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Deformation classes of real ruled manifolds

By *Jean-Yves Welschinger**) at Lyon

Abstract. A complete description of the deformation classes of real ruled manifolds is given. In particular, we prove that once the complex deformation class is fixed, the real deformation class is prescribed by the topology of the real structure.

Introduction

A *real algebraic manifold* (X, c_X) is a smooth complex algebraic manifold X equipped with an antiholomorphic involution c_X. The *real part* of X is the fixed point set of c_X. One of the main problems in real algebraic geometry nowadays is to understand the deformation classes (see §2.1 for the definition) of real algebraic manifolds. One can think of this problem as a modern version of a question of Hilbert in his 16th problem concerning the topology of smooth real quartics in the real projective 3-space. Several works have already been done to solve this problem: the cases of real curves, real rational surfaces, real minimal ruled surfaces and real minimal surfaces of Kodaira dimension 0 are known (see [8], [6], [10], [4], [3] and [5] for an extension of [4] to finite group actions on K_3's). The purpose of this paper is to extend the result of [10] to ruled manifolds of higher dimensions. Note that the complex deformation classes of ruled manifolds were studied in [9].

A *ruled manifold* is a smooth algebraic manifold equipped with a proper holomorphic submersion on a smooth compact irreducible curve B, whose fibers are projective spaces. In dimension two, these are geometrically ruled surfaces. The aim of this paper is to prove the following theorem:

Theorem 0.1. *Two real ruled manifolds are in the same real deformation class if and only if they are in the same complex deformation class and they are diffeomorphic via an equivariant diffeomorphism.*

Moreover, all the deformation classes of real ruled manifolds will be described (see §2.2). It is necessary here to fix the complex deformation class of the manifold since as was noticed by E. Brieskorn (see [2], Satz 3.1), there exist complex ruled manifolds which are diffeomorphic to each other but which are not deformation equivalent. However, once the

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complex deformation class is fixed, the topology of the involution is enough to describe the real deformation classes of the real manifolds, which is the case in all the known examples nowadays.

The paper is organized as follows: in the first section, we give basic facts and preliminary results on ruled manifolds, real structures on these manifolds and a notion of elementary transformations that can be performed on them. The second section is devoted to the statements of the results and the third to their proofs.

1. Real ruled manifolds and elementary transformations

1.1. Ruled manifolds. A smooth irreducible compact complex manifold X of dimension n is said to be *ruled* if there exists a smooth irreducible compact complex curve B and a proper holomorphic submersion $p : X \to B$ whose fibers are isomorphic to the projective space $\mathbb{C}P^{n-1}$. For example, let E be a complex vector bundle of rank n over the curve B and $X = P(E)$ be the associated projective bundle. Then X is a ruled manifold. Note that when $n = 2$, ruled manifolds are geometrically ruled surfaces. For these surfaces, it is well known that the curve B is unique, so as the ruling p except from $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ (see [1]). The following lemma extends this result.

Lemma 1.1. *Let X be a ruled manifold of dimension $n \geqq 3$. Then the ruling p and the curve B are unique.*

The curve B is called the *base* of X.

Proof. It follows from the fact that the only divisors of X isomorphic to $\mathbb{C}P^{n-1}$ are the fibers of p. Indeed, p would restrict otherwise to a surjective morphism from this divisor onto B, and the generic fibers of this morphism would give smooth disjoint complex hypersurfaces of $\mathbb{C}P^{n-1}$. Such hypersurfaces do not exist in dimension $n - 1 \geqq 2$. $\square$

The following proposition is mentioned in [9], p. 214.

Proposition 1.2. *Let X be a ruled manifold of dimension n over B. Then there exists a complex vector bundle E of rank n over B such that X is isomorphic to the projective bundle $P(E)$. Moreover, the projective bundle $P(E')$ is isomorphic to $P(E)$ if and only if $E' = E \otimes L$ for $L \in \mathrm{Pic}(B)$.* $\square$

Corollary 1.3. *Ruled manifolds are all projective algebraic.* $\square$

Remark 1.4. Let $L \in \mathrm{Pic}(B)$ and E be a complex vector bundle of rank n over B. Then $\deg(E \otimes L) = \deg(E) + n\deg(L)$, where $\deg(E)$ stands for the degree of E.

Let $X = P(E)$ be a ruled manifold of dimension n over B. We define the *degree of X* to be $\deg(E)$ reduced modulo n. It will be denoted by $\deg(X) \in \mathbb{Z}/n\mathbb{Z}$. Let L be a complex line bundle over B and L_0 be the trivial line bundle. The section $P(L)$ (resp. $P(L_0)$) of the ruled surface $P(L \oplus L_0)$ defines a divisor on this surface denoted by D_L (resp. D_{L_0}).

Lemma 1.5. (1) *Let p be the ruling $P(L \oplus L_0) \to B$ and $\mathcal{O}(D_L)$, $\mathcal{O}(D_{L_0})$ denote the invertible sheaves associated to the divisors D_L and D_{L_0}. Then*

$$\mathcal{O}(D_L) = \mathcal{O}(D_{L_0}) \otimes p^*(L^*).$$

(2) *Let F be a complex vector bundle over B, X be the ruled manifold $P(L \oplus L_0 \oplus F)$ and N be the normal bundle of $P(L \oplus L_0)$ in X. Then*:

$$N = p^*(F) \otimes \mathcal{O}(D_L).$$

Proof. Let $D = \sum_{i=1}^{k} n_i p_i$ be a divisor associated to L, where $p_i \in B$ and $n_i \in \mathbb{Z}$ for $i \in \{1, \dots, k\}$. Denote by $U_0 = B \backslash \{p_i \mid 1 \leqq i \leqq k\}$ and for every $i \in \{1, \dots, k\}$, choose some holomorphic chart (U_{p_i}, ϕ_{p_i}) such that $U_{p_i} \cap U_{p_j} = \emptyset$ if $i \neq j$ and $\phi_{p_i} : U_{p_i} \to \Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ is a biholomorphism satisfying $\phi_{p_i}(p_i) = 0$. For every $i \in \{1, \dots, k\}$, denote by ψ_i the morphism:

$$\begin{aligned} (U_{p_i} \backslash p_i) \times \mathbb{C}P^1 &\to U_0 \times \mathbb{C}P^1, \\ \big(x, (z_1 : z_0)\big) &\mapsto \big(x, \big(\phi_{p_i}(x)^{-n_i} z_1 : z_0\big)\big). \end{aligned}$$

The morphisms ψ_i allow to glue together the trivializations $U_{p_i} \times \mathbb{C}P^1$, $i \in \{0, \dots, k\}$, in order to define the ruled surface $P(L \oplus L_0)$.

Let $f : p^{-1}(U_0) \to \mathbb{C}$, $\big(x, (z_1 : 1)\big) \mapsto z_1$. Then f extends to a meromorphic function on $P(L \oplus L_0)$ such that $f^{-1}(0) = D_L + \sum_{n_i \geqq 0} n_i p^{-1}(p_i)$ and $f^{-1}(\infty) = D_{L_0} + \sum_{n_i \leqq 0} n_i p^{-1}(p_i)$. Hence $\operatorname{div}(f) = D_L - D_{L_0} + p^{-1}(D)$, so that $\mathcal{O}(D_L) = \mathcal{O}(D_{L_0}) \otimes p^*(L^*)$, which proves the first part of the lemma.

To prove the second part of the lemma, take a refinement of the covering (U_i) such that the bundle F is trivial over any element of this covering. The manifold X is then defined as the gluing of charts $\mathcal{U}_i^0$ and $\mathcal{U}_i^1$ isomorphic to $U_i \times \mathbb{C}^{n-1}$ with gluing maps: $\mathcal{U}_i^0 \cap \mathcal{U}_i^1 \to \mathcal{U}_i^1 \cap \mathcal{U}_i^0$, $(x, z_1, f) \mapsto \left(x, \frac{1}{z_1}, \frac{1}{z_1} f\right)$ where f is a local trivialization of F over U_i, and $\mathcal{U}_i^0 \cap \mathcal{U}_j^0 \to \mathcal{U}_j^0 \cap \mathcal{U}_i^0$, $(x, z_0, f) \mapsto \big(x, l_{ij}^{-1} z_0, l_{ij}^{-1} g_{ij}(f)\big)$ where l_{ij} and g_{ij} are the changes of trivialization of L and F respectively. We deduce that the normal bundle N of $P(L \oplus L_0)$ in $P(L \oplus L_0 \oplus F)$ is defined as the gluing of the trivializations:

$$\begin{aligned} \big(\mathcal{U}_i^0 \cap \mathcal{U}_i^1 \cap P(L \oplus L_0)\big) \times \mathbb{C}^{n-2} &\to \big(\mathcal{U}_i^1 \cap \mathcal{U}_i^0 \cap P(L \oplus L_0)\big) \times \mathbb{C}^{n-2}, \\ \big((x, z_1), v\big) &\mapsto \left(\left(x, \frac{1}{z_1}\right), \frac{1}{z_1} v\right) \end{aligned}$$

and

$$\begin{aligned} \big(\mathcal{U}_i^0 \cap \mathcal{U}_j^0 \cap P(L \oplus L_0)\big) \times \mathbb{C}^{n-2} &\to \big(\mathcal{U}_j^0 \cap \mathcal{U}_i^0 \cap P(L \oplus L_0)\big) \times \mathbb{C}^{n-2}, \\ \big((x, z_0), v\big) &\mapsto \big((x, l_{ij}^{-1} z_0), l_{ij}^{-1} g_{ij}(v)\big). \end{aligned}$$

