

Wellposedness of the cubic Gross-Pitaevskii equation with spatial white noise on \mathbb{R}^2

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Abstract

In this paper, we prove the global wellposedness of the Gross-Pitaevskii equation with white noise potential, i.e. a cubic nonlinear Schrödinger equation with harmonic confining potential and spatial white noise multiplicative term. This problem is ill-defined and a Wick renormalization is needed in order to give a meaning to solutions. In order to do this, we introduce a change of variables which transforms the original equation into one with less irregular terms. We construct a solution as a limit of solutions of the same equation but with a regularized noise. This convergence is shown by interpolating between a diverging bound in a high regularity Hermite-Sobolev space and a Cauchy estimate in $\mathbb{L}^2(\mathbb{R}^2)$.

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1 Introduction

In this paper, we are interested in solving the renormalized cubic Gross-Pitaevskii equation with spatial white noise potential in two dimensions of space

$$\begin{cases} i\partial_t u + Hu + \xi u + \lambda |u|^2 u = 0 \\ u(0) = u_0, \end{cases} \quad (1)$$

where $H = \Delta - |x|^2$ is the Hermite operator. We do not expect, in general, equation (1) to have solutions, even for smooth and localized initial datum, due to the low regularity of the spatial white noise. Instead, we are interested in solving the equation

$$\begin{cases} i\partial_t u + :H + \xi :u + \lambda |u|^2 u = 0 \\ u(0) = u_0, \end{cases} \quad (2)$$

where $:H + \xi:$ is the renormalized Hermite operator with spatial white noise potential, $\lambda \in \mathbb{R}$ and u_0 is in a certain function space which will be defined later. The unknown u is a random complex-valued function defined on $I \times \mathbb{R}^2$, where I is an interval of \mathbb{R} containing 0.

This renormalized equation is defined through the same kind of exponential transformation as the one used in [20] for the continuous parabolic Anderson model, and in [13] for the nonlinear Schrödinger equation on the 2d torus. Let $Y = (-H)^{-1}\xi$. Formally, the change of variable $u = ve^Y$ yields u is solution to equation (1) if and only if v is solution to the transformed equation

$$\begin{cases} i\partial_t v + Hv + 2\nabla Y \cdot \nabla v + xY \cdot xv + |\nabla Y|^2 v + \lambda |v|^2 ve^{2Y} = 0 \\ v(0) = u_0 e^{-Y} = v_0. \end{cases} \quad (3)$$

Since ξ is, a priori, only in $\mathcal{W}^{-1-s,q}$ a.s., with $q > 2$, $s > 0$ and $sq > 2$ (see Proposition 3.1), we only have $Y \in \mathcal{W}^{1-s,q}$ a.s., and thus $\nabla Y \in \mathcal{W}^{-s,q}$ and the term $|\nabla Y|^2$ is ill-defined. Here, spaces $\mathcal{W}^{s,q}$ denote Sobolev spaces associated with $-H$. We give more details on these spaces in Section 2.3. In order to give a meaning to this equation, we have to renormalize the term $|\nabla Y|^2$ using non-homogeneous Wick products associated to $-H$, which were introduced in [5]. We detail this renormalization in Section 3.2. We say that u is solution to equation (2) if $u = ve^Y$ and v is solution to the renormalized transformed equation

$$\begin{cases} i\partial_t v + Hv + 2\nabla Y \cdot \nabla v + xY \cdot xv + :|\nabla Y|^2: v + \lambda |v|^2 ve^{2Y} = 0 \\ v(0) = v_0, \end{cases} \quad (4)$$

where $:|\nabla Y|^2:$ is defined as the limit in a space of negative regularity of

$$:|\nabla Y_N|^2: = |\nabla Y_N|^2 - \mathbb{E} [|\nabla Y_N|^2]$$

and Y_N is a suitable regularization of Y defined in Section 2.4. We emphasize that equation (4) has, formally, two invariant quantities, a transformed mass for $v \in \mathbb{L}^2(\mathbb{R}^2)$

$$\tilde{M}(v) = \int_{\mathbb{R}^2} |v|^2 e^{2Y} dx = |u|_{\mathbb{L}^2_x}^2 \quad (5)$$

and a transformed energy for $v \in \mathcal{W}^{1,2}$

$$\tilde{\mathcal{E}}(v) = \frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla v|^2 + |xv|^2 - |xv|^2 Y \right) e^{2Y} dx - \frac{1}{2} \langle :|\nabla Y|^2:, |v|^2 e^{2Y} \rangle - \frac{\lambda}{4} \int_{\mathbb{R}^2} |v|^4 e^{4Y} dx, \quad (6)$$

where the bracket $\langle :|\nabla Y|^2:, |v|^2 e^{2Y} \rangle$ is defined as a duality bracket between suitable spaces.

This equation is in fact a nonlinear Schrödinger equation (NLS) on \mathbb{R}^2 with harmonic confining potential and spatial white noise potential. The equation without the harmonic potential was first studied by A. Debussche and H. Weber in [13] on the 2d torus, where the authors proved the well-posedness of a renormalized problem for cubic nonlinearities and well-chosen random initial data. In order to prove this result, they solved the renormalized equation for $v_\varepsilon = u_\varepsilon e^{-\Delta^{-1}\xi_\varepsilon}$, where ξ_ε is a mollification of the noise, and then passed to the limit in probability as ε goes to 0. In order to do this, they showed a diverging H^2 bound with logarithmic loss in ε and a converging $\mathbb{L}^2(\mathbb{T}^2)$

bound. This wellposedness result was extended to fourth order nonlinearities by N. Tzvetkov and N. Visciglia in [35] using modified energies. They further extended the result to any positive exponent in [34] using Strichartz estimates with losses for $-\Delta + \xi_\varepsilon$ in addition to modified energies.

The case of the full space, still without confining potential, was then studied in [12] for sub-cubic nonlinearities, where they used weighted Besov spaces in order to deal with the growth of the white noise at infinity. It was then extended in [14] to any polynomial nonlinearities using Strichartz estimates with losses for $-\Delta + \xi_\varepsilon$ on \mathbb{R}^2 in weighted Besov spaces. A similar result has been recently obtained for logarithmic nonlinearity in [9], also using weighted spaces.

Another approach is to work directly on the Anderson operator $-\Delta + \xi$, which has been widely studied recently, both on compact 2d manifolds (see [1, 2, 19, 23, 25, 26]) and on the full space (see [36]). Properties such as discrete spectrum, spectral gap or estimation of the eigenvalues have been obtained. Moreover some Strichartz estimates for this operator allow to solve in a more standard way NLS equations (see for example [27, 37]). Up to our knowledge, there is no comparable result for the Hermite-Anderson operator $H + \xi$ yet, hence we chose not to follow this approach.

On the other hand, the deterministic equation with confining potential is an extensively studied dispersive equation known as the Gross-Pitaevskii equation (GP). It appears in the study of Bose-Einstein condensates as a mean-field limit to the coupled linear many-body Schrödinger equation with confining potential (see [11]). The deterministic Gross-Pitaevskii equation is known to have different properties than the nonlinear Schrödinger equation due to the confining potential. For example, the harmonic potential allows standing waves to exist in the defocusing case (see [30]) whereas such solutions cannot exist without the potential. Moreover, in dimension 1, it was shown in [29] that the Gibbs measures with mass cut-off for the deterministic Gross-Pitaevskii equation are never normalizable at the critical exponent. This contrasts with the phase transition existing for the nonlinear Schrödinger equation on bounded domain. Up to our knowledge, there is no comparable result in 2d. A Gibbs measure which, formally, is invariant for the GP equation has been constructed in [5], but it was not proven to be invariant under the GP flow.

The two equations are linked by the so-called lens-transform that allows to pass from one to the other. This result can be used to prove various results for the NLS equation from the GP equation (see for example [7, 28, 32, 33]). However, in presence of a white noise potential ξ , the lens-transform changes the white noise into a time dependent stochastic process with white noise distribution at each fixed time. This time dependency makes the lens-transform unsuitable for our purpose.

The Gross-Pitaevskii equation with random potential can be seen as a model for Bose-Einstein condensates in presence of inhomogeneities (see for example [17, 18] and references therein). Hence the white noise potential can be seen as a toy model of random potential with null correlation length. However, we do not know exactly what physical meaning should the Wick renormalization have in this case.

Mathematically, this case is close to the periodic NLS with white noise potential, as H is a selfadjoint operator with compact resolvent on \mathbb{R}^2 , as the Laplacian is on the torus. In this paper,

inspired by the periodic case, we first prove the global wellposedness of the linear equation

$$\begin{cases} i\partial_t v + Hv + 2\nabla Y \cdot \nabla v + xY \cdot xv + :|\nabla Y|^2:v = 0 \\ v(0) = v_0 \end{cases} \quad (7)$$

in some Bochner spaces, and then the global wellposedness of the cubic case (i.e. equation (4)) for any well-chosen random initial data, in both linear and cubic cases, provided the \mathbb{L}_x^2 norm of the initial data is small in the focusing case (i.e. $\lambda > 0$). More precisely, we show the following results.

