u_2(S_2)$ is

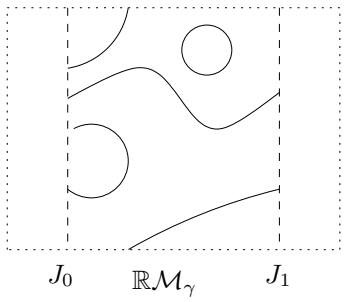
not transverse; or $u_1(S_1) \cup u_2(S_2)$ has only transversal double points as singularities, but with some double point in $y_1 \cup y_2$.

In the two first cases, the result follows from Proposition 2.7. In the two last cases, the proof is analogous to the one of cases 2 and 3 of Proposition 2.7, and mainly follows from Lemma 2.5 with $\nu = 1$. It is left to the reader. $\square$

Remark 2.12 Over a generic path γ of $\mathbb{R}\mathcal{J}_\omega$, elements of $\mathbb{R}\mathcal{M}^d(x)$ cannot degenerate onto a multiple curve or an element of the diagonal Δ of Proposition 2.9. Indeed, the condition for a curve having homology class $\frac{d}{j}$ to pass through $c_1(X)d - 1$ points is of codimension greater than one unless $j = 2$ and $c_1(X)d = 2$. In this case, a generic immersed rational curve having homology class $\frac{d}{2}$ has $\frac{1}{2}\big((\frac{d}{2})^2 + 1\big)$ transversal double points. After perturbation, each of these double points is reponsible for four double points of the doubled irreducible curve. In the case of a multiple curve, the doubled curve should then have at least $2\big((\frac{d}{2})^2 + 1\big) = \frac{d^2}{2} + 2$ transversal double points, which is impossible. In the case of a curve coming from Δ, one of these double points gets smoothed, but still, $\frac{1}{2}d^2 + 1$ is too much. Likewise, when such elements degenerate on a reduced curve given by Proposition 2.11, none of its irreducible components can be multiply covered.

2.3 Proof of Theorem 2.1

Let J_0, J_1 be regular values of the projections $\pi : \mathcal{M}^d(x) \to \mathcal{J}_\omega$ and $\pi_i : \mathcal{M}^{d_i}(x_i) \to \mathcal{J}_\omega$, for every $d_i \in H_2(X;\mathbb{Z})$ realized by a component of a reducible pseudo-holomorphic curve in the class d, and $x_i \subset x$ a real configuration of more than $c_1(X)d_i - 1$ points. Such values exist from Theorem 1.10. Let $\gamma : [0,1] \to \mathbb{R}\mathcal{J}_\omega$ be a path transversal to the projections $\pi_{\mathbb{R}} : \mathbb{R}\mathcal{M}^d(x) \to \mathbb{R}\mathcal{J}_\omega$ and $\pi : (\mathcal{M}^d(x) \setminus \mathbb{R}\mathcal{M}^d(x)) \to \mathcal{J}_\omega$ (see Proposition 1.11), joining J_0 to J_1. Then, $\mathbb{R}\mathcal{M}_\gamma = \pi_{\mathbb{R}}^{-1}(Im(\gamma))$ is a submanifold of dimension one of $\mathbb{R}\mathcal{M}^d(x)$, equipped with a projection $\pi_\gamma : \mathbb{R}\mathcal{M}_\gamma \to [0,1]$ induced by $\pi_{\mathbb{R}}$.



The path γ is chosen such that every element of $\mathbb{R}\mathcal{M}_\gamma$ is a curve with only transversal double points as singularities, with the exception of a finite number of them which may have a unique real ordinary cusp, a unique real triple point or a unique real tacnode. Moreover, this path is chosen such that when a sequence of elements of $\mathbb{R}\mathcal{M}_\gamma$ converges in Gromov topology to a reducible curve of X, then this curve has only two irreducible components, both real, and only ordinary double points as singularities. Finally, this path is chosen such that if $[u, J_S, J] \in \mathbb{R}\mathcal{M}_\gamma$ has a unique real ordinary cusp at the parameter $t_u \in \mathbb{R}S$, then the generator ψ_u of $H^0_{D^*}(S, K_S \otimes_{\mathbb{C}} N_u^*)_{-1}$ does not vanish at t_u. Such a choice of γ is possible from Propositions 2.7, 2.8, 2.9 and Remark 2.12.

Lemma 2.13 *The critical points of π_γ are the curves $[u, J_S, J] \in \mathbb{R}\mathcal{M}_\gamma$ having an ordinary cusp. Moreover, all these critical points are non-degenerated.*

Proof: From Proposition 1.9, at a point $[u, J_S, J] \in \mathbb{R}\mathcal{M}_\gamma$, the cokernel of $d\pi_\gamma$ is isomorphic to $H^1_D(S, N_{u,-z})_{+1}$ and its kernel to $H^0_D(S, \mathcal{N}_{u,-z})_{+1} = H^0_D(S, N_{u,-z})_{+1} \oplus H^0(S, \mathcal{N}^{sing}_{u,-z})_{+1}$. If $[u, J_S, J]$ is a critical point of π_γ, then $\dim H^1_D(S, N_{u,-z})_{+1} = +1$. Since S is rational and D is of generalized $\overline{\partial}$-type, this implies that $H^0_D(S, N_{u,-z})_{+1} = 0$ (see [4]). Since $\mathrm{ind}(d\pi_\gamma) = 0$, we have $\dim H^0(S, \mathcal{N}^{sing}_{u,-z})_{+1} = 1$ and thus u is not an immersion. From the hypothesis made on γ, this implies that u has a real ordinary cusp. Conversely, if $[u, J_S, J] \in \mathbb{R}\mathcal{M}_\gamma$ has a real ordinary cusp, then $\dim H^0(S, \mathcal{N}^{sing}_{u,-z})_{+1} = 1$. Thus, $\ker(d_{[u,J_S,J]}\pi_\gamma) \neq 0$ and since $\mathrm{ind}(d\pi_\gamma) = 0$, $\mathrm{coker}(d_{[u,J_S,J]}\pi_\gamma) \neq 0$, hence the first part of the lemma.

Now let $[u, J_S, J] \in \mathbb{R}\mathcal{M}_\gamma$ be a critical point of π_γ. Fix some lift $(u, J_S, J) \in \mathbb{R}\mathcal{P}^*(x)$ of this element, denote by $c_S \in \mathcal{D}iff^-(S, z)$ the associated element of order two and by $\mathbb{R}S \subset S$ the fixed point set of c_S. Let $t_u \in \mathbb{R}S$ be the point at which du is not injective. We have to prove that the second order differential

$$\nabla|_{[u,J_S,J]} d\pi_\gamma : H^0(S, \mathcal{N}^{sing}_{u,-z})_{+1} \times H^0(S, \mathcal{N}^{sing}_{u,-z})_{+1} \to H^1_D(S, N_{u,-z})_{+1}$$

is non-degenerated. The differential $d\pi_\gamma$ is identified with $(v, \dot{J}_S, \dot{J}) \in T|_{[u,J_S,J]}\mathbb{R}\mathcal{M}^d(x) \mapsto \dot{J} \circ du \circ J_S \in L^{k-1,p}(S, \Lambda^{0,1}S \otimes N_u)_{+1}$. From the relation $Dv + J \circ du \circ \dot{J}_S + \dot{J} \circ du \circ J_S = 0$, we deduce that $\nabla|_{[u,J_S,J]} d\pi_\gamma = -\nabla|_{[u,J_S,J]} \widehat{D}_\mathbb{R}$, where $\widehat{D}_\mathbb{R}(v, \dot{J}_S, \dot{J}) = D_\mathbb{R} v + J \circ du \circ \dot{J}_S$. Now, the result follows from [12], Lemmas 4.3.3, 4.3.4 and 4.3.5. A similar but more direct approach in this setting, as was communicated to me by J.-C. Sikorav, can be the following. Let ψ be a generator of $H^0_{D^*}(S, K_S \otimes_\mathbb{C} N^*_u)_{-1} = H^1_D(S, N_{u,-z})^*_{+1}$ and $(v, \dot{J}_S, 0)$ be a generator of $H^0(S, \mathcal{N}^{sing}_{u,-z})_{+1}$. Without loss of generality, we can assume that $\dot{J}_S$ vanishes in a neighbourhood of the cusp. Then v can be written $du(\tilde{v})$ where $\tilde{v} \in L^{k,p}(S, TS_{-z} \otimes_\mathbb{C} \mathcal{O}(t_u))_{+1}$ (see [12], Lemma 4.3.1), that is $\tilde{v}$ is a meromorphic real vector field of S having a simple pole at t_u and vanishing at $z \subset S$. After differentiation of the relation $D_\mathbb{R} \circ du = du \circ \overline{\partial}_S$, we deduce that $(\nabla_v D_\mathbb{R}) \circ du + D_\mathbb{R}(\nabla_v du) + (\nabla_{\dot{J}_S} D_\mathbb{R}) \circ du = (\nabla_v du) \circ \overline{\partial}_S \mod Im(du)$. Also, $\nabla_{(v,\dot{J}_S)} \widehat{D}_\mathbb{R}(v, \dot{J}_S, 0) = (\nabla_v D_\mathbb{R})(v) + (\nabla_{\dot{J}_S} D_\mathbb{R})(v) + (\nabla_v du) J_S \dot{J}_S \mod Im(du)$. Thus, for $v = du(\tilde{v})$, we get