Hence $N = p^*(F) \otimes \mathcal{O}(D_{L_0}) \otimes p^*(L^*) = p^*(F) \otimes \mathcal{O}(D_L)$. $\square$

Proposition 1.6. *Let B be a smooth irreducible compact complex curve and L be a complex line bundle over B such that $L \neq L^*$ if L is non-trivial. Let $E = (L \oplus L_0)^k$ and $X = P(E)$. Then every automorphism of X fibered over the identity of B which leaves the k ruled surfaces $P(L \oplus L_0)$ invariant lifts to an automorphism of E fibered over the identity of B.*

Proof. Let ϕ be such an automorphism of X. From proposition 2.1 of [10], we know that the restriction of ϕ to the jth ruled surface $P(L \oplus L_0)$ lifts to an automorphism ψ_j of the rank two vector bundle $L \oplus L_0$. Let Ψ be the automorphism $(\psi_1, \ldots, \psi_k)$ of E. This automorphism induces an automorphism ψ of X such that $\psi^{-1} \circ \phi$ is the identity once restricted to each ruled surface $P(L \oplus L_0)$. But such an automorphism lifts to a diagonal automorphism of E of the form $(\lambda_1 \operatorname{Id}, \ldots, \lambda_k \operatorname{Id})$, where $\lambda_j \in \mathbb{C}^*$. Hence the result. $\square$

1.2. Real structures on ruled manifolds. A *real structure* on the ruled manifold X is an antiholomorphic involution $c_X : X \to X$. The fixed point set of c_X is called the *real part* of X and is denoted by $\mathbb{R}X$.

Lemma 1.7. *Let $p : X \to B$ be a ruled manifold of dimension $n > 2$ and c_X be a real structure on X. Then there exists a real structure c_B on B such that $p \circ c_X = c_B \circ p$. Moreover, this real structure c_B is unique.*

The real structure c_X will be said to be *fibered* over c_B.

Proof. From the proof of lemma 1.1, we know that the only divisors of X isomorphic to $\mathbb{C}P^{n-1}$ are the fibers of p. So c_X preserves these fibers and hence induces a diffeomorphism c_B on the base. This diffeomorphism is antiholomorphic and is an involution. $\square$

We deduce from this lemma that the connected components of $\mathbb{R}X$ are $\mathbb{R}P^{n-1}$-bundles over the circle. For odd n, such a bundle is unique whereas for even n there are two such bundles, one which is orientable and the other one which is not. We define the *topological type* of a real ruled manifold (X, c_X) of even complex dimension n to be the quintuple of integers $(t, k, g, \mu, \varepsilon)$ where t is the number of orientable components of $\mathbb{R}X$, k is the number of non-orientable components of $\mathbb{R}X$ and (g, μ, ε) is the topological type of the real curve (B, c_B), that is the genus of B, the number of connected components of $\mathbb{R}B$ and the dividing or non-dividing type of (B, c_B). This definition extends the one given in [10] for $n = 2$.

Let us present now an important example of real ruled manifold. Let (B, c_B) be a real algebraic curve and L be a complex line bundle over B such that $c_B^*(L) = L^*$ where c_B^* is the real structure on $\operatorname{Pic}(B)$ induced by c_B (see [10], §1.1). Let D be a divisor associated to L and f_D be a meromorphic function on B such that $\operatorname{div}(f_D) = D + c_B(D)$ and $f_D = \overline{f_D \circ c_B}$ (it always exists, see [10], lemma 1.3). Note that the sign of f_D is constant on every component of $\mathbb{R}B$. The following proposition is analogous to proposition 1.6 of [10]:

Proposition 1.8. *Associated to every such couple (D, f_D) on (B, c_B), there exists a real structure c_{f_D} on $X = P\big((L \oplus L_0)^{\frac{n}{2}}\big)$ fibered over c_B, whose real part is orientable and maps surjectively onto the components of $\mathbb{R}B$ on which f_D is non-negative.*

Remark 1.9. When there will not be any ambiguity on the choice of the couple (D, f_D), we will denote by c_X^+ (resp. c_X^-) the real structure c_{f_D} (resp. c_{-f_D}).

Proof. Denote $D = \sum_{i=1}^{k} n_i p_i$ where $p_i \in B$ and $n_i \in \mathbb{Z}$ for $i \in \{1, \dots, k\}$. We can assume that the set $\{p_i \mid 1 \leqq i \leqq k\}$ is invariant under c_B. Let $U_0 = B \backslash \{p_i \mid 1 \leqq i \leqq k\}$ and (U_{p_i}, ϕ_{p_i}), $i \in \{1, \dots, k\}$, be an atlas compatible with the divisor D and the group $\langle c_B \rangle$ (see [10], page 3).

The morphisms:

$$(U_{p_i} \backslash p_i) \times \mathbb{C}P^{n-1} \to U_0 \times \mathbb{C}P^{n-1},$$
$$\left(x, (z_1^j : z_0^j)\right) \mapsto \left(x, \left(\phi_{p_i}(x)^{-n_i} z_1^j : z_0^j\right)\right)$$

$(i \in \{1, \dots, k\},\ j \in \{1, \dots, n/2\})$ allow to glue together the trivializations $U_{p_i} \times \mathbb{C}P^{n-1}$, $i \in \{0, \dots, k\}$, in order to define the ruled manifold X.

Now, the maps

$$U_0 \times \mathbb{C}P^{n-1} \to U_0 \times \mathbb{C}P^{n-1},$$
$$\left(x, (z_1^j : z_0^j)\right) \mapsto \left(c_B(x), \left(\overline{z_0^j} : f_D \circ c_B(x) \overline{z_1^j}\right)\right),$$

and for every $i \in \{1, \dots, k\}$,

$$U_{p_i} \times \mathbb{C}P^{n-1} \to U_{c_B(p_i)} \times \mathbb{C}P^{n-1},$$
$$\left(x, (z_1^j : z_0^j)\right) \mapsto \left(c_B(x), \left(\overline{z_0^j} : f_D \circ c_B(x) \overline{\phi_{p_i}(x)}^{-n_{c_B(p_i)} - n_{p_i}} \overline{z_1^j}\right)\right),$$

where $j \in \{1, \dots, n/2\}$, glue together to form an antiholomorphic map c_{f_D} on X. This map lifts c_B and is an involution.

Now, the fixed point set of c_{f_D} in $U_0 \times \mathbb{C}P^{n-1}$ is:

$$\left\{ \left(x, \left(\overline{\theta_j} : \sqrt{f_D(x)}\theta_j\right)\right) \in U_0 \times \mathbb{C}P^{n-1} \,\middle|\, x \in \mathbb{R}B, f_D(x) \geqq 0 \text{ and } \theta_j \in \mathbb{C} \text{ for } j \in \left\{1, \dots, \frac{n}{2}\right\} \right\}.$$

The connected components of this fixed point set are either an orientable $\mathbb{R}P^{n-1}$-bundle over a circle or an interval depending on whether the corresponding component of $\mathbb{R}B$ is completely included in U_0 or not. Similarly, the fixed point set of c_{f_D} in $U_{p_i} \times \mathbb{C}P^{n-1}$ is:

$$\left\{ \left(x, \left(\overline{\theta_j^i} : \sqrt{f_D(x) \times x_i^{-2n_i}}\theta_j^i\right)\right) \in U_{p_i} \times \mathbb{C}P^{n-1} \,\middle|\, x \in \mathbb{R}B, f_D(x) \geqq 0 \text{ and } \theta_j^i \in \mathbb{C}, j \in \left\{1, \dots, \frac{n}{2}\right\} \right\},$$

where $x_i = \phi_{p_i}(x)$. This fixed point set is a cylinder if $p_i \in \mathbb{R}B$ and is empty otherwise.

