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Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants

Jean-Yves Welschinger*

1st July 2004

Abstract:
Let X be a real algebraic convex 3-manifold whose real part is equipped with a Pin^- structure. We show that every irreducible real rational curve with non-empty real part has a canonical spinor state belonging to $\{\pm 1\}$. The main result is then that the algebraic count of the number of real irreducible rational curves in a given numerical equivalence class passing through the appropriate number of points does not depend on the choice of the real configuration of points, provided that these curves are counted with respect to their spinor states. These invariants provide lower bounds for the total number of such real rational curves independantly of the choice of the real configuration of points.

Introduction

A smooth complex algebraic projective manifold X is said to be *convex* when the vanishing $H^1(\mathbb{C}P^1; u^*TX) = 0$ occurs for every morphism $u : \mathbb{C}P^1 \to X$. Main examples are projective homogeneous spaces. These manifolds provide a suitable framework in order to define genus 0 algebraic Gromov-Witten invariants since the space of morphisms from $\mathbb{C}P^1$ to X in a given homology class $d \in H_2(X;\mathbb{Z})$ turns out to be a smooth manifold of the expected dimension $c_1(X)d + \dim_{\mathbb{C}} X$ (see [8], [2]). In particular, these Gromov-Witten invariants are enumerative. Let X be a convex manifold of dimension 3 and $d \in H_2(X;\mathbb{Z})$ be such that $c_1(X)d$ is even. Let k_d be half of this even integer. Then, through a generic configuration $\underline{x} = (x_1, \ldots, x_{k_d})$ of k_d distinct points of X passes only finitely many connected rational curves in the homology class d. These curves are all irreducible and immersed and their number N_d does not depend on the choice of $\underline{x}$. It is a Gromov-Witten invariant of the manifold X, often denoted by $GW_0(X, d, pt, \ldots, pt)$. From now on, assume that X is *real*, that is equipped with an antiholomorphic involution c_X. The fixed point set of c_X is denoted by $\mathbb{R}X$ and called the *real part* of X. Assume also that $(c_X)_* d = -d$ and that the configuration $\underline{x}$ is *real*, that is satisfies $\{c_X(x_1), \ldots, c_X(x_{k_d})\} = \{x_1, \ldots, x_{k_d}\}$. Then, the analogous result is no more true, the number $R_d(\underline{x})$ of real irreducible rational curves which realize the homology class d and pass through $\underline{x}$ depends in general on the choice of $\underline{x}$. The parity of this integer is however invariant, it is the one of N_d. The main purpose of this paper is to refine this $mod(2)$ invariant into an integer valued one, see Theorem 2.2.

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Fix a Pin_3^- structure $\mathfrak{p}$ on $\mathbb{R}X$ (see §2.1) and assume that $\underline{x} \cap \mathbb{R}X \neq \emptyset$. As soon as the choice of $\underline{x}$ is generic enough, this implies that for every curve $A \in \mathcal{R}_d(\underline{x})$, where $\mathcal{R}_d(\underline{x})$ denotes the set of real irreducible rational curves which realize the homology class d and pass through $\underline{x}$, the real part $\mathbb{R}A$ is nonempty. Also, as soon as $\underline{x}$ is generic enough, the normal bundle N_A of A in X is real and *balanced*, that is the direct sum of two isomorphic holomorphic line bundles. Choosing a real line subbundle of N_A of maximal degree $k_d - 1$, we can equip the knot $\mathbb{R}A \subset \mathbb{R}X$ with a canonical ribbon structure, see §2.2 for the detailed construction. This structure does depend on the way the complex curve A is immersed in the complex manifold X and it is inherited from this complex immersion. Associating a framing to this ribbon knot, we construct a loop in the $O_3(\mathbb{R})$-principal bundle F_X of orthonormal frames of $\mathbb{R}X$. The *spinor state* $sp(A) \in \{\pm 1\}$ of A is then defined to be the obstruction to lift this loop into a loop of the Pin_3^--principal bundle P_X given by the pin structure $\mathfrak{p}$. Denote by $(\mathbb{R}X)_1, \dots, (\mathbb{R}X)_N$ the connected components of $\mathbb{R}X$ and, for $i \in \{1, \dots, N\}$, by $r_i = \#(\underline{x} \cap (\mathbb{R}X)_i)$. Then define

$$\chi_r^{d,\mathfrak{p}}(\underline{x}) = \sum_{A \in \mathcal{R}_d(\underline{x})} sp(A),$$

where $r = (r_1, \dots, r_N)$.

Theorem 0.1 *1) As soon as $c_1(X)d \neq 4$, the integer $\chi_r^{d,\mathfrak{p}}(\underline{x})$ is independant of the choice of the real configuration $\underline{x} \in X^{k_d}$, it only depends on $d, \mathfrak{p}$ and the N-tuple r. The same holds when $c_1(X)d = 4$ and X does not have any non-immersed rational curve in the class d.*

2) The similar construction and result hold for the blown up projective space $Y = \mathbb{C}P^3 \# \overline{\mathbb{C}P}^3$ when the homology class d satisfies $Exc.d \geq 0$, where Exc is the exceptional divisor of Y.

Note that the blown up projective space is not convex, see §4 for the second part of this theorem 0.1. See also Theorem 2.2 for a slightly more general statement in the case $c_1(X)d = 4$.

This integer $\chi_r^{d,\mathfrak{p}}(\underline{x})$ is denoted by $\chi_r^{d,\mathfrak{p}}$ and we put $\chi_r^{d,\mathfrak{p}} = 0$ as soon as it is not well defined. This allows us to define the polynomial $\chi^{d,\mathfrak{p}}(T) = \sum_{r \in \mathbb{N}^N} \chi_r^{d,\mathfrak{p}} T^r \in \mathbb{Z}[T]$, where $T^r = T_1^{r_1} \dots T_N^{r_N}$. This polynomial is of the same parity as the integer k_d, that is each of its monomial $\chi_r^{d,\mathfrak{p}} T_1^{r_1} \dots T_N^{r_N}$ for which $r_1 + \dots + r_N \neq k_d \mod (2)$ has vanishing coefficients. Moreover, each of its monomials actually only depends on one indeterminate, that is only one of the r_j, $j \in \{1, \dots, N\}$, can be non zero. Theorem 0.1 means that the function $\chi^{\mathfrak{p}} : d \in H_2(X; \mathbb{Z}) \mapsto \chi^{d,\mathfrak{p}}(T) \in \mathbb{Z}[T]$ is an invariant associated to the isomorphism class of the real algebraic convex manifold (X, c_X). As an application, this invariant provides the following lower bounds in real enumerative geometry.

Corollary 0.2 *Under the assumptions of Theorem 0.1, we have*

$$|\chi_r^{d,\mathfrak{p}}| \leq R_d(\underline{x}) \leq N_d,$$

independantly of the choice of $\underline{x} \in X^{k_d}$.

Note that a similar invariant and similar lower bounds have already been obtained in [14], [15] using real rational J-holomorphic curves in real symplectic 4-manifolds. The question was

then raised whether there exist such invariants in higher dimensions. The results presented here thus provide a partial answer to this question.

Let us fix now $(X, c_X) = (\mathbb{C}P^3, conj)$ and a spin structure $\mathfrak{s}$ on $\mathbb{R}P^3$. Note that for orientable components of $\mathbb{R}X$, a spin structure is more convenient for us than a Pin^- structure, see Remark 2.4. The N-tuple r is then reduced to a single even integer between 2 and $2d = k_d$. Denote by $Y = \mathbb{C}P^3 \# \overline{\mathbb{C}P}^3$ the blow up of X at a real point x_0. Denote by $[l]$ the homology class of a line in the exceptional divisor Exc of Y and by $[f]$ the class of the strict transform of a line in $\mathbb{C}P^3$ passing through x_0. Set $d_Y = d([f] + [l]) - 2[l]$ and choose a spin structure $\overline{\mathfrak{s}}$ on $\overline{\mathbb{R}P}^3$. It induces a spin structure $\mathfrak{s} \# \overline{\mathfrak{s}}$ on $\mathbb{R}Y = \mathbb{R}P^3 \# \overline{\mathbb{R}P}^3$, see §4.2. From the second part of Theorem 0.1, the integer $\chi_r^{d_Y, \mathfrak{s} \# \overline{\mathfrak{s}}}$ is well defined for r even between 2 and $k_d - 2$ and it is an invariant associated to Y together with its real structure. The following theorem provides relations between the coefficients of the polynomial $\chi^{d,\mathfrak{s}}(T)$ for the projective space, see Theorem 4.4.

Theorem 0.3 *Let r be an even integer between 2 and $k_d - 2$, then*

$$\chi_{r+2}^{d,\mathfrak{s}} = \chi_r^{d,\mathfrak{s}} - 2\chi_r^{d_Y, \mathfrak{s} \# \overline{\mathfrak{s}}}.$$

This paper is divided in four paragraphs. In the first one, we introduce Kontsevich's space of genus 0 stable maps and its real structures, as well as preliminaries on the evaluation map and Gromov-Witten invariants. In the second one, we introduce the definition of spinor states and formulate the main results of this paper. In the third one, we give a proof of these results. Finally, the last paragraph is devoted to a further study of the polynomial $\chi^{d,\mathfrak{s}}(T)$ for the projective space, proving Theorem 0.3.

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1 Evaluation map and Gromov-Witten invariants

Let (X, c_X) be a real algebraic convex 3-manifold and $d \in H_2(X;\mathbb{Z})$ be such that $(c_X)_* d = -d$.

1.1 Moduli space of genus 0 real stable maps with k marked points

Let $\mathcal{M}or_d(X)$ be the set of morphisms u from $\mathbb{C}P^1$ to X *in the class* d, that is satisfying $u_*[\mathbb{C}P^1] = d$. It is a smooth quasi-projective manifold of pure dimension $c_1(X)d + 3$. Let $k \in \mathbb{N}^*$ and $Diag_k = \{(z_1, \ldots, z_k) \in (\mathbb{C}P^1)^k \,|\, \exists\, i \neq j\,,\, z_i = z_j\}$. The manifold $\mathcal{M}or_d(X) \times ((\mathbb{C}P^1)^k \setminus Diag_k)$ is equipped with an action of the Moebius group defined by:

$$\begin{array}{rcl} \mathcal{M}oeb \times \mathcal{M}or_d(X) \times ((\mathbb{C}P^1)^k \setminus Diag_k) & \to & \mathcal{M}or_d(X) \times ((\mathbb{C}P^1)^k \setminus Diag_k) \\ (\phi, (u, z_1, \ldots, z_k)) & \mapsto & \left\{ \begin{array}{l} (u \circ \phi^{-1}, \phi(z_1), \ldots, \phi(z_k)) \text{ if } \phi \in Aut(\mathbb{C}P^1), \\ (c_X \circ u \circ \phi^{-1}, \phi(z_1), \ldots, \phi(z_k)) \text{ otherwise.} \end{array} \right. \end{array}$$

It is also equipped with an action of the symmetric group $\mathcal{S}_k$ defined by $(\sigma, (u, z_1, \ldots, z_k)) \mapsto (u, z_{\sigma(1)}, \ldots, z_{\sigma(k)})$ for $(\sigma, (u, z_1, \ldots, z_k)) \in \mathcal{S}_k \times \mathcal{M}or_d(X) \times ((\mathbb{C}P^1)^k \setminus Diag_k)$. Denote by $\mathcal{M}^d_{0,k}(X)$ the quotient of $\mathcal{M}or_d(X) \times ((\mathbb{C}P^1)^k \setminus Diag_k)$ by $Aut(\mathbb{C}P^1)$. This moduli space is a quasi-projective manifold of pure dimension $c_1(X)d+k$ and a complex orbifold. It is equipped with an action of $\mathcal{S}_k$ and with an antiholomorphic involution $c_{\mathcal{M}}$ induced by the action of $\mathcal{M}oeb/Aut(\mathbb{C}P^1) \cong \mathbb{Z}/2\mathbb{Z}$. Denote by $\mathcal{M}^d_{0,k}(X)^*$ the complement of the set of maps u which can be written $u' \circ g$ where $u' : \mathbb{C}P^1 \to X$ and $g : \mathbb{C}P^1 \to \mathbb{C}P^1$ is a non-trivial ramified covering. It is included in the smooth locus of $\mathcal{M}^d_{0,k}(X)$. Denote by $U^d_{0,k}(X) \to \mathcal{M}^d_{0,k}(X)^*$ the universal curve, which is obtained as the quotient of $\mathcal{M}or_d(X) \times ((\mathbb{C}P^1)^k \setminus Diag_k) \times \mathbb{C}P^1$ by the action of $Aut(\mathbb{C}P^1)$ and then restricted over $\mathcal{M}^d_{0,k}(X)^*$. This universal curve is also equipped with an antiholomorphic involution $c_{\mathcal{U}}$ and with an action of $\mathcal{S}_k$ which lift $c_{\mathcal{M}}$ and the action of $\mathcal{S}_k$ on $\mathcal{M}^d_{0,k}(X)^*$ respectively. Finally, denote by $\overline{\mathcal{M}}^d_{0,k}(X)$ the moduli space of genus 0 stable maps with k marked points in the homology class d. This is the compactification due to M. Kontsevich of the space $\mathcal{M}^d_{0,k}(X)$, see [8], [2]. Denote by $\overline{\mathcal{M}}^d_{0,k}(X)^*$ the space of stable maps $(u, C, z_1, \ldots, z_k)$ for which u restricted to any irreducible component of C does not admit a factorization of the form $u' \circ g$ where $u' : \mathbb{C}P^1 \to X$ and $g : \mathbb{C}P^1 \to \mathbb{C}P^1$ is a non-trivial ramified covering. Note that this, in particular, avoids maps $(u, C, z_1, \ldots, z_k)$ mapping a component of C to a constant, which does not matter for us since these maps will play no rôle in this paper. We recall the following theorem.

Theorem 1.1 *1) The manifold $\overline{\mathcal{M}}^d_{0,k}(X)$ is projective normal of pure dimension $c_1(X)d+k$. It is a complex orbifold containing $\overline{\mathcal{M}}^d_{0,k}(X)^*$ in its smooth locus, which is equipped with*

an evaluation morphism $ev_k^d : \overline{\mathcal{M}}_{0,k}^d(X) \to X^k$. *The complement* $\mathcal{K} = \overline{\mathcal{M}}_{0,k}^d(X)^* \setminus \mathcal{M}_{0,k}^d(X)^*$ *is a divisor with normal crossings.*

2) The curve $U_{0,k}^d(X) \to \mathcal{M}_{0,k}^d(X)^*$ *extends to a universal curve* $\overline{U}_{0,k}^d(X) \to \overline{\mathcal{M}}_{0,k}^d(X)^*$.

3) The real structures $c_{\mathcal{M}}$ *and* $c_{\mathcal{U}}$ *extend to real structures* $c_{\overline{\mathcal{M}}}$ *and* $c_{\overline{\mathcal{U}}}$ *on* $\overline{\mathcal{M}}_{0,k}^d(X)$ *and* $\overline{U}_{0,k}^d(X)$ *respectively. The same holds for the actions of* $\mathcal{S}_k$ *on these spaces.* $\square$

This theorem is proved in [2], Theorems 2 and 3. The third part of this theorem follows from the fact that the construction presented in [2] can be carried out over the reals, see [9]. Note that the evaluation morphism restricted to $\mathcal{M}_{0,k}^d(X)$ is just the map induced by $(u, z_1, \dots, z_k) \in \mathcal{M}or_d(X) \times ((\mathbb{C}P^1)^k \setminus Diag_k) \mapsto (u(z_1), \dots, u(z_k)) \in X^k$.