Theorem 1.1. *Linear case*

Let $\sigma \in (1, 2)$, $p_0 \in (2, +\infty)$, $p \in (2, p_0)$, $a < 0 < b$ and $v_0 \in \mathbb{L}^{p_0}(\Omega, \mathcal{W}^{2,2})$. The following hold:

- There exists a unique solution v to equation (7) in $\mathbb{L}^p(\Omega, \mathcal{C}([a, b], \mathcal{W}^{\sigma,2}))$
- The sequence $(v^N)_{N \in \mathbb{N}}$ of solutions of the regularized linear equation (13) converges to v in $\mathbb{L}^p(\Omega, \mathcal{C}([a, b], \mathcal{W}^{\sigma,2}))$
- The transformed mass and energy, defined by equations (5) and (6) with $\lambda = 0$ respectively, are conserved.

Theorem 1.2. *Cubic defocusing case*

Let $\lambda \leq 0$. Almost surely, for every $a < 0 < b$, every $\sigma \in (1, 2)$ and every $v_0 \in \mathcal{W}^{2,2}$, it holds

- there exists a unique solution v to equation (4) in $\mathcal{C}([a, b], \mathcal{W}^{\sigma,2})$
- the sequence $(v^N)_{N \in \mathbb{N}}$ of solution of the regularized cubic equation (10) converges to v in $\mathcal{C}([a, b], \mathcal{W}^{\sigma,2})$
- transformed mass and energy, defined by equations (5) and (6) respectively, are conserved.

Theorem 1.3. *Cubic focusing case*

Let $\lambda > 0$ and $L > 0$, and let $\overline{\Omega}_L = \left\{ \lambda L^2 |e^{-2Y}|_{\mathbb{L}_x^\infty}^2 |e^{2Y}|_{\mathbb{L}_x^\infty} |e^{4Y}|_{\mathbb{L}_x^\infty} < 4 \right\}$. Then, on $\overline{\Omega}_L$, endowed with the conditional probability and σ -algebra, there exists a random variable $N_0 \in \mathbb{N}$ a.s. such that almost surely, for every $a < 0 < b$, every $\sigma \in (1, 2)$ and every $v_0 \in \mathcal{W}^{2,2}$ verifying $|v_0|_{\mathbb{L}_x^2} \leq L$, it holds

- there exists a unique solution v to equation (4) in $\mathcal{C}([a, b], \mathcal{W}^{\sigma,2})$
- the sequence $(v^N)_{N \geq N_0}$ of solution of the regularized cubic equation (10) converges to v in $\mathcal{C}([a, b], \mathcal{W}^{\sigma,2})$
- transformed mass and energy, defined by equations (5) and (6) respectively, are conserved.

In both linear and cubic cases, we will follow the main ideas of [13], with some technical modifications. First, we regularize the noise using the smooth spectral truncation introduced in [5] and detailed in Section 2.4 instead of using a mollification. This is a more natural regularization in this context because it has a better behaviour in Sobolev-Hermite spaces than the usual truncation and regularization procedure. Then, we show a diverging bound in sufficiently regular space. Contrary to the periodic case, we only have a polynomial (and not a logarithmic) loss, but with an exponent

as small as desired. We can obtain this diverging bound in $\mathcal{W}^{2,2}$ in the linear case but, in the nonlinear case, due to technical issues in the proof of Proposition 5.3, we only show a diverging bound in $\mathcal{W}^{\sigma,2}$ for any $\sigma \in (1,2)$. Finally, we obtain a converging \mathbb{L}_x^2 bound for the difference of approximations, and we use it along with the diverging bound and interpolation to prove the \mathbb{L}_ω^p or pathwise (depending on the case) convergence along a subsequence. The convergence along the whole sequence and conservation laws then follow easily.

Let us emphasize the differences with the periodic case treated in [13]. First, notice that the quantity we subtract in order to renormalize $|\nabla Y_N|^2$ is a function of the space variable going to infinity as N goes to infinity. In the periodic case, this quantity is constant in space, so the renormalization can be understood as an "infinite phase shift". This interpretation is no longer possible in presence of the harmonic potential. The renormalization should be thought as an "infinite potential" which prevents the solution from having infinite energy due to the high modes. Moreover, the eigenvalues of the Hermite operator grow slower than those of the periodic Laplacian. Thus, it is not sure that the diverging quantity goes to infinity as $\log(N)$ as it is the case in the periodic setting. In fact, one can show that it may diverge slower than N^ε for any $\varepsilon > 0$, but obtaining an optimal uniform upper bound on the divergence is still an open question. This leads to polynomial losses in Propositions 4.1, 4.2, 5.3 and 5.4 instead of the logarithmic loss in the periodic case. For this reason, it is necessary to take care not to have bounds of the form $N^{C\kappa}$ with a random variable C depending on κ .

The paper is structured as follows, the Section 2.2 recalls most results we need about the Hermite operator, the functional spaces we need in our analysis and their properties are introduced in Section 2.3, the smooth spectral truncation is detailed in Section 2.4. The regularity properties of the white noise and other random variables involved in our analysis are presented in Section 3.1, the renormalization procedure is explained in Section 3.2 and the global wellposedness of the regularized equation is shown in Section 3.3. Finally, Section 4 is devoted to the proof of Theorem 1.1 and Section 5 is devoted to the proof of Theorem 1.2. Most of the proofs of Sections 2 and 3 can be found in appendices A and B respectively.

2 Preliminaries

2.1 Notations and conventions

- $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ is the Laplacian
- For $x \in \mathbb{R}^2$, $|x|^2 = x_1^2 + x_2^2$ and $\langle x \rangle^2 = 1 + |x|^2$
- $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^* = \{1, 2, \dots\}$
- For $\alpha \in \mathbb{N}^2$, $|\alpha| = \alpha_1 + \alpha_2$ and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$
- For $s \in \mathbb{R}$ and $p \in (1, +\infty)$, $W^{s,p} = \{u \in \mathcal{S}'(\mathbb{R}^2), \mathcal{F}^{-1}(\langle \eta \rangle^s \mathcal{F}u) \in \mathbb{L}^p(\mathbb{R}^2)\}$ denotes the Bessel potential space endowed with the norm $|u|_{W^{s,p}} = |\mathcal{F}^{-1}(\langle \eta \rangle^s \mathcal{F}u)|_{\mathbb{L}_x^p}$, where \mathcal{F} denotes the usual Fourier transform. These spaces extend the usual Sobolev spaces by complex interpolation.

- We denote by $\mathcal{S}(\mathbb{R}^d)$ the space of smooth rapidly decaying functions on \mathbb{R}^d and by $\mathcal{S}'(\mathbb{R}^d)$ its continuous dual, the space of tempered distributions on \mathbb{R}^d . We endow the space of tempered distributions with the real bracket

$$\forall T \in \mathcal{S}'(\mathbb{R}^2) \cap \mathbb{L}_{loc}^1(\mathbb{R}^2), \forall \phi \in \mathcal{S}(\mathbb{R}^2), \langle T, \phi \rangle_{\mathcal{S}', \mathcal{S}} = \operatorname{Re} \left(\int_{\mathbb{R}^2} T(x) \overline{\phi(x)} dx \right)$$

so that for $T \in \mathbb{L}^2(\mathbb{R}^2)$, $\langle T, \phi \rangle_{\mathcal{S}', \mathcal{S}} = \langle T, \phi \rangle_{\mathbb{L}^2}$.

- We say that a sequence of random variables $(X_N)_{N \in \mathbb{N}}$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ verifies the Property (*) if
 - For almost all $\omega \in \Omega$, $\sup_{N \in \mathbb{N}} X_N(\omega) < +\infty$ and for all $N \in \mathbb{N}$, $X_N(\omega) > 0$.
 - For all $p \in [1, +\infty)$, $\sup_{N \in \mathbb{N}} \mathbb{E}[X_N^p] < +\infty$.

2.2 Hermite operator

We denote by $H = \Delta - |x|^2$ the Hermite operator. It is well-known that this operator has eigenfunctions $(h_k)_{k \in \mathbb{N}^2}$ which are given by $h_k = \psi_{k_1} \otimes \psi_{k_2}$ ($k = (k_1, k_2)$), where we denote by ψ_{k_1} the k_1 -th Hermite function. For any $k \in \mathbb{N}^2$, h_k verifies the relation $-Hh_k = \lambda_k^2 h_k$ where $\lambda_k^2 = 2|k| + 2$. Moreover, (h_k) is a complete orthonormal system of $\mathbb{L}^2(\mathbb{R}^2, \mathbb{R})$.

The Schwartz space $\mathcal{S}(\mathbb{R}^2, \mathbb{R})$ and its dual $\mathcal{S}'(\mathbb{R}^2, \mathbb{R})$ can be characterized using brackets against Hermite functions. The following proposition is a direct corollary of Theorem 1 in [31].