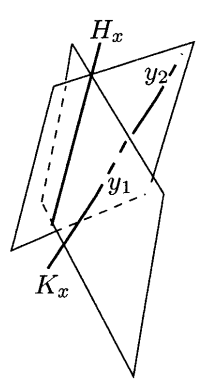
The gluing maps between these cylinders are given by $\theta_j = \sqrt{-1}\theta_j^i$ if $x_i = \phi_{p_i}(x) < 0$

and by $\theta_j = \theta_j^i$ if $x_i = \phi_{p_i}(x) > 0$. Since these two maps preserve the orientation of $\mathbb{R}P^{n-1}$, the results of these gluings are always orientable. Thus, the real part of (X, c_{f_D}) consists only of orientable components and these components stand exactly over the components of $\mathbb{R}B$ on which $f_D \geqq 0$. $\square$

Proposition 1.10. *Let (B, c_B) be a real algebraic curve with non-empty real part and $L \in \mathrm{Pic}(B)$ be such that $c_B^*(L) = L^*$ and $L \neq L^*$ if L is non-trivial. Let n be an even integer, $X = P\big((L \oplus L_0)^{\frac{n}{2}}\big)$ and c_X be a real structure on X which leaves the $n/2$ ruled surfaces $P(L \oplus L_0) \subset X$ invariant. Then c_X is conjugated to one of the two real structures $c_X^{\pm}$.*

Proof. If L is trivial, then $X = B \times \mathbb{C}P^{n-1}$ and $c_X^{\pm}$ are the two real structures $c_B \times \mathrm{conj}$ and $c_B \times c_0$, c_0 being the real structure of $\mathbb{C}P^{n-1}$ with empty real part. Now the automorphisms of X fibered over the identity of B are just the automorphisms of $\mathbb{C}P^{n-1}$, so every real structure of X fibered over c_B is the product of c_B with a real structure of $\mathbb{C}P^{n-1}$. Hence the result follows from the well known fact that the standard complex conjugation conj and c_0 are the only real structures of $\mathbb{C}P^{n-1}$ up to conjugation. Let us now assume that L is non-trivial. The real structure c_X can be written $c_X^+ \circ \phi$, where ϕ is an automorphism of X fibered over the identity of B and which leaves the $n/2$ ruled surfaces $P(L \oplus L_0) \subset X$ invariant. From proposition 1.6, there exists an automorphism Φ of the vector bundle $(L \oplus L_0)^{\frac{n}{2}}$ which lifts ϕ. Since L is non-trivial, $H^0(B; L) = H^0(B; L^*) = 0$, so that Φ is diagonal of the form $(a_1, b_1, a_2, b_2, \ldots, a_{\frac{n}{2}}, b_{\frac{n}{2}})$ with $a_j, b_j \in \mathbb{C}^*$. Since c_X^2 is the identity, $a_1\overline{b_1} = a_2\overline{b_2} = \cdots = a_{\frac{n}{2}}\overline{b_{\frac{n}{2}}} \in \mathbb{R}^*$. Thus, dividing Φ by a real constant if necessary, we can assume that $a_1\overline{b_1} = a_2\overline{b_2} = \cdots = a_{\frac{n}{2}}\overline{b_{\frac{n}{2}}} = \pm 1$. Moreover, we can assume that all the coefficients a_j are equal to one, replacing c_X by its conjugated with the automorphism $(a_1^{-1}, 1, a_2^{-1}, 1, \ldots, a_{\frac{n}{2}}^{-1}, 1)$ otherwise. Then, either all the coefficients b_j are equal to 1, or they are all equal to -1. In the first case, c_X is conjugated to c_X^+, in the second case, it is conjugated to c_X^-. $\square$

1.3. Elementary transformations. Let X be a ruled manifold of dimension n over the curve B. Let $x \in B$, $X_x = p^{-1}(x)$ and $H_x, K_x \subset X_x$ be two disjoint projective subspaces of X_x of dimensions k and $n-2-k$ respectively, where $0 \leqq k \leqq n-2$. The blowing up Y of X along H_x creates an exceptional divisor E_x isomorphic to $P(N_x \oplus L_0)$ where N_x is the normal bundle of H_x in X_x. The strict transform $\tilde{X}_x$ of X_x in Y intersects E_x in the submanifold $P(N_x) \subset P(N_x \oplus L_0)$. Moreover, $\tilde{X}_x$ is a ruled manifold over K_x. Indeed, for every $y \in K_x$ consider the projective subspace of dimension $k+1$ of X_x containing H_x and passing through y. These projective subspaces form a singular pencil parametrized by K_x. This pencil lifts on Y to a ruled manifold of base K_x and fiber $\mathbb{C}P^k$, which coincide with $\tilde{X}_x$.



The composition of the blowing up of X along H_x and the blowing down of $\tilde{X}_x$ on K_x is called the *elementary transformation* of X along H_x. For example if $n = 2$, then H_x and K_x are points in X_x and the elementary transformation of X along H_x is the blowing up of the point H_x composed with the blowing down of the strict transform of the fiber X_x. This notion of elementary transformation thus extends the one used in [10] for ruled surfaces.

Lemma 1.11. *Let* $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$, *the elementary transformation of* $\Delta \times P(\mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1})$ *along* $\{0\} \times P(\mathbb{C}^{k+1})$ *is given by the following quadratic transform*:

$$\Delta \times P(\mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1}) \to \Delta \times P(\mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1}),$$
$$\big(x, (y_0 : \cdots : y_k : z_0 : \cdots : z_{n-2-k})\big) \mapsto \big(x, (t_0 : \cdots : t_k : w_0 : \cdots : w_{n-2-k})\big)$$

with $t_i = y_i$ *for* $0 \leqq i \leqq k$ *and* $xw_j = z_j$ *for* $0 \leqq j \leqq n-k-2$.

Proof. It suffices to notice that the blowing up of $\Delta \times P(\mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1})$ along $\{0\} \times P(\mathbb{C}^{k+1})$ embeds into $\Delta \times \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}$. The image of this embedding is:

$$\begin{aligned} Y = \big\{ &\big(x, (y_0 : \cdots : y_k : z_0 : \cdots : z_{n-2-k}), (t_0 : \cdots : t_k : w_0 : \cdots : w_{n-2-k})\big) \in \Delta \times \mathbb{C}P^{n-1} \\ &\times \mathbb{C}P^{n-1} \mid (y_0 : \cdots : y_k) = (t_0 : \cdots : t_k) \text{ or } y_i = t_i = 0 \\ &\text{and } (xy_i : z_0 : \cdots : z_{n-2-k}) = (t_i : w_0 : \cdots : w_{n-2-k}) \text{ for } 0 \leqq i \leqq k \big\}. \end{aligned}$$

(The equations $(xy_i : z_0 : \cdots : z_{n-2-k}) = (t_i : w_0 : \cdots : w_{n-2-k})$ have to be ignored when they are not well defined.) The projection of Y onto the first factor $\Delta \times \mathbb{C}P^{n-1}_{(y,z)}$ (resp. onto the second factor $\Delta \times \mathbb{C}P^{n-1}_{(t,w)}$) is the blowing up of $\Delta \times P(\mathbb{C}^{k+1} \times \mathbb{C}^{n-k-1})$ along $\{0\} \times P(\mathbb{C}^{k+1})$ (resp. along $\{0\} \times P(\mathbb{C}^{n-k-1})$), hence the result. $\square$

Corollary 1.12. *Let* F, G *be complex vector bundles over a curve* B. *Let* $x \in B$ *and* $X = P(F \oplus G)$. *The ruled manifold obtained from* X *after an elementary transformation along* $P(F_x)$ *is the manifold* $P\big(F(x) \oplus G\big)$ *where* $F(x) = F \otimes \mathcal{O}(x)$. $\square$

The next corollary will be fundamental in what follows. It allows to break into pieces or to glue together some elementary transformations.

Corollary 1.13. *Let* F, G, H *be complex vector bundles over a curve* B, $x \in B$ *and* $X = P(F \oplus G \oplus H)$. *The ruled manifold obtained from* X *after an elementary transformation along* $P(F_x)$ *followed by an elementary transformation along* $P(G_x)$ *is the ruled manifold obtained from* X *after an elementary transformation along* $P(F_x \oplus G_x)$. $\square$

2. Statements of the results

2.1. Deformation over $\mathbb{C}$ of ruled manifolds. Let $\Delta \subset \mathbb{C}$ be the Poincaré's disk equipped with the complex conjugation conj. Remember that a *deformation* of complex manifolds of dimension n is a proper holomorphic submersion $\pi : Y \to \Delta$ where Y is an analytic manifold of dimension $n+1$. If Y is real and π satisfies $\pi \circ c_Y = \text{conj} \circ \pi$, then the deformation is said to be *real*. When $t \in]-1, 1[\in \Delta$, the fibers $Y_t = \pi^{-1}(t)$ are invariant under c_Y and are then compact real analytic manifolds. Two complex (resp. real) analytic

manifolds X' and X'' are said to be *in the same deformation class* or *deformation equivalent* if there exists a chain $X' = X_0, \ldots, X_k = X''$ of compact complex (resp. real) analytic manifolds such that for every $i \in \{0, \ldots, k-1\}$, the manifolds X_i and X_{i+1} are isomorphic to some complex (resp. real) fibers of a complex (resp. real) deformation.