$$\big(\nabla_{(v,\dot{J}_S)} \widehat{D}_\mathbb{R}\big)(v, \dot{J}_S, 0) + D_\mathbb{R}\big((\nabla_v du)(\tilde{v}), 0, 0\big) = (\nabla_v du)(\overline{\partial}_S \tilde{v} + J_S \dot{J}_S) \mod Im(du).$$

From $0 = Dv + J \circ du \circ \dot{J}_S = \overline{\partial}(du(\tilde{v})) + R \circ du(\tilde{v}) + du(J_S \dot{J}_S) = du(\overline{\partial}_S \tilde{v} + J_S \circ \dot{J}_S)$ (see [7], Lemma 1.3.1), we get

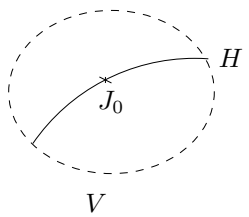
$$\big(\nabla_{(v,\dot{J}_S)} \widehat{D}_\mathbb{R}\big)(v, \dot{J}_S, 0) + D_\mathbb{R}\big((\nabla_v du)(\tilde{v}), 0, 0\big) \in Im(du).$$

Now, since ψ vanishes on $Im(du)$, we obtain:

$$\begin{aligned}
< \psi, \nabla|_{[u,J_S,J]} d\pi_\gamma((v, \dot{J}_S, 0), (v, \dot{J}_S, 0)) > &= - < \psi, \nabla|_{[u,J_S,J]} \widehat{D}_\mathbb{R}((v, \dot{J}_S), (v, \dot{J}_S)) > \\
&= < \psi, D_\mathbb{R}(\nabla_v du)(\tilde{v}) > \\
&= \Re e \int_S \psi \circ D_\mathbb{R}(\nabla_v du)(\tilde{v}) \\
&= \Re e \int_S \psi \circ \overline{\partial}(\nabla_v du)(\tilde{v}) + \Re e \int_S \psi \circ R(\nabla_v du)(\tilde{v}) \\
&= \Re e \lim_{\delta \to 0} \int_{S \setminus B(t_u, \delta)} d(\psi \circ (\nabla_v du)\tilde{v}) + \Re e \int_S D^*(\psi)(\nabla_v du)(\tilde{v}) \\
&= -\Re e \lim_{\delta \to 0} \int_{|\xi - t_u| = \delta} < \psi, (\nabla_v du)(\tilde{v}) > .
\end{aligned}$$

Note that the last sign comes from the fact that the orientation on the circle $|\xi - t_u| = \delta$ induced by $S \setminus B(t_u, \delta)$ is the opposite to the canonical one. Choose some local coordinates in a neighbourhood of $u(t_u) \in X$ so that u is conjugated to $t \mapsto (t^2, t^3) + t^3\epsilon(t) \in \mathbb{C}^2$, where $\epsilon : \overline{B}^2 \to \mathbb{C}^2$ satisfies $\epsilon(0) = 0$ (see, for example, Corollary 1.4.3 of [7]). The normal bundle of u can be identified with the ordinate axis and the connections ∇ with the standard ones, assuming the chosen metric on X is flat near $u(t_u)$. Then, writing $\tilde{v} = \frac{w}{t}$ for $w \in \mathbb{R}$, one has $(\nabla_v du)(\tilde{v}) = d_{\tilde{v}} v = d_{\frac{w}{t}}(2w, 3wt) = (0, 3\frac{w^2}{t})$. We thus finally deduce that the quadratic form $\nabla|_{[u,J_S,J]} d\pi_\gamma : H^0(S, \mathcal{N}^{sing}_{u,-z})_{+1} \times H^0(S, \mathcal{N}^{sing}_{u,-z})_{+1} \to H^1_D(S, N_{u,-z})_{+1}$ is equivalent to $w \in \mathbb{R} \mapsto -3\Re e \operatorname{Res}_{t=0}(\frac{w^2}{t} dt)$, which is non-degenerated. $\square$

Let C_0 be a real J_0-holomorphic curve having two irreducible components C_1 and C_2, only transversal double points as singularities, and which is the limit in Gromov topology of a sequence $[u_{\lambda_n}, J_S^{\lambda_n}, J^{\lambda_n}]$ of elements of $\mathcal{M}_\gamma$, where $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of $]0,1[$ converging to some parameter $\lambda_\infty \in]0,1[$. Denote by $d_1 \in H_2(X;\mathbb{Z})$ (resp. $d_2 \in H_2(X;\mathbb{Z})$) the homology class of C_1 (resp. of C_2) and by $x_1 = x \cap C_1$ (resp. $x_2 = x \cap C_2$), so that $d = d_1 + d_2$ and $x = x_1 \cup x_2$. From Propositions 2.9 and 1.9 we see that, exchanging C_1 and C_2 if necessary, we can assume that $\#(x_1) = c_1(X)d_1 - 1$ and $\#(x_2) = c_1(X)d_2$. Note that both C_1 and C_2 are real. Indeed, C_2 would otherwise coincide with $c_X(C_1)$ and since x_1 and x_2 are disjoint from each other and from the nodes of C_0, this would imply that they have no real points. In particular, $\#(x_1)$ and $\#(x_2)$ would have the same parity. Now $d_2 = -(c_X)_* d_1$ and $c_X^* c_1(X) = -c_1(X)$, so that $c_1(X)d_2 = c_1(X)d_1$. This together with the above expressions of $\#(x_1)$ and $\#(x_2)$ imply that $\#(x_1)$ and $\#(x_2)$ do not have the same parity, hence the contradiction. Let thus $[u_1, J_{S_1}, J_0]$ and $[u_2, J_{S_2}, J_0]$ be elements of $\mathbb{R}\mathcal{M}^{d_1}(x_1)$ and $\mathbb{R}\mathcal{M}^{d_2}(x_2)$ representing C_1 and C_2 respectively. From Proposition 1.9 we know that $\dim H^1_D(S_1, N_{u_1,-z_1})_{+1} \geq 0$ and $\dim H^1_D(S_2, N_{u_2,-z_2})_{+1} \geq 1$. Moreover, since S_1 and S_2 are rational and immersed, this Proposition 1.9 together with Theorem 1′ of [4] ensure that these inequalities are equalities. As a consequence, the projection $\pi^1_{\mathbb{R}} : \mathbb{R}\mathcal{M}^{d_1}(x_1) \to \mathbb{R}\mathcal{J}_\omega$ restricts in a neighbourhood of $[u_1, J_{S_1}, J_0]$ to a submersion onto a neighbourhood V_1 of J_0 in $\mathbb{R}\mathcal{J}_\omega$. Similarly, in a neighbourhood of $[u_2, J_{S_2}, J_0]$, the projection $\pi^2_{\mathbb{R}} : \mathbb{R}\mathcal{M}^{d_2}(x_2) \to \mathbb{R}\mathcal{J}_\omega$ maps onto a codimension one submanifold of a neighbourhood V_2 of J_0 in $\mathbb{R}\mathcal{J}_\omega$. Denote by $V = V_1 \cap V_2$ and by $H \subset V$ this codimension one submanifold. We can assume that V is connected and that $V \setminus H$ has two connected components.