Proposition 2.1. *If $T \in \mathcal{S}'(\mathbb{R}^2, \mathbb{R})$, then $\sum_{|k| \leq N} \langle T, h_k \rangle_{\mathcal{S}', \mathcal{S}} h_k$ converges weakly to T in $\mathcal{S}'(\mathbb{R}^2, \mathbb{R})$.*

Hence, for any $\alpha \in \mathbb{R}$, one can define $(-H)^\alpha$ on $\mathcal{S}'(\mathbb{R}^2, \mathbb{R})$ by

$$\forall T \in \mathcal{S}'(\mathbb{R}^2, \mathbb{R}), (-H)^\alpha T = \sum_{k \in \mathbb{N}^2} \lambda_k^{2\alpha} \langle T, h_k \rangle_{\mathcal{S}', \mathcal{S}} h_k$$

defined as the weak limit of $\sum_{|k| \leq N} \lambda_k^{2\alpha} \langle T, h_k \rangle_{\mathcal{S}', \mathcal{S}} h_k$. For complex valued tempered distributions, analogous results hold for expansions in terms of $\{h_k, ih_k\}_{k \in \mathbb{N}^2}$. It follows that $(-H)^{-1}$ has a kernel given by

$$K(x, y) = \sum_{k \in \mathbb{N}^2} \frac{h_k(x) h_k(y)}{\lambda_k^2} \quad (8)$$

in the sense that for any $\phi \in \mathcal{S}(\mathbb{R}^2)$, we have

$$\forall x \in \mathbb{R}^2, (-H)^{-1} \phi(x) = \int_{\mathbb{R}^2} K(x, y) \phi(y) dy.$$

This kernel has the following integrability properties.

Proposition 2.2. *(See proposition 5 in [5]) For any $n \in \mathbb{N}^*$, $r \geq 2$ and $\alpha < 1 - \frac{2}{r}$, $K^n \in \mathbb{L}_x^r \mathcal{W}_y^{\alpha, 2}$.*

This proposition will be useful to prove the convergence of the Wick products and to prove Proposition 3.1 about the white noise regularity.

2.3 Hermite-Sobolev spaces

Let $s \in \mathbb{R}$ and $p \in (1, +\infty)$ and define

$$\mathcal{W}^{s,p} = \{u \in \mathcal{S}'(\mathbb{R}^2), (-H)^{\frac{s}{2}} u \in \mathbb{L}^p(\mathbb{R}^2)\}$$

endowed with the norm

$$|u|_{\mathcal{W}_x^{s,p}} = |(-H)^{\frac{s}{2}} u|_{\mathbb{L}_x^p}.$$

It is known that these spaces are Banach spaces and that for q such that $\frac{1}{p} + \frac{1}{q} = 1$, we have $(\mathcal{W}^{s,p})' = \mathcal{W}^{-s,q}$ with equal norm. The subspace $\text{span} \{h_k, k \in \mathbb{N}^2\}$ is dense in these spaces (see for example [4] for the case $s \geq 0$) and for $s \geq 0$, an equivalent norm is given for $p \in (1, +\infty)$ by (see [15])

$$|\langle D \rangle^s u|_{\mathbb{L}_x^p} + |\langle x \rangle^s u|_{\mathbb{L}_x^p}.$$

This shows that $\mathcal{W}^{s,p} = \{u \in W^{s,p}(\mathbb{R}^2), \langle x \rangle^s u \in \mathbb{L}^p(\mathbb{R}^2)\}$. Using this equivalent norm, we define $\mathcal{W}^{k,p}$ for $k \in \mathbb{N}$ and $p \in \{1, +\infty\}$ as

$$\mathcal{W}^{k,p} = \{u \in W^{k,p}(\mathbb{R}^2), \langle x \rangle^k u \in \mathbb{L}^p(\mathbb{R}^2)\}$$

endowed with the norm

$$|u|_{\mathcal{W}_x^{k,p}} = |u|_{W_x^{k,p}} + |\langle x \rangle^k u|_{\mathbb{L}_x^p}$$

and extend this definition to $s \geq 0$ by complex interpolation between integer regularities. Moreover inspired by the relation $(\mathcal{W}^{s,p})' = \mathcal{W}^{-s,q}$, for $s < 0$, we define $\mathcal{W}^{s,\infty} = (\mathcal{W}^{-s,1})'$. As usual, we cannot define spaces of negative \mathbb{L}^1 regularity as dual spaces of positive \mathbb{L}^∞ regularity.

Remark 2.3.

- We do not claim that the first definition of $\mathcal{W}^{s,p}$ agrees with the one we give for spaces over \mathbb{L}^1 and \mathbb{L}^∞ . We use the same notation for simplicity as we will always control $\mathcal{W}^{s,\infty}$ norms by Sobolev embeddings.
- In the special case $\mathcal{W}^{1,2}$, we also use the notation Σ .

The choices we made ensure us that we can interpolate between spaces of positive regularity and some other convenient properties that we need in our analysis.

A useful property of these spaces is that differentiation and multiplication by x act directly on regularity. In order to see this, let us introduce the annihilation and creation operators:

$$\forall i \in \{1, 2\}, A_i = \partial_i + x_i \text{ and } A_{-i} = A_i^* = -\partial_i + x_i.$$

Proposition 2.4. *Let $s \in \mathbb{R}$, $p \in (1, +\infty)$ and $i \in \{\pm 1, \pm 2\}$, then A_i is a bounded linear operator from $\mathcal{W}^{s,p}$ to $\mathcal{W}^{s-1,p}$.*

Corollary 2.5. *Let $s \in \mathbb{R}$, $p \in (1, +\infty)$ and $i \in \{1, 2\}$, then ∂_i and $x_i \cdot$ are bounded linear operators from $\mathcal{W}^{s,p}$ to $\mathcal{W}^{s-1,p}$.*

The Sobolev-Hermite spaces verify some continuous embeddings as in the classical Sobolev framework. We will also refer to these embeddings as Sobolev embeddings.

Proposition 2.6. (Sobolev embeddings) Let $1 < p \leq q < +\infty$ and $s > \sigma$ such that $\frac{1}{p} - \frac{s}{2} \leq \frac{1}{q} - \frac{\sigma}{2}$. Then $\mathcal{W}^{s,p}$ is continuously embedded in $\mathcal{W}^{\sigma,q}$. Moreover if $1 < p < q \leq +\infty$ and $s > \sigma$ such that $\frac{1}{p} - \frac{s}{2} < \frac{1}{q} - \frac{\sigma}{2}$, then $\mathcal{W}^{s,p}$ is compactly embedded in $\mathcal{W}^{\sigma,q}$.

Moreover, our choices allow us to have a simple product rule on our spaces.

Lemma 2.7. Let $s \geq 0$ and $1 \leq p, q, r \leq +\infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. There exists a constant $C > 0$ such that for all $u \in \mathcal{W}^{s,p}$ and $v \in \mathcal{W}^{s,q}$, we have $uv \in \mathcal{W}^{s,r}$ and $|uv|_{\mathcal{W}^{s,r}} \leq C|u|_{\mathcal{W}^{s,p}}|v|_{\mathcal{W}^{s,q}}$.

Proof. For $s \in \mathbb{N}$, the claim follows from Leibniz's formula and Hölder's inequality. By bilinear interpolation (see for example theorem 4.4.1 of [3]), we obtain the claim for all $s \geq 0$. \square

We deduce the following product rule for negative-positive regularity products.

Lemma 2.8. Let $s \geq 0$, $1 < p < +\infty$ and $1 \leq q, r \leq +\infty$ such that $1 - \frac{1}{p} + \frac{1}{q} = 1 - \frac{1}{r}$. There exists a constant $C > 0$ such that for all $u \in \mathcal{W}^{-s,r}$ and $v \in \mathcal{W}^{s,q}$, we have $uv \in \mathcal{W}^{-s,p}$ and $|uv|_{\mathcal{W}^{-s,p}} \leq C|u|_{\mathcal{W}^{-s,r}}|v|_{\mathcal{W}^{s,q}}$.

Proof. Let p' and r' be such that $\frac{1}{p'} = 1 - \frac{1}{p}$ and $\frac{1}{r'} = 1 - \frac{1}{r}$. Let $\phi \in \mathcal{S}(\mathbb{R}^2)$, using Lemma 2.7 we have $\bar{v}\phi \in \mathcal{W}^{s,r'}$ whose dual is $\mathcal{W}^{-s,r}$. Then, we can define the tempered distribution uv as

$$\langle uv, \phi \rangle_{\mathcal{S}', \mathcal{S}} = \langle u, \bar{v}\phi \rangle_{\mathcal{W}^{-s,r}, \mathcal{W}^{s,r}}$$

and we have $|\langle uv, \phi \rangle_{\mathcal{S}', \mathcal{S}}| \lesssim |u|_{\mathcal{W}^{-s,r}}|\bar{v}\phi|_{\mathcal{W}^{s,r'}} \lesssim |u|_{\mathcal{W}^{-s,r}}|v|_{\mathcal{W}^{s,q}}|\phi|_{\mathcal{W}^{s,p'}}$. We conclude by density of Schwartz functions. \square

Let us recall another product rule on these spaces whose proof is in [5] and which will be useful in the proof of Proposition 3.12.

