Random process reconstruction from multiple noisy source observations

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Abstract

The problem addressed in this paper is the reconstruction of a continuous-time stationary random process from noisy sampled observations coming from different sources. An optimal solution in terms of linear filtering of observed samples is derived and the expression of the corresponding minimum reconstruction error power is given. Moreover, two equivalent reconstruction schemes are given. The first one is recursive, involving two filter banks. Its main interest is that adding or suppressing an input does not affect the whole scheme. The second scheme is symmetric and uses only one filter bank. However, to add a new input requires a complete modification of all the filter transfer functions. Simulation examples are given to prove the application of the reconstruction scheme.

Key words and phrases: Sampling, signal reconstruction, linear minimum mean square reconstruction

1 Introduction

Within the frame of radio communications, a method to overcome the effects of fading and circuit failures is to use some diversity techniques

[1]. The basic principle of diversity techniques is to send several versions of the same signal in order to obtain reliability and signal improvement. The receiver then combines the received signals that originate from two or more independent sources that have been modulated with identical information-bearing signals. Different methods can be used to provide the receiver with L independently fading replicas of the same information-bearing signal. One method is to employ frequency diversity, i.e., the same signal is transmitted on L carriers, sufficiently distant to be well-separated by the transmission channel. Another method is to transmit the signal in L different time slots, sufficiently distant to remain uncorrelated. As a consequence, whatever the diversity transmission method, the receiver has to retrieve the original information coming from different distorted versions of the same signal. The present paper is related to this kind of problem. Indeed, it addresses the problem of recovering a continuous-time random process from sampled noisy observations coming from several sources. The observations are modeled as follows:

$$U_k(n) = Z(n) + B_k(n), \quad n \in \mathbb{Z}, \quad k = 1, \ldots, L \tag{1}$$

where L represents the number of observed sources, $\mathbf{Z} = \{Z(t), t \in \mathbb{R}\}$ denotes the random process under interest and $\mathbf{B}_k = \{B_k(t), t \in \mathbb{R}\}$, $k = 1, \ldots, L$ the L different random processes modeling the L different noise sources. Note that the case $L = 1$ is well-known in the literature for many years [2, p.132]. However, the present paper deals with the L-fold extension ($L \neq 1$) of this model. The assumptions and notations required for the present study are summarized below:

- The random process $\mathbf{Z} = \{Z(t), t \in \mathbb{R}\}$ is a zero mean stationary process with finite second order moment. Its Power Spectral Density (PSD) $s_Z(\omega)$ is defined by [3]

$$K_Z(\tau) = E[Z(t) Z^*(t-\tau)] = \int_{\mathbb{R}} s_Z(\omega) e^{i\omega\tau} d\omega \tag{2}$$

  where $E[.]$ denotes the mathematical expectation, the superscript $*$ holds for complex conjugate, and $K_Z(\tau)$ denotes the random process autocorrelation function. Note in particular that the random process finite power is given by

$$E\left[|Z(t)|^2\right] < +\infty. \tag{3}$$

- The random noise processes $\mathbf{B}_k = \{B_k(t), t \in \mathbb{R}\}$, $k = 1, \ldots, L$ are zero mean stationary uncorrelated processes, uncorrelated with $\mathbf{Z}$, with respective PSD $s_{B_k}(\omega)$ as defined in (2).

- Without any loss of generality, the sampling period is assumed to be unitary, without any hypothesis on the signal and noise spectral bands.

- Each one of the L sampled noisy sequences defined in (1) is the sum of both sampled sequences $\tilde{\mathbf{Z}} = \{Z(n), n \in \mathbb{Z}\}$ and $\tilde{\mathbf{B}}_k = \{B_k(n), n \in \mathbb{Z}\}$, $k = 1, \ldots, L$ where the superscript $\tilde{.}$ is used to distinguish random processes from sequences of random variables. The respective PSD of these sequences $\tilde{s}_Z(\omega)$ and $\tilde{s}_{B_k}(\omega)$ are defined in the same manner (to simplify, only $\tilde{s}_Z(\omega)$ definition is given) [4]

$$E\left[Z(n) Z^*(n-m)\right] = \int_{-\pi}^{+\pi} \tilde{s}_Z(\omega) e^{i\omega m} d\omega, \quad (n,m) \in \mathbb{Z}^2. \quad (4)$$

  Sampling theory derives the relation between the continuous-time PSD and the folded one [5, p.426]

$$\tilde{s}_Z(\omega) = \sum_{n \in \mathbb{Z}} s_Z(\omega + 2\pi n). \quad (5)$$

  (the same relation holds between $\tilde{s}_{B_k}(\omega)$ and $s_{B_k}(\omega)$).

The problem developed in this paper consists of deriving the linear minimum mean square estimator (LMMSE) of $Z(t)$ from the L noisy sampled processes $\tilde{\mathbf{U}}_k = \{U_k(n), n \in \mathbb{Z}\}$, $k = 1, \ldots, L$ defined in (1). This LMMSE, denoted as $\hat{Z}(t)$, is a linear combination of the observed samples which minimizes the mean square error (MSE)

$$\sigma_t^2 = E\left[\left|Z(t) - \widehat{Z}(t)\right|^2\right]. \quad (6)$$

One of the main mathematical tools used in this paper is the Hilbert space geometry in $L^2(\Omega)$. The reader can refer directly to [3] or to [6], which contains the L_2-theoretical background for these aspects. (However, in [6], only bandlimited random processes are studied.)
The paper is organized as follows: Section 2 derives the expression of the

LMMSE. A suboptimal estimator is presented in section [3]. In section 4, it is shown that timing jitter can also be considered as a special case of application of the problem studied in this paper. Section 5 gives some examples and simulations highlight how the results of the Section 2 can be applied. Conclusions are reported in Section 6. Appendices are given in order to detail some theoretical developments necessary to derive the results of Section 2.