The following result can be found in [9].

Theorem 2.1. *Two complex ruled manifolds are in the same deformation class if and only if they have same degree and bases of same genus.* $\square$

(See §1.1 for the definition of the degree.)

Remember that ruled manifolds with opposite degrees and bases of same genus are diffeomorphic even though they are not deformation equivalent, see [2].

2.2. Deformation over $\mathbb{R}$ of ruled manifolds.

Theorem 2.2. *Two real ruled manifolds of odd dimension are in the same deformation class if and only if they have same degree and bases of same topological type. Similarly, two real ruled manifolds of even dimension are in the same deformation class if and only if they have same degree, same topological type and homeomorphic quotients.*

Remark 2.3. For real ruled manifolds of even dimension, as soon as the real part of the base is non-empty, the condition on the quotients can be removed. However, when the real part of the base is empty, there are two different deformation classes of real ruled manifolds with same degree and same topological type, see proposition 3.7. Note that when $n = 2$, this theorem 2.2 has already been obtained in [10], theorem 3.7.

Note that when n is odd, the ruled manifolds $P\big(L\big(x + c_B(x)\big)^d \oplus L_0^{n-d}\big)$, where $d \in \{0, \ldots, n-1\}$, $x \in B \backslash \mathbb{R}B$ and (B, c_B) is of any topological type, are all real. Hence there do exist odd-dimensional real ruled manifolds of any degree and any topological type (realized by a real algebraic curve). This together with theorem 2.2 completely describes the deformation classes of real ruled manifolds of odd dimension. A quintuple $(t, k, g, \mu, \varepsilon)$ of integers is called *allowable* if $t, k \geqq 0$, $t + k \leqq \mu$ and (g, μ, ε) is the topological type of a real curve. Obviously, the topological types of real ruled manifolds (see §1.2 for the definition) of even dimensions are allowable.

Proposition 2.4. *Any allowable quintuple $(t, k, g, \mu, \varepsilon)$ is realized as the topological type of a real ruled manifold of any even dimension and any degree d satisfying $d = k \bmod(2)$. There does not exist such real ruled manifold of degree $d \neq k \bmod(2)$.*

Note that in dimension two, this proposition has already been obtained in [10], proposition 3.4. Together with theorem 2.2 and remark 2.3, it completely describes the deformation classes of real ruled manifolds of even dimension.

Proof. Let $(t, k, g, \mu, \varepsilon)$ be an allowable quintuple. There exists a smooth compact connected real algebraic curve (B, c_B) whose topological type is (g, μ, ε) (see [8] for instance). If $\mu = 0$, the ruled manifold $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$, where conj is the standard real structure on $\mathbb{C}P^{n-1}$, is of topological type $(0, 0, g, 0, 0)$. If $\mu \neq 0$, choose a partition $\mathscr{P}$ of $\mathbb{R}B$

in two elements such that one of them contains $t+k$ components of $\mathbb{R}B$ and the other one $\mu - t - k$. It follows from [10], lemma 3.2, that there exists a line bundle L over B such that $c_B^*(L) = L^*$ and the partition associated to L is $\mathscr{P}$ (see [10], §3.1 for a definition). Thus, it follows from proposition 1.8 that there exists a real structure c_X^+ on the ruled manifold $X = P\big((L \oplus L_0)^{\frac{n}{2}}\big)$ such that the real part of X consists of $t+k$ orientable $\mathbb{R}P^{n-1}$ bundles. Choose k of these orientable bundles and make an elementary transformation on a point of each of them. The result is a real ruled manifold of degree k and topological type $(t, k, g, \mu, \varepsilon)$. To get any other ruled manifold of degree congruent to k modulo 2 and same topological type, it suffices to perform the suitable number of elementary transformations along complex conjugated points. The fact that the condition $d = k \bmod(2)$ is necessary will follow from the proof of theorem 2.2. Indeed, we will see that every real ruled manifold is deformation equivalent to a manifold obtained from a degree 0 real ruled manifold with orientable real part after a finite number of elementary transformations performed on real or complex conjugated points. Thus the degree modulo two of X is encoded by the topology of the real part, which finishes the proof. $\square$

3. Proof of theorem 2.2

Since the case of real ruled manifolds having bases with empty real parts requires special attention, the proof of theorem 2.2 in this case is postponed to §3.2.

3.1. When the base has non-empty real part. The next two propositions will allow us to reduce the study of real ruled manifolds to the study of some particular ones. Note that even if this paragraph is devoted to real ruled manifolds having bases with non-empty real parts, the assumption $\mathbb{R}B \neq \emptyset$ won't be made in proposition 3.1 and lemma 3.3.

Proposition 3.1. *Let (X, c_X) be a real ruled manifold over (B, c_B). Then there exist complex line bundles $L_1, \dots, L_k$ over B, complex vector bundles $F_1, \dots, F_l$ of rank two over B and a real structure c_Y on $Y = P(L_1 \oplus \cdots \oplus L_k \oplus F_1 \oplus \cdots \oplus F_l)$ such that:*

– *The sections $P(L_i)$, $i \in \{1, \dots, k\}$, and the ruled surfaces $P(F_j)$, $j \in \{1, \dots, l\}$, of (Y, c_Y) are all real.*

– *The real ruled surfaces $P(F_j) \subset Y$, $j \in \{1, \dots, l\}$, do not have any real holomorphic section.*

– *The real ruled manifolds (X, c_X) and (Y, c_Y) are in the same deformation class.*

The proof of this proposition is similar to the proof of proposition 3.8 in [10].

Proof. Let E be a complex vector bundle of rank n over B such that $X = P(E)$ (it exists from proposition 1.2). If X has a real holomorphic section, denote by M the sub-line bundle of E associated to such a section. Otherwise, let S be a holomorphic section of X. Then $c_X(S) \neq S$, so these two sections intersect in a finite number of points over the points $x_1, \dots, x_k$ of B say. Let $R \subset X$ be the ruled surface generated by S and $c_X(S)$, that is the closure of the surface whose fiber over $x \in B \backslash \{x_1, \dots, x_k\}$ is the line joining $S(x)$ to $c_X(S)(x)$ in X_x. By construction, R is a real ruled surface of X. Denote in this case by M the rank two sub-bundle of E associated to R. Now, let N be the quotient bundle E/M so

that E is an extension of N by M. Let $\mu \in H^1\big(B, \mathscr{H}om(N,M)\big)$ be the extension class of this bundle and let μ^1 be a 1-cocycle with coefficients in the sheaf $\mathscr{H}om(N,M)$, defined on a covering $\mathscr{U} = (U_i)_{i\in I}$ of B, realizing the cohomology class $\mu \in H^1\big(B, \mathscr{H}om(N,M)\big)$. The bundle E is then obtained as the gluing of the bundles $(M \oplus N)|_{U_i}$ by the gluing maps:

$$(M \oplus N)|_{U_i \cap U_j} \to (M \oplus N)|_{U_j \cap U_i},$$
$$(m,n) \mapsto \begin{bmatrix} 1 & \mu_{ij} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = (m + \mu_{ij} n, n).$$

We can assume that for every open set U_i of $\mathscr{U}$, there exists $\bar{\imath} \in I$ such that $U_{\bar{\imath}} = c_B(U_i)$ (add these open sets to $\mathscr{U}$ if not). We can also assume that there exists $J \subset I$ such that the open sets $(U_i)_{i\in J}$ cover B and such that the real structure $c_X : X|_{U_i} \to X|_{U_{\bar{\imath}}}$ lifts to an antiholomorphic map $E|_{U_i} \to E|_{U_{\bar{\imath}}}$ (take a refinement of $\mathscr{U}$ if not). Since by hypothesis the section or ruled surface of X associated to M is real, these antiholomorphic maps are of the form:

$$(M \oplus N)|_{U_i} \to (M \oplus N)|_{U_{\bar{\imath}}},$$
$$\big(x, (m,n)\big) \mapsto \big(c_B(x), \begin{bmatrix} a_i & b_i \\ 0 & d_i \end{bmatrix} \begin{pmatrix} m \\ n \end{pmatrix}\big),$$

where a_i (resp. b_i, resp. d_i) is an antiholomorphic morphism $M|_{U_i} \to M|_{U_{\bar{\imath}}}$ (resp. $N|_{U_i} \to M|_{U_{\bar{\imath}}}$, resp. $N|_{U_i} \to N|_{U_{\bar{\imath}}}$) which lifts c_B. Since c_X is an involution, we have for every $i \in J$, $a_{\bar{\imath}} \circ a_i = d_{\bar{\imath}} \circ d_i \in \mathscr{O}_B^*|_{U_i}$ and $a_{\bar{\imath}} \circ b_i + b_{\bar{\imath}} \circ d_i = 0 \in \mathscr{H}om(N,M)|_{U_i}$. Moreover, for $i, j \in J$ such that $U_i \cap U_j \neq \emptyset$, the gluing conditions are the following: $a_i = \lambda a_j$, $d_i = \lambda d_j$ and $b_i + \mu_{\bar{\imath}\bar{\jmath}} \circ d_i = \lambda(a_j \circ \mu_{ij} + b_j)$ where $\lambda \in \mathscr{O}_B^*|_{U_i \cap U_j}$.