Lemma 2.9. Let $r, r_1, r_2 > 1$, $p \geq 2$ and $0 < \alpha < \sigma$ such that $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r}$ and $\frac{\sigma-\alpha}{2} = \frac{1}{p}$. There exists a constant $C > 0$ such that for all $f, g \in \mathbb{L}_x^{r_1} \mathbb{L}_y^p \cap \mathbb{L}_x^{r_2} \mathcal{W}_y^{\sigma,2}$, it holds $fg \in \mathbb{L}_x^r \mathcal{W}_y^{\alpha,2}$ and

$$|fg|_{\mathbb{L}_x^r \mathcal{W}_y^{\alpha,2}} \leq C \left(|f|_{\mathbb{L}_x^{r_1} \mathbb{L}_y^p} |g|_{\mathbb{L}_x^{r_2} \mathcal{W}_y^{\sigma,2}} + |f|_{\mathbb{L}_x^{r_2} \mathcal{W}_y^{\sigma,2}} |g|_{\mathbb{L}_x^{r_1} \mathbb{L}_y^p} \right).$$

2.4 The smooth truncation

Choose $\chi \in \mathcal{S}(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\text{supp}(\chi) \subset [-1, 1]$ and $\chi|_{[0,1/2]} = 1$. We define the smooth truncation as the family $(S_N)_{N \in \mathbb{N}}$ with $S_N = \chi \left(\frac{-H}{\lambda_N^2} \right)$ where $\lambda_N = \lambda_{(N,0)} = \sqrt{2N+2}$. For $u \in \mathbb{L}^2(\mathbb{R}^2)$, S_N is well-defined and we have the formula

$$S_N u = \sum_{k \in \mathbb{N}^2} \chi_{k,N} \langle u, h_k \rangle_{\mathbb{L}^2} h_k,$$

where $\chi_{k,N} = \chi \left(\frac{\lambda_k^2}{\lambda_N^2} \right)$. It extends naturally to $\mathcal{W}^{s,2}$ with $s \geq 0$ and to the case $s < 0$ by duality. Using Sobolev embeddings, one can extend this definition to any $\mathcal{W}^{s,p}$ with $s \in \mathbb{R}$ and $p \in (1, 2)$ and finally to $\mathcal{W}^{s,q}$ with $s \in \mathbb{R}$ and $q \in (2, +\infty)$ by duality. One can show using Fourier-Hermite expansions that for $u \in \mathcal{W}^{s,p}$ ($s \in \mathbb{R}$ and $p \in (1, +\infty)$), we have

$$S_N u = \sum_{k \in \mathbb{N}^2} \chi_{k,N} \langle u, h_k \rangle_{\mathcal{S}', \mathcal{S}} h_k$$

which extends the previous \mathbb{L}^2 formula. We have the following important lemma which motivates the choice of a smooth truncation.

Lemma 2.10. (see Theorem 1 in [22]) *Let $\psi \in \mathcal{S}(\mathbb{R})$ and $p \in [1, +\infty]$. The family $(\psi(-\theta H))_{\theta \in [0,1]}$ is uniformly bounded in $\mathcal{L}(\mathbb{L}^p(\mathbb{R}^2))$.*

Using the fact that S_N commutes with any $(-H)^\alpha$, we obtain the following result, which is not verified by the spectral projector $\Pi_N u = \sum_{|k| \leq N} \langle u, h_k \rangle_{\mathbb{L}^2} h_k$.

Corollary 2.11. *Let $s \in \mathbb{R}$ and $p \in (1, +\infty)$. Then, $(S_N)_{N \in \mathbb{N}}$ is uniformly bounded in $\mathcal{L}(\mathcal{W}^{s,p})$.*

We will use the following lemmas, which are in fact corollaries of Lemma 2.10, to estimate norms of low-mode and high-mode functions.

Lemma 2.12. *Let $p \in (1, +\infty)$, $\alpha \in \mathbb{R}$ and $s > 0$, there exists $C > 0$ such that*

$$\forall N \in \mathbb{N}, \forall \phi \in \mathcal{W}^{\alpha,p}, |S_N \phi|_{\mathcal{W}_x^{\alpha+s,p}} \leq C \lambda_N^{s'} |\phi|_{\mathcal{W}_x^{\alpha,p}}.$$

Lemma 2.13. *Let $\alpha \in \mathbb{R}$ and $s > 0$. We have*

$$\forall N \in \mathbb{N}, \forall \phi \in \mathcal{W}^{\alpha+s,2}, |\phi - S_N \phi|_{\mathcal{W}_x^{\alpha,2}} \leq \lambda_{\lfloor \frac{N}{2} \rfloor}^{-s} |\phi|_{\mathcal{W}_x^{\alpha+s,2}}.$$

By density of Schwartz functions, Sobolev embeddings and Corollary 2.11, it follows that for any $s \in \mathbb{R}$, any $q \in [2, +\infty)$ and any $u \in \mathcal{W}^{s,q}$, $S_N u$ converges to u in $\mathcal{W}^{s,q}$.

3 The regularized noise

3.1 Noise regularity

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space such that there exists a sequence $(\xi_k)_{k \in \mathbb{N}^2}$ of standard normal random variables. We define the spatial white noise on \mathbb{R}^2 as $\xi = \sum_{k \in \mathbb{N}^2} \xi_k h_k$. Such random Hermite

series have been studied in [21]. As $(h_k)_{k \in \mathbb{N}^2}$ is a complete orthonormal system of $\mathbb{L}^2(\mathbb{R}^2)$, one recovers the fundamental property of white noises

$$\forall \phi, \psi \in \mathcal{S}(\mathbb{R}^2), \mathbb{E}[\langle \xi, \phi \rangle_{S',S} \langle \xi, \psi \rangle_{S',S}] = \langle \phi, \psi \rangle_{\mathbb{L}_x^2}.$$

Proposition 3.1. *Let $q > 2$ and $s > 0$ such that $sq > 2$. Then $\xi \in \mathcal{W}^{-1-s,q}$ a.s.*

Proof. Let $\alpha > 0$ and $q > 2$, we have $(-H)^{-\alpha/2} \xi = \sum_{k \in \mathbb{N}^2} \xi_k \lambda_k^{-\alpha} h_k$. Using Fubini's theorem and gaussianity, we obtain

$$\mathbb{E} \left[|\xi|_{\mathcal{W}_x^{-\alpha,q}}^q \right] = \int_{\mathbb{R}^2} \mathbb{E} \left[\left| (-H)^{-\alpha/2} \xi(x) \right|^q \right] dx \lesssim |\sigma_\alpha|_{\mathbb{L}_x^q}^q$$

where

$$\sigma_\alpha(x)^2 = \sum_{k \in \mathbb{N}^2} \lambda_k^{-2\alpha} |h_k(x)|^2 = \mathbb{E} \left[\left| (-H)^{-\alpha/2} \xi(x) \right|^2 \right].$$

Using the definition of the kernel K in equation (8), we obtain $\sigma_\alpha(x)^2 = |K(x, \cdot)|_{\mathcal{W}_y^{2-\alpha,2}}^2$. Then by Proposition 2.2, σ_α has finite \mathbb{L}_x^q norm when $2 - \alpha < 1 - \frac{2}{q}$ (i.e. $\alpha > 1 + \frac{2}{q}$). Let $s = \alpha - 1$, then $s > 0$ and $\xi \in \mathcal{W}^{-1-s,q}$ a.s. as long as $qs > 2$. \square

Remark 3.2. • This proof is not totally rigorous, but in fact, the following lemmas will show the convergence of $S_N \xi$ to ξ in $\mathcal{W}^{-1-s,q}$ ($q > 2$, $sq > 2$), both almost surely and in all \mathbb{L}_Ω^p spaces, with an explicit rate of convergence. Moreover, this regularity result is the same as the one given by applying Theorem 2.2 in [21] to $(-H)^{-1-s}\xi$.

- We do not know if this regularity is optimal but we strongly believe that for $q < +\infty$, ξ does not belong to $\mathcal{W}^{-1,q}$. This contrasts with the case of a radial white noise (i.e. with only radial Hermite functions in the expansion of ξ) which lives almost surely in $\mathcal{W}^{-1,q}$ for any $q \in (2, +\infty)$ according to Theorem 2.3 in [21].

Now, we introduce a probabilistic lemma that we will use several times in what follows.

Lemma 3.3. *Let $(X_N)_{N \in \mathbb{N}}$ be a sequence of real random variables defined on the same probability space. Assume there exists $\alpha > 0$ such that for every $p \in [1, +\infty)$, $\sup_{N \in \mathbb{N}} \lambda_N^\alpha |X_N|_{\mathbb{L}_\omega^p} < +\infty$. Then, for every $\beta < \alpha$, almost surely $\sup_{N \in \mathbb{N}} \lambda_N^\beta |X_N| < +\infty$.*

Let $N \in \mathbb{N}$ and set $Y = (-H)^{-1}\xi$ and $Y_N = S_N Y$. The proof of the following results are given in appendix.

Lemma 3.4. *Let $p \in [1, +\infty)$, $q > 2$ and $1 > s > s' > \frac{2}{q}$. There exists a constant $C > 0$ such that*

$$\sup_{N \in \mathbb{N}} \left[\lambda_N^{s-s'} |Y - Y_N|_{\mathbb{L}_\omega^p \mathcal{W}_x^{1-s,q}} \right] \leq C.$$

Using Sobolev embeddings and Lemma 3.3, this implies the following result.