Denote by $\overline{\mathcal{M}}^d(x)$ (resp. $\mathbb{R}\overline{\mathcal{M}}^d(x)$) the Gromov compactification of $\mathcal{M}^d(x)$ (resp. $\mathbb{R}\mathcal{M}^d(x)$), and by $\overline{\pi}$ (resp. $\overline{\pi}_{\mathbb{R}}$) the projection $\overline{\mathcal{M}}^d(x) \to \mathcal{J}_\omega$ (resp. $\mathbb{R}\overline{\mathcal{M}}^d(x) \to \mathbb{R}\mathcal{J}_\omega$). Restricting V if necessary, we can assume that there exists a neighbourhood W of C_0 in $\mathbb{R}\overline{\mathcal{M}}^d(x)$ such that $\overline{\pi}(W) = V$ and such that the image of reducible curves of W under this projection is exactly $H \subset V$. Finally, note that restricting W if necessary, we can assume that every irreducible J-holomorphic curve of W is topologically obtained from C_0 after smoothing one of the real intersection points of $C_1 \cap C_2$.

Proposition 2.14 *Let C_0 be a real reducible J_0-holomorphic curve of X passing through x and which is the limit of a sequence of elements of $\mathbb{R}\mathcal{M}_\gamma$. Let $J_0 = \gamma(\lambda_0)$ for $\lambda_0 \in]0,1[$ and C_1, C_2 be the two irreducible components of C_0. Let R be the number of real intersection points between C_1 and C_2. Then there exist a neighbourhood W of C_0 in the Gromov compactification $\mathbb{R}\overline{\mathcal{M}}^d(x)$ and $\eta > 0$ such that for every $\lambda \in]\lambda_0-\eta, \lambda_0+\eta[\setminus\{\lambda_0\}$, $\pi_\gamma^{-1}(\lambda)\cap W$ consists of exactly R real $\gamma(\lambda)$-holomorphic curves, each of them obtained topologically by smoothing a different real intersection point of $C_1 \cap C_2$.*

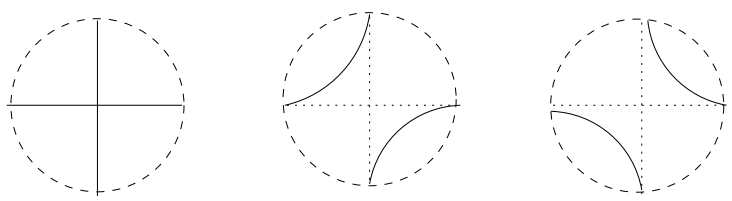
Proof:

We define as above a neighbourhood W of C_0 in $\mathbb{R}\overline{\mathcal{M}}^d(x)$, a neighbourhood V of J_0 in $\mathbb{R}\mathcal{J}_\omega$ and a submanifold H of codimension one of V, such that $\overline{\pi}_{\mathbb{R}}(W) = V$ and H coincide with the image under $\overline{\pi}_{\mathbb{R}}$ of the reducible curves of W. Let $\eta > 0$ be small enough, we will first prove that for $\lambda \in]\lambda_0 - \eta, \lambda_0 + \eta[\setminus\{\lambda_0\}$, $\pi_\gamma^{-1}(\lambda) \cap W$ contains at most one curve for each real intersection point of $C_1 \cap C_2$. Otherwise, let C' and C'' be two such curves associated to a same intersection point of $C_1 \cap C_2$ denoted by $y_0 \in X$. The irreducible curves C' and C'' intersect at a finite number of points, each local intersection being of positive multiplicity. If W has been chosen small enough, these curves have a double point in a neighbourhood of each double point of C_0 except y_0. In particular, in a neighbourhood of each such double point, these curves intersect each other in at least two points. This number of double points is $\frac{1}{2}(d^2 - c_1(X)d + 2)$ from adjunction formula. Moreover, since both C' and C'' pass through the configuration of points x, they intersect each other at each point of x with multiplicity at least one. Thus, one has

$$d^2 = C' \circ C'' \geq (d^2 - c_1(X)d + 2) + c_1(X)d - 1 = d^2 + 1,$$

which is impossible.

Let us now prove that $\pi_\gamma^{-1}(\lambda)\cap W$ actually contains exactly one curve for each intersection point of $C_1 \cap C_2$. Let $y \in X$ be such an intersection point. In a neighbourhood of y, the curve C_0 is biholomorphic to the standard real node $\mathcal{A}_0 = \{(z^+, z^-) \in B^2 \,|\, z^+z^- = 0\}$. Consider the cylinders $\mathcal{A}_\varphi = \{(z^+, z^-) \in B^2 \,|\, z^+z^- = \varphi\}$, for $\varphi \in B^2(\epsilon)$, $\epsilon > 0$. These cylinders form a partition of the real analytic space $\mathcal{A} = \{(z^+, z^-) \in B^2 \,|\, |z^+z^-| < \epsilon\}$. Note that when the parameter $\varphi \in B^2(\epsilon) \setminus \{0\}$ is real, the cylinders $\mathcal{A}_\varphi$ realize for $\varphi > 0$ and $\varphi < 0$ the two topological smoothings of the real node $\mathcal{A}_0$.



$\mathcal{A}_0$ $\qquad$ $\mathcal{A}_\varphi$, $\varphi < 0$ $\qquad$ $\mathcal{A}_\varphi$, $\varphi > 0$

From Theorem 5.4.1 of [12] (The map Φ given in this theorem is $\mathbb{Z}/2\mathbb{Z}$-equivariant for the real structures induced on $\mathcal{U} \times \Delta(\epsilon')$ and $\mathcal{P}(\mathcal{A})$), the real embedding of $\mathcal{A}_0$ in X given by C_0 deforms into a one parameter family of real embeddings of the cylinders $\mathcal{A}_\varphi$, for $\varphi \in]-\epsilon, \epsilon[$. There exists then a continuous family $(J_\varphi)_{\varphi\in]-\epsilon,\epsilon[}$ in $\mathbb{R}\mathcal{J}_\omega$ extending J_0, such that for every $\varphi \in]-\epsilon, \epsilon[\setminus\{0\}$, J_φ differs from J_0 only in a neighbourhood of $\partial\mathcal{A}_0 \subset X$, and a continuous family $(C_\varphi)_{\varphi\in]-\epsilon,\epsilon[}$ of real J_φ-holomorphic curves extending C_0, such that for $\varphi \neq 0$, C_φ is obtained topologically from C_0 by smoothing the real node y. Indeed, restricting a little bit

the real embedding of the cylinder $\mathcal{A}_\varphi$, the image of this embedding can be glued to the curve $C_0 \setminus \mathcal{A}_0$ by adding two small real annuli embedded in a neighbourhood of $\partial\mathcal{A}_0$. The real curve C_φ we thus obtain can be easily made J_φ-holomorphic for some almost-complex structure $J_\varphi \in \mathbb{R}\mathcal{J}_\omega$ close to J_0 and differing from the latter only in a neighbourhood of $\partial\mathcal{A}_0$.