2 LMMSE derivation

The LMMSE of $Z(t)$ belongs to the Hilbert space spanned by the set of observed random variables [3] and is defined by the orthogonal projection of $Z(t)$ onto this Hilbert space. Thus, $\hat{Z}(t)$ is defined by

$$\hat{Z}(t) = \underset{\oplus H_{\tilde{\mathbf{U}}_k}}{pr} [Z(t)] \tag{7}$$

where $pr[.]$ denotes an orthogonal projection and $H_{\tilde{\mathbf{U}}_k}$, $k = 1, \ldots, L$, the Hilbert spaces spanned respectively by $\tilde{\mathbf{U}}_k = \{U_k(n), n \in \mathbb{Z}\}$, $k = 1, \ldots, L$. The main problem of this projection arises from the fact that the different Hilbert spaces are not orthogonal. Thus, the projection on the sum of spaces is not the sum of projections on the different spaces. Therefore, before any projection, an orthogonalization is achieved, leading to derive L orthogonal random sequences $\tilde{\mathbf{C}}_k = \{C_k(n), n \in \mathbb{Z}\}$, $k = 1, \ldots, L$ from the L initial sequences $\tilde{\mathbf{U}}_k = \{U_k(n), n \in \mathbb{Z}\}$, $k = 1, \ldots, L$ such that their corresponding Hilbert spaces $H_{\tilde{\mathbf{C}}_k}$, $k = 1, \ldots, L$ are orthogonal and verify

$$\overset{L}{\underset{k=1}{\oplus}} H_{\tilde{\mathbf{U}}_k} = \overset{L}{\underset{k=1}{\oplus}} H_{\tilde{\mathbf{C}}_k}. \tag{8}$$

2.1 Orthogonalization

The orthogonalization procedure is analog to a Gram-Schmidt one and is iterative. First, consider that:

$$\tilde{\mathbf{C}}_1 = \tilde{\mathbf{U}}_1. \tag{9}$$

Then, the second random sequence $\tilde{\mathbf{C}}_2$ is defined by

$$C_2(n) = U_2(n) - \Phi_1[C_1](n), n \in \mathbb{Z} \tag{10}$$

where $\Phi_1[C_1](n)$ denotes the output of a linear time-invariant filter excited by the random sequence $\tilde{\mathbf{C}}_1$. The transfer function $\Phi_1(\omega)$ of this filter is such that both Hilbert spaces $H_{\tilde{\mathbf{C}}_1}$ and $H_{\tilde{\mathbf{C}}_2}$ are orthogonal. This is derived in *Example 1*, [2, p.132]

$$\Phi_1(\omega) = \frac{\tilde{s}_Z(\omega)}{\tilde{s}_Z(\omega) + \tilde{s}_{B_1}(\omega)}. \tag{11}$$

Then, the random sequences $\tilde{\mathbf{C}}_k, k = 2, \ldots, L$ are defined in an iterative manner

$$C_k(n) = U_k(n) - \sum_{m=1}^{k-1} \Phi_m[C_m](n) \quad k = 2, \ldots, L, \tag{12}$$

where $\Phi_m[C_m](n)$ denotes the output of a linear time-invariant filter excited by $\tilde{\mathbf{C}}_m$. Appendix A shows that the transfer function $\Phi_k(\omega)$ of the kth filter is such that

$$\Phi_k(\omega) = \frac{\tilde{s}_Z(\omega)}{\tilde{s}_{B_k}(\omega)} \frac{1}{1 + \sum_{m=1}^{k} \tilde{s}_Z(\omega) / \tilde{s}_{B_m}(\omega)} \quad for \quad k = 1, \ldots, L. \tag{13}$$

Hence, based on the observation of L (non-orthogonal) sequences $\tilde{\mathbf{U}}_1$,..., $\tilde{\mathbf{U}}_L$, this orthogonalization procedure leads to derive a set of orthogonal sequences $\tilde{\mathbf{C}}_1, \ldots, \tilde{\mathbf{C}}_L$ with respective PSD (see Appendix A)

$$\tilde{s}_{C_k}(\omega) = \tilde{s}_{B_k}(\omega) + \frac{\tilde{s}_Z(\omega)}{1 + \sum_{m=1}^{k-1} \tilde{s}_Z(\omega) / \tilde{s}_{B_m}(\omega)} \quad for \quad k = 2, \ldots, L. \tag{14}$$

2.2 Projection

The above orthogonalization procedure allows for the construction of a set of orthogonal sequences as a function of the observed ones $\tilde{\mathbf{U}}_1, \ldots, \tilde{\mathbf{U}}_L$ such that (8) is verified. Therefore, the projection (7) can be done on this sum of orthogonal spaces $\overset{L}{\underset{k=1}{\oplus}} H_{\tilde{\mathbf{C}}_k}$ and yields

$$\hat{Z}(t) = \sum_{k=1}^{L} pr_{H_{\tilde{\mathbf{C}}_k}}[Z(t)]. \tag{15}$$

Let us consider the different projections on each Hilbert space and denote

$$\hat{Z}_k(t+n) = \underset{H_{\tilde{\mathcal{C}}_k}}{pr}\left[Z(t+n)\right], \quad t \in \mathbb{R}, n \in \mathbb{Z}. \tag{16}$$

Appendix B shows that this projection can be viewed as the output of a linear time-invariant filter, t being fixed, characterized by its transfer function

$$\mu_k(t,\omega) = e^{i\omega t}\frac{\tilde{s}_{Z_t}(\omega)}{\tilde{s}_{B_k}(\omega)}\frac{1}{1+\sum_{m=1}^{k}\tilde{s}_Z(\omega)/\tilde{s}_{B_m}(\omega)} \tag{17}$$

where $\tilde{s}_{Z_t}(\omega)$ is the PSD corresponding to the random sequence $\tilde{\mathbf{Z}}_t = \{Z(t+n),\ n \in \mathbb{Z}\}$, t being fixed. The overall reconstruction procedure is represented in Fig.1, including both orthogonalization and projection steps. Both steps, orthogonalization and projection, correspond to two filter banks characterized by $\{\Phi_k(\omega), k = 1,\ldots,L\}$ and $\{\mu_k(t,\omega), k = 1,\ldots,L\}$.