Corollary 3.5. *Let $0 < \kappa' < \kappa < 1$, there exists a sequence of positive random variables $(C_N)_{N \in \mathbb{N}}$ verifying Property (*) such that almost surely, for all $N \in \mathbb{N}$,*

$$|Y - Y_N|_{\mathcal{W}_x^{1-\kappa,\infty}} \leq C_N \lambda_N^{-\kappa'}, \quad |\nabla Y - \nabla Y_N|_{\mathcal{W}_x^{-\kappa,\infty}} \leq C_N \lambda_N^{-\kappa'} \quad \text{and} \quad |xY - xY_N|_{\mathcal{W}_x^{-\kappa,\infty}} \leq C_N \lambda_N^{-\kappa'}.$$

Moreover, for any $p \in [1, +\infty)$, there exists $C > 0$ such that for all $N \in \mathbb{N}$,

$$|Y - Y_N|_{\mathbb{L}_\omega^p \mathcal{W}_x^{1-\kappa,\infty}} \leq C \lambda_N^{-\kappa'}, \quad |\nabla Y - \nabla Y_N|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-\kappa,\infty}} \leq C \lambda_N^{-\kappa'} \quad \text{and} \quad |xY - xY_N|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-\kappa,\infty}} \leq C \lambda_N^{-\kappa'}.$$

Using Corollary 2.11 and Propositions 2.6 and 3.1, we obtain the following lemma.

Corollary 3.6. *Let $\kappa \in (0, 1)$, there exists a sequence of positive random variables $(C_N)_{N \in \mathbb{N}}$ verifying Property (*) such that almost surely*

$$\forall N \in \mathbb{N}, \quad |\nabla Y_N|_{\mathbb{L}_x^\infty} \leq C_N \lambda_N^\kappa \quad \text{and} \quad |xY_N|_{\mathbb{L}_x^\infty} \leq C_N \lambda_N^\kappa.$$

Moreover, for any $p \in [1, +\infty)$, there exists $C > 0$ such that

$$\forall N \in \mathbb{N}, \quad |\nabla Y_N|_{\mathbb{L}_\omega^p \mathbb{L}_x^\infty} \leq C \lambda_N^\kappa \quad \text{and} \quad |xY_N|_{\mathbb{L}_\omega^p \mathbb{L}_x^\infty} \leq C \lambda_N^\kappa.$$

Using Hölder's inequality, Sobolev embeddings and Lemma 3.3, we deduce the following result.

Corollary 3.7. *Let $\kappa \in (0, 1)$, $q \in (1, +\infty)$ and a random variable $X \in \mathcal{W}^{1,q}$ a.s., there exists a sequence of positive random variables $(C_N)_{N \in \mathbb{N}}$ verifying Property (*) such that almost surely*

$$\forall N \in \mathbb{N}, |\nabla X \cdot \nabla Y_N|_{\mathbb{L}_x^q} \leq C_N |X|_{\mathcal{W}_x^{1,q}} \lambda_N^\kappa \text{ and } |xX \cdot xY_N|_{\mathbb{L}_x^q} \leq C_N |X|_{\mathcal{W}_x^{1,q}} \lambda_N^\kappa.$$

Moreover, let $p_0 > p \geq 1$ and assume that $X \in \mathbb{L}^{p_0}(\Omega, \mathcal{W}^{1,q})$, then there exists $C > 0$ such that

$$\forall N \in \mathbb{N}, |\nabla X \cdot \nabla Y_N|_{\mathbb{L}_\omega^p \mathbb{L}_x^q} \leq C \lambda_N^\kappa |X|_{\mathbb{L}_\omega^{p_0} \mathcal{W}^{1,q}} \text{ and } |xX \cdot xY_N|_{\mathbb{L}_\omega^p \mathbb{L}_x^q} \leq C \lambda_N^\kappa |X|_{\mathbb{L}_\omega^{p_0} \mathcal{W}^{1,q}}.$$

Using the algebra property of $\mathcal{W}^{1-s,q}$ for $q > 4$ and $sq > 2$, $e^{aY} - 1 \in \mathcal{W}^{1-s,q}$ for any $a \in \mathbb{R}$. Using the gaussianity of Y_N and Y , we obtain the following moment estimates for exponentials of Y_N and Y .

Lemma 3.8. *Let $p \geq 1$, $q > 4$, $s \in (\frac{2}{q}, 1 - \frac{2}{q})$ and $a \in \mathbb{R}$, there exists $C > 0$ such that*

$$|e^{aY} - 1|_{\mathbb{L}_\omega^p \mathcal{W}_x^{1-s,q}} \leq C \text{ and } \sup_{N \in \mathbb{N}} |e^{aY_N} - 1|_{\mathbb{L}_\omega^p \mathcal{W}_x^{1-s,q}} \leq C.$$

Using Sobolev embeddings, we obtain the following corollary.

Corollary 3.9. *Let $p \geq 1$, $\kappa \in (0, 1)$ and $a \in \mathbb{R}$, there exists $C > 0$ such that*

$$|e^{aY} - 1|_{\mathbb{L}_\omega^p \mathcal{W}_x^{1-\kappa,\infty}} \leq C \text{ and } \sup_{N \in \mathbb{N}} |e^{aY_N} - 1|_{\mathbb{L}_\omega^p \mathcal{W}_x^{1-\kappa,\infty}} \leq C.$$

We deduce a control on exponential moments in \mathbb{L}^∞ Hermite-Sobolev spaces.

Lemma 3.10. *Let $0 < \kappa' < \kappa < 1$, $a \in \mathbb{R}$ and $p \geq 1$. There exists $C > 0$ such that*

$$\sup_{N \in \mathbb{N}} \left[\lambda_N^{\kappa'} |e^{aY_N} - e^{aY}|_{\mathbb{L}_\omega^p \mathcal{W}_x^{1-\kappa,\infty}} \right] \leq C.$$

Moreover, there exists a positive random variable C almost surely finite such that, almost surely, it holds

$$\forall N \in \mathbb{N}, |e^{aY_N} - e^{aY}|_{\mathcal{W}_x^{1-\kappa,\infty}} \leq C \lambda_N^{-\kappa'} \text{ and } |e^{aY_N} - 1|_{\mathcal{W}_x^{1-\kappa,\infty}} \leq C.$$

We introduce the shorten notation $|f|_{1+\mathcal{W}_x^{\sigma,p}} = 1 + |f - 1|_{\mathcal{W}_x^{\sigma,p}}$ for any $f \in \mathcal{S}'(\mathbb{R}^2)$ verifying $f - 1 \in \mathcal{W}^{\sigma,p}$ ($\sigma \in \mathbb{R}$ and $p \in [1, +\infty]$). We emphasize that this is just a notation for the seek of compactness and that it does not denote the norm of any space. It allows to formulate in a compact fashion the following straightforward corollary of Lemma 2.7.

Lemma 3.11. *Let $s \in (0, 1)$ and $p \in [1, +\infty]$. There exists a deterministic constant $C > 0$ such that almost surely, for any $v \in \mathcal{W}^{s,p}$ and any $a \in \mathbb{R}$, $ve^{aY} \in \mathcal{W}^{s,p}$ and it holds*

$$|ve^{aY}|_{\mathcal{W}_x^{s,p}} \leq C |v|_{\mathcal{W}_x^{s,p}} |e^{aY}|_{1+\mathcal{W}_x^{s,\infty}}.$$

3.2 Renormalization

Let $N \in \mathbb{N}^*$, define $|\nabla Y_N|^2 := |\nabla Y_N|^2 - C_N^2$ where $C_N(x)^2 = \mathbb{E} [|\nabla Y_N(x)|^2] = \sum_{k \in \mathbb{N}^2} \chi_{k,N}^2 \frac{|\nabla h_k(x)|^2}{\lambda_k^4}$.

We now prove the following result on the convergence of $|\nabla Y_N|^2$.

Proposition 3.12. *Let $q > 2$ and $s > \frac{2}{q}$. Then $(:|\nabla Y_N|^2:)_N \in \mathbb{N}$ is a Cauchy sequence in the Banach space $\mathbb{L}^q(\Omega, \mathcal{W}^{-s,q})$, and thus converges to an element $:|\nabla Y|^2: \in \mathbb{L}^q(\Omega, \mathcal{W}^{-s,q})$. Moreover for all $0 < \delta < \frac{2s}{3} - \frac{4}{3q}$ there exists a constant $C > 0$ such that for all $M \in \mathbb{N}^*$*

$$|:|\nabla Y_M|^2: - :|\nabla Y|^2:|_{\mathbb{L}^q \mathcal{W}_x^{-s,q}} \leq C \lambda_M^{-\delta}.$$

Let us recall some results about Wiener chaos and renormalization. For an introduction to Wick products see [10] and for more details about Wick products adapted to $-H$, see [5]. First recall the probabilist's Hermite polynomials:

$$\forall x \in \mathbb{R}, \forall n \in \mathbb{N}, H_n(x) = \frac{(-1)^n e^{\frac{x^2}{2}}}{\sqrt{n!}} \frac{d^n}{dx^n} \left[e^{-\frac{x^2}{2}} \right].$$

They form a complete orthogonal system of $\mathbb{L}^2(\mathbb{R}, \mu)$ (with $d\mu(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}} dx$).

Let $W : \mathbb{L}^2(\mathbb{R}^2) \rightarrow \mathbb{L}^2(\Omega)$ be the unique isometry such that $W_{h_k} = \xi_k$ for all $k \in \mathbb{N}^2$. We define the n -th chaos as

$$\mathcal{H}_n = \overline{\{H_n(W_f), |f|_{\mathbb{L}_x^2} = 1\}}^{\mathbb{L}^2(\Omega)}.$$

It is well known that $\mathbb{L}^2(\Omega, \mathcal{G}) = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$, where $\mathcal{G} = \sigma(\{W_f, f \in \mathbb{L}^2(\mathbb{R}^2)\}) = \sigma(\{\xi_k, k \in \mathbb{N}^2\})$ is the σ -field generated by the white noise W . Thanks to the orthonormality of $(H_n)_{n \in \mathbb{N}}$, we have the following relation.