Let us fix now $\varphi_+ \in]0, \epsilon[$ and $\varphi_- \in]-\epsilon, 0[$ such that $J_{\varphi_+}, J_{\varphi_-} \in V \setminus H$ and $C_{\varphi_+}, C_{\varphi_-} \in W$. Deforming locally $C_{\varphi_+}, C_{\varphi_-}$ if necessary, we can assume that $J_{\varphi_+}, J_{\varphi_-}$ are regular values of $\pi_{\mathbb{R}}$. The almost-complex structures J_{φ_+} and J_{φ_-} do not belong to the same connected component of $V \setminus H$. Indeed, it would otherwise be possible to join them by a path of $V \setminus H$ transversal to the projection $\pi_{\mathbb{R}}$, and there would then be no obstruction to isotop C_{φ_-} along this path into a continuous family of W to end up with a J_{φ_+}-holomorphic curve denoted by C'_{φ_-}. The lack of such an obstruction follows from the fact that C_{φ_-} can neither degenerate onto a reducible curve nor onto a cuspidal curve along this path. Since these curves are rational and immersed, they are all regular points of the projection (see Proposition 1.9 and [4]). This provides the contradiction since C_{φ_+} and C'_{φ_-} are two J_{φ_+}-holomorphic curves in W obtained topologically by smoothing the same node of C_0, which is impossible from the computation done at the beginning of this proof. Now the result follows similarly. Let $\lambda_+ \in]0, \epsilon[$ and $\lambda_- \in]-\epsilon, 0[$, the almost-complex structures $\gamma(\lambda_+)$ and $\gamma(\lambda_-)$ are not in the same component of $V \setminus H$. Each of them can be joined to J_{φ_+} or J_{φ_-} by a path of $V \setminus H$ transversal to the projection $\pi_{\mathbb{R}}$. There is then no obstruction to isotop C_{φ_-} or C_{φ_+} along these paths to get a $\gamma(\lambda_+)$ or $\gamma(\lambda_-)$-holomorphic curve which is obtained topologically by smoothing the real node y of C_0. Hence the result. $\square$

Remark 2.15 Note that there is actually no need of Theorem 5.4.1 of [12] to deduce the local gluing argument we used in the proof of Proposition 2.14. Indeed, this local gluing argument is obvious if the almost complex structure is integrable in a neighbourhood of the node y. Now, since the space of almost complex structures $J \in \mathbb{R}\mathcal{J}_\omega$ for which C_0 is J-holomorphic is contractible, it can be deduced in general from the isotopy argument we used at the end of this proof of Proposition 2.14.

Proposition 2.16 *Let $C_{\lambda_0} \in \mathbb{R}\mathcal{M}_\gamma$ be a critical point of π_γ which is a local maximum (resp. minimum). Then there exist a neighbourhood W of C_{λ_0} in $\mathbb{R}\mathcal{M}_\gamma$ and $\eta > 0$ such that for every $\lambda \in]\lambda_0 - \eta, \lambda_0[$ (resp. for every $\lambda \in]\lambda_0, \lambda_0 + \eta[$), $\pi_\gamma^{-1}(\lambda) \cap W$ consists of two curves C_λ^+ and C_λ^- satisfying $m(C_\lambda^+) = m(C_\lambda^-) + 1$, and for every $\lambda \in]\lambda_0, \lambda_0 + \eta[$ (resp. for every $\lambda \in]\lambda_0 - \eta, \lambda_0[$), $\pi_\gamma^{-1}(\lambda) \cap W = \emptyset$.*

Proof:

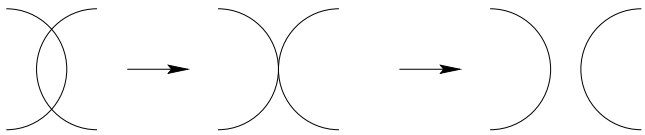
Let us assume that C_{λ_0} is a local maximum of π_γ, and let us denote C_{λ_0} by $[u_{\lambda_0}, J_S^{\lambda_0}, J_{\lambda_0}]$. Since $\mathbb{R}\mathcal{M}_\gamma$ is one dimensional and C_{λ_0} is a non-degenerated critical point (see Lemma 2.13), it is clear that there exists $\eta > 0$ such that in a neighbourhood W of $[u_{\lambda_0}, J_S^{\lambda_0}, J_{\lambda_0}]$, $\pi_\gamma^{-1}(\lambda) \cap W$ consists of two curves if $\lambda \in]\lambda_0 - \eta, \lambda_0[$, and $\pi_\gamma^{-1}(\lambda) \cap W = \emptyset$ if $\lambda \in]\lambda_0, \lambda_0 + \eta[$. The only thing to prove is that if η is small enough, the two curves C_λ^+ and C_λ^- of $\pi_\gamma^{-1}(\lambda) \cap W$ satisfy $m(C_\lambda^+) = m(C_\lambda^-) + 1$. From the choice of γ we know that the only singularities of C_{λ_0} are transversal double points and a unique real ordinary cusp. If η is small enough, the two curves C_λ^+ and C_λ^- are close enough to C_{λ_0} so that they have a double point in a neighbourhood of each double point of C_{λ_0} plus a double point in a neighbourhood of the cusp of C_{λ_0}. Since the double points which are close to double points of C_{λ_0} are of the same nature, we have to prove that for one of the curves C_λ^+ or C_λ^-, the real double point close to the cusp of C_{λ_0}

is non-isolated, and for the other one, it is isolated. The result indeed follows then from the definition of the mass.

So let us fix a parameterization $\mu \in]-\epsilon, \epsilon[\mapsto C_\mu \in \mathbb{R}\mathcal{M}_\gamma \cap W$, such that $C_0 = C_{\lambda_0}$ and $\pi_\gamma(C_\mu) = \pi_\gamma(C_{-\mu})$. Considering the restriction of these curves to a small neighbourhood of the cusp of C_0 diffeomorphic to the ball B^4 of $\mathbb{C}^2$, we deduce, with the notations of Lemma 2.6, a path $(C'_\mu)_{\mu \in]-\epsilon,\epsilon[}$ of $\mathbb{R}\mathcal{P}'$ such that $C'_0 \in \mathbb{R}\mathcal{P}'_s$. We have to prove that this path is transversal to $\mathbb{R}\mathcal{P}'_s$ at $\mu = 0$. From the hypothesis, we know that $v_{\lambda_0} = \frac{d}{d\mu}(C_\mu)|_{\mu=0} \in H^0_D(S, \mathcal{N}^{sing}_{u,-z})_{+1}$. Denoting by $t_0 \in \mathbb{R}S$ the point at which du_{λ_0} is not injective, we deduce that $v_{\lambda_0} = du_{\lambda_0}(w_{\lambda_0})$ for a section w_{λ_0} of class $L^{k-1,p}$ of the bundle $TS_{-z} \otimes_{\mathbb{C}} \mathcal{O}(t_0)$, that is for a vector field of S vanishing at z and with a simple pole at t_0 (see Lemma 4.3.1 of [12]). Restricting ourselves to the ball B^4 defined above, we can write $w_{\lambda_0} = \frac{1}{t} w'_{\lambda_0}$. Moreover, from Corollary 1.4.3 of [7], the diffeomorphism onto this ball can be chosen in order that u_{λ_0} writes $t \mapsto (t^2, t^3) + t^3\epsilon(t)$ with $\epsilon \in L^{k,p}(\overline{B}^2, \mathbb{C}^2)$ and $\epsilon(0) = 0$. Thus $v_{\lambda_0} = (2w'_{\lambda_0}, 3tw'_{\lambda_0}) + o(t)$ and $\nabla|_{t=0} v_{\lambda_0} = (2\nabla|_{t=0} w'_{\lambda_0}, 3w'_{\lambda_0} \frac{d}{dt})$. Hence $Im(\nabla|_{t=0} v_{\lambda_0})$ is not the tangent of C_{λ_0} at the cusp and the transversality condition of Lemma 2.6 is satisfied, which proves the result. $\square$

Proof of Theorem 2.1:

Let $J_0, J_1 \in \mathbb{R}\mathcal{J}_\omega$ be two regular values of the projection $\pi : \mathcal{M}^d(x) \to \mathcal{J}_\omega$, and $\gamma : [0,1] \to \mathbb{R}\mathcal{J}_\omega$ be the path fixed at the beginning of §2.3 joining them. The integer $\chi^d_r(x, \gamma(\lambda))$ is then well defined for every $\lambda \in [0,1]$ but a finite number of values $0 < \lambda_1 < \cdots < \lambda_j < 1$ corresponding either to reducible curves, to cuspidal curves, or to curves having a real triple point or tacnode. Since the function $\lambda \mapsto \chi^d_r(x, \gamma(\lambda))$ is obviously constant between these values, we just have to prove that for $i \in \{1, \ldots, j\}$, $\chi^d_r(x, \gamma(\lambda_i^-)) = \chi^d_r(x, \gamma(\lambda_i^+))$ where λ_i^- (resp. λ_i^+) is the left limit (resp. right limit) of λ at λ_i. If λ_i corresponds to a curve having a real triple point or tacnode, it is straightforward and illustrated by the following pictures.