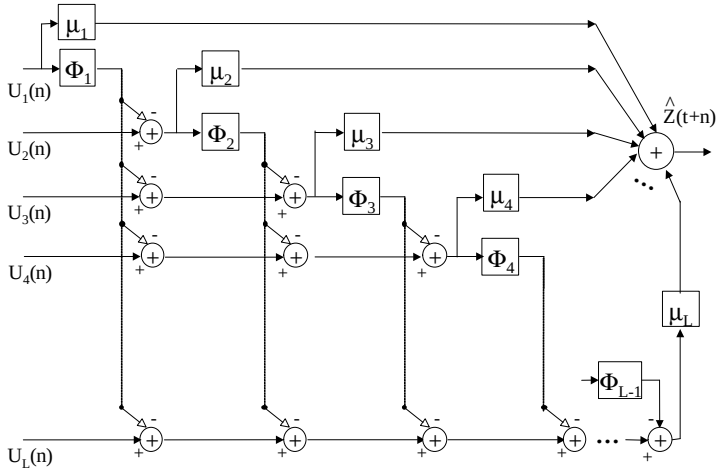

Figure 1: LMMSE of $Z(t+n)$ from the observations of $U_1(n), U_2(n), ..., U_L(n)$.

Note that

$$\mu_k(0,\omega) = \Phi_k(\omega). \tag{18}$$

This means that, in the case of a reconstruction of the random process on $\mathbb{Z}$ rather than on $\mathbb{R}$, the scheme presented in Fig.1 is simplified. Since the projection filters are identical to the orthogonalization ones, only one filter bank is needed. Therefore, in Fig.1, the projection filter bank (μ_k) can be suppressed and the output of the orthogonalization filter bank (Φ_k) can be directly linked to the L-term sum yielding $\hat{Z}(n)$, as shown in Fig.2.

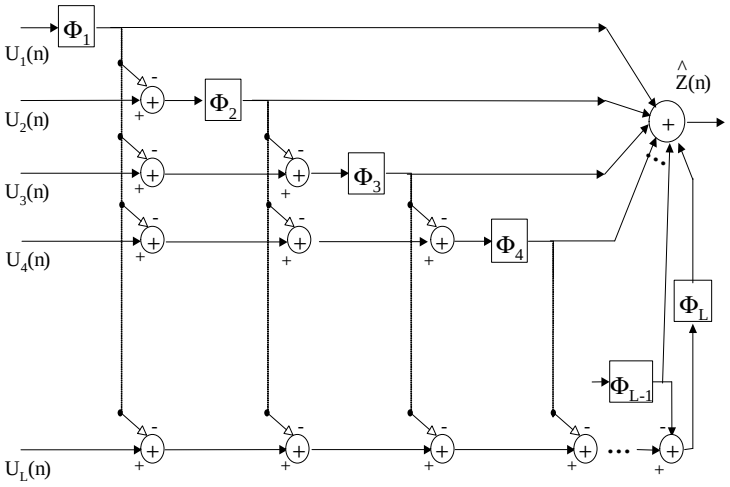

Figure 2: LMMSE of $Z(n)$ from the observations of $U_1(n), U_2(n), ..., U_L(n)$.

Moreover, it is interesting to note that the general scheme of Fig.1 is recursive. Therefore, if a new source $\tilde{\mathbf{U}}_{L+1}$ is available, the scheme can be modified easily by adding a new line with two filters and without changing the other filters. However, this recursive property of the general scheme allows one to assume that an equivalent symmetric scheme can be found. Appendix C derives this equivalent scheme, presented in Fig.3. In this equivalent scheme, only one filter bank is necessary, characterized by its transfer function

$$\Gamma_k(t,\omega) = e^{i\omega t}\frac{\tilde{s}_{Z_t}(\omega)}{\tilde{s}_{B_k}(\omega)}\frac{1}{1+\sum_{m=1}^{L}\tilde{s}_Z(\omega)/\tilde{s}_{B_m}(\omega)} \quad for \quad k=1,\ldots,L, \tag{19}$$

as shown in Appendix C. The main advantage of this symmetric scheme is

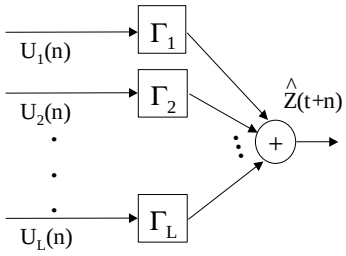

Figure 3: LMMSE derivation symmetric scheme.

to take into account only one bank of linear time-invariant filters (t being fixed) to be applied directly on the observed sequences $\tilde{\mathbf{U}}_1, \ldots, \tilde{\mathbf{U}}_L$, in order to yield the LMMSE of $Z(t)$. However, the outputs of the different filters are correlated. But the main drawback of this symmetric scheme is that the different filters depend on the number L of observed sources. To add or suppress a new observed source leads to complete modification of all the involved filters. This is not the case when dealing with the recursive scheme of Fig.1.

2.3 Reconstruction error power

The reconstruction error power (6) is given by

$$\sigma_t^2 = \int_{-\pi}^{+\pi} \left(\tilde{s}_Z(\omega) - \sum_{k=1}^{L} |\mu_k(t,\omega)|^2 \, \tilde{s}_{C_k}(\omega) \right) d\omega. \tag{20}$$

Appendix B shows that this MSE can be viewed as the sum of two terms $\left(\sigma_t^2\right)_1$ and $\left(\sigma_t^2\right)_2$ where

$$\left(\sigma_t^2\right)_1 = \int_{-\pi}^{+\pi} \left(\tilde{s}_Z(\omega) - \frac{\left(\tilde{s}_{Z_t}(\omega)\right)^2}{\tilde{s}_Z(\omega)} \right) d\omega \tag{21}$$

represents the error due to aliasing effects (no aliasing yields $\left(\sigma_t^2\right)_1 = 0$) and

$$\left(\sigma_t^2\right)_2 = \int_{-\pi}^{+\pi} \frac{\left(\tilde{s}_{Z_t}(\omega)\right)^2}{\tilde{s}_Z(\omega)} \frac{1}{1 + \sum_{m=1}^{L} \tilde{s}_Z(\omega) / \tilde{s}_{B_m}(\omega)} d\omega \tag{22}$$

the error due to the presence of noise in the observations (1). If one of the noise sources is negligible, this second term equals zero.