Lemma 3.13. *Let $k, n \in \mathbb{N}$ and $f, g \in \mathbb{L}^2(\mathbb{R}^2)$ such that $|f|_{\mathbb{L}_x^2} = 1 = |g|_{\mathbb{L}_x^2}$. Then, we have $\mathbb{E}[H_k(W_f)H_n(W_g)] = k! \langle f, g \rangle_{\mathbb{L}_x^2}^k \delta_{n,k}$, where δ denote the Kronecker symbol.*

We can now define the renormalization of the n -th power of white noise images as a projection on the n -th chaos. Denote by Π_n the orthogonal projection from $\mathbb{L}^2(\Omega, \mathcal{G})$ to \mathcal{H}_n .

Lemma 3.14. *Let $\rho \geq 0$ and $\eta \in \mathbb{L}^2(\mathbb{R}^2)$ such that $|\eta|_{\mathbb{L}_x^2} = 1$ and define $Z = \rho W_\eta$. For all $n \in \mathbb{N}$, we have $\Pi_n(Z^n) = \rho^n \sqrt{n!} H_n(W_\eta)$.*

We define $:Z^n:$ as $\Pi_n(Z^n)$. For example, in the case $n = 2$ and Z as in the previous lemma, it gives $:Z^2: = Z^2 - \rho^2 = Z^2 - \mathbb{E}[Z^2]$.

To prove the convergence of Wick product, first recall Nelson's estimate.

Proposition 3.15. *(Nelson's estimate) Let $n \in \mathbb{N}^*$ and $F \in \mathcal{H}_n$. Then, for all $p \geq 2$, we have $\mathbb{E}[|F|^p] \leq (p-1)^{\frac{np}{2}} \mathbb{E}[|F|^2]^{\frac{p}{2}}$.*

The proof of Proposition 3.12 is very similar to the proof of Proposition 4 in [5]. Thus, we only give here the technical argument specific to our case.

Proof. of Proposition 3.12.

Let $i \in \{1, 2\}$ and $N \in \mathbb{N}$, define $Z_{i,N} = \partial_i Y_N$ where Y_N was defined in Section 3.1,

$$\forall x \in \mathbb{R}^2, \rho_N^i(x)^2 = \sum_{k \in \mathbb{N}^2} \chi_{k,N}^2 \frac{(\partial_i h_k(x))^2}{\lambda_k^4} = \mathbb{E}[Z_{i,N}^2] \quad \text{and} \quad \eta_N^i(x) = \frac{1}{\rho_N^i(x)} \sum_{k \in \mathbb{N}^2} \chi_{k,N} \frac{\partial_i h_k(x)}{\lambda_k^2} h_k$$

so that $|\eta_N^i(x)|_{\mathbb{L}_{\mathbb{R}^2}^2} = 1$ and $Z_{i,N}(x) = \rho_N^i(x)W_{\eta_N^i(x)}$. Then, we can define $:Z_{i,N}^2: = Z_{i,N}^2 - \mathbb{E}[Z_{i,N}^2]$ so that $:\nabla Y_N|^2: = :Z_{1,N}^2: + :Z_{2,N}^2:$.

We will prove the convergence of each $:Z_{i,N}^2:$ separately. So fix $i \in \{1, 2\}$ and let $q > 2$ and $s > 0$ which will be fixed later. As in the proof of proposition 4 in [5], we can show

$$\mathbb{E} \left[| :Z_{i,N}^2: - :Z_{i,M}^2: |_{\mathcal{W}_x^{-s,q}}^q \right] \lesssim |K_{i,N,N}^2 - 2K_{i,N,M}^2 + K_{i,M,M}^2|_{\mathbb{L}_{x_1}^{\frac{q}{2}} \mathcal{W}_{x_2}^{1-s+\delta,2}},$$

with $0 < \delta < s$, $\frac{2}{q} - \frac{s}{2} < \frac{1}{r} < \frac{2}{q}$ and for all $x, y \in \mathbb{R}^2$:

$$K_{i,N,M}(x, y) = \sum_{k \in \mathbb{N}^2} \chi_{k,N} \chi_{k,M} \frac{\partial_i h_k(x) \partial_i h_k(y)}{\lambda_k^4}.$$

Let $s' = s - \delta$. Using that for any $N, M \in \mathbb{N}^*$ we have

$$K_{i,N,N}^2 - K_{i,N,M}^2 = (K_{i,N,N} - K_{i,N,M})(K_{i,N,N} + K_{i,N,M})$$

and applying Lemma 2.9 with $r_1 = r_2 = 2r$, $\alpha = 1 - s'$, $\sigma = 1 - \frac{s'}{2}$ and $p = \frac{4}{s'}$ and the Sobolev embedding of $\mathcal{W}^{1-\frac{s'}{2},2}$ in \mathbb{L}^p we obtain

$$\begin{aligned} E \left[| :Z_{i,N}^2: - :Z_{i,M}^2: |_{\mathcal{W}_x^{-s,q}}^q \right]^{\frac{2}{q}} &\lesssim |K_{i,N,N} - K_{i,N,M}|_{\mathbb{L}_x^{2r} \mathcal{W}_y^{1-\frac{s'}{2},2}} |K_{i,N,N} + K_{i,N,M}|_{\mathbb{L}_x^{2r} \mathcal{W}_y^{1-\frac{s'}{2},2}} \\ &\quad + |K_{i,M,M} - K_{i,N,M}|_{\mathbb{L}_x^{2r} \mathcal{W}_y^{1-\frac{s'}{2},2}} |K_{i,M,M} + K_{i,N,M}|_{\mathbb{L}_x^{2r} \mathcal{W}_y^{1-\frac{s'}{2},2}}. \end{aligned} \quad (9)$$

Recall that Hermite functions verify the formula

$$\partial_i h_k = \sqrt{\frac{k_i}{2}} h_{k-e_i} - \sqrt{\frac{k_i+1}{2}} h_{k+e_i}$$

where $e_i = (\delta_{1,i}, \delta_{2,i})$. This gives the following result:

$$\begin{aligned} K_{i,N,N}(x, y) - K_{i,N,M}(x, y) &= \sum_{k \in \mathbb{N}^2} \left[\sqrt{\frac{k_i+1}{2}} \frac{\chi_{k+e_i,N}(\chi_{k+e_i,N} - \chi_{k+e_i,M}) \partial_i h_{k+e_i}(x)}{\lambda_{k+e_i}^4} \right. \\ &\quad \left. - \sqrt{\frac{k_i}{2}} \frac{\chi_{k-e_i,N}(\chi_{k-e_i,N} - \chi_{k-e_i,M}) \partial_i h_{k-e_i}(x)}{\lambda_{k-e_i}^4} \right] h_k(y). \end{aligned}$$

Let $\alpha \in \mathbb{R}$, we deduce

$$|K_{i,N,N}(x, \cdot) - K_{i,N,M}(x, \cdot)|_{\mathcal{W}_y^{\alpha,2}}^2 \lesssim \sum_{k \in \mathbb{N}^2} \frac{\chi_{k,N}^2 (\chi_{k,N} - \chi_{k,M})^2}{\lambda_k^8} (\partial_i h_k(x))^2 (k_i \lambda_{k-e_i}^{2\alpha} + (k_i+1) \lambda_{k+e_i}^{2\alpha}).$$

As $\lambda_k^2 = 2|k| + 2$, we obtain

$$|K_{i,N,N}(x, \cdot) - K_{i,N,M}(x, \cdot)|_{\mathcal{W}_y^{\alpha,2}}^2 \lesssim \sum_{\lfloor \frac{M}{2} \rfloor \leq |k| \leq N} \lambda_k^{2\alpha-6} (\partial_i h_k(x))^2 \lesssim \sum_{\lfloor \frac{M}{2} \rfloor - 1 \leq |k| < +\infty} \lambda_k^{2\alpha-4} h_k(x)^2.$$

Hence, for $M \geq 2$, we have

$$|K_{i,N,N}(x, \cdot) - K_{i,N,M}(x, \cdot)|_{\mathcal{W}_y^{\alpha,2}}^2 \lesssim \lambda_M^{-2\delta} \sum_{k \in \mathbb{N}^2} \lambda_k^{2\delta - 2\alpha - 4} h_k(x)^2 \lesssim \lambda_M^{-2\delta} |K(x, \cdot)|_{\mathcal{W}_y^{\alpha+\delta,2}}^2.$$

The same holds for $|K_{i,N,M}(x, \cdot) - K_{i,M,M}(x, \cdot)|_{\mathcal{W}_y^{\alpha,2}}^2$ and an analogous proof shows

$$\max \left(|K_{i,N,N} + K_{i,N,M}|_{\mathbb{L}_x^{2r} \mathcal{W}_y^{1-\frac{s'}{2},2}}^2, |K_{i,N,M} + K_{i,M,M}|_{\mathbb{L}_x^{2r} \mathcal{W}_y^{1-\frac{s'}{2},2}}^2 \right) \lesssim |K|_{\mathbb{L}_x^{2r} \mathcal{W}_y^{1-\frac{s'}{2}+\delta,2}}^2.$$