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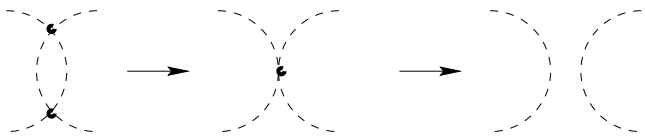
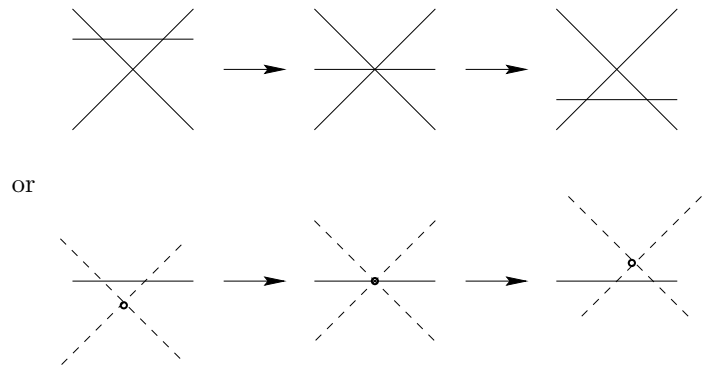


Passing through a real curve with tacnode



Passing through a curve with a real triple point

If λ_i corresponds to a reducible curve, it follows from Proposition 2.14, and if λ_i corresponds to a cuspidal curve, it follows from Proposition 2.16. Hence, the integer $\chi_r^d(x, J)$ does not depend on the choice of $J \in \mathbb{R}\mathcal{J}_\omega$. Note that in contrast to the previous cases, the coefficient $(-1)^m$ in the definition of $\chi_r^d(x, J)$ plays in this last case a crucial rôle to get the invariance. This integer $\chi_r^d(x, J)$ also does not depend on the choice of x, since the group of equivariant symplectomorphisms of X acts transitively on the real configurations of points as long as they have the same number of real points in each connected component of $\mathbb{R}X$. Theorem 2.1 is thus proved. $\square$

3 Further study of the generating function $\chi^d(T)$

The aim of the first three sub-paragraphs is to provide relations between the coefficients of the generating function $\chi^d(T)$ in term of a new invariant θ. In the last sub-paragraph, we will prove the non-triviality of this polynomial function in degrees 4 and 5 for $(\mathbb{C}P^2, \omega_{std}, \text{conj})$ (for degree less than four, it has already been computed in the example given in §2.1). We will assume throughout this paragraph that $\mathbb{R}X$ is connected, for convenience.

3.1 The invariant θ

Let $y = (y_1, \ldots, y_{c_1(X)d-2})$ be a real configuration of $c_1(X)d - 2$ distinct points of X, and s be the number of those which are real. We assume that $y_{c_1(X)d-2}$ is real, so that s does not vanish. Let $J \in \mathbb{R}\mathcal{J}_\omega$ be generic enough. Then there are only finitely many rational J-holomorphic curves in X which realize the homology class d, pass through y and have a transversal double point at $y_{c_1(X)d-2}$. These curves are all irreducible and have only transversal double points as singularities. For every integer m ranging from 0 to δ, denote by $\hat{n}_d^+(m)$ (resp. $\hat{n}_d^-(m)$) the total number of these curves which are real, of mass m and with a non-isolated (resp. isolated) real double point at $y_{c_1(X)d-2}$. Define then:

$$\theta_s^d(y, J) = \sum_{m=0}^{\delta} (-1)^m (\hat{n}_d^+(m) - \hat{n}_d^-(m)).$$

Theorem 3.1 *Let (X, ω, c_X) be a real symplectic 4-manifold such that $\mathbb{R}X$ is connected, and $d \in H_2(X; \mathbb{Z})$ such that $c_1(X)d > 1$. Let $y \subset X$ be a real configuration of $c_1(X)d - 2$ distinct points and $s \neq 0$ be the cardinality of $y \cap \mathbb{R}X$. Finally, let $J \in \mathbb{R}\mathcal{J}_\omega$ be an almost*

complex structure generic enough, so that the integer $\theta_s^d(y,J)$ is well defined. Then, this integer $\theta_s^d(y,J)$ neither depends on the choice of J nor on the choice of y.

For convenience, this integer $\theta_s^d(y,J)$ will be denoted by θ_s^d, and we set $\theta_s^d = 0$ when s does not have the same parity as $c_1(X)d$. This invariant makes it possible to give relations between the coefficients of the polynomial χ^d, namely:

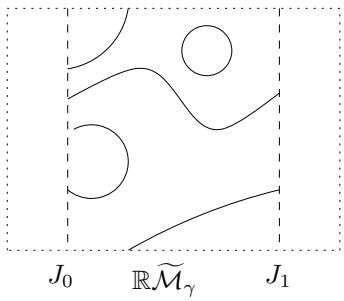
Theorem 3.2 *Let (X,ω,c_X) be a real symplectic 4-manifold such that $\mathbb{R}X$ is connected, $d \in H_2(X;\mathbb{Z})$ and r be an integer between 0 and $c_1(X)d-3$. Then $\chi_{r+2}^d = \chi_r^d + 2\theta_{r+1}^d$.*

Remark 3.3 When $\mathbb{R}X$ is not connected, then as before s should be an n-tuple encodding the number of points of y in each connected components of $\mathbb{R}X$. Under this modification, Theorem 3.1 and the analog of Theorem 3.2 are still valid, the proofs being the same.

3.2 Proof of Theorem 3.1

To begin with, we construct as in §1 the moduli space $\mathcal{M}^d(y)$ of real rational pseudo-holomorphic maps $u : S \to X$ realizing the homology class d, mapping the marked points $z_1,\dots,z_{c_1(X)d-2}$ of S to the corresponding points $y_1,\dots,y_{c_1(X)d-2}$ of X and mapping $z_{c_1(X)d-1}$ also to $y_{c_1(X)d-2}$. This moduli space is obtained by taking the quotient of the space of such maps by the group $\mathcal{D}iff^+(S,z)$ acting by reparameterization. Since now $u(z_{c_1(X)d-2}) = u(z_{c_1(X)d-1})$, there is a degree two extension of this group acting by reparameterization, namely the group of orientation preserving diffeomorphisms of S which fix the points $z_1,\dots,z_{c_1(X)d-3}$, and fix or exchange the points $z_{c_1(X)d-2}$ and $z_{c_1(X)d-1}$. This degree two extension induces a $\mathbb{Z}/2\mathbb{Z}$-action on $\mathcal{M}^d(y)$ which has no fixed point, since its effect is to exchange the two local branches at the double point $y_{c_1(X)d-2}$. Denote by $\widetilde{\mathcal{M}}^d(y)$ the orbit space of this action. It is a separable Banach manifold of class C^{l-k} equipped with an index zero Fredholm projection $\tilde{\pi}$ on $\mathcal{J}_\omega$. Denote by $\tilde{\pi}_{\mathbb{R}} : \mathbb{R}\widetilde{\mathcal{M}}^d(y) \to \mathbb{R}\mathcal{J}_\omega$ the restriction of $\tilde{\pi}$. The Theorem of regular values 1.10 applies also in this situation, so that the set of regular values of $\tilde{\pi}$ intersects $\mathbb{R}\mathcal{J}_\omega$ in a dense set of the second category.

Let then $J_0, J_1 \in \mathbb{R}\mathcal{J}_\omega$ be regular values of the projection $\tilde{\pi} : \widetilde{\mathcal{M}}^d(y) \to \mathcal{J}_\omega$ such that no reducible J_0 or J_1-holomorphic curve in the class d passes through y with a double point at $y_{c_1(X)d-2}$. Let $\gamma : [0,1] \to \mathbb{R}\mathcal{J}_\omega$ be a path transversal to the projection $\tilde{\pi}_{\mathbb{R}} : \mathbb{R}\widetilde{\mathcal{M}}^d(y) \to \mathbb{R}\mathcal{J}_\omega$, joining J_0 to J_1. Hence, $\mathbb{R}\widetilde{\mathcal{M}}_\gamma = \tilde{\pi}_{\mathbb{R}}^{-1}(Im(\gamma))$ is a submanifold of dimension one of $\mathbb{R}\widetilde{\mathcal{M}}^d(y)$, equipped with a projection $\tilde{\pi}_\gamma : \mathbb{R}\widetilde{\mathcal{M}}_\gamma \to [0,1]$ induced by $\tilde{\pi}_{\mathbb{R}}$.