3 A suboptimal procedure

A suboptimal procedure can be proposed, rather than the optimal one given in Fig.1. The idea is to use the optimal reconstruction of $\{Z(n), n \in \mathbb{Z}\}$ presented in Fig.2. This reconstructed sequence $\left\{\hat{Z}(n), n \in \mathbb{Z}\right\}$ spans a Hilbert space, denoted as $H_{\hat{\mathbf{Z}}}$. An estimation of $Z(t)$ can be defined by

$$\hat{Z}_{subopt}(t) = \underset{H_{\hat{\mathbf{Z}}}}{pr}\left[Z(t)\right]. \tag{23}$$

The main interest of this method compared to the optimal one is its reduced computational cost. Indeed, in this case, L time-invariant linear filters (the Φ_k of Fig.2) and one linear filter depending on t (the one derived from (23)) are needed. This must be compared with the optimal reconstruction which requires $L-1$ time-invariant linear filters and L linear filters depending on t. However, this method is suboptimal, except for $L=1$. In this case, $\left\{\hat{Z}(n), n \in \mathbb{Z}\right\}$ is the output of the filter $\Gamma_1(0,\omega) = \Phi_1(\omega)$ given in (11). Then, the projection (23) is equivalent to a linear filter characterized by the following transfer function:

$$G_{L=1}(t,\omega) = e^{i\omega t}\frac{\tilde{s}_{Z_t}(\omega)}{\tilde{s}_Z(\omega)}. \tag{24}$$

Both steps are equivalent to the optimal filter $\Gamma_1(t,\omega)$. This property is no more true for $L > 1$.

4 Special case of timing jitter

Consider a case where the observations are not of the form given by (1), but are corrupted by timing jitter. In this case, the observations can be written as

$$U_k(n) = Z(n - A_k(n)), \quad n \in \mathbb{Z}, \quad k = 1, \ldots, L \tag{25}$$

where timing jitter is modeled by random sequences $\tilde{\mathbf{A}}_k = \{A_k(n), n \in \mathbb{Z}\}$, $k = 1, \ldots, L$ mutually independent and independent of $\mathbf{Z}$. The jitter is assumed to be stationary in the sense that the following characteristic functions [7] are independent of n:

$$\begin{aligned} \Psi_k(\omega) &= E\left[e^{i\omega A_k(n)}\right] \\ \Lambda_k(m,\omega) &= E\left[e^{i\omega(A_k(n) - A_k(n-m))}\right]. \end{aligned} \tag{26}$$

Since sampling rates become higher and higher, the jitter effect on system performance can no longer be neglected. This may be the reason why the problem of signal reconstruction in the presence of timing jitter has received an increasing attention in recent years: see for example [8], [9], [10]. In [11] and [12], it has been shown that the jittered observations can be written in an orthogonal decomposition form

$$U_k(n) = G_k(n) + B_k(n) \tag{27}$$

where $G_k(n)$ is the output of a linear time-invariant filter, $\Psi_k^*(\omega)$ being the transfer function of this filter and $\mathbf{Z}$ the input. Moreover, $B_k(n)$ is linked to the timing jitter and can be viewed as an additive noise, independent of $\mathbf{Z}$. Therefore, the problem of recovering the original process based on observations corrupted by timing jitter (25) can be viewed as the problem (1) under interest in this paper. The LMMSE derived in (2) can be applied. This will lead the recovering of a filtered version of $\mathbf{Z}$. Then, some inverse filtering method must be used in order to reconstruct the original process $\mathbf{Z}$.

5 Simulation results

5.1 Random process under the sampling condition

This first simulation example considers the case where the sampling condition is verified. In order to derive easily analytical expressions of the reconstruction filters given in (64), the random processes involved in the observed sequences in (1) are chosen as pure white processes on $(-\pi, +\pi)$, i.e.,

$$\begin{array}{ll} s_Z(\omega) = \frac{1}{2\pi} \text{ and } s_{B_k}(\omega) = \frac{1}{2\pi b_k} & \text{for } \omega \in (-\pi, +\pi) \text{ and } k = 1, \ldots, L \\ s_Z(\omega) = s_{B_k}(\omega) = 0 & \text{for } \omega \notin (-\pi, +\pi) \text{ and } k = 1, \ldots, L \end{array} \tag{28}$$

and with $b_k > 0$ for $k = 1, \ldots, L$. Note that this parameter b_k represents the inverse of the noise variance σ_k^2. In this case, the expression of the reconstruction filters of the equivalent scheme of Fig.3 is given by

$$\Gamma_k(t, \omega) = e^{i\omega t} \frac{b_k}{1 + \sum_{m=1}^{L} b_m} \quad for \quad k = 1, \ldots, L. \tag{29}$$

It is important to note that, when the sampling condition is verified, the reconstruction filters are time-invariant related to the sampled sequences

$\tilde{\mathbf{U}}_k$. In the present example, the impulse response of these filters is of the form

$$\gamma_k(t) = \frac{b_k}{1 + \sum_{m=1}^{L} b_m} \mathrm{sinc}(\pi t) \quad for \quad k = 1, \ldots, L \tag{30}$$

where the notation $\mathrm{sinc}(x)$ is used for $\frac{\sin(x)}{x}$. Using (60) and (62), the MSE is directly linked to the noise variances

$$\sigma_t^2 = \left(\sigma_t^2\right)_2 = \frac{1}{+\sum_{m=1}^{L} b_m}. \tag{31}$$