Going back to equation (9), we obtain

$$|:Z_{i,N}^2: - :Z_{i,M}^2:|_{\mathbb{L}_\omega^q \mathcal{W}_x^{-s,q}} \lesssim \lambda_M^{-\delta} |K|_{\mathbb{L}_x^{2r} \mathcal{W}_y^{1-\frac{s'}{2}+\delta,2}}^2.$$

Moreover, $|K|_{\mathbb{L}_x^{2r} \mathcal{W}_y^{1-\frac{s'}{2}+\delta,2}}$ is finite as soon as $1 - \frac{s'}{2} + \delta < 1 - \frac{1}{r}$ according to Proposition 2.2. There exists such a δ if $sq > 2$. In order to have $0 < s < 1$, we need $q > 2$. This proves the sequence is Cauchy. Then let N go to infinity and the estimate follows. We can choose $\delta < \frac{s}{3} - \frac{2}{3r}$. Taking $\frac{1}{r}$ arbitrarily close to $\frac{2}{q} - \frac{s}{2}$, we obtain the announced range for δ . \square

Remark 3.16. An analogous proof shows $|\nabla Y|^2$ does not depend on the choice of $\chi \in \mathcal{S}(\mathbb{R})$ (as long as $\chi(0) = 1$), but choosing a χ which is compactly supported and equal to one on a neighborhood of the origin makes the proof of Proposition 3.12 easier and gives an explicit convergence speed. This independence, together with Proposition 5.1 below, ensures that the renormalized solution we construct does not depend on the choice of χ .

To estimate \mathbb{L}_x^q norms of Wick products, we use the following lemmas. First see that Nelson's estimate allows us to estimate higher order moments of $\mathcal{W}_x^{\alpha,q}$ norms of the Wick product.

Lemma 3.17. *Let $2 \leq q < q' < +\infty$ and $\alpha \in \mathbb{R}$. It holds,*

$$\forall N \in \mathbb{N}, \quad |:\nabla Y_N|^2:|_{\mathbb{L}_\omega^{q'} \mathcal{W}_x^{\alpha,q}} \leq (q' - 1) |:\nabla Y_N|^2:|_{\mathbb{L}_\omega^q \mathcal{W}_x^{\alpha,q}}.$$

Using Borel-Cantelli lemma, we obtain the following corollary.

Corollary 3.18. *Let $q \in (2, +\infty]$, $\frac{2}{q} < s < 1$ and $0 < \delta < \frac{2s}{3} - \frac{4}{3q}$. Almost surely*

$$\sup_{N \in \mathbb{N}} \left[\lambda_N^\delta |:\nabla Y_N|^2: - |:\nabla Y|^2:|_{\mathcal{W}_x^{-s,q}} \right] < +\infty.$$

Moreover, for any $p \in [1, +\infty)$ there exists a constant $C > 0$ such that

$$\forall N \geq M, \quad |:\nabla Y_N|^2: - |:\nabla Y_M|^2:|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}} \leq C \lambda_M^{-\delta}$$

and

$$\max \left(|:\nabla Y|^2:|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}}, \sup_{N \in \mathbb{N}} |:\nabla Y_N|^2:|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}} \right) \leq C.$$

Remark 3.19. This shows that, up to restricting ourselves to an event of probability 1, we can assume in what follows that for all $\omega \in \Omega$, $:\nabla Y_N|^2:(\omega)$ and $:\nabla Y|^2:(\omega)$ are well-defined and for every $q \in (2, +\infty]$ and $s > \frac{2}{q}$, $:\nabla Y_N|^2:(\omega)$ converges to $:\nabla Y|^2:(\omega)$ in $\mathcal{W}^{-s,q}$.

Finally, in order to control the divergence of spatial \mathbb{L}^q -norms of Wick products, we will need the following lemma.

Lemma 3.20. *Let $2 \leq q < +\infty$ and $2 \leq p_1, q_1, p_2, q_2 < +\infty$ such that $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{q}$. There exists $C > 0$ such that*

$$\forall N \in \mathbb{N}, \quad |:\nabla Y_N|^2:|_{\mathbb{L}_\omega^q \mathbb{L}_x^q} \leq C |\nabla Y_N|_{\mathbb{L}_\omega^{p_1} \mathbb{L}_x^{p_2}} |\nabla Y_N|_{\mathbb{L}_\omega^{q_1} \mathbb{L}_x^{q_2}}.$$

We deduce the following corollary.

Corollary 3.21. *Let $1 \leq p < +\infty$, $2 < q < +\infty$ and $1 > s > \frac{2}{q}$. There exists $C > 0$ such that for all $N \in \mathbb{N}$, we have*

$$|:\nabla Y_N|^2:|_{\mathbb{L}_\omega^p \mathbb{L}_x^q} \leq C \lambda_N^s.$$

Hölder's inequality, Sobolev embeddings and Lemma 3.3 then give an immediate corollary.

Corollary 3.22. *Let $\kappa \in (0, 1)$, $q \in (1, +\infty)$ and a random variable $X \in \mathcal{W}^{1,q}$ a.s. There exists a sequence of random variables $(C_N)_{N \in \mathbb{N}}$ verifying Property (*) such that almost surely*

$$\forall N \in \mathbb{N}, \quad |X:|\nabla Y_N|^2:|_{\mathbb{L}_x^q} \leq C_N |X|_{\mathcal{W}_x^{1,q}} \lambda_N^\kappa.$$

Moreover, let $p_0 > p \geq 1$ and assume $X \in \mathbb{L}^{p_0}(\Omega, \mathcal{W}^{1,q})$, then there exists $C > 0$ such that

$$\sup_{N \in \mathbb{N}} [\lambda_N^{-\kappa} |X:|\nabla Y_N|^2:|_{\mathbb{L}_\omega^p \mathbb{L}_x^q}] \leq C |X|_{\mathbb{L}_\omega^{p_0} \mathcal{W}_x^{1,q}}.$$

3.3 The regularized equation

In order to solve equation (4) we are going to study a regularized transformed equation

$$\begin{cases} i\partial_t v^N + H v^N + 2\nabla Y_N \cdot \nabla v^N + x Y_N \cdot x v^N + :|\nabla Y_N|^2: v^N + \lambda |v^N|^2 v^N e^{2Y_N} = 0 \\ v(0) = v_0 \end{cases} \quad (10)$$

where $N \in \mathbb{N}$ and $Y_N = S_N Y$ as previously. Formally, this renormalized transformed equation has two invariant quantities, first a transformed mass

$$\tilde{M}_N(v) = \int_{\mathbb{R}^2} |v(x)|^2 e^{2Y_N(x)} dx \quad (11)$$

and a transformed energy

$$\tilde{\mathcal{E}}_N(v) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla v|^2 + |xv|^2 - |xv|^2 Y_N - :|\nabla Y_N|^2: |v|^2) e^{2Y_N} dx - \frac{\lambda}{4} \int_{\mathbb{R}^2} |v|^4 e^{4Y_N} dx. \quad (12)$$

It is well-known that equation (10) has a unique maximal solution for $v_0 \in \Sigma$ (see for example [8]). Here we give a complete statement.

Theorem 3.23. *(Local wellposedness) Let $\lambda \in \mathbb{R}$ and $N \in \mathbb{N}$. Almost surely, for every $v_0 \in \Sigma$, there exists a unique maximal solution v^N to equation (10) in $\mathcal{C}((-T_N^-, T_N^+), \Sigma)$. Moreover we have almost surely, the blow-up alternative*

$$\lim_{t \rightarrow T_N^\pm} |v^N(t)|_\Sigma = +\infty \text{ or } T_N^\pm = +\infty,$$

transformed mass and energy, defined by equations (11) and (12), are conserved and if $v_0 \in \mathcal{W}^{2,2}$, then $v^N \in \mathcal{C}((-T_+^N, T_+^N), \mathcal{W}^{2,2}) \cap \mathcal{C}^1((-T_-^N, T_+^N), \mathbb{L}^2(\mathbb{R}^2))$.

The mass conservation allows us to control the \mathbb{L}^2 norm of v^N :

$$|v^N(t)|_{\mathbb{L}_x^2}^2 \leq |e^{-2Y_N}|_{\mathbb{L}_x^\infty} \tilde{M}_N(v^N(t)) \leq |e^{-2Y_N}|_{\mathbb{L}_x^\infty} \tilde{M}_N(v_0) \leq |e^{-2Y_N}|_{\mathbb{L}_x^\infty} |e^{2Y_N}|_{\mathbb{L}_x^\infty} |v_0|_{\mathbb{L}_x^2}.$$

Moreover the energy conservation allows us to control the Σ norm of v^N in $\mathbb{L}^p(\Omega)$ in the linear case (for some $p \geq 2$ depending on the integrability of v_0), almost surely in the defocusing case and on a non-zero probability event in the focusing case.

Proposition 3.24. *Assume that $v_0 \in \Sigma$ a.s. and $\lambda = 0$. Then, for any $N \in \mathbb{N}$, v^N is global almost surely. Moreover, let $p_0 \in (2, +\infty)$ and assume $v_0 \in \mathbb{L}^{p_0}(\Omega, \Sigma)$, then for any $p \in [2, p_0)$, there exists $C > 0$ such that*

$$\forall N \in \mathbb{N}, |v^N|_{\mathbb{L}_t^p \mathbb{L}_x^\infty \Sigma} \leq C |v_0|_{\mathbb{L}^{p_0} \Sigma}.$$

In the nonlinear case, we will need the following Gagliardo-Nirenberg inequality.