The path γ is chosen such that every element of $\mathbb{R}\widetilde{\mathcal{M}}_\gamma$ is a curve with only transversal double points as singularities, with the exception of a finite number of them which may have a unique real ordinary cusp, a unique real triple point or a unique real tacnode. This path is

also chosen such that when a sequence of elements of $\mathbb{R}\widetilde{\mathcal{M}}_\gamma$ converges in Gromov topology to a reducible curve of X, then this curve has only two irreducible components, both real, and only transversal double points as singularities. Moreover, this path is chosen such that if $[u, J_S, J] \in \mathbb{R}\widetilde{\mathcal{M}}_\gamma$ has a unique real ordinary cusp at the parameter $t_u \in \mathbb{R}S$, then the generator ψ_u of $H^0_{D^*}(S, K_S \otimes_{\mathbb{C}} N_u^*)_{-1}$ does not vanish at t_u. Finally, it is chosen such that when a sequence of elements of $\mathbb{R}\widetilde{\mathcal{M}}_\gamma$ converges in Gromov topology to an irreducible curve of X not in $\mathbb{R}\widetilde{\mathcal{M}}_\gamma$, thus a cuspidal curve, then this curve has a real ordinary cusp at $y_{c_1(X)d-2}$, and only transversal double points as remaining singularities. Such a choice of γ is possible from Propositions 2.7, 2.8 and 2.9. The integer $\theta^d_s(y, \gamma(\lambda))$ is then well defined for every $\lambda \in [0, 1]$ but a finite number of values $0 < \lambda_1 < \cdots < \lambda_j < 1$ corresponding either to reducible curves, to cuspidal curves, or to curves having a real triple point or tacnode. Since the function $\lambda \mapsto \theta^d_s(y, \gamma(\lambda))$ is obviously constant between these values, we just have to prove that for $i \in \{1, \ldots, j\}$, $\theta^d_s(y, \gamma(\lambda_i^-)) = \theta^d_s(y, \gamma(\lambda_i^+))$ where λ_i^- (resp. λ_i^+) is the left limit (resp. right limit) of λ at λ_i. The only cases to consider is the appearance of a cuspidal curve, the cusp being at $y_{c_1(X)d-2}$, or the appearance of a curve with a tacnode, the tacnode being at $y_{c_1(X)d-2}$. Indeed, all the other cases follow along the same lines as in the proof of Theorem 2.1, the only additional thing to remark is that the topology of the node at $y_{c_1(X)d-2}$ does not change under these moves. We will only consider the case of a cuspidal curve, since the other one can be treated exactly in the same way.

So, let $(C_\lambda)_{\lambda \in]\lambda_i - \epsilon, \lambda_i[}$ be a smooth family of $\gamma(\lambda)$-holomorphic curves in $\mathbb{R}\widetilde{\mathcal{M}}_\gamma$ which converges in Gromov topology to a real irreducible cuspidal $\gamma(\lambda_i)$-holomorphic curve, the cusp being at $y_{c_1(X)d-2}$. All these curves have only transversal double points as singularities as soon as ϵ is small enough. Moreover, such a family is unique. Indeed, if for $\lambda \in]\lambda_i - \epsilon, \lambda_i[$, there were two $\gamma(\lambda)$-holomorphic curves C' and C'' close to the cuspidal curve, then they would have two intersection points in the neighbourhood of each double point of the cuspidal curve, plus four intersection points at $y_{c_1(X)d-2}$ and moreover, they would intersect each other at each point of y. This would give

$$(c_1(X)d - 3) + 4 + 2(\frac{1}{2}(d^2 - c_1(X)d + 2) - 1) = d^2 + 1$$

intersection points, which is to much since all multiplicities are positive. In particular, this family is made of real curves, and since the parity of the number of real curves does not change, this family does extend to a smooth family $(C_\lambda)_{\lambda \in]\lambda_i - \epsilon, \lambda_i + \epsilon[}$ of real $\gamma(\lambda)$-holomorphic curves. Then, after the transformation, either the topology of the real double point at $y_{c_1(X)d-2}$ is unchanged and then the mass of the curve is also unchanged, or it has changed, but then the mass of the curve also has changed. In both cases, the integer $\theta^d_s(y, \gamma(\lambda))$ is left invariant, hence the result. $\square$

3.3 Proof of Theorem 3.2

Let $y = (y_1, \ldots, y_{c_1(X)d-2})$ be a real configuration of distinct points of X, such that $y_{c_1(X)d-2} \in \mathbb{R}X$ and $\#(y \cap \mathbb{R}X) = r + 1$. Denote by $\mathcal{M}^d(y)$ the moduli space of rational pseudo-holomorphic curves of X which pass through y and realize the homology class d. Similarly, denote by $\widetilde{\mathcal{M}}^d(y)$ the moduli space of such curves which have a transversal double point at $y_{c_1(X)d-2}$. This space has been introduced in §3.2. Denote by $P(T_{y_{c_1(X)d-2}}X)$ the space of tangent lines of X at $y_{c_1(X)d-2}$. Then the projection $[u, J_S, J] \in \mathcal{M}^d(y) \mapsto (J, d|_{z_{c_1(X)d-2}} u(T_{z_{c_1(X)d-2}}S)) \in \mathcal{J}_\omega \times P(T_{y_{c_1(X)d-2}}X)$ is Fredholm of vanishing index. Let

$(J,\tau) \in \mathbb{R}\mathcal{J}_\omega \times P(T_{y_{c_1(X)d-2}}\mathbb{R}X)$ be a regular value of this projection. We also assume that J is a regular value of the projection $\widetilde{\mathcal{M}}^d(y) \to \mathcal{J}_\omega$ and that there exists no reducible or cuspidal rational J-holomorphic curve which pass through y, realize the homology class d and have a double point or τ as a tangent line at $y_{c_1(X)d-2}$. There are then only finitely many elements of $\mathbb{R}\mathcal{M}^d(y)$ having τ as a tangent line at $y_{c_1(X)d-2}$. These curves are all irreducible and have only transversal double points as singularities. For every integer m between 0 and δ, denote by $\tilde{n}_d(m)$ the number of such curves which are real and of mass m. Denote then by:

$$\widetilde{\chi}_r^d(y,J) = \sum_{m=0}^{\delta} (-1)^m \tilde{n}_d(m).$$

Proposition 3.4 *Under the above assumptions, we have the relations:*

$$\chi_{r+2}^d = \widetilde{\chi}_r^d(y,J) + 2\sum_{m=0}^{\delta} (-1)^m \hat{n}_d^+(m),$$

$$\chi_r^d = \widetilde{\chi}_r^d(y,J) + 2\sum_{m=0}^{\delta} (-1)^m \hat{n}_d^-(m).$$

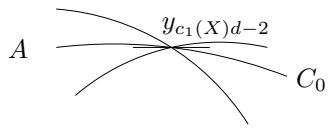
The integers $\hat{n}_d^+(m)$ and $\hat{n}_d^-(m)$ have been defined in §3.1. The Theorem 3.2 follows easily from this Proposition 3.4 and the definition of the invariant θ.