For every element $\tau \in \mathcal{S}_k$, denote by Φ_τ the associated automorphism of $\overline{\mathcal{M}}_{0,k}^d(X)$. This automorphism commutes with $c_{\overline{\mathcal{M}}}$. When $\tau^2 = id$, put $c_{\overline{\mathcal{M}},\tau} = \Phi_\tau \circ c_{\overline{\mathcal{M}}}$ the associated real structure. Note that if $\tau' = \sigma \circ \tau \circ \sigma^{-1}$, with $\tau', \sigma \in \mathcal{S}_k$, then $c_{\overline{\mathcal{M}},\tau'} = \Phi_\sigma \circ c_{\overline{\mathcal{M}},\tau} \circ \Phi_\sigma^{-1}$. Thus, the isomorphism class of $(\overline{\mathcal{M}}_{0,k}^d(X), c_{\overline{\mathcal{M}},\tau})$ only depends on the conjugacy class of τ in $\mathcal{S}_k$. Similarly, the group $\mathcal{S}_k$ acts on X^k by permutation of the factors. Denote by c_{id} the real structure $c_X \times \cdots \times c_X$ on X^k. The evaluation morphism ev_k^d is obviously equivariant for the actions of $\mathcal{S}_k$ and $\mathbb{Z}/2\mathbb{Z}$. For every element $\tau \in \mathcal{S}_k$ such that $\tau^2 = id$, we set c_τ to be the real structure $\Psi_\tau \circ c_{id}$ on X^k, where Ψ_τ is the automorphism of X^k associated to τ. Then, the isomorphism class of (X^k, c_τ) only depends on the conjugacy class of τ in $\mathcal{S}_k$ and $ev_k^d : (\overline{\mathcal{M}}_{0,k}^d(X), c_{\overline{\mathcal{M}},\tau}) \to (X^k, c_\tau)$ is a real morphism between real algebraic varieties.

1.2 Evaluation map and balanced curves

1.2.1 Balanced rational curves

Let $(u, z_1, \dots, z_k) \in \mathcal{M}or_d(X) \times ((\mathbb{C}P^1)^k \setminus Diag_k)$. The differential of u induces a morphism of sheaves $0 \to \mathcal{O}_{\mathbb{C}P^1}(T\mathbb{C}P^1) \to \mathcal{O}_{\mathbb{C}P^1}(u^*TX)$. Denote by $\mathcal{N}_u$ the quotient sheaf. This quotient sheaf admits a decomposition $\mathcal{O}_{\mathbb{C}P^1}(N_u) \oplus \mathcal{N}_u^{sing}$, where N_u is the normal bundle of u and $\mathcal{N}_u^{sing}$ is the skyscraper sheaf having the critical points $p_i \in \mathbb{C}P^1$ of du as support, and $\mathbb{C}^{n_i}$ as fibers, where n_i is the vanishing order of du at p_i. The bundle N_u is holomorphic of rank two over $\mathbb{C}P^1$. From a theorem of Grothendieck, it splits as the direct sum of two line bundles $\mathcal{O}_{\mathbb{C}P^1}(a) \oplus \mathcal{O}_{\mathbb{C}P^1}(b)$, see §2.1 of [11] for example. This normal bundle is said to be *balanced* when $a = b$. The rational curve $(u, z_1, \dots, z_k)$ is said to be *balanced* when u is an immersion and N_u is balanced. Note that when u is an immersion, the adjunction formula imposes $a + b = c_1(X)d - 2$. Thus a necessary condition for the curve to be balanced is that $c_1(X)d$ is even.

Examples:

1) The rational curve $u : (t_0, t_1) \in \mathbb{C}P^1 \mapsto (t_0^d, t_0^{d-1}t_1, t_0t_1^{d-1}, t_1^d) \in \mathbb{C}P^3$ is balanced as soon as $d \in \mathbb{N}^*$ is different from 2. A proof of this fact is given in Lemma 1.2 below, even though we will make no essential use of it throughout this paper.

2) If $A \subset X^3$ is balanced and Y is the blow up of X^3 at a point $x_0 \in A$, then the strict transform of A in Y is balanced, see Lemma 4.5.

Lemma 1.2 *The rational curve* $u : (t_0, t_1) \in \mathbb{C}P^1 \mapsto (t_0^d, t_0^{d-1}t_1, t_0t_1^{d-1}, t_1^d) \in \mathbb{C}P^3$ *is balanced as soon as* $d \in \mathbb{N}^*$ *is different from* 2.

Proof:

Denote by (x_0, x_1, x_2, x_3) the homogeneous coordinates of $\mathbb{C}P^3$. In the affine chart $x_0 = 1$, the curve is given by the parameterization $t_1 \in \mathbb{C} \mapsto (t_1, t_1^{d-1}, t_1^d) \in \mathbb{C}^3$. In the affine chart $x_3 = 1$, it is given by the parameterization $t_0 \in \mathbb{C} \mapsto (t_0^d, t_0^{d-1}, t_0) \in \mathbb{C}^3$, where $t_0 t_1 = 1$. Its normal bundle is then generated by the vector fields $\frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial x_3}$ on the open set $t_0 \neq 0$, and by the vector fields $\frac{\partial}{\partial x_0}$ and $\frac{\partial}{\partial x_1}$ on the open set $t_1 \neq 0$. In these trivializations, the change of trivializations $\mathbb{C}_{t_1} \times \mathbb{C}^2_{(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})} \to \mathbb{C}_{t_0} \times \mathbb{C}^2_{(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1})}$ is given by the matrix

$$M(t_0) \quad = \quad \begin{bmatrix} -dt_0^{2d-1} & (d-1)t_0^{2d} \\ -(d-1)t_0^{2d-2} & (d-2)t_0^{2d-1} \end{bmatrix}.$$

From the relation

$$\begin{bmatrix} 1 & -(\frac{d-1}{d-2})t_0 \\ 0 & 1 \end{bmatrix} M(t_0) \begin{bmatrix} 1 & 0 \\ (\frac{d-1}{d-2})t_1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{d-2}t_0^{2d-1} & 0 \\ 0 & (d-2)t_0^{2d-1} \end{bmatrix},$$

we deduce the isomorphism $N_u \cong \mathcal{O}_{\mathbb{C}P^1}(2d-1) \oplus \mathcal{O}_{\mathbb{C}P^1}(2d-1)$. $\square$

1.2.2 Dominance of the evaluation map

From now on, we assume that $c_1(X)d$ is even and we set $k_d = \frac{1}{2}c_1(X)d$. Note that this condition is automatically fulfilled when $\mathbb{R}X$ is orientable and d is realized by a real 2-cycle A, since then $c_1(X)[A] = w_1(\mathbb{R}X)[\mathbb{R}A] = 0 \mod (2)$.

Let $(u, \mathbb{C}P^1, z_1, \ldots, z_{k_d}) \in \mathcal{M}^d_{0,k_d}(X)^*$ and set $N_{u,-z} = N_u \otimes \mathcal{O}_{\mathbb{C}P^1}(-z)$, where $z = (z_1, \ldots, z_{k_d}) \in (\mathbb{C}P^1)^{k_d}$. Then, set $\mathcal{N}_{u,-z} = \mathcal{O}_{\mathbb{C}P^1}(N_{u,-z}) \oplus \mathcal{N}_u^{sing}$.

Lemma 1.3 *Let $(u, \mathbb{C}P^1, z_1, \ldots, z_{k_d}) \in \mathcal{M}^d_{0,k_d}(X)^*$. Then, the following isomorphisms occur:*

$$\ker(d|_{(u,\mathbb{C}P^1,z)} ev^d_{k_d}) \cong H^0(\mathbb{C}P^1; \mathcal{N}_{u,-z}) \cong H^0(\mathbb{C}P^1; N_{u,-z}) \oplus H^0(\mathbb{C}P^1; \mathcal{N}_u^{sing}),$$

$$\textit{and } \operatorname{coker}(d|_{(u,\mathbb{C}P^1,z)} ev^d_{k_d}) \cong H^1(\mathbb{C}P^1; N_{u,-z}).$$

In particular, balanced curves are regular points of the evaluation map $ev^d_{k_d}$.

Proof:

Denote by $eval^d_{k_d}$ the map $(u, z_1, \ldots, z_{k_d}) \in \mathcal{M}or_d(X) \times ((\mathbb{C}P^1)^{k_d} \setminus Diag_{k_d}) \mapsto (u(z_1), \ldots, u(z_{k_d})) \in X^{k_d}$. The differential of this map at the point (u, z) is given by $(v, \dot{z}) \in H^0(\mathbb{C}P^1; u^*TX) \times T_z(\mathbb{C}P^1)^{k_d} \mapsto v(z) + d|_z u(\dot{z}) \in T_{u(z)}X^{k_d}$. Denote by $\mathcal{N}_{u|_z}$ the skyscraper sheaf with support $\{z_1, \ldots, z_{k_d}\}$ and fiber $T_{u(z_i)}X/d|_{z_i}u(T\mathbb{C}P^1)$ over z_i. The cokernel of $\dot{z} \in T_z(\mathbb{C}P^1)^{k_d} \mapsto d|_z u(\dot{z}) \in T_{u(z)}X^{k_d}$ is isomorphic to $H^0(\mathbb{C}P^1; \mathcal{N}_{u|_z})$. Thus, the cokernel of $d|_{(u,z)} eval^d_{k_d}$ is identified with the cokernel of the composition:

$$\begin{array}{ccccc} H^0(\mathbb{C}P^1; u^*TX) & \to & H^0(\mathbb{C}P^1; \mathcal{N}_u) & \to & H^0(\mathbb{C}P^1; \mathcal{N}_{u|_z}) \quad (*) \\ v & \mapsto & [v] & \mapsto & [v(z)]. \end{array}$$

From the exact sequence of sheaves $0 \to T\mathbb{C}P^1 \to u^*TX \to \mathcal{N}_u \to 0$ we deduce that the first morphism of $(*)$ is surjective since $H^1(\mathbb{C}P^1; T\mathbb{C}P^1) = 0$ and that $H^1(\mathbb{C}P^1; N_u) = 0$ since X is

convex. From the short exact sequence of sheaves $0 \to \mathcal{N}_{u,-z} \to \mathcal{N}_u \to \mathcal{N}_{u|_z} \to 0$, we deduce the long exact sequence

$$\cdots \to H^0(\mathbb{C}P^1;\mathcal{N}_u) \to H^0(\mathbb{C}P^1;\mathcal{N}_{u|_z}) \to H^1(\mathbb{C}P^1;\mathcal{N}_{u,-z}) \to H^1(\mathbb{C}P^1;\mathcal{N}_u) \to \dots$$

Since $H^1(\mathbb{C}P^1;\mathcal{N}_u) = 0$, the cokernel of the second morphism of $(*)$ is isomorphic to $H^1(\mathbb{C}P^1;\mathcal{N}_{u,-z})$. Thus, the cokernel of $d|_{(u,z)}eval^d_{k_d}$ and hence the cokernel of $d|_{(u,\mathbb{C}P^1,z)}ev^d_{k_d}$ is isomorphic to $H^1(\mathbb{C}P^1;\mathcal{N}_{u,-z})$.

In the same way, the kernel of the map $d|_{(u,\mathbb{C}P^1,z)}ev^d_{k_d}$ is isomorphic to the quotient of the kernel of the composition $(*)$ with the image of the morphism $du : H^0(\mathbb{C}P^1;T\mathbb{C}P^1) \to H^0(\mathbb{C}P^1;u^*TX)$. Indeed, the latter coincides with the infinitesimal action of $Aut(\mathbb{C}P^1)$ on $\mathcal{M}or_d(X)$. Since the quotient of $H^0(\mathbb{C}P^1;u^*TX)$ by $du(H^0(\mathbb{C}P^1;T\mathbb{C}P^1))$ is isomorphic to $H^0(\mathbb{C}P^1;\mathcal{N}_u)$, we deduce that the kernel of $d|_{(u,\mathbb{C}P^1,z)}ev^d_{k_d}$ is isomorphic to the kernel of the second morphism of $(*)$, that is to $H^0(\mathbb{C}P^1;\mathcal{N}_{u,-z})$. $\square$

Proposition 1.4 *Let $(u,C,z) \in \overline{\mathcal{M}}^d_{0,k_d}(X)^*$ be such that C has two irreducible components C_1 and C_2 for which $u(C_1)$, $u(C_2)$ are balanced and $u(C_1 \cap C_2)$ is an ordinary double point. Assume moreover that for $i \in \{1,2\}$, $k_{d_i} = \#(C_i \cap z) = \frac{1}{2}c_1(X)d_i$ where $d_i = u_*[C_i] \in H_2(X;\mathbb{Z})$. Then, (u,C,z) is a regular point of $ev^d_{k_d}$. The same holds if $u(C_2)$ has normal bundle isomorphic to $\mathcal{O}_{\mathbb{C}P^1}(k_{d_2}-2) \oplus \mathcal{O}_{\mathbb{C}P^1}(k_{d_2})$ and cannot be deformed into a balanced curve, and if the tangent line of $u(C_1)$ at $u(C_1 \cap C_2)$ does not belong to the line subbundle of degree k_{d_2} of its normal bundle.*

Proof:

For $i \in \{1,2\}$, set $u_i = u|_{C_i}$ and $z^i = C_i \cap z$. Denote by $\mathcal{K} = \overline{\mathcal{M}}^d_{0,k_d}(X)^* \setminus \mathcal{M}^d_{0,k_d}(X)^*$ the divisor of stable maps having reducible domain. Then, (u,C,z) is a smooth point of $\mathcal{K}$. Indeed, for $i \in \{1,2\}$, denote by $e_i : \overline{\mathcal{M}}^{d_i}_{0,k_{d_i}\cup\{\bullet\}}(X) \to X$ the evaluation map associated to the additional marked point $\bullet$. Since X is convex, $d|_{(u_i,C_i,z^i\cup(C_1\cap C_2))}e_i$ is surjective. Thus, $\big((u_1,C_1,z^1\cup(C_1\cap C_2)),(u_2,C_2,z^2\cup(C_1\cap C_2))\big)$ is a smooth point of $\widetilde{\mathcal{K}} = (e_1\times e_2)^{-1}(Diag_X)$, where $Diag_X \subset X \times X$ is the diagonal. From Lemma 12(i) of [2], the map $\widetilde{\mathcal{K}} \to \mathcal{K}$ is an isomorphism, hence (u,C,z) is a smooth point of $\mathcal{K}$.

Now, the restriction of $ev^d_{k_d}$ to $\mathcal{K}$ in the neighborhood of (u,C,z) writes as the composition of the morphism of restriction $(u,C_1\cup C_2,z) \in \mathcal{K} \mapsto ((u_1,C_1,z^1),(u_2,C_2,z^2)) \in \mathcal{M}^{d_1}_{0,k_{d_1}}(X) \times \mathcal{M}^{d_2}_{0,k_{d_2}}(X)$ and the morphism of evaluation $ev^{d_1}_{k_{d_1}} \times ev^{d_2}_{k_{d_2}} : \mathcal{M}^{d_1}_{0,k_{d_1}}(X) \times \mathcal{M}^{d_2}_{0,k_{d_2}}(X) \to X^{k_d}$. Since $u(C_1\cap C_2)$ is an ordinary double point, the first morphism is injective in a neighborhood of (u,C,z). If $u(C_1)$ and $u(C_2)$ are balanced, from Lemma 1.3, the second morphism is an isomorphism in the neighborhood of $((u_1,C_1,z^1),(u_2,C_2,z^2))$. Hence, to prove Proposition 1.4, it suffices to prove that $ev^d_{k_d}$ is injective when restricted to a transversal of $\mathcal{K}$ at the point (u,C,z). The same holds if $u(C_2)$ has normal bundle isomorphic to $\mathcal{O}_{\mathbb{C}P^1}(k_{d_2}-2)\oplus\mathcal{O}_{\mathbb{C}P^1}(k_{d_2})$ and cannot be deformed into a balanced curve, since the image of $ev^d_{k_d}$ is then the product of $X^{k_{d_1}}$ and the image of $ev^{d_2}_{k_{d_2}}$ near $u_2(z^2)$. Since $ev^d_{k_d}(\mathcal{K})$ is smooth in a neighborhood of $u(z)$, a transversal to this divisor can be chosen in one factor of X^{k_d}, that is of the form $\gamma : t \in \Delta \mapsto (x_1(t),\dots,x_{k_d}(t)) \in X_{k_d}$ where only one point $x_i(t)$, say $x_{k_d}(t) \in u(C_2)$, indeed depends on t. Since such a path is transversal to $ev^d_{k_d}$, its inverse image is a smooth curve B in $\overline{\mathcal{M}}^d_{0,k_d}(X)^*$ transversal to $\mathcal{K}$ at the point (u,C,z). Denote by $U \to B$ the restriction

of $\overline{U}^d_{0,k_d}(X) \to \overline{\mathcal{M}}^d_{0,k_d}(X)^*$ over B. This universal curve has one unique reducible fiber over (u, C, z) (choosing a smaller B if necessary) and k_d tautological sections $s_1, \ldots, s_{k_d} : B \to U$. The evaluation morphism $ev_U : U \to X$ contracts the curves $Im(s_1), \ldots, Im(s_{k_d-1})$ and we have to prove that it is injective when restricted to $Im(s_{k_d})$. This follows from the following Lemma 1.5. □