Figure.4 shows this MSE as a function of one of the noise variance in

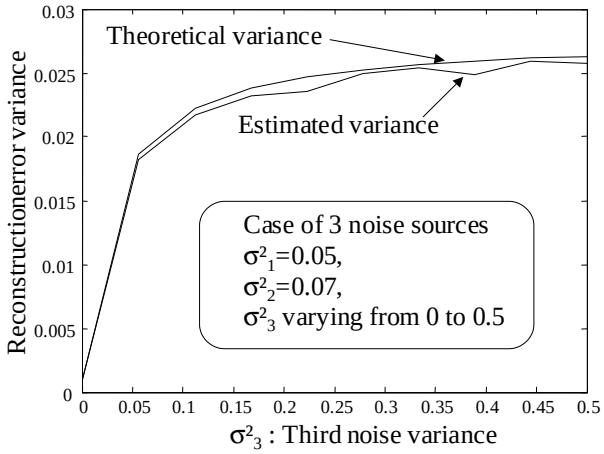

Figure 4: MSE as a function of variance of one noise source.

order to highlight the fact that this MSE tends to zero when any noise source variance tends to zero. In this figure, the source number is chosen equal to $L = 3$, two noise variances are fixed and the third one varies from 0 to 0.5. Both theoretical (31) and simulated MSE are presented. The simulated MSE is computed first by generating 1024 noisy observations (1), then by estimating the reconstructed random process as the output of the reconstruction filters given by (30) and finally by estimating the reconstruction error variance. Moreover, the expression of the MSE in (31) shows that the MSE has to decrease when the number of noise sources increases. This is confirmed by simulations in Fig.5.

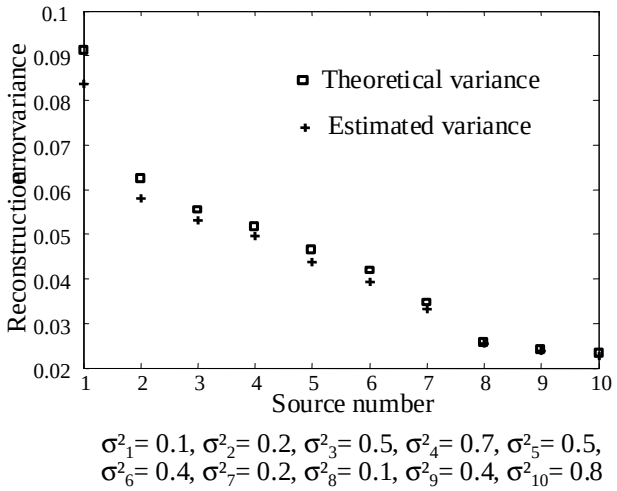

Figure 5: MSE as a function of noise source number.

5.2 NRZ random process

The aim of the second example is to consider a case where aliasing is present. Therefore, the random process $\mathbf{Z}$ is assumed to be a NRZ process [1] and the noise sequences $\tilde{\mathbf{B}}_k$ are assumed to be white sequences

$$\begin{aligned} s_Z(\omega) &= \frac{1}{2\pi}\operatorname{sinc}^2\left(\frac{\omega}{2}\right), \\ \widetilde{s}_{B_k}(\omega) &= \frac{1}{2\pi b_k} \quad b_k > 0 \text{ and } k = 1, \ldots, L. \end{aligned} \tag{32}$$

The Poisson summation formula [13] allows to derive the expression of $\widetilde{s}_{Z_t}(\omega)$ given in (57). Indeed, using the Poisson formula,

$$e^{it\omega}\sum_{k\in\mathbb{Z}} e^{i2\pi kt}\operatorname{sinc}^2\left(\frac{\omega+2\pi k}{2}\right) = \sum_{n\in\mathbb{Z}} F(t+n)\, e^{-in\omega} \tag{33}$$

where

$$F(t) = \int_{\mathbb{R}} e^{it\omega}\operatorname{sinc}^2\left(\frac{\omega}{2}\right) d\omega. \tag{34}$$

Hence,

$$\widetilde{s}_{Z_t}(\omega) = \frac{e^{-i\omega t}}{2\pi}\left[(1-\underline{t})\, e^{i\omega\bar{t}} + \underline{t} e^{i\omega\left(1+\bar{t}\right)}\right] \tag{35}$$

where $\bar{t}$ and $\underline{t}$ denote respectively the entire and fractional parts of the real instant t. The expression of the reconstruction filters (64) is given by

$$\Gamma_k(t,\omega) = \frac{b_k}{1+\sum_{m=1}^{L} b_m}\left[(1-\underline{t})\, e^{i\omega\bar{t}} + \underline{t} e^{i\omega(1+\bar{t})}\right] \quad for \quad k = 1,\ldots,L. \tag{36}$$

These time-varying filters leads to a reconstruction of the form

$$\widehat{Z}(t) = \sum_{k=1}^{L} (1-\underline{t})\, U_k\left(\bar{t}\right) + \underline{t} U_k\left(\bar{t}+1\right). \tag{37}$$

In this case, the MSE is the sum of two terms, as shown in section 2.3 with a first term due to the aliasing effect

$$\left(\sigma_t^2\right)_1 = 2\underline{t}\,(1-\underline{t}) \tag{38}$$

and a second one due to the noisy effect

$$\left(\sigma_t^2\right)_2 = \frac{1}{1+\sum_{m=1}^{L} b_m}\left(1 - 2\underline{t}\,(1-\underline{t})\right). \tag{39}$$

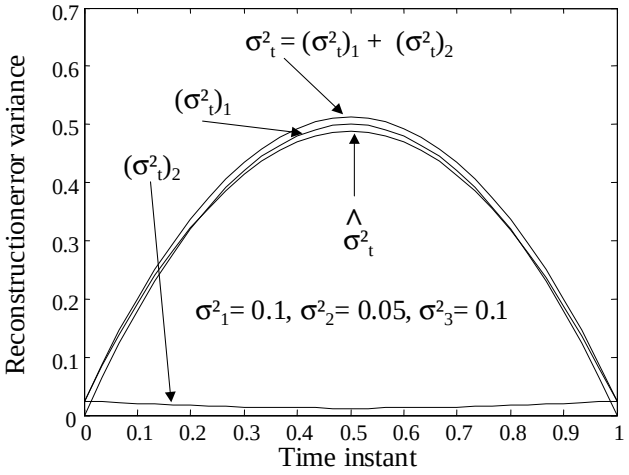

Figure 6: MSE as a function of time instant.