Proof of Proposition 3.4:

To begin with, let us prove the first relation. For this purpose, let us fix a path $\mu :]-\epsilon,\epsilon[\to \mathbb{R}X$ of class C^2 such that $\mu(0) = y_{c_1(X)d-2}$ and $\mu'(0) \in \tau$. For every $\lambda \in]-\epsilon,\epsilon[\setminus\{0\}$, denote by y_λ the set $(y_1,\ldots,y_{c_1(X)d-2},\mu(\lambda))$. Denote then by $\mathbb{R}\mathcal{M}^d(y_\lambda)$ the moduli space of real rational pseudo-holomorphic curves of X which pass through y_λ and realize the homology class d. Then J is a regular value of the projection $\pi_{\mathbb{R}}^\lambda : \mathbb{R}\mathcal{M}^d(y_\lambda) \to \mathbb{R}\mathcal{J}_\omega$ as soon as λ is close enough to zero. Indeed, from Gromov compactness theorem, as soon as λ is close enough to zero, the elements of $\mathbb{R}\mathcal{M}^d(y_\lambda)$ are close, in Gromov topology, either to elements of $\mathbb{R}\widetilde{\mathcal{M}}^d(y)$ having a non-isolated real double point at $y_{c_1(X)d-2}$, or to elements of $\mathbb{R}\mathcal{M}^d(y)$ having τ as a tangent line at $y_{c_1(X)d-2}$. As a consequence, these curves are neither cuspidal, nor reducible, and thus J is a regular value of $\pi_{\mathbb{R}}^\lambda$ from Proposition 1.9. The set $\{(\pi_{\mathbb{R}}^\lambda)^{-1}(J), \lambda \in]-\epsilon,0[\}$ is thus the union of the images of finitely many continuous functions $C_1(\lambda),\ldots,C_j(\lambda)$. Each of these functions converges as λ goes to zero either to an irreducible real J-holomorphic curve having a non-isolated real double point at $y_{c_1(X)d-2}$, or to an irreducible curve having τ as a tangent line at $y_{c_1(X)d-2}$. We will prove that each curve of the first kind (resp. second kind) is the limit of exactly two (resp. one) such functions $C_{i_1}(\lambda), C_{i_2}(\lambda)$. The first relation of Proposition 3.4 follows, since $\chi_{r+2}^d = \chi_{r+2}^d(y_\lambda,J)$ for λ close enough to zero, and since the masses of the curves are unchanged while passing to the limit $\lambda \to 0$.

Let then C_0 be an element of $\mathbb{R}\mathcal{M}^d(y)$ having τ as a tangent line at $y_{c_1(X)d-2}$. Since by hypothesis, (J,τ) is a regular value of the projection $\mathcal{M}^d(y) \to \mathcal{J}_\omega \times P(T_{y_{c_1(X)d-2}}X)$, the J-holomorphic curves in a neighbourhood of C_0 in $\mathbb{R}\mathcal{M}^d(y)$ are exactly parameterized by their tangent lines at $y_{c_1(X)d-2}$. Once we move this tangent line, we see that these curves provide

a partition of an angular neighbourhood A of $y_{c_1(X)d-2}$ in $\mathbb{R}X$.



Since the path $\mu :]-\epsilon,\epsilon[\to \mathbb{R}X$ satisfies $\mu(0) = y_{c_1(X)d-2}$ and $\mu'(0) \in \tau$, restricting ϵ if necessary, we can assume that its image is completely included in A. For every $\lambda \in]-\epsilon,\epsilon[\setminus\{0\}$, there exists thus one and only one J-holomorphic curve in a neighbourhood of C_0, which passes through y_λ, which was the announced result.

Now let C_0 be an element of $\mathbb{R}\widetilde{\mathcal{M}}^d(y)$. This element lifts into two elements C_1 and C_2 of the moduli space $\mathbb{R}\mathcal{M}^d(\overline{y})$, where $\overline{y} = (y_1, \ldots, y_{c_1(X)d-2}, y_{c_1(X)d-2})$, see the beginning of §3.2. Denote by $\mathbb{R}\widehat{\mathcal{M}}^d(y)$ the moduli space of real rational pseudo-holomorphic maps which have $c_1(X)d-1$ distinct marked points $z_1, \ldots, z_{c_1(X)d-1}$ at the source, realize the homology class d and such that $u(z_i) = y_i$ for $1 \le i \le c_1(X)d-2$. The map $\hat{\pi}_{\mathbb{R}} : [u, J_S, J] \in \mathbb{R}\widehat{\mathcal{M}}^d(y) \mapsto (J, u(z_{c_1(X)d-1})) \in \mathbb{R}\mathcal{J}_\omega \times \mathbb{R}X$ is Fredholm of index zero. The value $(J, y_{c_1(X)d-2})$ is regular for this projection. Thus the curves C_1 and C_2 in $(\hat{\pi}_{\mathbb{R}})^{-1}(J, y_{c_1(X)d-2})$ extend in a unique way into two families $C_1(\lambda)$ and $C_2(\lambda)$ of $(\hat{\pi}_{\mathbb{R}})^{-1}(J, \mu(\lambda)) = (\pi^\lambda_{\mathbb{R}})^{-1}(J)$. These are the two families we were looking for.

The second relation of Proposition 3.4 can be proved in a similar way. We choose this time a path $\mu :]-\epsilon,\epsilon[\to X$ of class C^2 such that $\mu(0) = y_{c_1(X)d-2}$, $\mu'(0) \in \tau$, and for every $\lambda \in]-\epsilon,\epsilon[$, $c_X(\mu(\lambda)) = \mu(-\lambda)$. For every $\lambda \in]-\epsilon,\epsilon[\setminus\{0\}$, denote by y_λ the set $(y_1, \ldots, y_{c_1(X)d-3}, \mu(\lambda), \mu(-\lambda))$, and by $\mathbb{R}\mathcal{M}^d(y_\lambda)$ the corresponding moduli space. Now, from Gromov compactness theorem, as soon as λ is close enough to zero, the elements of $\mathbb{R}\mathcal{M}^d(y_\lambda)$ are close, in Gromov topology, either to elements of $\mathbb{R}\widetilde{\mathcal{M}}^d(y)$ having a real isolated double point at $y_{c_1(X)d-2}$ or to elements of $\mathbb{R}\mathcal{M}^d(y)$ having τ as a tangent line at $y_{c_1(X)d-2}$. Now, each curve of the first kind (resp. second kind) is limit of exactly two (resp. one) families $C_{i_1}(\lambda), C_{i_2}(\lambda)$ of elements of $\mathbb{R}\mathcal{M}^d(y_\lambda)$. Since the masses of these curves are unchanged while passing to the limit $\lambda \to 0$, the second relation follows from the fact that $\chi^d_r = \chi^d_r(y_\lambda, J)$. $\square$

3.4 Non-triviality of $\chi^4(T)$ and $\chi^5(T)$ for the complex projective plane

3.4.1 Generalization of the invariant θ

The invariant θ has been defined by fixing the position of one of the double points of the pseudo-holomorphic curves in the homology class d. More generally, one can define such an invariant by fixing the position of σ double points of these curves, where $0 \le \sigma \le \frac{1}{2}[c_1(X)d-1]$. More precisely, let $y = (y_1, \ldots, y_{c_1(X)d-1-\sigma})$ be a real configuration of $c_1(X)d-1-\sigma$ distinct points of X, and s be the number of those which are real. We assume that $y_{c_1(X)d-2\sigma}, y_{c_1(X)d-2\sigma+1}, \ldots, y_{c_1(X)d-1-\sigma}$ are real, so that $s \ge \sigma$. Let $J \in \mathbb{R}\mathcal{J}_\omega$ be generic enough. Then there are only finitely many rational J-holomorphic curves in X which realize the homology class d, pass through y and have a transversal double point at each of the points $y_{c_1(X)d-2\sigma}, y_{c_1(X)d-2\sigma+1}, \ldots, y_{c_1(X)d-1-\sigma}$. These curves are all irreducible and have only transversal double points as singularities. For every integer m ranging from 0 to δ, denote by $\hat{n}^+_d(m)$ (resp. $\hat{n}^-_d(m)$) the total number of these curves which are real, of mass m and with an even (resp. odd) number of real isolated double points at

$y_{c_1(X)d-2\sigma}, y_{c_1(X)d-2\sigma+1}, \dots, y_{c_1(X)d-1-\sigma}$. Define then:

$$\theta_s^{d,\sigma}(y,J) = \sum_{m=0}^{\delta} (-1)^m (\hat{n}_d^+(m) - \hat{n}_d^-(m)).$$

These definitions extend the ones given in paragraph 3.1. In particular, $\theta_s^{d,0}(y,J) = \chi_s^d(y,J)$ and $\theta_s^{d,1}(y,J) = \theta_s^d(y,J)$.