Lemma 1.5 *The differential of the evaluation morphism $ev_U : U \to X$ defined above is injective at every point of the central fiber $C \setminus \{z_1, \ldots, z_{k_d-1}\}$.*

Proof:

Without loss of generality, we can assume that $z^1 = \{z_1, \ldots, z_{k_{d_1}}\}$ and $z^2 = \{z_{k_{d_1}+1}, \ldots, z_{k_d}\}$. Since $u|_{C_1}$ and $u|_{C_2}$ are immersions, the rank of dev_U is at least one at each point of C. Now, the normal bundle of C_1 in U is isomorphic to $\mathcal{O}_{C_1}(-1)$. The morphism dev_U thus induces a morphism of sheaves $0 \to \mathcal{O}_{C_1}(-1) \to \mathcal{O}_{C_1}(N_{u_1})$ where $u_1 = u|_{C_1}$. Since this morphism vanishes at the points $z_1, \ldots, z_{k_{d_1}}$, its image is a line subbundle of N_{u_1} of degree at least $k_{d_1} - 1$. From the isomorphism $N_{u_1} \cong \mathcal{O}_{C_1}(k_{d_1} - 1) \oplus \mathcal{O}_{C_1}(k_{d_1} - 1)$, we deduce that this line subbundle has degree exactly $k_{d_1} - 1$. This implies that the morphism of sheaves vanishes nowhere else than at the points $z_1, \ldots, z_{k_{d_1}}$ and thus dev_U is of rank two at each point of $C_1 \setminus z^1$. Similarly, the normal bundle of C_2 in U is isomorphic to $\mathcal{O}_{C_2}(-1)$. The morphism dev_U thus induces a morphism of sheaves $0 \to \mathcal{O}_{C_2}(-1) \to \mathcal{O}_{C_2}(N_{u_2})$, where $u_2 = u|_{C_2}$, which vanishes at the points $z_{k_{d_1}+1}, \ldots, z_{k_d-1}$. The image of this morphism is thus a line subbundle L of N_{u_2} of degree at least $k_{d_2} - 2$ which contains the tangent line of $u(C_1)$ at $u(C_1 \cap C_2)$. In case $u(C_2)$ is not balanced, we conclude as before. Otherwise, denote by L_1 the unique line subbundle of $N_{u_2} \cong \mathcal{O}_{C_2}(k_{d_2} - 1) \oplus \mathcal{O}_{C_2}(k_{d_2} - 1)$ which is of degree $k_{d_2} - 1$ and contains the tangent line of $u(C_1)$ at $u(C_1 \cap C_2)$. Denote by M_1 the quotient N_{u_2}/L_1, it is a line bundle of degree $k_{d_2} - 1$. The bundle L_1 corresponds to infinitesimal deformations of the curve $u(C_2)$ into a curve passing through $u(z_{k_{d_1}+1}), \ldots, u(z_{k_d-1})$ and $u(C_1)$. Since by construction B is transversal to the divisor $\mathcal{K}$ of reducible curves of $\overline{\mathcal{M}}^d_{0,k_d}(X)^*$, one has $L \neq L_1$. However, L also contains the tangent line of $u(C_1)$ at $u(C_1 \cap C_2)$. It follows that the morphism of sheaves $0 \to \mathcal{O}_{C_2}(L) \to \mathcal{O}_{C_2}(M_1)$ induced by the projection $N_{u_2} \to M_1$ vanishes at the point $u(C_1 \cap C_2)$. This implies that $\deg(L) \leq \deg(M_1) - 1 = k_{d_2} - 2$. Hence, the line bundle L is of degree exactly $k_{d_2} - 2$ and dev_U is of rank two at each point of $C_2 \setminus \{z_{k_{d_1}+1}, \ldots, z_{k_d-1}\}$. The result follows. □

Proposition 1.6 *Let $(u, C, z) \in \overline{\mathcal{M}}^d_{0,k_d}(X)^*$ be such that C has two irreducible components C_1 and C_2 for which $u(C_1)$, $u(C_2)$ are immersed and $u(C_1 \cap C_2)$ is an ordinary double point. For $i \in \{1, 2\}$, denote by $z^i = C_i \cap z$, $d_i = u_*[C_i] \in H_2(X; \mathbb{Z})$, $k_{d_1} = E(\frac{1}{2}c_1(X)d_1)$ and $k_{d_2} = E(\frac{1}{2}c_1(X)d_2) + 1$ where $E()$ denotes the integer part. Assume that $N_{u|_{C_1}} \cong \mathcal{O}_{C_1}(k_{d_1} - 1) \oplus \mathcal{O}_{C_1}(k_{d_1})$, $N_{u|_{C_2}} \cong \mathcal{O}_{C_2}(k_{d_2} - 2) \oplus \mathcal{O}_{C_2}(k_{d_2} - 1)$ and that the tangent line to $u(C_1)$ (resp. to $u(C_2)$) at the point $u(C_1 \cap C_2)$ is not mapped to the unique line subbundle of degree $k_{d_2} - 1$ (resp. k_{d_1}) of $N_{u|_{C_2}}$ (resp. $N_{u|_{C_1}}$). Then, (u, C, z) is a regular point of $ev^d_{k_d}$.*

Proof:

As in the proof of proposition 1.4, the divisor $\mathcal{K} = \overline{\mathcal{M}}^d_{0,k_d}(X)^* \setminus \mathcal{M}^d_{0,k_d}(X)^*$ is smooth in a neighborhood of (u, C, z). Moreover, $ev^d_{k_d}$ restricted to $\mathcal{K}$ is injective in the neighborhood of (u, C, z). Indeed, this restriction writes as the composition of the morphism $(u, C_1 \cup C_2, z) \in$

$\mathcal{K} \mapsto ((u_1, C_1, z^1), (u_2, C_2, z^2)) \in \mathcal{M}^{d_1}_{0,k_{d_1}}(X) \times \mathcal{M}^{d_2}_{0,k_{d_2}}(X)$ and the morphism of evaluation $ev^{d_1}_{k_{d_1}} \times ev^{d_2}_{k_{d_2}} : \mathcal{M}^{d_1}_{0,k_{d_1}}(X) \times \mathcal{M}^{d_2}_{0,k_{d_2}}(X) \to X^{k_d}$, where $u_1 = u|_{C_1}$ and $u_2 = u|_{C_2}$. It follows from the hypothesis that the first morphism is injective. Also, from Lemma 1.3, the morphism $d|_{(u_2,C_2,z^2)} ev^{d_2}_{k_{d_2}}$ is injective and $d|_{(u_1,C_1,z^1)} ev^{d_1}_{k_{d_1}}$ is surjective. We deduce from the hypothesis that the composition of these morphisms, that is $ev^d_{k_d}|_{\mathcal{K}}$, is injective. It suffices thus to prove that the restriction of $ev^d_{k_d}$ to a transversal of $\mathcal{K}$ at the point (u, C, z) is injective. Without loss of generality, we can assume that $z^1 = \{z_1, \ldots, z_{k_{d_1}}\}$ and $z^2 = \{z_{k_{d_1}+1}, \ldots, z_{k_d}\}$. Since $ev^d_{k_d}(\mathcal{K})$ is smooth in a neighborhood of $u(z)$ and its projection on the first k_{d_1} factors of X^{k_d} is onto, a transversal to this divisor can be chosen of the form $\gamma : t \in \Delta \mapsto (x_1(t), \ldots, x_{k_d}(t)) \in X_{k_d}$ where only one point $x_i(t)$, say $x_{k_d}(t)$, depends on t. The inverse image of this path is a smooth curve B in $\overline{\mathcal{M}}^d_{0,k_d}(X)^*$ transversal to $\mathcal{K}$ at the point (u, C, z). Denote by $U \to B$ the restriction of $\overline{U}^d_{0,k_d}(X) \to \overline{\mathcal{M}}^d_{0,k_d}(X)^*$ over B and by $s_1, \ldots, s_{k_d} : B \to U$ the tautological sections. The evaluation morphism $ev_U : U \to X$ contracts the curves $Im(s_1), \ldots, Im(s_{k_d-1})$ and we have to prove that it is injective once restricted to $Im(s_{k_d})$. But the normal bundle of C_1 (resp. C_2) in U is isomorphic to $\mathcal{O}_{C_1}(-1)$ (resp. $\mathcal{O}_{C_2}(-1)$). The morphism dev_U maps this line bundle onto a line subbundle of degree at least $k_{d_1} - 1$ (resp. $k_{d_2} - 2$) of N_{u_1} (resp. N_{u_2}) since this morphism vanishes exactly at the points $z_1, \ldots, z_{k_{d_1}}$ (resp. $z_{k_{d_1}+1}, \ldots, z_{k_d-1}$). Now this line subbundle contains the tangent line of $u(C_2)$ (resp. $u(C_1)$) at the point $u(C_1 \cap C_2)$. Thus, it cannot be the unique line subbundle of degree k_{d_1} (resp. $k_{d_2} - 1$) of N_{u_1} (resp. N_{u_2}). It follows that this subline bundle is of degree exactly $k_{d_1} - 1$ (resp. $k_{d_2} - 2$) which implies that dev_U is injective at every point of $C \setminus \{z_1, \ldots, z_{k_d-1}\}$. Hence the result. $\square$

Remark 1.7 With the help of an expression of the cokernel of the evaluation map at reducible curves analogous to the one given in Lemma 1.3, it would be possible to reduce the proofs of Propositions 1.4 and 1.6 to some vanishing results, compare §I.2 of [7].

1.3 Relation with the Gromov-Witten invariants

Proposition 1.8 *Let X be a convex manifold of dimension 3 and $d \in H_2(X; \mathbb{Z})$ be an effective homology class such that $c_1(X)d$ is even. Denote by $k_d = \frac{1}{2}c_1(X)d$. Then, the following are equivalents:*

i) The morphism $ev^d_{k_d}$ is dominant.

ii) There exists an irreducible rational curve in the class d which is balanced.

iii) The genus 0 Gromov-Witten invariant $GW_0(X, d, pt, \ldots, pt)$ does not vanish.

Proof:

i) $\Rightarrow$ ii) Let $\underline{x} \in X^{k_d}$ be a regular value of $ev^d_{k_d}$ which does not belong to the image of $\overline{\mathcal{M}}^d_{0,k_d}(X) \setminus \mathcal{M}^d_{0,k_d}(X)^*$. Then by hypothesis there exists $(u, C, z) \in \mathcal{M}^d_{0,k_d}(X)^*$ such that $u(z) = \underline{x}$ and $\operatorname{coker} d|_{(u,C,z)} ev^d_{k_d} = 0$. From Lemma 1.3, this means that $H^1(C, N_{u,-z}) = 0$. Now $N_{u,-z}$ is isomorphic to $\mathcal{O}_C(a - k_d) \oplus \mathcal{O}_C(b - k_d)$ with $a + b \leq 2k_d - 2$. This condition thus implies that $a = b = k_d - 1$, which means that u is immersed and balanced.

ii) $\Rightarrow$ i) Follows from Lemma 1.3 and the implicit function theorem.

iii) $\Leftrightarrow$ i) Let $\underline{x} \in X^{k_d}$ be a regular value of $ev^d_{k_d}$ which does not belong to $ev^d_{k_d}(\mathcal{K})$. Then, by definition, $GW_0(X, d, pt, \ldots, pt) = \#(ev^d_{k_d})^{-1}(\underline{x})$. Hence the equivalence. $\square$

In particular, we deduce from Lemma 1.2 the following well known corollary.

Corollary 1.9 *If $X = \mathbb{C}P^3$ and $d \neq 2[\mathbb{C}P^1] \in H_2(X;\mathbb{Z})$ is effective, then the evaluation map $ev^d_{k_d}$ is dominant and the genus 0 Gromov-Witten invariant $GW_0(X, d, pt, \ldots, pt)$ does not vanish.* $\square$

For some computations of such Gromov-Witten invariants, see [12], [2] and references therein.

2 Main results

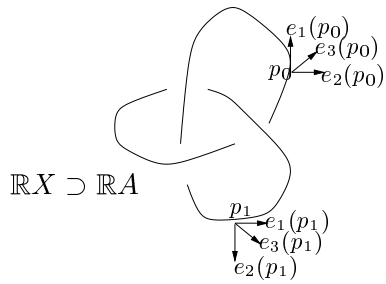
2.1 Choice of a Pin^- structure $\mathfrak{p}$ on $\mathbb{R}X$

Let (X, c_X) be a real algebraic convex 3-manifold. Equip the real part $\mathbb{R}X$ with a riemannian metric g and denote by F_X the $O_3(\mathbb{R})$-principal bundle of orthonormal frames of $T\mathbb{R}X$. Remember that the universal covering of $O_3(\mathbb{R})$ can be equipped with two different group structures turning the covering map into a morphism. The one for which the lift of a reflexion is of order 4 is denoted by Pin_3^-, see [1]. The obstruction to lift the $O_3(\mathbb{R})$-principal bundle F_X into a Pin_3^--principal bundle is given by the expression $w_2(\mathbb{R}X) + w_1^2(\mathbb{R}X) \in H^2(\mathbb{R}X;\mathbb{Z}/2\mathbb{Z})$ where $w_1(\mathbb{R}X)$ (resp. $w_2(\mathbb{R}X)$) stands for the first (resp. second) Stiefel-Whitney class of $\mathbb{R}X$, see [6] for instance. Now from Wu relations (see [10], p. 132 for instance), the obstruction $w_2(M) + w_1^2(M)$ vanishes for every compact 3-manifold M. From now on, we will thus fix such a Pin_3^- structure $\mathfrak{p}$ on $\mathbb{R}X$ and denote by P_X the associated Pin_3^--principal bundle. Note that on orientable components of $\mathbb{R}X$, the choice of an orientation allows to reduce the Pin_3^- structure to a $Spin_3$ structure.

2.2 Spinor states of balanced real rational curves

Let $(A, c_A) \subset (X, c_X)$ be a balanced irreducible real rational curve realizing the homology class d and with nonempty real part $\mathbb{R}A$. In particular, $\mathbb{R}A$ is an immersed knot in $\mathbb{R}X$ and the normal bundle N_A of A is a real holomorphic vector bundle isomorphic to $\mathcal{O}_A(k_d - 1) \oplus \mathcal{O}_A(k_d - 1)$ where $k_d = \frac{1}{2}c_1(X)d \in \mathbb{N}^*$. Assume that $\mathbb{R}A$ has a marked point x_0 and fix an orientation on $T_{x_0}\mathbb{R}X$. Since $w_1(\mathbb{R}X)[\mathbb{R}A] = c_1(X)[A] = 0 \mod (2)$, this orientation induces an orientation on $T\mathbb{R}X|_{\mathbb{R}A}$. The real ruled surface $P(N_A) \cong \mathbb{C}P^1 \times \mathbb{C}P^1$ has a real part homeomorphic to a torus, it is thus isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ equipped with the product of standard complex conjugation $conj \times conj$. Put an orientation on $\mathbb{R}A$, the real rank two vector bundle $\mathbb{R}N_A$ has an induced orientation such that a direct orthonormal frame of $T_x\mathbb{R}A$ followed by a direct orthonormal frame of $\mathbb{R}N_A|_x$ provides a direct orthonormal frame of $T_x\mathbb{R}X$ for every $x \in \mathbb{R}A$. Denote by $[h]$ the class of the real part of a section of $P(N_A)$ having vanishing self-intersection, which is equipped with the orientation induced by the one of $\mathbb{R}A$. Similarly, denote by $[f]$ the class of a real fiber of $P(\mathbb{R}N_A)$, which is equipped with the orientation induced by the one of $\mathbb{R}N_A$. Then the couple $([h], [f])$ provides a basis of the lattice $H_1(P(\mathbb{R}N_A);\mathbb{Z})$. Choose a real holomorphic line subbundle $L \subset N_A$ such that $P(L) \subset P(N_A)$ is a section with vanishing self-intersection if k_d is odd and a section of bidegree $(1, 1)$ whose real part is homologous to $\pm([h] + [f]) \in H_1(P(\mathbb{R}N_A);\mathbb{Z})$ otherwise. The line bundle L is then of degree $k_d - 1$ if k_d is odd and $k_d - 2$ otherwise. In both case, the real part $\mathbb{R}L$ is an orientable real line bundle. Denote by $\mathbb{R}E \supset T\mathbb{R}A$ the real rank two sub-bundle of $T\mathbb{R}X$ which projects onto $\mathbb{R}L$. This bundle is equipped with a riemannian metric induced by the one of $T\mathbb{R}X$. Choose then $(e_1(p), e_2(p))_{p\in\mathbb{R}A}$ a loop of orthonormal

frames of $\mathbb{R}E$ such that $(e_1(p))_{p\in\mathbb{R}A}$ is a loop of direct orthonormal frames of $T\mathbb{R}A$. Note that this choice is not unique since $(e_1(p), -e_2(p))_{p\in\mathbb{R}A}$ provides another such choice. Let $(e_3(p))_{p\in\mathbb{R}A}$ be the unique section of $T\mathbb{R}X|_{\mathbb{R}A}$ such that $(e_1(p), e_2(p), e_3(p))_{p\in\mathbb{R}A}$ is a loop of direct orthonormal frames of $T\mathbb{R}X|_{\mathbb{R}A}$. Define then $sp(A) = +1$ if this loop of F_X lifts to a loop of the bundle P_X and $sp(A) = -1$ otherwise. This integer is called the *spinor state* of the curve A.