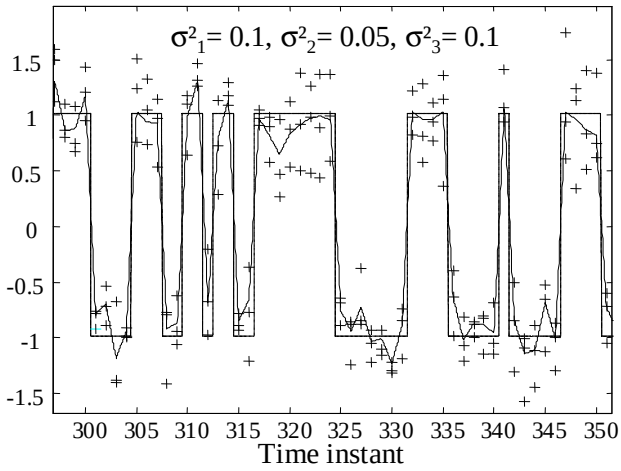

Figure 7: Portion of reconstructed signal (continuous line), given the observed samples ($L = 3$ + for each sampling time) and the original NRZ process (dotted line).

The overall MSE is periodic with a unit time period. Figure.6 shows this MSE and its two components as a function of the time instant t. As waited, the MSE is maximum when the considered instant t is in the middle of two consecutive samples. In order to clearly illustrate the result of the reconstruction, Fig.7 presents a portion of the reconstructed signal. This result has to be compared with the NRZ realization which is represented with a dotted line, taking into account that the observed samples are represented by "+". In this simulation example, $L = 3$.

6 Conclusion

This paper has considered the problem of reconstruction of a continuous-time random process when several noisy sampled versions of this process are available. This kind of problem can be found in telecommunications when diversity techniques are used. The linear minimum mean square estimator is derived in this paper. Two implementation schemes are given. The first one is recursive and involves two filter banks; the main interest of this scheme is its recursive property, i.e., adding or suppress-

ing an input does not affect the whole scheme. The second equivalent scheme is symmetric and is composed solely of one filter bank. However, any change in the input number leads to modify the whole filter bank. Simulation results have shown the performances of the proposed estimator.

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Appendix A: Orthogonalization procedure

The definition of the second sequence $\tilde{\mathbf{C}}_2$ given by (10) leads to derive a linear filter such that

$$E\left[\left(U_2\left(n\right)-\Phi_1\left[C_1\right]\left(n\right)\right)C_1^*\left(k\right)\right]=0,\quad\forall\left(k,n\right)\in\mathbb{Z}^2. \tag{40}$$

Noting that this problem is exactly the one derived in Example 1, [2, p.132], the expression of this first filter can be directly given

$$\Phi_1\left(\omega\right)=\frac{\tilde{s}_Z\left(\omega\right)}{\tilde{s}_Z\left(\omega\right)+\tilde{s}_{B_1}\left(\omega\right)}. \tag{41}$$

The PSD of the sequence $\tilde{\mathbf{C}}_2$ defined in (10) is computed via

$$\begin{aligned}
& E\left[C_2\left(n\right)C_2^*\left(n-m\right)\right] \qquad (42)\\
= \;& E\left[\left(Z\left(n\right)+B_2\left(n\right)-\Phi_1\left[Z+B_1\right]\left(n\right)\right)\right.\\
& \left.\left(Z\left(n-m\right)+B_2\left(n-m\right)-\Phi_1\left[Z+B_1\right]\left(n-m\right)\right)^*\right]\\
= \;& \int_{-\pi}^{+\pi}e^{i\omega m}\left(\tilde{s}_Z\left(\omega\right)-2\Phi_1\left(\omega\right)\tilde{s}_Z\left(\omega\right)+\tilde{s}_{B_2}\left(\omega\right)\right.\\
& \left.+\Phi_1^2\left(\omega\right)\left(\tilde{s}_Z\left(\omega\right)+\tilde{s}_{B_1}\left(\omega\right)\right)\right)d\omega\\
= \;& \int_{-\pi}^{+\pi}e^{i\omega m}\left(\tilde{s}_{B_2}\left(\omega\right)+\frac{\tilde{s}_Z\left(\omega\right)}{1+\tilde{s}_Z\left(\omega\right)/\tilde{s}_{B_1}\left(\omega\right)}\right)d\omega.
\end{aligned}$$

Therefore

$$\tilde{s}_{C_2}\left(\omega\right)=\tilde{s}_{B_2}\left(\omega\right)+\frac{\tilde{s}_Z\left(\omega\right)}{1+\tilde{s}_Z\left(\omega\right)/\tilde{s}_{B_1}\left(\omega\right)}. \tag{43}$$

The expression of the filters $\Phi_2\left(\omega\right),\ldots,\Phi_{L-1}\left(\omega\right)$ necessary to define the next orthogonal sequences $\tilde{\mathbf{C}}_3,\ldots,\tilde{\mathbf{C}}_L$ is derived by induction. Assume that m orthogonal sequences are defined such that

$$C_1\left(n\right) = U_1\left(n\right) \tag{P(1)}$$

$$C_k\left(n\right) = U_k\left(n\right)-\sum_{j=1}^{k-1}\Phi_j\left[C_j\right]\left(n\right)\quad k=2,\ldots,m \tag{P(k)}$$

with

$$\Phi_j(\omega) = \frac{\tilde{s}_Z(\omega)}{\tilde{s}_{B_j}(\omega)} \frac{1}{1 + \sum_{k=1}^{j} \tilde{s}_Z(\omega) / \tilde{s}_{B_k}(\omega)} \quad for \quad j = 1, \ldots, m-1. \tag{44}$$