Now let Y be the complex analytic manifold of dimension $n+1$ defined as the gluing of the charts $\mathbb{C} \times P(M \oplus N)|_{U_i}$, $i \in J$, with change of charts given by the maps:

$$\mathbb{C} \times P(M \oplus N)|_{U_i} \to \mathbb{C} \times P(M \oplus N)|_{U_j},$$
$$\big(t, x, (m:n)\big) \mapsto (t, x, \begin{bmatrix} 1 & t\mu_{ij} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} m \\ n \end{pmatrix}) = \big(t, x, (m + t\mu_{ij} n : n)\big).$$

The projection on the first coordinate defines a holomorphic submersion $\pi : Y \to \mathbb{C}$. The surface $\pi^{-1}(0)$ is isomorphic to the ruled manifold $P(M \oplus N)$, whereas, as soon as $t \in \mathbb{C}^*$, the fiber $Y_t = \pi^{-1}(t)$ is isomorphic to the ruled manifold $X = P(E)$. Such an isomorphism $\psi_t : Y_t \to X$ is given in the charts $P(M \oplus N)|_{U_i}$, $i \in J$, by:

$$P(M \oplus N)|_{U_i} \to P(M \oplus N)|_{U_i},$$
$$\big(x, (m:n)\big) \mapsto \big(x, (m : tn)\big).$$

Denote by c_Y the real structure on Y defined on charts $\mathbb{C} \times P(M \oplus N)|_{U_i}$ by:

$$\mathbb{C} \times P(M \oplus N)|_{U_i} \to \mathbb{C} \times P(M \oplus N)|_{U_{\bar{\imath}}},$$
$$\big(t, x, (m:n)\big) \mapsto \big(\bar{t}, c_B(x), \begin{bmatrix} a_i & \bar{t} b_i \\ 0 & d_i \end{bmatrix} \begin{pmatrix} m \\ n \end{pmatrix}\big).$$

This real structure satisfies $\pi \circ c_Y = \mathrm{conj} \circ \pi$ where conj is the complex conjugation on $\mathbb{C}$. Moreover, when $t \in \mathbb{R}^*$, ψ_t gives an isomorphism between the real ruled manifolds $(Y_t, c_Y|_{Y_t})$ and (X, c_X). Hence, the restriction of $\pi : Y \to \mathbb{C}$ over $\Delta \subset \mathbb{C}$ defines a real deformation and the real ruled manifold $P(M \oplus N)$ is thus deformation equivalent to (X, c_X). By iteration of this process to $P(N) \subset P(M \oplus N)$, we obtain the desired result. $\square$

From now on, we will assume that $X = P(L_1 \oplus \cdots \oplus L_k \oplus F_1 \oplus \cdots \oplus F_l)$ and that c_X satisfies the conditions of proposition 3.1. It follows from proposition 3.1 that this can be done without changing the deformation class of (X, c_X).

Proposition 3.2. *Let $X = P(L_1 \oplus \cdots \oplus L_k \oplus F_1 \oplus \cdots \oplus F_l)$ be a ruled manifold over B and c_X be a real structure satisfying the conditions of proposition* 3.1. *Assume that the real part of B is non-empty, then*:

(1) *If $k \geqq 1$, without changing the deformation class of (X, c_X), we can assume that $l = 0$.*

(2) *If $k = 0$, without changing the deformation class of (X, c_X), we can assume that the real ruled surfaces $P(F_j) \subset X$, $j \in \{1, \ldots, l\}$, are obtained from a same real decomposable ruled surface after at most one elementary transformation on each component of its real part if $\mathbb{R}X \neq \emptyset$ and after at most one couple of elementary transformations on complex conjugated points if $\mathbb{R}X = \emptyset$.*

Lemma 3.3. *Let $L \in \mathrm{Jac}(B)$ be a complex line bundle belonging to the same component of the real part of $\big(\mathrm{Jac}(B), -c_B^*\big)$ as the trivial line bundle. Then there exists a divisor D associated to L such that $c_B(D) = -D$. The same conclusion holds for any line bundle in the real part of $\big(\mathrm{Jac}(B), -c_B^*\big)$ when $\mathbb{R}B$ is empty.*

(In this lemma, the Jacobian $\mathrm{Jac}(B)$ is identified with the degree zero part of the Picard manifold of B.)

Proof. Let L be a line bundle belonging to the same component of the real part of $\big(\mathrm{Jac}(B), -c_B^*\big)$ as the trivial line bundle. Then there exists a line bundle $\tilde{L}$ in this component such that $\tilde{L} \otimes \tilde{L} = L$. Let $\tilde{D}$ be a divisor associated to $\tilde{L}$. Then $-c_B(\tilde{D})$ is also associated to $\tilde{L}$ and $D = \tilde{D} - c_B(\tilde{D})$ is suitable. Now assume that $\mathbb{R}B$ is empty. Remember that if the genus of B is even (resp. is odd), then the real part of $\big(\mathrm{Jac}(B), -c_B^*\big)$ is connected (resp. has two connected components), see [7], proposition 3.3. Assume thus that the genus of B is odd and pick up a point x in B. The line bundle $L\big(x - c_B(x)\big)$ associated to the divisor $x - c_B(x)$ does not belong to the same component of the real part of $\big(\mathrm{Jac}(B), -c_B^*\big)$ as L_0. Indeed the quotients of the real ruled surfaces $\big(P\big(L(x - c_B(x)) \oplus L_0\big), c_X^{\pm}\big)$ by the real structures $c_X^{\pm}$ are not spin whereas they are for any real ruled surfaces $\big(P(L \oplus L_0), c_X^{\pm}\big)$ for L in the same component as L_0, see proposition 3.7. Making the tensor product with the line bundle $L\big(x - c_B(x)\big)$ if necessary, we get the result for any line bundle of the real part of $\big(\mathrm{Jac}(B), -c_B^*\big)$ in this case. $\square$

Proof of proposition 3.2. To begin with, let us prove the first part of proposition 3.2. For this, we suppose that the integer l given by proposition 3.1 is non-zero. Since $k \geqq 1$, X admits some real sections and so the projection $\mathbb{R}X \to \mathbb{R}B$ is surjective. It follows that every real fiber of X has a non-empty real part and thus is isomorphic to $\mathbb{C}P^{n-1}$ equipped

with the standard complex conjugation. Let $j \in \{1, \dots, l\}$, the ruled surface $P(F_j)$ in X is real and has no holomorphic section. Every real fiber of this surface is a real line in a real fiber of X, it thus has real points and so the real part of $P(F_j)$ has as many components as $\mathbb{R}B$. It follows then from [10], proposition 3.10, that $P(F_j)$ can be obtained from a decomposable ruled surface after a finite number of elementary transformations on real or complex conjugated points. This decomposable ruled surface is of the form $\big(P(M \oplus L_0), c_X^{\pm}\big)$, where $M \in \mathrm{Pic}(B)$ and $c_B^*(M) = M^*$ (see [10]). Since the real part of $P(F_j)$ has as many components as $\mathbb{R}B$, it follows from [10], lemma 3.2, that M belongs to the same component of the real part of $\big(\mathrm{Jac}(B), -c_B^*\big)$ as L_0. From lemma 3.3, we deduce that there exists a divisor D associated to M such that $c_B(D) = -D$. So there exist some points $x_i \in B$ and integers n_i such that $D = \sum_i n_i\big(x_i - c_B(x_i)\big)$. From corollary 1.12 then follows that $\big(P(M \oplus L_0), c_X^{\pm}\big)$ is obtained from $(B \times \mathbb{C}P^1, c_B \times \mathrm{conj})$ after a finite number of elementary transformations in complex conjugated points. It suffices indeed to make elementary transformations on the section $P(M)$ over the points $c_B(x_i)$ and on the section $P(L_0)$ over the points x_i to pass from $\big(P(M \oplus L_0), c_X^{\pm}\big)$ to $(B \times \mathbb{C}P^1, c_B \times \mathrm{conj})$, since $c_B \times \mathrm{conj}$ is the only real structure on $B \times \mathbb{C}P^1$ fibered over c_B and with non-empty real part. In conclusion, the real ruled manifold (X, c_X) is obtained from a manifold

$$\big(Y = P(L_1 \oplus \cdots \oplus L_k \oplus N \oplus N \oplus F_2 \oplus \cdots \oplus F_l), c_Y\big)$$

where c_Y satisfies the conditions of proposition 3.1, after a finite number of elementary transformations on real and complex conjugated points. Using a real deformation, these points can be brought to the real section $P(L_1) \subset Y$. This gives a new real ruled manifold in the same deformation class as (X, c_X) but with a lower l. After an iteration of this process, we get the result.