It neither depends on the choice of L nor on the choice of the orientation on $\mathbb{R}A$ or on the metric g. It only depends on the pin structure $\mathfrak{p}$ and on the choice of the orientation on $T_{x_0}\mathbb{R}X$ when k_d is even. Indeed, reversing this orientation changes $sp(A)$ into its opposite in this case. Note that this spinor state is inherited from the complex immersion of A in X and does depend on this immersion, not just on the immersion of $\mathbb{R}A$ in $\mathbb{R}X$.

Remark 2.1 1) A spinor state for non-balanced real rational curve can be defined as well, but only the balanced case will be relevant for our purpose.

2) A more geometric explanation of our construction can be given as follows. The choice of a real holomorphic section $h \in P(N_A)$ with vanishing self-intersection provides a framing by lines on our knot, which is uniquely defined up to homotopy. This provides a canonical - homotopy class of - ribbon structure on our knot. In case our ribbon is a Mœbius strip, we have to cut this ribbon and perform half a twist in some direction before gluing it again in order to get cylinder and be able to associate some framing. This happens exactly when k_d is even. The choice of the homology class $\pm([h] + [f])$ instead of $\pm([h] - [f])$ in the above construction is equivalent to the choice of such a direction. The point is that fixing an orientation on the tangent bundle over our knot is enough to fix a choice of such a direction without ambiguity.

2.3 Statement of the results

Let (X, c_X) be a smooth real algebraic convex 3-manifold. Denote by $(\mathbb{R}X)_1, \ldots, (\mathbb{R}X)_N$ the connected components of its real part $\mathbb{R}X$ and equip them with a Pin_3^- structure $\mathfrak{p}$, see §2.1. Let $d \in H_2(X;\mathbb{Z})$ be a class realized by real rational curves, such that $c_1(X)d$ is even and $k_d = \frac{1}{2}c_1(X)d \in \mathbb{N}^*$.

Let $\underline{x}$ be a *real configuration* of k_d distinct points of X, that is a configuration of k_d distinct points which are either real or exchanged by the involution c_X. For $i \in \{1, \ldots, N\}$, denote by $r_i = \#(\underline{x} \cap (\mathbb{R}X)_i)$ and assume that $\sum_{i=1}^N r_i \neq 0$. Note that the N-tuple $r = (r_1, \ldots, r_N)$ encods the equivariant isotopy class of $\underline{x}$. As soon as the choice of $\underline{x}$ is generic enough, there are only finitely many connected rational curves in the homology class d passing through $\underline{x}$. Moreover, these curves are all irreducible and balanced and their number does not depend on the generic choice of $\underline{x}$, it is equal to the genus 0 Gromov-Witten invariant $N_d = GW_0(X, d, pt, \ldots, pt)$. Denote by $\mathcal{R}_d(\underline{x})$ the subset of these curves which are real and

by $R_d(\underline{x})$ its cardinality. Since $\sum_{i=1}^N r_i \neq 0$ and $\underline{x}$ is generic, every curve $A \in \mathcal{R}_d(\underline{x})$ has nonempty real part. We can assume that all the real points of $\underline{x}$ are in a same connected component of $\mathbb{R}X$, say $(\mathbb{R}X)_1$. Indeed, otherwise $\mathcal{R}_d(\underline{x})$ is empty, since real rational curves have connected real part and this real part cannot interpolate points which are not in a same connected component of $\mathbb{R}X$. Choose such a real point, say x_{k_d}, as well as an orientation of $T_{x_{k_d}}\mathbb{R}X$, and denote by $\underline{r}$ the equivariant isotopy class of $\underline{x}$ together with x_{k_d} enriched with an orientation on $T_{x_{k_d}}\mathbb{R}X$. Then, every curve $A \in \mathcal{R}_d(\underline{x})$ has a well defined spinor state. Define finally:

$$\chi_{\underline{r}}^{d,\mathfrak{p}}(\underline{x}) = \sum_{A \in \mathcal{R}_d(\underline{x})} sp(A) \in \mathbb{Z}.$$

Theorem 2.2 *Let (X, c_X) be a smooth real algebraic convex 3-manifold and $\mathfrak{p}$ be a Pin_3^- structure on $\mathbb{R}X$. Let $d \in H_2(X;\mathbb{Z})$ be such that $c_1(X)d$ is even and different from 4, and $k_d = \frac{1}{2}c_1(X)d \in \mathbb{N}^*$. Let $\underline{x} = (x_1, \ldots, x_{k_d})$ be a real configuration of k_d distinct points with at least one of them real, say x_{k_d}, and $\underline{r}$ be the equivariant isotopy class of $\underline{x}$ together with x_{k_d} enriched with an orientation on $T_{x_{k_d}}\mathbb{R}X$ if k_d is even. Then the integer $\chi_{\underline{r}}^{d,\mathfrak{s}}(\underline{x})$ does not depend on the choice of $\underline{x}$, it only depends on $d, \mathfrak{p}$ and $\underline{r}$. The same holds when $c_1(X)d = 4$ and the set of non-immersed rational curves of X in the class d is of codimension at least 2 in the moduli space $\mathcal{M}_{0,k_d}^d(X)$.*

(See Remark 3.2 for a discussion on the condition $c_1(X)d \neq 4$. In fact, I don't know any convex 3-fold having a non-immersed curve whose homology class d satisfies $c_1(X)d = 4$. Similarly, I don't know any real convex 3-manifold having non-vanishing Gromov-Witten invariants and a non-connected real part.)

This integer $\chi_r^{d,\mathfrak{p}}(\underline{x})$ is denoted by $\chi_r^{d,\mathfrak{p}}$ and we set $\chi_r^{d,\mathfrak{p}} = 0$ as soon as it is not well defined.

Corollary 2.3 *Under the assumptions of Theorem 2.2, assume in addition that k_d is even and that $\mathbb{R}X$ has a non-orientable component $(\mathbb{R}X)_i$, $i \in \{1, \ldots, n\}$. If $r = (r_1, \ldots, r_N)$ with $r_i \neq 0$, then $\chi_{\underline{r}}^{d,\mathfrak{p}} = 0$. In particular, the genus 0 Gromov-Witten invariant $GW_0(X, d, pt, \ldots, pt)$ is even.*

Proof:
Let $\underline{x}(t) = (x_1(t), \ldots, x_{k_d}(t))$, $t \in [0, 2\pi]$, be a loop of configurations of points of X such that $x_1(t), \ldots, x_{k_d-1}(t)$ are fixed and $x_{k_d}(t)$ defines a loop γ of $\mathbb{R}X$ which is non-trivial against the first Stiefel-Whitney class of $\mathbb{R}X$. Equip $T_{x_{k_d}(t)}\mathbb{R}X$ with an orientation depending continuously on t, then the orientations of $T_{x_{k_d}(0)}\mathbb{R}X = T_{x_{k_d}(2\pi)}\mathbb{R}X$ are opposite. Thus $\chi_{\underline{r}}^{d,\mathfrak{p}}(\underline{x}(0)) = -\chi_{\underline{r}}^{d,\mathfrak{p}}(\underline{x}(2\pi))$. Now from Theorem 2.2, $\chi_{\underline{r}}^{d,\mathfrak{p}}(\underline{x}(0)) = \chi_{\underline{r}}^{d,\mathfrak{p}}(\underline{x}(2\pi))$. This forces the vanishing $\chi_{\underline{r}}^{d,\mathfrak{p}} = 0$. $\square$

Denote by $\mathcal{R}_d^+(\underline{x})$ (resp. $\mathcal{R}_d^-(\underline{x})$) the subset of $\mathcal{R}_d(\underline{x})$ consisting of curves having positive (resp. negative) spinor state. Corollary 2.3 means that these two subsets have exactly same cardinality, they are mirror to each other. One example of such a manifold is the quadric in $\mathbb{C}P^4$ equipped with the real structure whose real part is non-orientable.

Remark 2.4 1) It is convenient to fix an orientation on orientable components of $\mathbb{R}X$ and to use it to define the integer $\chi_{\underline{r}}^{d,\mathfrak{p}}$. When k_d is even, the sign of $\chi_{\underline{r}}^{d,\mathfrak{p}}$ depends on this choice of an

orientation. We can then get rid of the necessity to put an orientation on $T_{x_{k_d}}\mathbb{R}X$. Indeed, for orientable components, it is given by their orientation and for non-orientable components, either k_d is odd and the integer $\chi_{\underline{r}}^{d,\mathfrak{p}}$ is anyway well defined, or k_d is even and $\chi_{\underline{r}}^{d,\mathfrak{p}} = 0$ from Corollary 2.3. Note that the pin structure $\mathfrak{p}$ together with the orientation induce then a spin structure on orientable components of $\mathbb{R}X$. **From now on**, every orientable components of $\mathbb{R}X$ will be equipped with a spin structure $\mathfrak{s}$, which is more convenient for our need. In case $\mathbb{R}X$ is orientable, the integer $\chi_{\underline{r}}^{d,\mathfrak{p}}$ will be then rather denoted by $\chi_r^{d,\mathfrak{s}}$.

2) Such a vanishing result as the one given by Corollary 2.3 can also be given for curves in $\mathbb{C}P^3$ of even degree. Indeed, the integers $\chi_r^{d,\mathfrak{s}}(\underline{x}_0)$ and $\chi_r^{d,\mathfrak{s}}(\underline{x}_1)$ must then have opposite signs if $\underline{x}_0$ and $\underline{x}_1$ are images to each other under a real reflexion of $\mathbb{C}P^3$. This has just been noticed and communicated to me by G. Mikhalkin. Once more, this implies that the associated Gromov-Witten invariant is even in even degree. The first degree where the values of these invariants of the projective space are not known is thus 5. The associated Gromov-Witten invariant is then 105 (and 122129 in degree 7 as taken out from [3]). Note that similarly, the action of the group of real automorphisms of X on the space of real configuration of points provides symetries in the invariant $\chi^{\mathfrak{p}}$.

Thanks to this remark, we can define the polynomial $\chi^{d,\mathfrak{p}}(T) = \sum_{r\in\mathbb{N}^N} \chi_r^{d,\mathfrak{p}} T^r \in \mathbb{Z}[T]$, where $T^r = T_1^{r_1}\dots T_N^{r_N}$. This polynomial is of the same parity as the integer k_d, that is each of its monomial $\chi_r^{d,\mathfrak{p}} T_1^{r_1}\dots T_N^{r_N}$ for which $r_1+\dots+r_N \neq k_d \mod (2)$ has vanishing coefficients. Moreover, each of its monomials actually only depends on one indeterminate, that is only one of the r_j, $j\in\{1,\dots,N\}$, can be non zero. Indeed, since the real part of a real rational curve is connected, it cannot interpolate points which do not lye in a same connected component of $\mathbb{R}X$. Theorem 2.2 means that the function $\chi^{\mathfrak{p}} : d \in H_2(X;\mathbb{Z}) \mapsto \chi^{d,\mathfrak{p}}(T) \in \mathbb{Z}[T]$ is an invariant associated to the isomorphism class of the real algebraic convex 3-manifold (X, c_X). As an application, this invariant provides the following lower bounds in real enumerative geometry.

Corollary 2.5 *Under the assumptions of Theorem 2.2, denote by $R_d(\underline{x})$ the number of real irreducible rational curves passing through $\underline{x}$ in the class d and by N_d the associated Gromov-Witten invariant. Then, $|\chi_r^{d,\mathfrak{p}}| \leq R_d(\underline{x}) \leq N_d$.* $\square$

Note that a similar invariant and similar lower bounds have already been obtained in [14], [15] using real rational J-holomorphic curves in real symplectic 4-manifolds. The question was then raised whether there exists such invariants in higher dimensions. The results presented here thus provide a partial answer to this question. Extension of this work in dimension greater than three is in preparation.

The following natural questions arise from Corollary 2.5. Are the upper and lower bounds given by this corollary sharps ? How to compute the invariant $\chi_r^{d,\mathfrak{p}}$? See [13] for a discussion of related problems in real enumerative geometry and [5] for an estimation of the similar invariants constructed in [14], [15].

Finally, note that it is possible to understand the dependence of $\chi_r^{d,\mathfrak{p}}$ with respect to r, see §4.2.

Remark 2.6 G. Mikhalkin found a configuration of 8 generic points in $\mathbb{R}P^3$ such that none of the four irreducible rational curves of degree 4 passing through them are real. Hence, though the invariant $\chi_8^{4,\mathfrak{s}}$ vanishes, the lower bound 0 given by Corollary 2.5 is sharp. This contrasts with the complex dimension 2, see [5].

3 Proof of Theorem 2.2

Let $\tau \in \mathcal{S}_{k_d}$ having $\sum_{i=1}^N r_i$ fixed points in $\{1, \dots, k_d\}$ and such that $\tau^2 = id$. Remember that $ev^d_{k_d} : (\overline{\mathcal{M}}^d_{0,k_d}(X), c_{\overline{\mathcal{M}},\tau}) \to (X^{k_d}, c_\tau)$ is a real morphism between real algebraic varieties, see §1.1. Denote by $\mathbb{R}_\tau X^{k_d}$ (resp. $\mathbb{R}_\tau \overline{\mathcal{M}}^d_{0,k_d}(X)$) the real part of (X^{k_d}, c_τ) (resp. $(\overline{\mathcal{M}}^d_{0,k_d}(X), c_{\overline{\mathcal{M}},\tau})$) and by $\mathbb{R}_\tau ev^d_{k_d}$ the restriction of $ev^d_{k_d}$ to $\mathbb{R}_\tau \overline{\mathcal{M}}^d_{0,k_d}(X) \to \mathbb{R}_\tau X^{k_d}$.