Theorem 3.5 *Let (X,ω,c_X) be a real symplectic 4-manifold such that $\mathbb{R}X$ is connected, $d \in H_2(X;\mathbb{Z})$ such that $c_1(X)d > 0$ and $0 \leq \sigma \leq \frac{1}{2}[c_1(X)d-1]$. Let $y \subset X$ be a real configuration of $c_1(X)d-1-\sigma$ distinct points and $s \geq \sigma$ be the cardinality of $y \cap \mathbb{R}X$. Finally, let $J \in \mathbb{R}\mathcal{J}_\omega$ be an almost complex structure generic enough, so that the integer $\theta_s^{d,\sigma}(y,J)$ is well defined. Then, this integer $\theta_s^{d,\sigma}(y,J)$ neither depends on the choice of J nor on the choice of y.* $\square$

The proof of this theorem is the same as the one of Theorem 3.1. As usual, this integer $\theta_s^{d,\sigma}(y,J)$ will be denoted by $\theta_s^{d,\sigma}$, and we set $\theta_s^{d,\sigma} = 0$ when s does not have the suitable parity.

Theorem 3.6 *Let (X,ω,c_X) be a real symplectic 4-manifold such that $\mathbb{R}X$ is connected, and $d \in H_2(X;\mathbb{Z})$. Let σ be an integer such that $0 \leq 2\sigma \leq c_1(X)d-3$, and s be an integer between σ and $c_1(X)d-3-\sigma$. Then $\theta_{s+2}^{d,\sigma} = \theta_s^{d,\sigma} + 2\theta_{s+1}^{d,\sigma+1}$.* $\square$

The proof of this theorem is the same as the one of Theorem 3.2.

3.4.2 Non-triviality of $\chi^4(T)$ and $\chi^5(T)$

In this subparagraph, the real symplectic 4-manifold (X,ω,c_X) is the complex projective plane equipped with its standard symplectic form ω_{st} and the complex conjugation conj. We use the canonical identification of $H_2(\mathbb{C}P^2;\mathbb{Z})$ with $\mathbb{Z}$. We defined in §2.1 an invariant $\chi : d \in \mathbb{Z} \mapsto \chi^d(T) \in \mathbb{Z}[T]$ and have computed it for $d \leq 3$ in the example of this paragraph.

Proposition 3.7 *Let (X,ω,c_X) be $(\mathbb{C}P^2,\omega_{st},\mathrm{conj})$. Then the polynomials $\chi^4(T)$ and $\chi^5(T)$ of $\mathbb{Z}[T]$ do not vanish.*

Lemma 3.8 *Let (X,ω,c_X) be $(\mathbb{C}P^2,\omega_{st},\mathrm{conj})$. Then $\theta_q^3 = 1$ for every odd $1 \leq q \leq 7$ $\theta_r^{4,3} = 1$ for every even $4 \leq r \leq 8$ and $\theta_s^{5,6} = 1$ for every even $6 \leq s \leq 8$.*

Proof:
The proofs are the same in all the cases, so we will prove only the degree 4 case. Let y be a real configuration of 8 distinct points in the plane, $r \geq 3$ of which being real. Let $J \in \mathcal{J}_\omega$ be generic enough. There exists then only one rational J-holomorphic curve of degree 4 in $\mathbb{C}P^2$ which passes through y and have its 3 double points at y_6, y_7, y_8. Indeed, if there were two of them, they would intersect at each point $y_1, \dots, y_5$ with multiplicity at least one, and at each point y_6, y_7, y_8 with multiplicity at least four. This would give an intersection index greater than 16 which is impossible. This implies that the corresponding Gromov-Witten invariant is one, since it is obviously not zero. Now let $J \in \mathbb{R}\mathcal{J}_\omega$ be generic enough, this unique curve is real. Denote by m its mass, we have $\theta_r^{4,3}(x,J) = (-1)^m(-1)^m = 1$. $\square$

Proof of Proposition 3.7:
It is a consequence of Theorem 3.6 and Lemma 3.8. □

For instance, the coefficients of the polynomial $\chi^4(T)$ satisfy the relations
$\chi_3^4 = \chi_1^4 + 2\theta_2^4$,
$\chi_5^4 = \chi_1^4 + 4\theta_2^4 + 4\theta_3^{4,2}$,
$\chi_7^4 = \chi_1^4 + 6\theta_2^4 + 12\theta_3^{4,2} + 8$,
$\chi_9^4 = \chi_1^4 + 8\theta_2^4 + 24\theta_3^{4,2} + 32$, and
$\chi_{11}^4 = \chi_1^4 + 10\theta_2^4 + 40\theta_3^{4,2} + 80$.
Hence, all these coefficients cannot vanish simultaneously. Similarly,
$\chi_{14}^5 = \chi_0^5 + 14\theta_1^5 + 84\theta_2^{5,2} + 280\theta_3^{5,3} + 560\theta_4^{5,4} + 672\theta_5^{5,5} + 448$.

Remark 3.9 Using the results of this paragraph together with a modification of [10] to configurations admitting complex conjugated points due to E. Shustin (see [13]), I. Itenberg, V. Kharlamov and E. Shustin have been able to compute the polynomials $\chi^4(T)$ and $\chi^5(T)$, see [6]. The values are:
$\chi^4(T) = 16T^3 + 40T^5 + 80T^7 + 144T^9 + 240T^{11}$
$\chi^5(T) = 64 + 64T^2 + 248T^4 + 744T^6 + 1872T^8 + 4272T^{10} + 9096T^{12} + 18264T^{14}$.

References

[1] M. Audin and J. Lafontaine, editors. *Holomorphic curves in symplectic geometry*, volume 117 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994.

[2] A. I. Degtyarev and V. M. Kharlamov. Topological properties of real algebraic varieties: Rokhlin's way. *Russ. Math. Surv.*, 55(4):735–814, 2000.

[3] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[4] H. Hofer, V. Lizan, and J.-C. Sikorav. On genericity for holomorphic curves in four-dimensional almost-complex manifolds. *J. Geom. Anal.*, 7(1):149–159, 1997.

[5] I. Itenberg, V. Kharlamov, and E. Shustin. Welschinger invariant and enumeration of real rational curves. *Internat. Math. Res. Notices*, 49:2639–2653, 2003.

[6] I. Itenberg, V. Kharlamov, and E. Shustin. Logarithmic equivalence of Welschinger and Gromov-Witten invariants. *Preprint Math.AG/0407188*, 2004. To appear in Russian Math. Surveys.

[7] S. Ivashkovich and V. Shevchishin. Structure of the moduli space in a neighborhood of a cusp-curve and meromorphic hulls. *Invent. Math.*, 136(3):571–602, 1999.

[8] M. Kontsevich and Y. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.*, 164(3):525–562, 1994.

[9] D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[10] G. Mikhalkin. Counting curves via lattice paths in polygons. *C. R. Math. Acad. Sci. Paris*, 336(8):629–634, 2003.

[11] Y. Ruan and G. Tian. A mathematical theory of quantum cohomology. *J. Differential Geom.*, 42(2):259–367, 1995.

[12] V. V. Shevchishin. Pseudoholomorphic curves and the symplectic isotopy problem. *preprint math.SG/0010262*, 2000. Habilitation thesis at Ruhr-University, Bochum, Germany.

[13] E. Shustin. A tropical calculation of the Welschinger invariants of real toric Del Pezzo surfaces. *Preprint Math.AG/0406099*, 2004.

[14] J.-C. Sikorav. The gluing construction for normally generic J-holomorphic curves. In *Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001)*, volume 35 of *Fields Inst. Commun.*, pages 175–199. Amer. Math. Soc., Providence, RI, 2003.

[15] S. Smale. An infinite dimensional version of Sard's theorem. *Amer. J. Math.*, 87:861–866, 1965.

[16] F. Sottile. Enumerative real algebraic geometry. In *Algorithmic and quantitative real algebraic geometry (Piscataway, NJ, 2001)*, volume 60 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 139–179. Amer. Math. Soc., Providence, RI, 2003.

[17] J.-Y. Welschinger. Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry. *C. R. Acad. Sci. Paris Sér. I Math.*, 336(4):341–344, 2003.

[18] J.-Y. Welschinger. Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants. *Prépublication de l'école normale supérieure de Lyon*, 321, 2003. To appear in Duke Math. J.

[19] E. Witten. Two-dimensional gravity and intersection theory on moduli space. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 243–310. Lehigh Univ., Bethlehem, PA, 1991.

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