The next sequence $\tilde{\mathbf{C}}_{m+1}$ is defined as the previous ones such that

$$E\left[\left(U_{m+1}(n) - \sum_{j=1}^{m} \Phi_j[C_j](n)\right) C_r^*(k)\right] = 0 \tag{45}$$
$$for \quad r = 1, \ldots, m, \forall (k,n) \in \mathbb{Z}^2.$$

Noting that

$$C_{m+1}(n) = C_m(n) - B_m(n) + B_{m+1}(n) - \Phi_m[C_m](n), \tag{46}$$

(45) is true for $r = 1, \ldots, m-1$. For $r = m$, this equation (45) is equivalent to

$$E[(Z(n) - \Phi_m[C_m](n)) C_m^*(k)] = 0. \tag{47}$$

The expression of $E[Z(n) C_m^*(k)]$ can be found by induction. Indeed, obviously,

$$E[Z(n) C_1^*(k)] = E[Z(n) Z^*(k)] = \int_{-\pi}^{+\pi} e^{i\omega(n-k)} \tilde{s}_Z(\omega) \, d\omega. \tag{48}$$

Then, taking into account this first result,

$$\begin{aligned} E[Z(n) C_2^*(k)] &= E[Z(n) (Z(k) + B_2(k) - \Phi_1[C_1](k))^*] \quad (49) \\ &= \int_{-\pi}^{+\pi} e^{i\omega(n-k)} \tilde{s}_Z(\omega) (1 - \Phi_1(\omega)) \, d\omega. \end{aligned}$$

In the same manner,

$$\begin{aligned} E[Z(n) C_3^*(k)] &= E[Z(n) (Z(k) + B_3(k) - \Phi_1[C_1](k) - \Phi_2[C_2](k))^*] \\ &= \int_{-\pi}^{+\pi} e^{i\omega(n-k)} \tilde{s}_Z(\omega) (1 - \Phi_1(\omega)) (1 - \Phi_2(\omega)) \, d\omega. \quad (50) \end{aligned}$$

Therefore, it can be easily deduced by induction that

$$E[Z(n) C_m^*(k)] = \int_{-\pi}^{+\pi} e^{i\omega(n-k)} \tilde{s}_Z(\omega) \prod_{j=1}^{m-1} (1 - \Phi j(\omega)) \, d\omega \tag{51}$$

with

$$\prod_{j=1}^{m-1}\left(1-\Phi j\left(\omega\right)\right)=\frac{1}{1+\sum_{k=1}^{m-1}\tilde{s}_{Z}\left(\omega\right)/\tilde{s}_{B_{k}}\left(\omega\right)}. \tag{52}$$

Furthermore, using orthogonal properties of the sequences,

$$\begin{aligned} E\left[C_{m}\left(n\right)C_{m}^{*}\left(k\right)\right] &= E\left[U_{m}\left(n\right)C_{m}^{*}\left(k\right)\right] \\ &= E\left[Z\left(n\right)C_{m}^{*}\left(k\right)\right]+E\left[B_{m}\left(n\right)B_{m}^{*}\left(k\right)\right]. \end{aligned} \tag{53}$$

This leads to derive the expression of the PSD of the sequence $\tilde{\mathbf{C}}_{m}$

$$\tilde{s}_{C_{m}}\left(\omega\right)=\tilde{s}_{B_{m}}\left(\omega\right)+\frac{\tilde{s}_{Z}\left(\omega\right)}{1+\sum_{k=1}^{m-1}\tilde{s}_{Z}\left(\omega\right)/\tilde{s}_{B_{k}}\left(\omega\right)}. \tag{54}$$

As a result, (47) yields the expression of the mth filter

$$\Phi_{m}\left(\omega\right)=\frac{\tilde{s}_{Z}\left(\omega\right)\prod_{j=1}^{m-1}\left(1-\Phi j\left(\omega\right)\right)}{\tilde{s}_{C_{m}}\left(\omega\right)} \tag{55}$$

which can be written as in (13).

Appendix B: Orthogonal projection

Let $W_{k,t}\left(n\right)$ denotes the orthogonal projection of $Z\left(t+n\right)$ on the Hilbert space $H_{\tilde{\mathbf{C}}_{k}}$ and $\mu_{k}\left(t,\omega\right)$ the transfer function of the equivalent linear filter such that $W_{k,t}\left(n\right)$ is the output of this filter excited by the sequence $\tilde{\mathbf{C}}_{k}$. Therefore, $W_{k,t}\left(n\right)$ verifies

$$E\left[\left(W_{k,t}\left(n\right)-Z\left(t+n\right)\right)C_{k}^{*}\left(m\right)\right]=0\quad\forall\left(n,m\right)\in\mathbb{Z}^{2}. \tag{56}$$

The correlation term $E\left[Z\left(t+n\right)C_{k}^{*}\left(m\right)\right]$ can be derived as (51). The only difference is that it involves the inter-spectrum corresponding to the intercorrelation function between the random sequence $\tilde{\mathbf{Z}}_{t}=\{Z(t+n),$ $n\in\mathbb{Z}\}$, t being fixed and $\tilde{\mathbf{Z}}=\{Z(n),n\in\mathbb{Z}\}$ rather than the PSD of $\tilde{\mathbf{Z}}$. Let $\tilde{s}_{Z_{t}}\left(\omega\right)$ denote this interspectrum. It can be shown that

$$\tilde{s}_{Z_{t}}\left(\omega\right)=\sum_{k\in\mathbb{Z}}e^{i2\pi kt}s_{Z}\left(\omega+2\pi k\right). \tag{57}$$

Taking this into account, (56) yields

$$\int_{-\pi}^{+\pi} \left(e^{i\omega(n-m)} \mu_k(t,\omega)\, \tilde{s}_{C_k}(\omega) - e^{i\omega(n-m+t)} \frac{\tilde{s}_{Z_t}(\omega)}{1+\sum_{k=1}^{m-1} \tilde{s}_Z(\omega)/\tilde{s}_{B_k}(\omega)} \right) d\omega = 0 \tag{58}$$

$$\forall\, (n,m) \in \mathbb{Z}^2.$$

This leads to the expression of the filter $\mu_k(t,\omega)$ given in (17).