3.1 Genericity arguments

Proposition 3.1 *Let X be a smooth algebraic convex 3-manifold and $u_0 : \mathbb{C}P^1 \to X$ be a morphism having a unique cuspidal point at $z_0 \in \mathbb{C}P^1$. Assume that the holomorphic vector bundle $u_0^*TX \otimes \mathcal{O}_{\mathbb{C}P^1}(-z_0)$ is generated by its global sections and denote by $d = (u_0)_*[\mathbb{C}P^1] \in H_2(X;\mathbb{Z})$. Then, the locus of non-immersed curves is a subvariety of codimension at least 2 of $\mathcal{M}^d_{0,0}(X)$ in the neighborhood of u_0.*

Proof:
Denote by T^* the holomorphic vector bundle of rank 3 over $\mathcal{M}^d_{0,1}(X)$ whose fiber over $(u, \mathbb{C}P^1, z) \in \mathcal{M}^d_{0,1}(X)$ is the vector space $Hom(T_z\mathbb{C}P^1, T_{u(z)}X)$. This bundle is equipped with a tautological section $\sigma : (u, \mathbb{C}P^1, z) \in \mathcal{M}^d_{0,1}(X) \mapsto d|_z u \in Hom_{\mathbb{C}}(T_z\mathbb{C}P^1, T_{u(z)}X)$. The vanishing locus of σ coincides with the locus of curves $(u, \mathbb{C}P^1, z) \in \mathcal{M}^d_{0,1}(X)$ having a cuspidal point at z. Let us prove that σ vanishes transversely at the point $(u_0, \mathbb{C}P^1, z_0)$. For this purpose, we fix some holomorphic local coordinates in the neighborhood of $z_0 \in \mathbb{C}P^1$ and in the neighborhood of $u_0(z_0) \in X$. These coordinates induce a connection ∇ on the bundle u_0^*TX in the neighborhood of z_0, as well as a connection ∇^{T^*} on the bundle T^* in the neighborhood of $(u_0, \mathbb{C}P^1, z_0)$. Let $\xi \in T_{z_0}\mathbb{C}P^1$ and $\zeta \in T_{u_0(z_0)}X$. By hypothesis, there exists a section v of the bundle u_0^*TX such that $v(z_0) = 0$ and $\nabla v|_{z_0} = \xi^* \otimes \zeta$. Then, $(v, \dot{z}= 0) \in T_{(u_0,\mathbb{C}P^1,z_0)}\mathcal{M}^d_{0,1}(X)$ and $\nabla^{T^*}_{(v,\dot{z}=0)}\sigma = \nabla v|_{z_0} = \xi^* \otimes \zeta$. Hence $\nabla^{T^*}\sigma|_{(u_0,\mathbb{C}P^1,z_0)}$ is surjective, the result follows. $\square$

Remark 3.2 1) Let $X = \mathbb{C}P^1 \times \mathbb{C}P^2$ and $d = k[l]$ where $k \geq 3$ and $[l]$ is the class of a line in a fiber $\{z\} \times \mathbb{C}P^2$ of X. Then the locus of cuspidal curves is of codimension 1 in $\mathcal{M}^d_{0,0}(X)$. However, if $(u, \mathbb{C}P^1)$ is such a curve and $z \in \mathbb{C}P^1$ is a cuspidal point of u, then the bundle $u^*TX \otimes \mathcal{O}_{\mathbb{C}P^1}(-z)$ is not generated by its global sections.

2) Let $(u, \mathbb{C}P^1, z) \in \mathcal{M}^d_{0,k_d}(X)$ be such that $\dim H^1(\mathbb{C}P^1; N_u \otimes \mathcal{O}_{\mathbb{C}P^1}(-z)) = 1$. Then, $N_u \cong \mathcal{O}_{\mathbb{C}P^1}(k_d-2) \oplus \mathcal{O}_{\mathbb{C}P^1}(k_d-1)$, $\dim H^0(\mathbb{C}P^1; \mathcal{N}_u \otimes \mathcal{O}_{\mathbb{C}P^1}(-z)) = \dim H^0(\mathbb{C}P^1; \mathcal{N}_u^{sing}) = 1$, so that u has a unique cuspidal point $z_0 \in \mathbb{C}P^1$. From the long exact sequence associated to the short exact sequence $0 \to T\mathbb{C}P^1 \to u^*TX \otimes \mathcal{O}_{\mathbb{C}P^1}(-z_0) \to N_u \otimes \mathcal{O}_{\mathbb{C}P^1}(-z_0) \to 0$ we deduce that the bundle $u^*TX \otimes \mathcal{O}_{\mathbb{C}P^1}(-z_0)$ is generated by its global sections as soon as $k_d \geq 3$. When $k_d = 1$, $\mathcal{M}^d_{0,k_d}(X)$ does not contain any non-immersed curve since $\deg(u^*TX) = 2$, which implies that any morphism of sheaves $T\mathbb{C}P^1 \to u^*TX$ is injective. For $k_d = 2$, see the next remark.

3) Let A be a cuspidal cubic curve in $\mathbb{C}P^2$. Denote by Y the blown up of $\mathbb{C}P^2$ at 5 distinct points of A outside its cuspidal point and by $\widetilde{A}$ the strict transform of A in Y. Then $\widetilde{A} \subset X = Y \times \mathbb{C}P^1$ satisfies $c_1(X)\widetilde{A} = 4$, but the locus of non-immersed curves of X in the

class $[\widetilde{A}]$ is of codimension one. In this case however, though $H^1(\widetilde{A}; TX|_{\widetilde{A}}) = 0$, X is not convex.

The divisor $\mathbb{R}_\tau\mathcal{K} = \mathbb{R}_\tau\overline{\mathcal{M}}^d_{0,k_d}(X)^* \setminus \mathbb{R}_\tau\mathcal{M}^d_{0,k_d}(X)^*$ is real and made of reducible curves. Denote by $\mathbb{R}_\tau\mathcal{K}_{reg}$ the locus of curves $(u, C, z) \in \mathbb{R}_\tau\mathcal{K}$ such that C has two irreducible components C_1 and C_2 for which $u(C_1)$, $u(C_2)$ are real and immersed, the double point $u(C_1 \cap C_2)$ is an ordinary double point, and which have one of these two properties:

- Either for $i \in \{1, 2\}$, $z^i = C_i \cap z$ has cardinality $k_{d_i} = \frac{1}{2}c_1(X)d_i$ where $d_i = u_*[C_i] \in H_2(X; \mathbb{Z})$, $u(C_1)$ is balanced and $u(C_2)$ is either balanced or has normal bundle isomorphic to $\mathcal{O}_{C_2}(k_{d_2} - 2) \oplus \mathcal{O}_{C_2}(k_{d_2})$ and no balanced curves nearby. In the latter case moreover, the tangent line of $u(C_1)$ at $u(C_1 \cap C_2)$ does not belong to $\mathcal{O}_{C_2}(k_{d_2}) \subset \mathcal{O}_{C_2}(k_{d_2} - 2) \oplus \mathcal{O}_{C_2}(k_{d_2})$.

- Or $N_{u_1} \cong \mathcal{O}_{C_1}(k_{d_1} - 1) \oplus \mathcal{O}_{C_1}(k_{d_1})$ and $N_{u_2} \cong \mathcal{O}_{C_2}(k_{d_2} - 2) \oplus \mathcal{O}_{C_2}(k_{d_2} - 1)$ where $k_{d_1} = E(\frac{1}{2}c_1(X)d_1)$, $k_{d_2} = E(\frac{1}{2}c_1(X)d_2) + 1$, $u_1 = u|_{C_1}$ and $u_2 = u|_{C_2}$. Moreover, $z^i = C_i \cap z$ has cardinality k_{d_i} for $i \in \{1, 2\}$, and the tangent line to $u(C_1)$ (resp. to $u(C_2)$) at the point $u(C_1 \cap C_2)$ is not mapped to the unique line subbundle of degree $k_{d_2} - 1$ (resp. k_{d_1}) of N_{u_2} (resp. N_{u_1}).

Proposition 3.3 *The image of the complement $\mathbb{R}_\tau\mathcal{K} \setminus \mathbb{R}_\tau\mathcal{K}_{reg}$ under $\mathbb{R}_\tau ev^d_{k_d}$ is of codimension at least two in $\mathbb{R}_\tau X^{k_d}$.*

Proof:

From Theorem 1.1, $\mathcal{K}$ is a divisor with normal crossings of $\overline{\mathcal{M}}^d_{0,k_d}(X)$. As a consequence, the locus of curves having more than two irreducible components codimension at least two in $\overline{\mathcal{M}}^d_{0,k_d}(X)$. It suffices then to prove the result for curves (u, C, z) such that C has two irreducible components C_1 and C_2. For $i \in \{1, 2\}$, denote by $z^i = C_i \cap z$, $d_i = u_*[C_i] \in H_2(X; \mathbb{Z})$ and $k_{d_i} = \#z^i$. The morphism $ev^d_{k_d}$ restricted to these curves is the composition of the restriction morphism $(u, C, z) \in \mathcal{K} \mapsto ((u_1, C_1, z^1), (u_2, C_2, z^2)) \in \overline{\mathcal{M}}^{d_1}_{0,k_{d_1}}(X) \times \overline{\mathcal{M}}^{d_2}_{0,k_{d_2}}(X)$ and of the evaluation morphism $ev^{d_1}_{k_{d_1}} \times ev^{d_2}_{k_{d_2}} : \overline{\mathcal{M}}^{d_1}_{0,k_{d_1}}(X) \times \overline{\mathcal{M}}^{d_2}_{0,k_{d_2}}(X) \to X^{k_d}$. Two cases are now to be considered depending on whether $c_1(X)d_i$ is even or odd. In the first case, we can assume that $k_{d_i} = \frac{1}{2}c_1(X)d_i$. Indeed, if $k_{d_1} > \frac{1}{2}c_1(X)d_1$ for example, then $\dim \overline{\mathcal{M}}^{d_1}_{0,k_{d_1}}(X) = c_1(X)d_1 + k_{d_1} \leq 3k_{d_1} - 2 = \dim(X^{k_d}) - 2$ so that already the image of $ev^{d_1}_{k_{d_1}} \times ev^{d_2}_{k_{d_2}}$ is of codimension 2 in X^{k_d}. For the same reason, from Lemma 1.3, we can assume that at least one of the two curves $u(C_1)$, $u(C_2)$, say $u(C_1)$, is balanced. Note that perturbing the k_{d_1} points $u(z^1)$ in $X^{k_{d_1}}$, we can deform $u(C_1)$ in order to separate it from $u(C_2)$. It follows that the image of the restriction morphism $\mathcal{K} \to \overline{\mathcal{M}}^{d_1}_{0,k_{d_1}}(X) \times \overline{\mathcal{M}}^{d_2}_{0,k_{d_2}}(X)$ is of codimension at least one in $\overline{\mathcal{M}}^{d_1}_{0,k_{d_1}}(X) \times \overline{\mathcal{M}}^{d_2}_{0,k_{d_2}}(X)$. Moreover, from Lemma 1.3, the image of non-balanced curves of $\overline{\mathcal{M}}^{d_2}_{0,k_{d_2}}(X)$ under $ev^{d_2}_{k_{d_2}}$ is of codimension at least 1 in $X^{k_{d_2}}$. It follows from these two remarks that we can also assume that $u(C_2)$ is balanced or has normal bundle isomorphic to $\mathcal{O}_{C_2}(k_{d_2} - 2) \oplus \mathcal{O}_{C_2}(k_{d_2})$ and no balanced curves nearby, since in this case non-balanced curves come in a one parameter family. Moreover, in the latter case, we can assume that the curve $u(C_1)$ is transversal to this one parameter family, which means that the tangent line of $u(C_1)$ at $u(C_1 \cap C_2)$ does not belong to $\mathcal{O}_{C_2}(k_{d_2}) \subset \mathcal{O}_{C_2}(k_{d_2} - 2) \oplus \mathcal{O}_{C_2}(k_{d_2})$. In both cases, the double point $u(C_1 \cap C_2)$ is generically an ordinary double point. The only remaining thing to prove in this first case is that we can assume that both $u(C_1)$ and $u(C_2)$ are real. If this would not be the case, they would be exchanged by the involution c_X. They would thus meet

at each real point in the configuration $\underline{x}$. By hypothesis, there exists such real points. Now since a marked stable map does not have a marked point at the nodal point of $C = C_1 \cup C_2$, this cannot happen in codimension one in $\overline{\mathcal{M}}^d_{0,k_d}(X)$, hence we are done.

In the second case, we can for the same reason assume that $k_{d_1} = E(\frac{1}{2}c_1(X)d_1)$, $k_{d_2} = E(\frac{1}{2}c_1(X)d_2) + 1$ and then that $N_{u_1} \cong \mathcal{O}_{C_1}(k_{d_1} - 1) \oplus \mathcal{O}_{C_1}(k_{d_1})$ and $N_{u_2} \cong \mathcal{O}_{C_2}(k_{d_2} - 2) \oplus \mathcal{O}_{C_2}(k_{d_2} - 1)$. Moreover, we can assume that both $u(C_1)$ and $u(C_2)$ are real. Finally, the condition on the tangent line of $u(C_1)$ (resp. of $u(C_2)$) at the point $u(C_1 \cap C_2)$ can be obtained by perturbation of the configuration of points $u(z^1)$ and $u(z^2)$ in $X^{k_{d_1}}$ and $X^{k_{d_2}}$. It follows that the locus of curves which do not have all these properties is of codimension at least two in $\overline{\mathcal{M}}^d_{0,k_d}(X)$. Hence the result. $\square$

3.2 Proof of Theorem 2.2

Let $\underline{x}^0, \underline{x}^1 \in \mathbb{R}_\tau X^{k_d}$ be two regular values of $\mathbb{R}_\tau ev^d_{k_d}$ which belong to the same connected component of $\mathbb{R}_\tau X^{k_d}$ outside the divisor $\mathbb{R}D_{reg} = \mathbb{R}_\tau ev^d_{k_d}(\mathbb{R}_\tau \mathcal{K}_{reg})$. We have to prove that $\chi^{d,\mathfrak{p}}_{\underline{r}}(\underline{x}^0) = \chi^{d,\mathfrak{p}}_{\underline{r}}(\underline{x}^1)$. Let $\gamma : [0,1] \to \mathbb{R}_\tau X^{k_d}$ be a generic piecewise analytic path joining $\underline{x}^0$ to $\underline{x}^1$ and transversal to $\mathbb{R}_\tau ev^d_{k_d}$. Denote by $\mathbb{R}\mathcal{M}_\gamma$ the fiber product $\mathbb{R}_\tau \overline{\mathcal{M}}^d_{0,k_d}(X) \times_\gamma [0,1]$ and by $\mathbb{R}\pi_\gamma$ the associated projection $\mathbb{R}\mathcal{M}_\gamma \to [0,1]$. The integer $\chi^{d,\mathfrak{p}}_{\underline{r}}(\gamma(t))$ is well defined for every $t \in [0,1]$ but a finite number of parameters $0 < t_0 < \cdots < t_j < 1$ corresponding either to critical values of $\mathbb{R}\pi_\gamma$, or to the crossing of the wall $\mathbb{R}D_{reg} = \mathbb{R}_\tau ev^d_{k_d}(\mathbb{R}_\tau \mathcal{K}_{reg})$. We can assume that the latter is crossed transversely by γ. Note also that from Lemma 1.3 and Proposition 3.1, critical points of $\mathbb{R}\pi_\gamma$ correspond to immersed irreducible curves (u, C, z) with normal bundle isomorphic to $\mathcal{O}_C(k_d - 2) \oplus \mathcal{O}_C(k_d)$.

Lemma 3.4 *The path γ can be chosen such that when it crosses the wall of critical values of $\mathbb{R}\pi_\gamma$, only one real point $x_i(t)$ of $\gamma(t) = (x_1(t), \ldots, x_{k_d}(t))$ depends on t. Similarly, it can be chosen such that when it crosses the wall $\mathbb{R}D_{reg}$, either only one real point of $\gamma(t)$ depends on t, or only two points exchanged by c_X depend on t. In the last case, this choice can be done such that in addition to the $k_d - 2$ constant points of γ, the corresponding family of curves in $\mathbb{R}\mathcal{M}_\gamma$ has a common fixed real point in $\mathbb{R}X$.*

Proof:

Let (u, C, z), $C = C_1 \cup C_2$, be a stable map in $\mathbb{R}_\tau \mathcal{K}_{reg}$. The tangent space to $\mathbb{R}D_{reg}$ at the point $u(z)$ consists of infinitesimal deformations of $u(z)$ for which $u(C)$ deforms into a reducible connected curve passing through this configuration of points. From Proposition 3.3, three cases are then to be considered. If (u_1, C_1, z^1) and (u_2, C_2, z^2) are balanced, where $z^i = z \cap C_i$ and $u_i = u|_{C_i}$, we can assume that $u(C_2)$ has a real point in the configuration $u(z)$, say $u(z_{k_d})$. There exists then a real section v_2 of the normal bundle N_{u_2} which vanishes at every point of z^2 except z_{k_d} and which at the point $u(C_1 \cap C_2)$ does not belong to the image of the tangent line of $u(C_1)$ in N_{u_2}. Then, the infinitesimal deformation of $u(z_{k_d})$ in the direction $v_2(z_{k_d})$, the other points $u(z_1), \ldots, u(z_{k_d-1})$ being fixed, provides a vector transversal to $\mathbb{R}D_{reg}$ at the point $u(z)$. This proves the lemma in this case.