Now, let us prove the second part of proposition 3.2. As in the first part, we can prove that every real ruled surface $P(F_j)$ is obtained from a decomposable real ruled surface $\big(P(M_j \oplus L_0), c_X^{\pm}\big)$ after a finite number of elementary transformations on real or complex conjugated points. Moreover, the line bundle M_j satisfies $c_B^*(M_j) = M_j^*$. Also, using the same trick as in the beginning, we deduce that the real parts of these surfaces project exactly on the same subset of $\mathbb{R}B$. Thus, using the terminology of [10], we obtain that the partition of $\mathbb{R}B$ in two elements associated to the bundles M_j are the same. From [10], lemma 3.2, it follows that the bundles M_j are in the same component of the real part of $\big(\mathrm{Jac}(B), -c_B^*\big)$ and thus are of the form $L \otimes \tilde{M}_j$ with $L, \tilde{M}_j \in \mathrm{Jac}(B)$, $c_B^*(L) = L^*$ and $\tilde{M}_j$ in the same real component of $\big(\mathrm{Jac}(B), -c_B^*\big)$ as L_0. Hence we deduce as before that every surface $P(F_j)$ is obtained from a same decomposable real ruled surface $\big(P(L \oplus L_0), c_X^{\pm}\big)$ after a finite number of elementary transformations on real or complex conjugated points. Now if $\mathbb{R}X$ is non-empty, from corollary 1.12 every couple of complex conjugated points can be deformed into a double real point and then into two real points. Moreover, every couple of real points lying in a same real component of $P(L \oplus L_0)$ can be removed, making the elementary transformation at the first point and bringing the second on the image of the contracted fiber. This gives the result if $\mathbb{R}X$ is non-empty, and otherwise the result is deduced from the fact that every pair of elementary transformations done on couples of complex conjugated points of $P(L \oplus L_0)$ can be similarly removed. $\square$

Remark 3.4. From this proof follows that the common decomposable real ruled surface can be chosen of the form $\big(P(L \oplus L_0), c_X^{\pm}\big)$ with $L \neq L^*$ if L is non-trivial.

We can now prove theorem 2.2 when the bases of the manifolds have non-empty real parts (the case of empty real part is postponed to next subparagraph). We separate the cases of odd and even dimensions.

Proof of theorem 2.2 *in odd dimension.* Let (X, c_X) be a real ruled manifold of odd dimension n and base (B, c_B) with non-empty real part. Since the integer k given by proposition 3.1 has the same parity as n, it is non-zero. From proposition 3.2, we can thus assume that $k = n$ and $X = P(L_1 \oplus \cdots \oplus L_n)$, every section $P(L_i)$ being real. We can assume that L_1 is trivial, making the tensor product by L_1^* otherwise. For every $i \in \{2, \ldots, n\}$, the ruled surface $X_i = P(L_1 \oplus L_i)$ is real. The normal bundle of $P(L_1)$ in X_i is real and isomorphic to L_i. There thus exists a divisor D_i on B such that $c_B(D_i) = D_i$ and $L_i = L(D_i)$. Hence, from corollary 1.12, the real ruled manifold (X, c_X) is obtained from $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$ after a finite number of elementary transformations on real or complex conjugated points (remember that $c_B \times \mathrm{conj}$ is the only real structure on $B \times \mathbb{C}P^{n-1}$ fibered over c_B, since n is odd). We can assume that on every component of $\mathbb{R}B \times \mathbb{R}P^{n-1}$, the number of such points is even. Indeed, we can add n to this number otherwise, making an elementary transformation on every section $P(L_i)$ over a same point of $\mathbb{R}B$, which does not affect the isomorphism class of X from corollary 1.13 and proposition 1.2. So from corollary 1.12 we can assume that all these points are complex conjugated, deforming every couple of real points on a same component to a double real point and then to two complex conjugated points. Finally, from corollary 1.13, the number of elementary transformations done on complex conjugated points can be reduced modulo $2n$ to an even number in between zero and $2n - 2$. Indeed, n couples of elementary transformations done on complex conjugated points can be removed bringing these points to the n sections $P(L_i)$ over a same pair of complex conjugated points of B. Note that the ruled manifold obtained from $B \times \mathbb{C}P^{n-1}$ after $2d$ elementary transformations has degree $2d$, so when d ranges from 0 to $n - 1$, this gives ruled manifolds non deformation equivalent to each other.

The proof of theorem 2.2 is now clear. Let (X_1, c_{X_1}) and (X_2, c_{X_2}) be two real ruled manifolds of odd dimension n with same degree $d \in \mathbb{Z}/n\mathbb{Z}$ and bases (B_1, c_{B_1}) and (B_2, c_{B_2}) of same topological type. We know that (X_1, c_{X_1}) (resp. (X_2, c_{X_2})) is in the same deformation class as the manifold obtained from $(B_1 \times \mathbb{C}P^{n-1}, c_{B_1} \times \mathrm{conj})$ (resp. $(B_2 \times \mathbb{C}P^{n-1}, c_{B_2} \times \mathrm{conj})$) after d elementary transformations done on complex conjugated points. Since (B_1, c_{B_1}) and (B_2, c_{B_2}) have same topological type, from [8] follows that they are deformation equivalent and there exists a path (B_t, c_{B_t}) joining these curves. The path (X_t, c_{X_t}), where (X_t, c_{X_t}) is obtained from $(B_t \times \mathbb{C}P^{n-1}, c_{B_t} \times \mathrm{conj})$ after making d elementary transformations on complex conjugated points, gives a real deformation in between (X_1, c_{X_1}) and (X_2, c_{X_2}). $\square$

Proof of theorem 2.2 *in even dimension.* Let (X, c_X) be a real ruled manifold of even dimension n and base (B, c_B) with non-empty real part. Without changing the deformation class of (X, c_X), we can assume that $X = P(L_1 \oplus \cdots \oplus L_k \oplus F_1 \oplus \cdots \oplus F_l)$ and that c_X satisfies the conditions of proposition 3.1. From proposition 3.2, either k or l vanishes.

First case: $l = 0$. In this case, we proceed as in the proof of theorem 2.2 in odd dimension. It follows that the manifold (X, c_X) is obtained from $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$ after a finite number of elementary transformations done on real or complex conjugated points. As in the proof of proposition 3.2, the number of elementary transformations can be reduced to zero or one on every component of $\mathbb{R}B \times \mathbb{R}P^{n-1}$. As in the proof of theo-

rem 2.2 in odd dimension, the number of elementary transformations done on complex conjugated points can be reduced modulo $2n$ to an even number between 0 and $2n-2$ and even between 0 and $n-2$ since $\mathbb{R}B \neq \emptyset$ and so complex conjugated points can be brought to a same real fiber of X and reduced modulo n. Let (X_1, c_{X_1}) and (X_2, c_{X_2}) be two real ruled manifolds of even dimension n with same degree $d \in \mathbb{Z}/n\mathbb{Z}$ and same topological type. Then the number of components of $\mathbb{R}B_1 \times \mathbb{R}P^{n-1}$ (resp. $\mathbb{R}B_2 \times \mathbb{R}P^{n-1}$) on which is done one elementary transformation is given by the number of non-orientable components of $\mathbb{R}X_1$ (resp. $\mathbb{R}X_2$), so it is the same for X_1 and X_2. Since (B_1, c_{B_1}) and (B_2, c_{B_2}) have same topological type, from [8] follows that there exists a path (B_t, c_{B_t}) joining these curves. Moreover, this path can be chosen so that the component of $\mathbb{R}B_1$ over which are the non-orientable components of $\mathbb{R}X_1$ are mapped to the components of $\mathbb{R}B_2$ over which are the non-orientable components of $\mathbb{R}X_2$. This follows from the presentation in [8] of a real algebraic curve as the gluing of a Riemann surface with boundary with its conjugate, the gluing maps being either identity or antipodal. Hence, we can assume that (X_1, c_{X_1}) and (X_2, c_{X_2}) are obtained from a same real ruled manifold (Y, c_Y) after making an even number of elementary transformations on complex conjugated points, these numbers being less than $n-1$. Since by hypothesis these manifolds have same degree, these numbers are the same for (X_1, c_{X_1}) and (X_2, c_{X_2}) and these two manifolds are deformation equivalent.