This orthogonal projection allows to minimize the reconstruction error power given by (20). Using (17) and (54), it can be shown that

$$\begin{array}{l} \int_{-\pi}^{+\pi} |\mu_k(t,\omega)|^2\, \tilde{s}_{C_k}(\omega)\, d\omega = \int_{-\pi}^{+\pi} \frac{\left(\tilde{s}_{Z_t}(\omega)\right)^2}{\tilde{s}_Z(\omega)} \times \\ \left(\frac{1}{1+\sum_{j=1}^{k-1} \tilde{s}_Z(\omega)/\tilde{s}_{B_j}(\omega)} - \frac{1}{1+\sum_{j=1}^{k} \tilde{s}_Z(\omega)/\tilde{s}_{B_j}(\omega)} \right) d\omega. \end{array} \tag{59}$$

Therefore (20) reduces to

$$\sigma_t^2 = \int_{-\pi}^{+\pi} \tilde{s}_Z(\omega) - \frac{\left(\tilde{s}_{Z_t}(\omega)\right)^2}{\tilde{s}_Z(\omega)} \left(1 - \frac{1}{1+\sum_{j=1}^{L} \tilde{s}_Z(\omega)/\tilde{s}_{B_j}(\omega)} \right) d\omega. \tag{60}$$

Appendix C: Equivalent symmetric scheme

The LMMSE derivation of $Z(t+n)$ is summarized on the Fig.1. However, this scheme presents an iterative structure, leading to find a symmetric equivalent scheme as the one presented in Fig.3 Comparing both schemes (Fig.1 and Fig.3) yields immediately

$$\Gamma_L(t,\omega) = \mu_L(t,\omega) = e^{i\omega t} \frac{\tilde{s}_{Z_t}(\omega)}{\tilde{s}_{B_L}(\omega)} \frac{1}{1+\sum_{m=1}^{L} \tilde{s}_Z(\omega)/\tilde{s}_{B_m}(\omega)}. \tag{61}$$

Taking into consideration the contribution of $U_{L-1}(n)$ in the Fig.1 allows to isolate the part illustrated in Fig. 8, which must be taken into account to derive the expression of $\Gamma_{L-1}(t,\omega)$.

$$\Gamma_{L-1}(t,\omega) = \mu_{L-1}(t,\omega) - \phi_{L-1}(\omega)\, \mu_L(t,\omega). \tag{62}$$

Given the expressions of the different filters (13) and (17),

$$\Gamma_{L-1}(t,\omega) = e^{i\omega t} \frac{\tilde{s}_{Z_t}(\omega)}{\tilde{s}_{B_{L-1}}(\omega)} \frac{1}{1+\sum_{m=1}^{L} \tilde{s}_Z(\omega)/\tilde{s}_{B_m}(\omega)}. \tag{63}$$

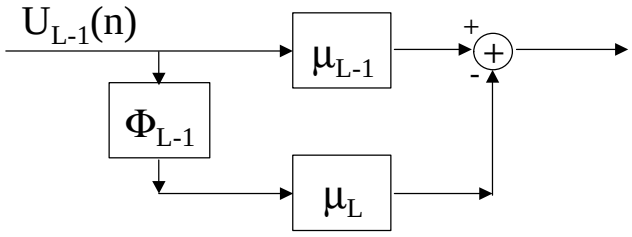

Figure 8: Contribution of $U_{L-1}(n)$ for the LMMSE derivation.

This leads us to propose the following expression for any $\Gamma_k(t,\omega)$ filter of Fig.3:

$$\Gamma_k(t,\omega) = e^{i\omega t}\frac{\tilde{s}_{Z_t}(\omega)}{\tilde{s}_{B_k}(\omega)}\frac{1}{1+\sum_{m=1}^{L}\tilde{s}_Z(\omega)/\tilde{s}_{B_m}(\omega)} \quad for \quad k=1,\ldots,L. \tag{64}$$

This can be proved by induction. It has already been shown that the result (64) holds for $k = L$ and $k = L-1$. Now, assume that the result holds for $k+1,\ldots,L$. Thus, the contribution of $U_k(n)$ in Fig.1 can be represented as illustrated on Fig.9. The expression of the equivalent

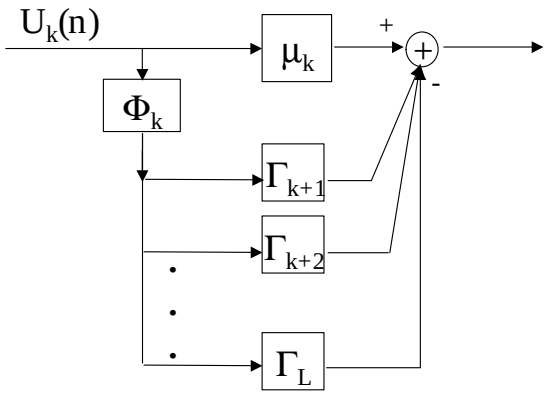

Figure 9: Contribution of $U_k(n)$ in the LMMSE derivation.

filter $\Gamma_k(t,\omega)$ is therefore given by:

$$\Gamma_k(t,\omega) = \mu_k(t,\omega) - \phi_k(\omega) \sum_{m=k+1}^{L} \Gamma_m(t,\omega). \tag{65}$$

Inserting the different filter expressions yields the expression of $\Gamma_k(t,\omega)$ of the form given by (64).