Assume now that $N_{u_1} \cong \mathcal{O}_{C_1}(k_{d_1} - 1) \oplus \mathcal{O}_{C_1}(k_{d_1})$ and $N_{u_2} \cong \mathcal{O}_{C_2}(k_{d_2} - 2) \oplus \mathcal{O}_{C_2}(k_{d_2} - 1)$, where $k_{d_i} = \#(z^i)$. If $u(z^2)$ has a real point, say $u(z_{k_d})$, then it suffices to deform $u(z_{k_d})$ in a real direction which does not project onto a fiber of the unique line subbundle of degree $k_{d_2} - 1$ of N_{u_2}, the other points $u(z_1), \ldots, u(z_{k_d-1})$ being fixed, to define an infinitesimal

deformation of $u(z)$ transversal to $\mathbb{R}D_{reg}$. Otherwise, $u(z^2)$ has at least two points exchanged by c_X, say $u(z_{k_d-1})$ and $u(z_{k_d})$. Choose two distinct real points z'_{k_d-1} and z'_{k_d} in the real part of $C_2 \setminus z^2$, and denote by $z' = (z_1, \dots, z_{k_d-2}, z'_{k_d-1}, z'_{k_d})$. Then $(u, C, z') \in \mathbb{R}_{\tau'}\mathcal{K}_{reg}$, where $\tau' = \tau \circ (k_d - 1\ k_d)$. Let $(u_t, C_t, z'_t) \in \mathbb{R}_{\tau'}\overline{\mathcal{M}}^d_{0,k_d}(X)$ be a path transversal to $\mathbb{R}_{\tau'}\mathcal{K}_{reg}$, such that only $u_t(z'_{k_d}(t))$ depends on t. Such a path exists from what we have already done. Choose a deformation $z_{k_d-1}(t)$, $z_{k_d}(t)$ of the points z_{k_d-1}, z_{k_d} of $C_0 = C$ such that $u_t(z_{k_d-1}(t))$ and $u_t(z_{k_d}(t))$ are exchanged by c_X. Then the path $(u_t, C_t, z(t)) \in \mathbb{R}_\tau\overline{\mathcal{M}}^d_{0,k_d}(X)$, where $z(t) = (z_1(t), \dots, z_{k_d}(t))$, is transversal to $\mathbb{R}_\tau\mathcal{K}_{reg}$ at the point (u, C, z). Moreover, only the points $u_t(z_{k_d-1}(t))$ and $u_t(z_{k_d}(t))$ depend on t, and the curves $u_t(C_t)$ have an additional real fixed point, namely $u_t(z'_{k_d-1}(t))$. The lemma is thus proved in this case.

If now $u(C_1)$ is balanced and $u(C_2)$ has normal bundle isomorphic to $\mathcal{O}_{C_2}(k_{d_2} - 2) \oplus \mathcal{O}_{C_2}(k_{d_2})$, then if it has a real point, any move of this real point in a direction not belonging to $\mathcal{O}_{C_2}(k_{d_2}) \subset \mathcal{O}_{C_2}(k_{d_2} - 2) \oplus \mathcal{O}_{C_2}(k_{d_2})$, the other points being fixed, provides a path transversal to $\mathbb{R}D_{reg}$. If not, we can proceed as before by moving two complex conjugated points.

Finally, assume that (u, C, z) is a critical point of $\mathbb{R}\pi_\gamma$ and that $u(z_{k_d})$ is real. Then $N_u \cong \mathcal{O}_C(k_d - 2) \oplus \mathcal{O}_C(k_d)$. Choose any infinitesimal deformation of $u(z)$ which fixes $u(z_1), \dots, u(z_{k_d-1})$ and deforms $u(z_{k_d})$ in a real direction different from the one given by the unique line subbundle of degree k_d of N_u. This deformation is transversal to the wall of critical values of $\mathbb{R}_\tau ev^d_{k_d}$. Hence the result. $\square$

Proof of Theorem 2.2:

Choose a path γ given by Lemma 3.4. We have to prove that the value of the integer $\chi_r^{d,\mathfrak{p}}(\gamma(t))$ does not change when t crosses the parameters $t_0 < \dots < t_j$. When this parameter corresponds to a critical value of $\mathbb{R}\pi_\gamma$, this is given by the following Proposition 3.5. When this parameter corresponds to the crossing of the wall $\mathbb{R}_\tau D_{reg}$, this is given by the following Proposition 3.7. $\square$

Proposition 3.5 *Let $(u, C, z) \in \mathbb{R}\mathcal{M}_\gamma$ be a critical point of $\mathbb{R}\pi_\gamma$, $l_0 \in \mathbb{N}^*$ be the vanishing order of $d|_{(u,C,z)}\mathbb{R}\pi_\gamma$ and $t_0 = \mathbb{R}\pi_\gamma(u, C, z) \in]0, 1[$. Then, there exists $\eta > 0$ and a neighborhood W_0 of (u, C, z) in $\mathbb{R}\mathcal{M}_\gamma$ such that the following alternative occurs:*

1) Either l_0 is odd, then for every $t \in]t_0 - \eta, t_0[$, $W_0 \cap \mathbb{R}\pi_\gamma^{-1}(t) = \{(u_t^+, C_t^+, z_t^+), (u_t^-, C_t^-, z_t^-)\}$ with $sp(u_t^+(C_t^+)) = -sp(u_t^-(C_t^-))$ and for every $t \in]t_0, t_0 + \eta[$, $W_0 \cap \mathbb{R}\pi_\gamma^{-1}(t) = \emptyset$, or the converse.

2) Or l_0 is even, then for every $t \in]t_0 - \eta, t_0 + \eta[$, $W_0 \cap \mathbb{R}\pi_\gamma^{-1}(t) = \{(u_t, C_t, z_t)\}$ and $sp(u_t(C_t))$ does not depend on $t \neq t_0$.

Proof:

Let us complexify the path γ in the neighborhood of $t_0 \in]0, 1[$ in order to get an analytic path $\gamma_{\mathbb{C}} : t \in \Delta_{t_0}(\eta) = \{z \in \mathbb{C} \,|\, |z - t_0| < \eta\} \to X^{k_d}$ such that for every $t \in \Delta_{t_0}(\eta)$, $\gamma_{\mathbb{C}}(\bar{t}) = c_\tau \circ \gamma_{\mathbb{C}}(t)$ and $\gamma_{\mathbb{C}}|_{]t_0-\eta, t_0+\eta[} = \gamma$. Denote then by $\mathcal{M}_\gamma$ the fiber product $\overline{\mathcal{M}}^d_{0,k_d}(X) \times_\gamma [0, 1]$ and by π_γ the associated projection $\mathcal{M}_\gamma \to \Delta_{t_0}(\eta)$. Denote also by $U \to \mathcal{M}_\gamma$ the restriction of $\overline{U}^d_{0,k_d}(X) \to \overline{\mathcal{M}}^d_{0,k_d}(X)^*$. Hence, U is a ruled surface over the smooth curve $\mathcal{M}_\gamma$. Denote by c_γ and c_U the real structures on $\mathcal{M}_\gamma$ and U, so that the submersion $U \to \mathcal{M}_\gamma$ is $\mathbb{Z}/2\mathbb{Z}$-equivariant. Fix a real parameterization $\lambda \in \Delta \mapsto (u_\lambda, C_\lambda, z^\lambda) \in \mathcal{M}_\gamma$ so that the projection $\mathcal{M}_\gamma \to \Delta_{t_0}(\eta)$ writes $\lambda \in \Delta \mapsto \eta\lambda^{l_0+1} + t_0$. Denote then by N the holomorphic rank two vector bundle over U whose fiber over C_λ is the normal bundle N_{u_λ}. For $\lambda \neq 0$, we have

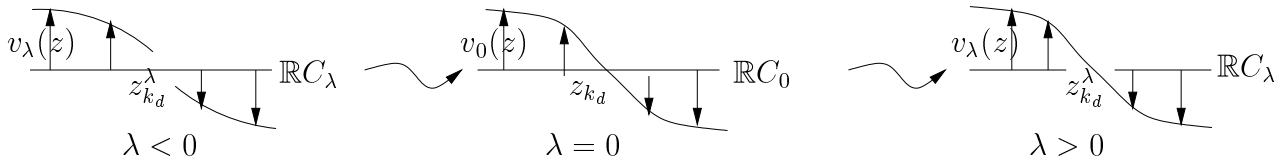
$N_{u_\lambda} \cong \mathcal{O}_{C_\lambda}(k_d-1) \oplus \mathcal{O}_{C_\lambda}(k_d-1)$ and for $\lambda = 0$, $N_{u_0} \cong \mathcal{O}_{C_0}(k_d-2) \oplus \mathcal{O}_{C_0}(k_d)$. Thus, the projectivization $P(N)$ is a deformation of ruled surfaces over the disk Δ such that the fibers $P(N_{u_\lambda})$ over $\lambda \in \Delta\backslash\{0\}$ are isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the fiber $P(N_{u_0})$ is isomorphic to the ruled surface $P(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(2))$. The path $\gamma_{\mathbb{C}}$ can be written $(x_1(t), \dots, x_{k_d}(t))$ where only the last point $x_{k_d}(t) \in \mathbb{R}X$ depends on t and $x'_{k_d}(t)$ is never tangent to a rational curve of $\mathcal{M}_\gamma$. It follows that each ruled surface $P(N_{u_\lambda})$ has a marked point w_λ which is the projectivization of the complex line generated by $x'_{k_d}(t_\lambda) \in N_{u_\lambda}$, for $t_\lambda = \eta\lambda^{l_0+1} + t_0$. Since $\gamma_{\mathbb{C}}$ is transversal to the wall of critical values of $ev^d_{k_d}$, w_0 does not belong to the exceptional section of $P(N_{u_0})$. Let h_0 be a real section of $P(N_{u_0})$ disjoint from the exceptional section and passing through w_0. This section can be deformed into an analytic family $(h_\lambda)_{\lambda\in\Delta}$ of sections of $P(N_{u_\lambda})$ such that h_λ passes through w_λ, $h_{\overline{\lambda}} = \overline{h_\lambda}$ and h_λ is of bidegree $(1,1)$ in $P(N_{u_\lambda})$ for $\lambda \neq 0$. Denote by $L_\lambda \subset N_{u_\lambda}$ the line subbundle associated to h_λ and by $M_\lambda = N_{u_\lambda}/L_\lambda$. We have $\deg(L_\lambda) = k_d - 2$ and $\deg(M_\lambda) = k_d$. Moreover, the extension $0 \to L_\lambda \to N_{u_\lambda} \to M_\lambda \to 0$ is non-trivial for $\lambda \neq 0$ and trivial for $\lambda = 0$. Fix an identification $U \cong \Delta \times \mathbb{C}P^1$ such that z^λ is constant and let δ be a real generator of $H^1(\mathbb{C}P^1; \mathcal{O}_{\mathbb{C}P^1}(-2))$. There exists a holomorphic function $f : \Delta \to \mathbb{C}$ such that the extension class of $0 \to L_\lambda \to N_{u_\lambda} \to M_\lambda \to 0$ is $f(\lambda)\delta \in H^1(\mathbb{C}P^1; M^*_\lambda \otimes L_\lambda) = H^1(\mathbb{C}P^1; \mathcal{O}_{\mathbb{C}P^1}(-2))$. Then, $f(0) = 0$, $f(\lambda) \neq 0$ if $\lambda \neq 0$ and we define $l_f \in \mathbb{N}^*$ to be the vanishing order of f at 0. Let V_0, V_1 be the two standard affine charts of $\mathbb{C}P^1$, chosen such that $z_{k_d}(t_0)$ belongs to V_0. The deformation N is then obtained as the gluing of the trivializations $\mathcal{V}_0 = \Delta \times V_0 \times \mathcal{O}_{\mathbb{C}P^1}(k_d-2) \oplus \mathcal{O}_{\mathbb{C}P^1}(k_d)$ and $\mathcal{V}_1 = \Delta \times V_1 \times \mathcal{O}_{\mathbb{C}P^1}(k_d-2) \oplus \mathcal{O}_{\mathbb{C}P^1}(k_d)$ with gluing maps:

$$(\lambda, z, (s_1, s_2)) \in \mathcal{V}_0 \cap \mathcal{V}_1 \mapsto \big(\lambda, z, (s_1, s_1 + f(t)\delta s_2)\big) \in \mathcal{V}_1 \cap \mathcal{V}_0.$$

Now, note that the bundle N has a tautological section $v^\lambda = \frac{d}{d\lambda}u_\lambda$. This section vanishes at the fixed points $z^\lambda_1, \dots, z^\lambda_{k_d-1}$. Denote its coordinates in the trivialization $\mathcal{V}_0$ by $v^\lambda(z) = (\lambda, z, (s^\lambda_1(z), s^\lambda_2(z)))$. Since (u_0, C_0, z^0) is a critical point of π_γ, from Lemma 1.3, v_0 defines a non-zero element of $H^0(C_0; N_{u_0} \otimes \mathcal{O}_{C_0}(-z_0)) = H^0(\mathbb{C}P^1; \mathcal{O}_{\mathbb{C}P^1}(-2) \oplus \mathcal{O}_{\mathbb{C}P^1})$. It follows that the component s^0_1 of v^0 vanishes. Let us prove that the vanishing order of $s^\lambda_1(z_{k_d})$ at $\lambda = 0$ is l_f. Indeed, in the Taylor expansion of s^λ_1 all the terms of order less than l_f define sections of L_λ which vanish at the points $z^\lambda_1, \dots, z^\lambda_{k_d-1}$. These are thus the zero section. Denote by $(s_1)^{(l_f)}$ (resp. $(s_2)^{(0)}$) the term of order l_f (resp. 0) in the Taylor expansion of s^λ_1 (resp. s^λ_2). Then the triple $(\lambda, z, ((s_1)^{(l_f)}(z^\lambda_{k_d}), (s_2)^{(0)}(z^\lambda_{k_d}))$ defines a section of the bundle N. If $(s_1)^{(l_f)}(z_{k_d}) = 0$, then this section vanishes at the k_d points $z_1, \dots, z_{k_d}$. Since when $\lambda \neq 0$, $N_{u_\lambda} \cong \mathcal{O}_{C_\lambda}(k_d-1) \oplus \mathcal{O}_{C_\lambda}(k_d-1)$, this is impossible and thus the vanishing order of $s^\lambda_1(z_{k_d})$ at $\lambda = 0$ is exactly l_f. Note that since by construction L_λ contains $x'_{k_d}(t_\lambda)$, where $t_\lambda = \eta\lambda^{l_0+1} + t_0$, the second component s^λ_2 is identically zero at the point $z^\lambda_{k_d}$. From $\pi_\gamma(u_\lambda, C_\lambda, z^\lambda) = u_\lambda(z^\lambda)$, we deduce that $d|_{(u_\lambda, C_\lambda, z^\lambda)}\pi_\gamma(v_\lambda) = v_\lambda(z^\lambda) = (0, \dots, 0, v_\lambda(z^\lambda_{k_d}))$. Hence the vanishing order l_0 of $d|_{(u,C,z)}\pi_\gamma$ is the one of $s^\lambda_1(z^\lambda_{k_d})$ at the point $\lambda = 0$, that is l_f.

Now, fix an orientation of the curves $\mathbb{R}C_\lambda$, $\lambda \in]-1, 1[$, and denote by $[\mathbb{R}v_\lambda]$ the section of $P(\mathbb{R}N_{u_\lambda})$ associated to the line bundle generated by v_λ. The latter is equipped with the orientation induced by the one of $\mathbb{R}C_\lambda$. Note that it is the real part of a section of $P(N_{u_\lambda})$ with self-intersection zero, since the holomorphic line bundle generated by v_λ has degree $k_d - 1$. Similarly, the real rank two vector bundle $\mathbb{R}N_{u_\lambda}$ has an orientation induced by the ones of $\mathbb{R}C_\lambda$ and $T_{x_{k_d}}\mathbb{R}X$. Let $[f_\lambda]$ be the class of a fiber of $P(\mathbb{R}N_{u_\lambda})$ equipped with the orientation induced by the one of $\mathbb{R}N_{u_\lambda}$. Then, for $\lambda \neq 0$, $([\mathbb{R}v_\lambda], [f_\lambda])$ provides a basis of the lattice $H_1(P(\mathbb{R}N_{u_\lambda}); \mathbb{Z})$. The section $\mathbb{R}h_\lambda = P(\mathbb{R}L_\lambda)$ has bidegree $(1,1)$. Let

$\epsilon \in \{\pm 1\}$ be such that for $\lambda < 0$, $[\mathbb{R}h_\lambda] = [\mathbb{R}v_\lambda] + \epsilon[f_\lambda] \in H_1(P(\mathbb{R}N_{u_\lambda}); \mathbb{Z})$. Then, for $\lambda > 0$, $[\mathbb{R}h_\lambda] = [\mathbb{R}v_\lambda] + (-1)^{l_f}\epsilon[f_\lambda] \in H_1(P(\mathbb{R}N_{u_\lambda}); \mathbb{Z})$. This follows from the local expression of v_λ in the neighborhood of $z^\lambda_{k_d}$, which is of the form $z \in \mathbb{R}C_\lambda \mapsto (\lambda^{l_f}, z)$, whereas the classes $[\mathbb{R}h_\lambda]$ and $[f_\lambda]$ continuously depend on $\lambda \in]-1, 1[$.