Second case: $k = 0$. Then, from proposition 3.2, without changing the deformation class of (X, c_X) and making a finite number of elementary transformations on real points if necessary, we can assume that there exists $L \in \mathrm{Pic}(B)$ such that $c_B^*(L) = L^*$ and for $1 \leqq j \leqq n/2$, $P(F_j) = P(L \oplus L_0)$. Denote by M_j the line bundle such that $F_j = (M_j \otimes L) \oplus M_j$. We can assume that M_1 is trivial and we denote by c_1 the real structure on $P(F_1)$ induced by c_X. The real structure c_X induces a real structure on the normal bundle N_j of $P(F_1)$ in $P(F_1 \oplus F_j)$. Thus $c_1^*(N_j) = N_j$. But from lemma 1.5, we know that

$$N_j = p^*(F_j) \otimes \mathcal{O}(D_L) = p^*\big((M_j \otimes L) \oplus M_j\big) \otimes \mathcal{O}(D_L) = \big(\mathcal{O}(D_L) \oplus \mathcal{O}(D_{L_0})\big) \otimes p^*(M_j).$$

So $c_1^*(N_j) = N_j$ implies that $c_B^*(M_j) = M_j$. Since $\mathbb{R}B \neq \emptyset$, this implies that there exists a divisor D_j on B associated to M_j such that $c_B(D_j) = D_j$. We then deduce from corollary 1.12 and proposition 1.10 that (X, c_X) is obtained from $\big(Y = P\big((L \oplus L_0)^{\frac{n}{2}}\big), c_Y^{\pm}\big)$ after a finite number of elementary transformations on real or complex conjugated points (we can indeed assume that $L \neq L^*$ if L is non-trivial, see remark 3.4). As in the first case, the number of elementary transformations done on real points can be reduced to 0 or 1 for each component of $\mathbb{R}Y$ and on complex conjugated points, they can be reduced modulo n to an even number between 0 and $n-2$. We conclude exactly as in the first case. $\square$

3.2. When the base has empty real part. The aim of this paragraph is to prove theorem 2.2 assuming the bases of the manifolds have empty real parts. This is the only remaining case to consider, after §3.1.

Lemma 3.5. *Let $\big(X = P(L_0 \oplus L \oplus L), c_X\big)$ be a real ruled manifold having a base (B, c_B) with empty real part. Assume that the real structure c_X fixes the section $P(L_0) \subset X$ and exchanges the two sections $P(L) \subset X$. Then X is obtained from $(B \times \mathbb{C}P^3, c_B \times \mathrm{conj})$ after a finite number of elementary transformations done on real or complex conjugated points.*

Proof. Denote by D_L the divisor $P(L) \subset P(L \oplus L)$, where $P(L)$ is one of the two sections of $P(L \oplus L)$, and by c_1 the real structure of $P(L \oplus L)$ induced by c_X. The normal bundle of $P(L \oplus L)$ in X is real and isomorphic to $p^*(L^*) \otimes \mathcal{O}(D_L)$ from lemma 1.5. Thus, $c_1^*\big(p^*(L^*) \otimes \mathcal{O}(D_L)\big) = p^*(L^*) \otimes \mathcal{O}(D_L)$. Now $c_1^*\big(p^*(L^*) \otimes \mathcal{O}(D_L)\big) = p^*\big(c_B^*(L^*)\big) \otimes \mathcal{O}(D_L)$, so that $c_B^*(L) = L$. Let D be a divisor associated to L. Then $c_B(D)$ is also associated to L and one can write $\big(X = P\big(L_0 \oplus L(D) \oplus L\big(c_B(D)\big)\big)\big)$. Making elementary transformations on $P(L_0)$ over the points of $D + c_B(D)$, on $P\big(L(D)\big)$ over the points of $c_B(D)$ and on $P\big(L\big(c_B(D)\big)\big)$ over the points of D, we obtain the real ruled manifold $(B \times \mathbb{C}P^3, c_B \times \mathrm{conj})$. Hence the result. $\square$

Proposition 3.6. *Let $X = P(L_1 \oplus \cdots \oplus L_k \oplus F_1 \oplus \cdots \oplus F_l)$ be a ruled manifold over B and c_X be a real structure satisfying the conditions of proposition* 3.1. *Assume that the real part of B is empty, then*:

(1) *If $k \geqq 1$, without changing the deformation class of (X, c_X), we can assume that $l = 0$.*

(2) *If $k = 0$, without changing the deformation class of (X, c_X), we can assume that the real ruled surfaces $P(F_j) \subset X$, $j \in \{1, \ldots, l\}$, are obtained from the trivial ruled surface after at most one couple of elementary transformations done on complex conjugated points.*

Proof. Thanks to lemma 3.3, we can proceed as in the proof of proposition 3.2 to get that all the real ruled surfaces $P(F_j)$ are obtained from the trivial ruled surface after some elementary transformations done on complex conjugated points. If $k \geqq 1$, all the elementary transformations can be brought to the section $P(L_1)$ without changing the real deformation class of X. If $k = 0$, every two such couples can be cancelled, making the first two elementary transformations and bringing the two remaining ones on the images of the contracted fibers. This already proves the second part of proposition 3.6. Now if $k, l \geqq 1$, we can assume that L_1 is the trivial line bundle L_0. From what has just be done, we know that $F_j = M_j \oplus M_j$ for some line bundle M_j over B. From lemma 3.5, we deduce that the real ruled surfaces $P(F_j) = P(M_j \oplus M_j)$ have a real section, which contradicts the hypothesis (see proposition 3.1). $\square$

Proposition 3.7. *Let $n \geqq 2$ be an even integer and (B, c_B) be a real compact irreducible algebraic curve with empty real part. Let c_0 be the real structure with empty real part on $\mathbb{C}P^{n-1}$.*

(1) *If $g(B)$ is odd, then the real ruled manifolds $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$ and $(B \times \mathbb{C}P^{n-1}, c_B \times c_0)$ are deformation equivalent.*

(2) *Let (X, c_X) be the real ruled manifold obtained from $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$ after a couple of elementary transformations done along two projective subspaces of dimension $n/2 - 1$ belonging to two complex conjugated fibers. Then the quotients X/c_X and $B \times \mathbb{C}P^{n-1}/c_B \times \mathrm{conj}$ are not homeomorphic to each other.*

Corollary 3.8. *Let $n \geqq 2$ be an even integer and (B, c_B) be a real compact irreducible algebraic curve with empty real part and even genus. Then the quotients $B \times \mathbb{C}P^{n-1}/c_B \times \mathrm{conj}$ and $B \times \mathbb{C}P^{n-1}/c_B \times c_0$ are not homeomorphic to each other.*

Proof. From proposition 3.7, we know that the real ruled manifold (X, c_X) obtained from $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$ after a couple of elementary transformations done along two projective subspaces of dimension $n/2-1$ of two complex conjugated fibers has a quotient non homeomorphic to $B \times \mathbb{C}P^{n-1}/c_B \times \mathrm{conj}$. Let us write $B \times \mathbb{C}P^{n-1} = P(L_0^{\frac{n}{2}} \oplus L_0^{\frac{n}{2}})$, where the two sub-ruled manifolds $P(L_0^{\frac{n}{2}})$ are exchanged by $c_B \times \mathrm{conj}$. We can perform the elementary transformations along these subspaces over two complex conjugated points x and $c_B(x)$ of B. From corollary 1.12 we then deduce that $X = P\big(\big(L\big(x - c_B(x)\big) \oplus L_0\big)^{\frac{n}{2}}\big)$ and from proposition 1.10, we deduce that c_X is one of the two real structures $c_X^{\pm}$. Now since the genus of B is even, the real part of $\big(\mathrm{Jac}(B), -c_B^*\big)$ is connected and hence (X, c_X) deforms onto $B \times \mathbb{C}P^{n-1}$ equipped with a real structure having a quotient homeomorphic to X/c_X. From proposition 3.7 it cannot be $c_B \times \mathrm{conj}$, hence the result. $\square$