It follows that if $l_0 = l_f$ is even, the spinor states of $(u_\lambda, C_\lambda, z^\lambda)$ and $(u_{-\lambda}, C_{-\lambda}, z^{-\lambda})$ coincide for every $\lambda \neq 0$, whereas they are opposite when $l_0 = l_f$ is odd. Indeed, in this last case, the homotopy classes of the loops of the bundle F_X of orthonormal frames that we construct in order to compute the spinor state (see §2.2) exactly differ from the class of a non-trivial loop in a fiber of F_X. The result now follows from the fact that the projection π_γ has been identified in the neighborhood of $(u, C, z) \in \mathbb{R}\mathcal{M}_\gamma$ with the map $\lambda \in]-1, 1[\mapsto \eta\lambda^{l_0+1} + t_0 \in]t_0 - \eta, t_0 + \eta[$. $\square$

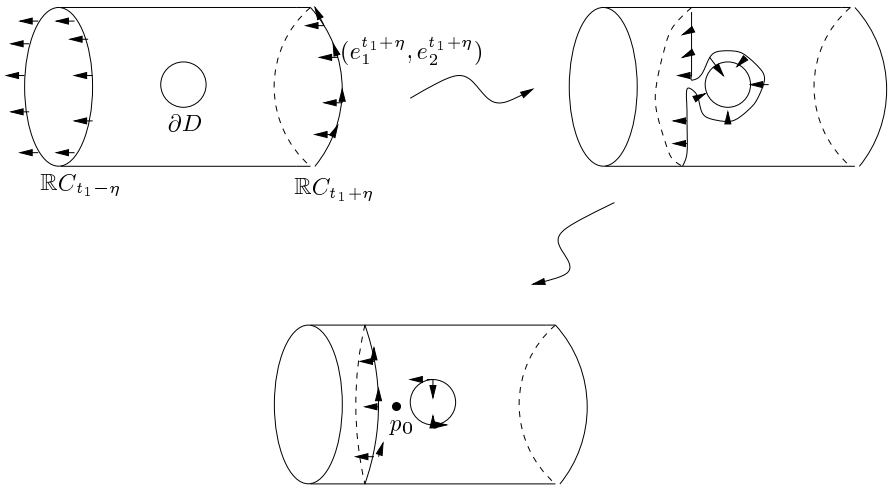
Remark 3.6 From the proof of Proposition 3.5 arises the following question. Are the generic critical points of the evaluation map $ev^d_{k_d} : \overline{\mathcal{M}}^d_{0,k_d}(X) \to X^{k_d}$ non-degenerate when X is convex, e.g. when $X = \mathbb{C}P^3$? With the notations of Proposition 3.5, this means: Is $l_0 = 1$ for every generic critical point of the evaluation map ?

Proposition 3.7 *Let $(u, C, z) \in \mathbb{R}\mathcal{M}_\gamma$ be such that C is reducible and $t_1 = \mathbb{R}\pi_\gamma(u, C, z) \in]0, 1[$. Then, there exists $\eta > 0$ and a neighborhood W_1 of (u, C, z) in $\mathbb{R}\mathcal{M}_\gamma$ such that for every $t \in]t_1 - \eta, t_1 + \eta[$, $W_1 \cap \mathbb{R}\pi_\gamma^{-1}(t) = \{(u_t, C_t, z_t)\}$. Moreover, $sp(u_t(C_t))$ does not depend on $t \neq t_1$.*

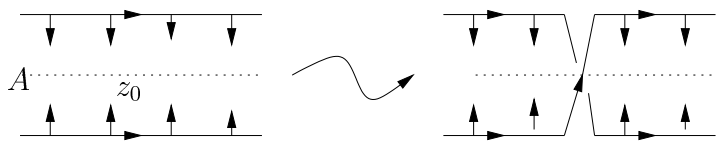
Proof:

From Propositions 1.4, 1.6 and the choice of γ, (u, C, z) is a regular point of $\mathbb{R}\pi_\gamma$. Thus there exists $\eta > 0$ and a neighborhood W_1 of (u, C, z) in $\mathbb{R}\mathcal{M}_\gamma$ such that for every $t \in]t_1 - \eta, t_1 + \eta[$, $W_1 \cap \mathbb{R}\pi_\gamma^{-1}(t) = \{(u_t, C_t, z_t)\}$. Denote by $U \to W_1$ the universal curve, it has a unique singular fiber over $t = t_1$. The real part $\mathbb{R}U$ is thus homeomorphic to a cylinder Cyl blown up at one point, that is to the connected sum of a cylinder and a Moebius strip. From Lemma 3.4, the family $u_t(C_t)$ passes through $k_d - 1$ fixed points. For every $t \neq t_1$, the restriction $T\mathbb{R}U|_{\mathbb{R}C_t}$ projects onto a line subbundle of the normal bundle of $\mathbb{R}C_t$ in $\mathbb{R}X$ which is the real part of a holomorphic line bundle of degree $k_d - 1$. Assume that k_d is odd, so that this line bundle can be used in order to compute the spinor state of $\mathbb{R}C_t$, $t \neq t_1$. The case k_d even will follow along the same lines. Fix a continuous family $(e^t_1(z), e^t_2(z))$, $t \neq t_1$, $z \in \mathbb{R}C_t$, of orthonormal frames of the tangent space at $u_t(z)$ of the image of $\mathbb{R}U$ in $\mathbb{R}X$. This family is completed in a family $(e^t_1(z), e^t_2(z), e^t_3(z))$ of orthonormal frames of $T_{u_t(z)}\mathbb{R}X$. Remove a small disk D around the blown up point of Cyl. In $H_1(Cyl \setminus D; \mathbb{Z}/2\mathbb{Z})$, the class of the fiber $[\mathbb{R}C_{t_1+\eta}]$ equals the sum of the class of the fiber $[\mathbb{R}C_{t_1-\eta}]$ and the class of the boundary ∂D. Equip these loops with a family of orthonormal frames of $T\mathbb{R}X$ as before and denote by $p(\mathbb{R}C_t)$ and $p(\partial D)$ the loops of F_X thus obtained. These loops are related by $p(\mathbb{R}C_{t_1+\eta}) = p(\mathbb{R}C_{t_1-\eta}) + p(\partial D) + p_0 \in H_1(F_X; \mathbb{Z}/2\mathbb{Z})$, where p_0 is a non-trivial loop in a

fiber of F_X. This is suggested by the following picture.



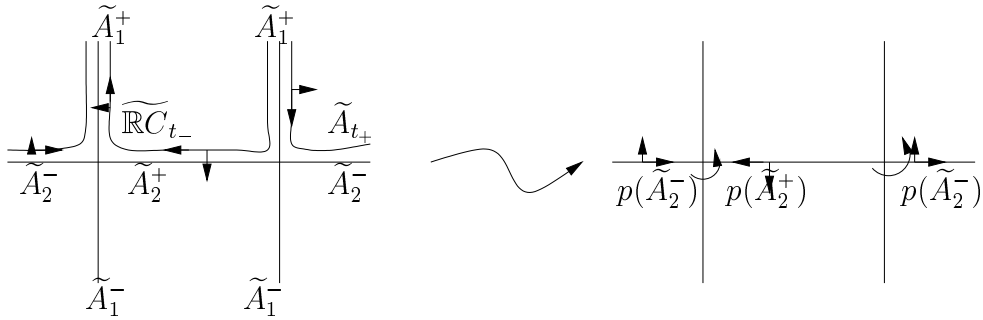
Assume now that $\mathbb{R}X$ is orientable, or that if C_1, C_2 denote the two components of C, then the curve $(u|_{C_1}, C_1, z \cap C_1)$ is balanced. In the Moebius strip that is glued instead of the disk D, the boundary ∂D is homologous to two times the core A of the Moebius strip. Denote by $p(A)$ a loop in F_X obtained by equipping every point $z \in A$ with a orthonormal frame $(e_1(z), e_2(z), e_3(z))$ of $T_{u_t(z)}\mathbb{R}X$, such that $e_1(z) \in T_zA$ and $(e_1(z), e_2(z))$ generates the tangent space of the Moebius strip at z except in a neighborhood of a point $z_0 \in A$ where $(e_1(z), e_2(z), e_3(z))$ does a half twist around the axis TA (which is necessary since the Moebius strip is not orientable). Then, one observes the relation $p(\partial D) = 2p(A) + p_0 \in H_1(F_X; \mathbb{Z}/2\mathbb{Z})$.



We finally deduce that $p(\mathbb{R}C_{t_1+\eta}) = p(\mathbb{R}C_{t_1-\eta}) \in H_1(F_X; \mathbb{Z}/2\mathbb{Z})$, so that $sp(u_{t_1+\eta}(C_{t_1+\eta})) = sp(u_{t_1-\eta}(C_{t_1-\eta}))$.

It remains to consider the case when $\mathbb{R}X$ is not orientable and the curves $(u|_{C_1}, C_1, z \cap C_1)$ and $(u|_{C_2}, C_2, z \cap C_2)$ are not balanced. In this case, the double cover of orientations $\widetilde{\mathbb{R}X} \to \mathbb{R}X$ is non-trivial over $A_1 = u(\mathbb{R}C_1)$ and $A_2 = u(\mathbb{R}C_2)$. Denote by $\widetilde{A}_1$ and $\widetilde{A}_2$ the lifts of A_1 and A_2 in $\widetilde{\mathbb{R}X}$. These are two immersed circles in $\widetilde{\mathbb{R}X}$ which intersects in two ordinary double points a_1 and a_2. These points divide $\widetilde{A}_1$ (resp. $\widetilde{A}_2$) in two connected components $\widetilde{A}_1^+$ and $\widetilde{A}_1^-$ (resp. $\widetilde{A}_2^+$ and $\widetilde{A}_2^-$). Let $\tilde{x}_{k_d}$ be the lift of x_{k_d} in $\widetilde{\mathbb{R}X}$ given by the choice of an orientation on $T_{x_{k_d}}\mathbb{R}X$. For $t \neq t_1$ in the neighborhood of t_1, denote by $\widetilde{\mathbb{R}C}_t$ the unique lift of $u(\mathbb{R}C_t)$ in $\widetilde{\mathbb{R}X}$ which passes through $\tilde{x}_{k_d}$. Without loss of generality, we can assume that $\tilde{x}_{k_d} \in \widetilde{A}_1^+$ and that for $t_+ \in]t_1, t_1+\eta[$ (resp. $t_- \in]t_1-\eta, t_1[$), $\widetilde{\mathbb{R}C}_{t_+}$ is in the neighborhood of $\widetilde{A}_1^+ \cup \widetilde{A}_2^+$ (resp. $\widetilde{A}_1^+ \cup \widetilde{A}_2^-$). Denote by $p(\widetilde{\mathbb{R}C}_{t_\pm})$ the loop of F_X obtained by equipping the curve $\widetilde{\mathbb{R}C}_{t_\pm}$ with a family of direct orthonormal frames $(e_1(z), e_2(z), e_3(z))_{z \in \widetilde{\mathbb{R}C}_{t_\pm}}$ as before. Then, the difference $\lim_{t_+ \to t_1} p(\widetilde{\mathbb{R}C}_{t_+}) - \lim_{t_- \to t_1} p(\widetilde{\mathbb{R}C}_{t_-})$ is observed to be a loop of F_X over $\widetilde{A}_2$ which is invariant under the involution $inv_R : (e_1, e_2, e_3) \in F_X|_x \mapsto (d|_x \operatorname{inv}(e_1), d|_x \operatorname{inv}(e_2), -d|_x \operatorname{inv}(e_3)) \in F_X|_{inv(x)}$ where inv is the involution of the covering $\widetilde{\mathbb{R}X} \to \mathbb{R}X$. Indeed, its restrictions

$p(\widetilde{A}_2^+)$ and $p(\widetilde{A}_2^-)$ over $\widetilde{A}_2^+$ and $\widetilde{A}_2^-$ respectively are paths exchanged by inv_R.



It follows that

$$\begin{aligned} sp(\widetilde{\mathbb{R}C}_{t_+}) - sp(\widetilde{\mathbb{R}C}_{t_-}) &= < \pi^* w(P_X), p(\widetilde{\mathbb{R}C}_{t_+}) - p(\widetilde{\mathbb{R}C}_{t_-}) > \\ &= < w(P_X), \pi_*(p(\widetilde{A}_2^+) + p(\widetilde{A}_2^-)) > \quad (t_+, t_- \to t_1) \\ &= 0, \end{aligned}$$

where $w(P_X)$ is the characteristic class of the covering $P_X \to F_X$. $\square$

Remark 3.8 It would be interesting to extend Theorem 2.2 to the case $\sum_{i=1}^N r_i = 0$. The non-vanishing of this sum has been used to define the spinor state of real balanced rational curves of class d when $\mathbb{R}X$ is not orientable and $k_d = \frac{1}{2}c_1(X)d$ is even. It has also been used to make sure that all the real rational curves under consideration have non-empty real parts. In the case of real curves with empty real parts, I cannot define a spinor state yet, and I cannot prevent the presence of critical points as the one described in Proposition 3.5. Unfortunatly, it is not possible to get rid of these curves with empty real part, since over a generic path, a one parameter family of curves with non-empty real part can break into a reducible curve with two complex conjugated irreducible components and then deform into a one parameter family of curves with empty real part. Note that this problem did not appear in dimension 2, see [15].

4 A study of the polynomial $\chi^{d,\mathfrak{s}}$

As was done in [15], it is possible to understand how the invariant $\chi_r^{d,\mathfrak{s}}$ depends on the N-tuple r in terms of a new invariant. The aim of this last paragraph is to explain this phenomenum in the case of $(X, c_X) = (\mathbb{C}P^3, conj)$.

4.1 Invariants of the blown up projective space

4.1.1 Statement of the results

Let (Y, c_Y) be the real algebraic 3-manifold obtained after blowing up a real point x_0 of $(\mathbb{C}P^3, conj)$. Denote by $Exc \subset Y$ the exceptional divisor of the blow up, $[l]$ the class of a line in $Exc \cong \mathbb{C}P^2$ and $[f]$ the class of the strict transform of a line in $\mathbb{C}P^3$ passing through x_0. The cone $NE(Y)$ of effective curves of Y is of dimension 2, is closed and generated by $[l]$ and $[f]$. Note that Y is not convex. Indeed, if $u : \mathbb{C}P^1 \to Y$ has an image contained in Exc which is not a line, then $H^1(\mathbb{C}P^1; u^*TY) \neq 0$. Note also that if $A \subset Y$ is an irreducible curve not contained in Exc, then A is equivalent to $a[f] + b[l]$ with $0 \leq b \leq a$, since $0 \leq A.Exc = a - b$.

Let $0 \leq k \leq d$ and $d_Y = d([f]+[l]) - k[l]$, so that $c_1(Y).d_Y = 4d - 2k$. Denote by $k_{d_Y} = 2d - k$. Let $\underline{y} \subset Y^{k_{d_Y}}$ be a generic real configuration of k_{d_Y} distinct points of Y, and r be the number of real points in this configuration which is assumed to be non-zero. There are then only finitely many connected rational curves in Y passing through $\underline{y}$ in the homology class d_Y. Moreover, these curves are all irreducible and balanced and their number does not depend on the generic choice of $\underline{y}$, it is equal to the genus 0 Gromov-Witten invariant $N_{d_Y} = GW_0(Y, d_Y, pt, \ldots, pt)$ (see [3]). Denote by $\mathcal{R}_{d_Y}(\underline{y})$ the subset of these curves which are real and by $R_{d_Y}(\underline{y})$ its cardinality. Equip $\mathbb{R}Y = \mathbb{R}P^3 \# \overline{\mathbb{R}P}^3$ with a spin structure $\mathfrak{s}_Y$. This spin structure allows us to define a spinor state for each real curve $A \subset \mathcal{R}_{d_Y}(\underline{y})$, as in §2.2. Put then:

$$\chi_r^{d_Y, \mathfrak{s}_Y}(\underline{y}) = \sum_{A \in \mathcal{R}_{d_Y}(\underline{y})} sp(A) \in \mathbb{Z}.$$

Theorem 4.1 *Let (Y, c_Y) be the blown up of $(\mathbb{C}P^3, conj)$ at a real point x_0 and $\mathfrak{s}_Y$ be a spin structure on $\mathbb{R}Y$. Let $0 \leq k \leq d$, $d_Y = d([f]+[l]) - k[l]$ and $k_{d_Y} = 2d - k$. Let $\underline{y}$ be a generic real configuration of k_{d_Y} distinct points of Y such that $r = \#(\underline{y} \cap \mathbb{R}Y)$ does not vanish. Then the integer $\chi_r^{d_Y, \mathfrak{s}_Y}(\underline{y})$ does not depend on the choice of $\underline{y}$.*

This integer $\chi_r^{d_Y, \mathfrak{s}_Y}(\underline{y})$ is denoted by $\chi_r^{d_Y, \mathfrak{s}_Y}$. We deduce from this Theorem 4.1 the following lower bounds in real enumerative geometry.

Corollary 4.2 *Under the assumptions of Theorem 4.1, denote by $R_{d_Y}(\underline{y})$ the number of real irreducible rational curves passing through $\underline{y}$ in the class d_Y and by N_{d_Y} the associated Gromov-Witten invariant. Then, $|\chi_r^{d_Y, \mathfrak{s}_Y}| \leq R_{d_Y}(\underline{y}) \leq N_{d_Y}$.* $\square$

4.1.2 Proof of Theorem 4.1

Lemma 4.3 *Let Y be the blown up of $\mathbb{C}P^3$ at a point and $Exc \subset Y$ be the associated exceptional divisor. Then every morphism $u : \mathbb{C}P^1 \to Y$ such that $H^1(\mathbb{C}P^1; u^*TY) \neq 0$ has an image contained in Exc which is a curve of degree greater than one.*

Proof:

The manifold Y has a submersion $p : Y \to \mathbb{C}P^2 = Exc$ whose fibers are rational curves. In fact, Y is isomorphic to the projectivization $P(\mathcal{O}_{\mathbb{C}P^2} \oplus \mathcal{O}_{\mathbb{C}P^2}(1))$. Denote by $H = \mathcal{O}_{\mathbb{C}P^2} \oplus \mathcal{O}_{\mathbb{C}P^2}(1)$, $L \subset p^*H$ the tautological line subbundle over Y and $M = p^*H/L$. Denote by F the kernel of the morphism $dp : TY \to T\mathbb{C}P^2$, one has the isomorphism $F \cong Hom(L, M) \cong L^* \otimes M$. Let $u : \mathbb{C}P^1 \to Y$, from the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{C}P^1}(u^*F) \to \mathcal{O}_{\mathbb{C}P^1}(u^*TY) \to \mathcal{O}_{\mathbb{C}P^1}(u^*T\mathbb{C}P^2) \to 0$$

we deduce the long exact sequence

$$\cdots \to H^1(\mathbb{C}P^1; u^*(L^* \otimes M)) \to H^1(\mathbb{C}P^1; u^*TY) \to H^1(\mathbb{C}P^1; (p \circ u)^*(T\mathbb{C}P^2) \to \ldots$$

Since $\mathbb{C}P^2$ is convex, $H^1(\mathbb{C}P^1; (p \circ u)^*(T\mathbb{C}P^2) = 0$. Consider then the exact sequence $0 \to u^*L \to (p \circ u)^*H \to u^*M \to 0$. Denote by b the degree of $p \circ u : \mathbb{C}P^1 \to \mathbb{C}P^2$, one has $(p \circ u)^*H \cong \mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(b)$. We deduce the following alternative. Either u^*L is the unique line subbundle of $(p \circ u)^*H$ of degree b, which implies $\deg(u^*(L^* \otimes M)) = -b$ and

$H^1(\mathbb{C}P^1; u^*TY) \neq 0$ as soon as $b \geq 2$. In this case, $Im(u)$ is contained in Exc. Or u^*L is not the unique line subbundle of $(p \circ u)^*H$ of degree b, then $\deg(u^*(L)) \leq 0$ and $\deg(u^*(M)) \geq b$. This implies that $H^1(\mathbb{C}P^1; u^*(L^* \otimes M)) = 0$ and thus $H^1(\mathbb{C}P^1; u^*TY) = 0$. $\square$

Proof of Theorem 4.1:

Let $\tau \in \mathcal{S}_{k_{d_Y}}$ having r fixed points in $\{1, \ldots, k_{d_Y}\}$ and such that $\tau^2 = id$. Let $(\overline{\mathcal{M}}^{d_Y}_{0,k_{d_Y}}(Y), c_{\overline{\mathcal{M}},\tau})$ be the space of genus 0 stable maps of Y having k_{d_Y} marked points and realizing the homology class d_Y. This is a real algebraic subvariety of the variety $(\overline{\mathcal{M}}^{b}_{0,k_{d_Y}}(\mathbb{C}P^N), c_{\overline{\mathcal{M}},\tau})$ associated to a real embedding $\pi : (Y, c_Y) \to (\mathbb{C}P^N, conj)$ such that $\pi_* d_Y = b$, see [2]. Denote by $\overline{\mathcal{M}}^{d_Y}_{0,k_{d_Y}}(Y)^{\#}$ the complement in $\overline{\mathcal{M}}^{d_Y}_{0,k_{d_Y}}(Y)$ of the locus of curves having a multiple component or a component included in Exc. This variety is smooth of dimension $c_1(Y)d_Y + k_{d_Y} = 3k_{d_Y}$, but not compact. However, the evaluation morphism $\overline{\mathcal{M}}^{d_Y}_{0,k_{d_Y}}(Y)^{\#} \to Y^{k_{d_Y}}$ is proper over a generic path in $Y^{k_{d_Y}}$. Indeed, a reducible curve having a component included in Exc projects onto a rational curve having a point of multiplicity greater than k at x_0. This does not happen in codimension one in $Y^{k_{d_Y}}$. Theorem 4.1 is then proven in the same way as Theorem 2.2. $\square$

4.2 Relations between the coefficients of the polynomial $\chi^{d,\mathfrak{s}}(T)$

Denote once more by (Y, c_Y) the real algebraic 3-manifold obtained after blowing up a real point x_0 of $(\mathbb{C}P^3, conj)$. Assume that the riemannian metric g of $\mathbb{R}P^3$ fixed in §2.1 is flat in a neighborhood of x_0. Denote by $\overline{g}$ the associated metric of $\overline{\mathbb{R}P}^3$ and by $\overline{F}_X$ the $SO_3(\mathbb{R})$-principal bundle of direct orthonormal frames of $\overline{\mathbb{R}P}^3$. Let $\overline{P}_X \to \overline{F}_X$ be a $Spin_3$-principal bundle defining a spin structure $\overline{\mathfrak{s}}$ on $\overline{\mathbb{R}P}^3$. Making a surgery in a neighborhood of x_0 which is small enough, the manifold $\mathbb{R}Y = \mathbb{R}P^3 \# \overline{\mathbb{R}P}^3$ comes equipped with a riemannian metric $g \# \overline{g}$. Denote by F_Y the associated $SO_3(\mathbb{R})$-principal bundle of direct orthonormal frames. The bundles P_X and $\overline{P}_X$ glue together to form a $Spin_3$-principal bundle P_Y over F_Y which defines a spin structure on $\mathbb{R}Y$ denoted by $\mathfrak{s} \# \overline{\mathfrak{s}}$.

Theorem 4.4 *Let (Y, c_Y) be the blown up of $(\mathbb{C}P^3, conj)$ at a real point x_0, $\mathfrak{s}$ be a spin structure on $\mathbb{R}P^3$ and $\mathfrak{s} \# \overline{\mathfrak{s}}$ be the associated spin structure on $\mathbb{R}Y$. Let $d \in \mathbb{N}^*$, $d_Y = d([f] + [l]) - 2[l] \in H_2(Y; \mathbb{Z})$ and $k_d = 2d$. Then, for every integer r between 2 and $k_d - 2$, we have $\chi^{d,\mathfrak{s}}_{r+2} = \chi^{d,\mathfrak{s}}_r - 2\chi^{d_Y,\mathfrak{s}\#\overline{\mathfrak{s}}}_r$.*

(The classes $[f]$ and $[l]$ in $H_2(Y; \mathbb{Z})$ have been defined in §4.1.1.)

Lemma 4.5 *Let $u : \mathbb{C}P^1 \to \mathbb{C}P^3$ be a morphism passing through $x_0 \in \mathbb{C}P^3$, which is an immersion in the neighborhood of $u^{-1}(x_0)$. Denote by N_u the normal bundle of u in $\mathbb{C}P^3$ and by $z = u^{-1}(x_0) \subset \mathbb{C}P^1$. Let Y be the blown up of $\mathbb{C}P^3$ at x_0, $\tilde{u} : \mathbb{C}P^1 \to Y$ be the strict transform of u and $N_{\tilde{u}}$ be its normal bundle in Y. Then one has the isomorphism:*

$$N_{\tilde{u}} \cong N_u \otimes \mathcal{O}_{\mathbb{C}P^1}(-z).$$

It follows in particular from this lemma that if u is balanced, then so is $\tilde{u}$.

Proof:

The morphism $Y \to \mathbb{C}P^3$ induces a morphism between the normal sheaves of $\tilde{u}$ and u. This morphism is an isomorphism everywhere outside z and vanishes at z. The fact that it vanishes at order one can either be seen on local coordinates or from adjunction formula, since $\deg N_{\tilde{u}} = \deg N_u - 2$. $\square$

Proof of Theorem 4.4:

The begining of this proof is the same as the one of Theorem 3.2 of [15]. Let $\underline{x}^+ = (x_1, \dots, x_{k_d})$ be a generic real configuration of k_d distinct points of $\mathbb{C}P^3 \setminus \{x_0\}$, $r+2$ of which are real. Let two such real points, say x_{k_d-1} and x_{k_d}, converge to x_0 along a generic real tangency $\tau_0 \in P(T_{x_0}\mathbb{R}P^3)$. Denote by $\underline{x}_\infty$ the configuration of points $(x_1, \dots, x_{k_d-2}, x_0)$ in the limit. The rational curves in the set $\mathcal{R}_d(\underline{x}^+)$ converge to immersed irreducible real rational curves passing through $\underline{x}_\infty$ and either having the tangency τ_0 at x_0, or having a real ordinary double point at x_0 which is the local intersection of two real branches. (For instance, as soon as $\underline{x}_\infty$ is generic, there passes no reducible or cuspidal curves through $\underline{x}_\infty$ that have a singular point at x_0.) The latter are the limit of exactly two curves of $\mathcal{R}_d(\underline{x}^+)$, denote by $\mathcal{L}im^+(\underline{x}_\infty)$ their set. Similarly, $\underline{x}_\infty$ is the limit of a generic real configuration $\underline{x}^-$ of k_d distinct points of $\mathbb{C}P^3 \setminus \{x_0\}$, r of which being real, when two complex conjugated points of $\underline{x}^-$ converge to x_0 along a generic real tangency $\tau_0 \in P(T_{x_0}\mathbb{R}P^3)$. The rational curves in the set $\mathcal{R}_d(\underline{x}^-)$ converge to immersed irreducible real rational curves passing through $\underline{x}_\infty$ and either having the tangency τ_0 at x_0, or having a real ordinary double point at x_0 which is the local intersection of two complex conjugated branches. Indeed, the branches of the real rational curves at the two complex conjugated points of $\underline{x}^-$ that converge to x_0 are complex conjugated, so that the curves in the limit having a double point at x_0 also have complex conjugated branches at x_0. These are actually the limit of exactly two curves of $\mathcal{R}_d(\underline{x}^-)$, denote by $\mathcal{L}im^-(\underline{x}_\infty)$ their set. It follows easily that

$$\chi_{r+2}^{d,\mathfrak{s}} - \chi_r^{d,\mathfrak{s}} = 2\Big(\sum_{A \in \mathcal{L}im^+(\underline{x}_\infty)} sp(A) - \sum_{A \in \mathcal{L}im^-(\underline{x}_\infty)} sp(A) \Big).$$

For every curve $A \in \mathcal{L}im(\underline{x}_\infty) = \mathcal{L}im^+(\underline{x}_\infty) \cup \mathcal{L}im^-(\underline{x}_\infty)$, denote by A_Y its strict transform in Y. The curves A_Y, $A \in \mathcal{L}im(\underline{x}_\infty)$, are exactly the elements of the set $\mathcal{R}_{d_Y}(\underline{y})$, for $\underline{y} = (x_1, \dots, x_{k_d-2}) \in Y^{k_{d_Y}}$. When $A \in \mathcal{L}im^-(\underline{x}_\infty)$, it follows from the construction of the spin structure $\mathfrak{s} \# \overline{\mathfrak{s}}$ that $sp(A_Y) = sp(A)$. We have thus to prove that when $A \in \mathcal{L}im^+(\underline{x}_\infty)$, $sp(A_Y) = -sp(A)$.

Let $(e_1(p), e_2(p), e_3(p))_{p \in \mathbb{R}A_Y}$ be a loop in the bundle F_Y of direct orthonormal frames of $\mathbb{R}Y$ constructed in §2.2, so that $sp(A_Y)$ is the obstruction to lift this loop as a loop in P_Y. Fix an orientation of $\mathbb{R}A_Y$, which induces also an orientation of the normal bundle of $\mathbb{R}A_Y$ in $\mathbb{R}Y$, and let $\mathbb{R}A_Y \cap \mathbb{R}Exc = \{p_0, p_1\}$. We construct then the loop $(\tilde{e}_1(p), \tilde{e}_2(p), \tilde{e}_3(p))$ of F_Y as the concatenation of the four following paths. The path $(e_1(p), e_2(p), e_3(p))_{p \in [p_0, p_1]}$, then a path completely included in the fiber of F_Y over p_1. This path is obtained from $(e_1(p_1), e_2(p_1), e_3(p_1))$ by having this frame turning of half a twist in the positive direction around the axis generated by $e_1(p_1)$. The end point of this path is thus the frame $(e_1(p_1), -e_2(p_1), -e_3(p_1))$. Then the path $(e_1(p), -e_2(p), -e_3(p))_{p \in [p_1, p_0]}$. Finally, a path completely included in the fiber of F_Y over p_0. This path is obtained from $(e_1(p_0), -e_2(p_0), -e_3(p_0))$ by having this frame turning of half a twist in the negative direction around the axis generated by $e_1(p_0)$. The end point of this path is thus the frame $(e_1(p_0), e_2(p_0), e_3(p_0))$, so that we have indeed constructed a loop in F_Y which is ho-

motopic to the initial loop $(e_1(p), e_2(p), e_3(p))_{p\in\mathbb{R}A_Y}$. Remember that $\mathbb{R}Y$ is obtained as the connected sum of $\mathbb{R}P^3$ and $\overline{\mathbb{R}P}^3$ in the neighborhood of x_0. We may choose the radius of the ball used to perform the connected sum as small as we want, and have this radius converging to zero. In this process, the curve $\mathbb{R}A_Y$ degenerates to the union of the curve $\mathbb{R}A \subset \mathbb{R}P^3$ and two lines $\mathbb{R}D_1, \mathbb{R}D_2$ of $\overline{\mathbb{R}P}^3$ passing through x_0. The loop $(\tilde{e}_1(p), \tilde{e}_2(p), \tilde{e}_3(p))$ of F_Y degenerates to the union of a loop $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)|_{\mathbb{R}A}$ of F_X and two loops $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)|_{\mathbb{R}D_1}$, $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)|_{\mathbb{R}D_2}$ of $\overline{F}_X$. By construction, the obstruction to lift $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)|_{\mathbb{R}A}$ to P_X is exactly $sp(A)$. Similarly, the homotopy class of the loop $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)|_{\mathbb{R}D_1}$ differs from the one of $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)|_{\mathbb{R}D_2}$ by a non-trivial loop in a fiber of $\overline{P}_X$. We deduce that $sp(\mathbb{R}D_1) = -sp(\mathbb{R}D_2)$ independantly of the choice of a spin structure on $\overline{\mathbb{R}P}^3$. It follows that $sp(A_Y) = sp(A)sp(\mathbb{R}D_1)sp(\mathbb{R}D_2) = -sp(A)sp(\mathbb{R}D_1)^2 = -sp(A)$. $\square$

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