Proof of proposition 3.7. The proof of the first part of this proposition is analogous to the one of proposition 2.8 of [10], so we will give only a sketch of it. Without changing the deformation class of the manifolds, we can assume that the base (B, c_B) is the curve constructed in corollary 2.8 of [10]. Let L be the line bundle given by this corollary, it satisfies $c_B^*(L) = L = L^*$. Moreover, there exist a divisor D associated to L, an automorphism φ of B, a meromorphic function f_D on B such that $\overline{f \circ c_B} = f$, $\mathrm{div}(f) = D + c_B(D)$ and $f \circ \varphi = f$, as well as a meromorphic function g on B such that $\mathrm{div}(g) = \varphi(D) - D$ and $g(\overline{g \circ c_B}) = -1$ (see [10]). We define, as in the proof of proposition 2.6 of [10], an automorphism Φ of $P\big((L \oplus L_0)^{\frac{n}{2}}\big)$ defined over the open set $U_0 = B \backslash D$, by:

$$U_0 \times \mathbb{C}P^{n-1} \to U_0 \times \mathbb{C}P^{n-1},$$

$$\big(x, (z_1^j : z_0^j)\big) \mapsto \big(\varphi(x), \big(g \circ \varphi(x) z_1^j : z_0^j\big)\big),$$

where $j \in \{1, \ldots, n/2\}$. This automorphism conjugates the two real structures c_X^+ and c_X^- on X. Now if L is in the same component of the real part of $\big(\mathrm{Jac}(B), -c_B^*\big)$ as the trivial line bundle, the real ruled manifolds $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$ and $(B \times \mathbb{C}P^{n-1}, c_B \times c_0)$ are in the same deformation class as $\big(P\big((L \oplus L_0)^{\frac{n}{2}}\big), c_X^{\pm}\big)$ and we are done. Otherwise, we perform an elementary transformation along the subspace $P(L^{\frac{n}{2}}) \subset P\big((L \oplus L_0)^{\frac{n}{2}}\big)$ over a point x of B, and an elementary transformation along the subspace $c_X^+\big(P(L^{\frac{n}{2}})\big) = c_X^-\big(P(L^{\frac{n}{2}})\big)$ over the point $c_B(x)$. The ruled manifold obtained is of the form $P\big((M \oplus L_0)^{\frac{n}{2}}\big)$ with M in the same component of the real part of $\big(\mathrm{Jac}(B), -c_B^*\big)$ as L_0. The real structures $c_X^{\pm}$ lift to the real structures $c_Y^{\pm}$, and the automorphism Φ lifts to an automorphism of Y which conjugates them (see [10], lemma 3.14). Hence we conclude as before.

Now let us prove the second part of the proposition. We will prove that there is no $\mathbb{Z}/2\mathbb{Z}$-equivariant diffeomorphism between $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$ and (X, c_X). Let H be a real hyperplane of $(\mathbb{C}P^{n-1}, \mathrm{conj})$ and $s = B \times H$. Then s is a real divisor of $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$ and $[s]^n = 0 \in H_0(B \times \mathbb{C}P^{n-1}; \mathbb{Z})$. If s' is any other real $(2n-2)$-cycle of $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$, then $[s'] = [s] + 2k[f]$ where $[f]$ is the homology class of a fiber. Hence $[s']^n = [s]^n + 2kn[f][s]^{n-1} = \pm 2kn = 0 \bmod (2n)$. Now, without changing the diffeomorphism class of X/c_X, we can assume that the blown up projective subspaces of dimension $n/2-1$ are real once projected onto the second factor $(\mathbb{C}P^{n-1}, \mathrm{conj})$, so that $X = P\big(L\big(x + c_B(x)\big)^{\frac{n}{2}} \oplus L_0^{\frac{n}{2}}\big)$ and c_X is the standard real structure on this manifold. Denote by s_X the real divisor on X associated to the dual of the tautological line bundle $\mathcal{O}(-1) \subset p^*\big(L\big(x + c_B(x)\big)^{\frac{n}{2}} \oplus L_0^{\frac{n}{2}}\big)$. From [9], p. 215, we see that

$$[s_X]^n = -\deg\big(L\big(x + c_B(x)\big)^{\frac{n}{2}} \oplus L_0^{\frac{n}{2}}\big) = -n \neq 0 \bmod(2n),$$

hence the result. $\square$

Proof of theorem 2.2 *when the base has empty real part.* Let (X, c_X) be a real ruled manifold of dimension n and base (B, c_B) with empty real part. If n is odd, from propositions 3.1 and 3.6, we know that without changing the deformation class of (X, c_X), we can assume that $X = P(L_1 \oplus \cdots \oplus L_n)$, all the sections $P(L_j)$ being real. We can also assume that L_1 is the trivial line bundle. Then, for $2 \leqq j \leqq n$, the normal bundle of $P(L_0)$ in $P(L_0 \oplus L_j)$ is real and isomorphic to L_j. Thus there exists a real divisor D_j associated to L_j and from corollary 1.12, (X, c_X) is obtained from $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$ after a finite number of elementary transformations done on complex conjugated points. Note that since n is odd, $c_B \times \mathrm{conj}$ is the only real structure on $B \times \mathbb{C}P^{n-1}$ fibered over c_B. Now from corollary 1.12, without changing the deformation class of (X, c_X), n couples of elementary transformations done on complex conjugated points can be removed, bringing these points on the n sections over a same couple $\big(x, c_B(x)\big)$ of $B \times B$. Hence we can assume that (X, c_X) is obtained from $(B \times \mathbb{C}P^{n-1}, c_B \times \mathrm{conj})$ after an even number of elementary transformations between 0 and $2n-2$. Since n is odd, this number is prescribed by the degree of X, and since there is only one deformation class of smooth real irreducible compact algebraic curve with empty real part (see [8]), the result is proved in odd dimension.

Now if n is even and the integer k given by proposition 3.6 is non-zero, the same proof shows that without changing the deformation class of (X, c_X), we can assume that it is obtained from $B \times \mathbb{C}P^{n-1}$ equipped with one of its real structures after an even number of elementary transformations done on complex conjugated points, between 0 and $2n-2$. In particular, the degree of X is even. If $g(B)$ is even, then from corollary 3.8, the two real structures on $B \times \mathbb{C}P^{n-1}$ have non-diffeomorphic quotients, and from proposition 3.7, one passes from the deformation class of one of these real structures to the deformation class of the other one making n elementary transformations on complex conjugated points. Thus we can assume that the number of elementary transformations necessary to obtain (X, c_X) from $B \times \mathbb{C}P^{n-1}$ with one of its real structures is even, between 0 and $n-2$. This number is then prescribed by the degree of X, and the real structure on $B \times \mathbb{C}P^{n-1}$ by the topology of the quotient X/c_X. If $g(B)$ is odd, then from proposition 3.7, the two real structures on $B \times \mathbb{C}P^{n-1}$ are in the same deformation class, hence the number modulo n of elementary transformations necessary to obtain (X, c_X) from $B \times \mathbb{C}P^{n-1}$ is prescribed by the degree of X, and then the total number by the topology of the quotient X/c_X. We can then conclude as before.

It thus only remains to consider the case when n is even, but the integer k given by proposition 3.6 vanishes. From proposition 3.6, we can assume that (X, c_X) is obtained from a real ruled manifold $P(F_1 \oplus \cdots \oplus F_l)$ after some elementary transformations done on complex conjugated points. Moreover, all the ruled surfaces $P(F_j)$ are real, and every bundle F_j is of the form $M_j \oplus M_j$ for some $M_j \in \mathrm{Pic}(B)$. We can then assume that M_1 is trivial and since the normal bundle of $P(F_1)$ in $P(F_1 \oplus F_j)$ is real, we deduce from lemma 1.5 that $c_B^*(M_j) = M_j$, as in the proof of lemma 3.5. Let D_j be a divisor associated to M_j. Then $c_B(D_j)$ is also associated to M_j and we can write

$$P(F_1 \oplus \cdots \oplus F_l) = P\big(L_0 \oplus L_0 \oplus L(D_2) \oplus L\big(c_B(D_2)\big) \oplus \cdots \oplus L(D_l) \oplus L\big(c_B(D_l)\big)\big).$$

Making elementary transformations on $P(L_0 \oplus L_0)$ over the points of

$$D_2 + c_B(D_2) + \cdots + D_l + c_B(D_l),$$

on $P\big(L(D_j)\big)$ over the points of $c_B(D_j)$ and on $P\big(L\big(c_B(D_j)\big)\big)$ over the points of D_j (with appropriate multiplicities), we obtain the real ruled manifold $B \times \mathbb{C}P^{n-1}$ with one of its real structures. Hence once more, (X, c_X) is in the same deformation class as a manifold obtained from $B \times \mathbb{C}P^{n-1}$ with one of its real structures after a finite number of elementary transformations done on complex conjugated points and we conclude as before. $\square$

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