Lemma 3.25. *(See [6]) For all $u \in W^{1,2}$, $|u|_{\mathbb{L}_x^4}^4 \leq \frac{1}{2} |u|_{\mathbb{L}_x^2}^2 |\nabla u|_{\mathbb{L}_x^2}^2$.*

Proposition 3.26. *Assume $\lambda \leq 0$. Then, there exists a positive random variable C almost surely finite such that almost surely, for any $v_0 \in \Sigma$ and any $N \in \mathbb{N}$, v^N is global and*

$$\forall N \in \mathbb{N}, |v^N|_{\mathbb{L}_t^\infty \Sigma} \leq C(1 + |v_0|_\Sigma)^2.$$

Proposition 3.27. *Assume $\lambda > 0$ and let $L > 0$. Let $\bar{\Omega}_L = \left\{ \lambda L^2 |e^{-2Y}|_{\mathbb{L}_x^\infty}^2 |e^{2Y}|_{\mathbb{L}_x^\infty} |e^{4Y}|_{\mathbb{L}_x^\infty} < 4 \right\}$. Then on $\bar{\Omega}_L$ (endowed with the conditional probability and σ -algebra), there exists a random variable $N_0 \in \mathbb{N}$ a.s. and a positive random variable C almost surely finite such that almost surely, for any $v_0 \in \Sigma$ with $|v_0|_{\mathbb{L}_x^2} \leq L$ and any $N \geq N_0$, v^N is global and*

$$\forall N \geq N_0, |v^N|_{\mathbb{L}_t^\infty \Sigma} \leq C |v_0|_\Sigma.$$

Unfortunately, this bound is not sufficient to obtain a uniform bound on $(\partial_t v^N)$ because of the negative regularity of the noisy terms.

4 The linear case

In what follows, we denote by v^N the unique global solution of the linear equation

$$\begin{cases} i\partial_t v^N + H v^N + 2\nabla Y_N \cdot \nabla v^N + x Y_N \cdot x v^N + :|\nabla Y_N|^2 : v^N = 0 \\ v(0) = v_0 \in \mathbb{L}^{p_0}(\Omega, \mathcal{W}^{2,2}) \end{cases} \quad (13)$$

with $p_0 \in (2, +\infty)$. First, as in [13], we prove a diverging bound on v^N in $\mathcal{W}^{2,2}$.

Proposition 4.1. *Let $\kappa \in (0, 1)$ and $2 \leq p < p_0$. There exists a constant $C > 0$ such that for any $N \in \mathbb{N}$,*

$$|v^N|_{\mathbb{L}_t^p \mathbb{L}_x^\infty \mathcal{W}_x^{2,2}} \leq C \lambda_N^\kappa |v_0|_{\mathbb{L}^{p_0} \mathcal{W}_x^{2,2}}$$

Proof. Define $w^N = \partial_t v^N$. Using equation (13), we have

$$Hv^N = - (iw^N + 2\nabla Y_N \cdot \nabla v^N + xY_N \cdot xv^N + :|\nabla Y_N|^2:v^N).$$

Taking the \mathbb{L}_x^2 norm, we obtain

$$|v^N|_{\mathcal{W}_x^{2,2}} \leq |w^N|_{\mathbb{L}_x^2} + 2|\nabla Y_N \cdot \nabla v^N|_{\mathbb{L}_x^2} + |xY_N \cdot xv^N|_{\mathbb{L}_x^2} + :|\nabla Y_N|^2:v^N|_{\mathbb{L}_x^2}.$$

Applying Corollaries 3.7 and 3.22 and Proposition 3.24, we have for any $p < p' < p_0$

$$|v^N|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathcal{W}_x^{2,2}} \lesssim |w^N|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathbb{L}_x^2} + \lambda_N^\kappa |v^N|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \Sigma} \lesssim |w^N|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathbb{L}_x^2} + \lambda_N^\kappa |v_0|_{\mathbb{L}_\omega^{p_0} \Sigma}. \quad (14)$$

To bound the \mathbb{L}_x^2 norm of w^N , we use the fact that w^N verifies at least formally equation (13) with initial data $i(Hv_0 + 2\nabla v_0 \cdot \nabla Y_N + xv_0 \cdot xY_N + :|\nabla Y_N|^2:v_0) \in \mathbb{L}^2(\mathbb{R}^2)$ a.s. Approximating v_0 by a sequence of Schwartz functions, one can show that w^N has its transformed mass conserved

$$\forall t \in \mathbb{R}, \int_{\mathbb{R}^2} |w^N(t)| e^{2Y_N} dx = \int_{\mathbb{R}^2} |w^N(0)| e^{2Y_N} dx.$$

Thus, by Hölder's inequality and Corollary 3.9, we have for any $p < p' < p_0$:

$$\begin{aligned} |w^N|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathbb{L}_x^2} &\lesssim |w^N(0)|_{\mathbb{L}_\omega^{p'} \mathbb{L}_x^2} \\ &\lesssim |Hv_0|_{\mathbb{L}_\omega^{p'} \mathbb{L}_x^2} + 2|\nabla Y_N \cdot \nabla v_0|_{\mathbb{L}_\omega^{p'} \mathbb{L}_x^2} + |xY_N \cdot xv_0|_{\mathbb{L}_\omega^{p'} \mathbb{L}_x^2} + :|\nabla Y_N|^2:v_0|_{\mathbb{L}_\omega^{p'} \mathbb{L}_x^2} \\ &\lesssim |v_0|_{\mathbb{L}_\omega^{p_0} \mathcal{W}_x^{2,2}} + \lambda_N^\kappa |v_0|_{\mathbb{L}_\omega^{p_0} \Sigma} \end{aligned}$$

using Corollaries 3.7 and 3.22. Going back to equation (14), we finally obtain

$$|v^N|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathcal{W}_x^{2,2}} \lesssim |v_0|_{\mathbb{L}_\omega^{p_0} \mathcal{W}_x^{2,2}} + \lambda_N^\kappa |v_0|_{\mathbb{L}_\omega^{p_0} \Sigma} \lesssim \lambda_N^\kappa |v_0|_{\mathbb{L}_\omega^{p_0} \mathcal{W}_x^{2,2}}.$$

□

In order to manage this divergence, we prove a Cauchy estimate in $\mathbb{L}^p(\Omega, \mathcal{C}([a, b], \mathbb{L}^2(\mathbb{R}^2)))$.

Proposition 4.2. *Let $N \geq M > 0$ be two integers, $p \in [2, p_0)$, $\delta \in (0, \frac{2}{3})$ and $a < 0 < b$. Then there exists a constant $C > 0$ such that we have*

$$|v^N - v^M|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathbb{L}_x^2}^2 \leq C \lambda_M^{-\delta} |v_0|_{\mathbb{L}_\omega^{p_0} \mathcal{W}_x^{2,2}}^2.$$

Proof. Let $N \geq M > 0$ and set $R = v^N - v^M$. It verifies

$$\begin{cases} -i\partial_t R = & HR + 2\nabla Y_N \cdot \nabla R + 2\nabla(Y_N - Y_M) \cdot \nabla v^M + xY_N \cdot xR \\ & + x(Y_N - Y_M) \cdot xv^M + :|\nabla Y_N|^2:R + (:|\nabla Y_N|^2: - :|\nabla Y_M|^2:) v^M \\ R(0) = & 0 \end{cases} \quad (15)$$

and it belongs to $\mathcal{C}^1([a, b], \mathbb{L}^2(\mathbb{R}^2))$ almost surely. Taking the time derivative of $\tilde{M}_N(R)$, we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |R|^2 e^{2Y_N} dx &= \langle \partial_t R, R e^{2Y_N} \rangle_{\mathbb{L}^2} \\
&= -\operatorname{Im} \int_{\mathbb{R}^2} (HR + 2\nabla Y_N \cdot \nabla R + 2\nabla(Y_N - Y_M) \cdot \nabla v^M) \bar{R} e^{2Y_N} dx \\
&\quad - \operatorname{Im} \int_{\mathbb{R}^2} (xY_N \cdot xR + x(Y_N - Y_M) \cdot xv^M) \bar{R} e^{2Y_N} dx \\
&\quad - \operatorname{Im} \int_{\mathbb{R}^2} (|\nabla Y_N|^2 : R + (|\nabla Y_N|^2 - |\nabla Y_M|^2) v^M) \bar{R} e^{2Y_N} dx \\
&= -\operatorname{Im} \int_{\mathbb{R}^2} (2\nabla(Y_N - Y_M) \cdot \nabla v^M + x(Y_N - Y_M) \cdot xv^M) \bar{R} e^{2Y_N} dx \\
&\quad - \operatorname{Im} \int_{\mathbb{R}^2} (|\nabla Y_N|^2 - |\nabla Y_M|^2) v^M \bar{R} e^{2Y_N} dx,
\end{aligned}$$

where we used

$$\begin{aligned}
\operatorname{Im} \int_{\mathbb{R}^2} (HR + 2\nabla Y_N \cdot \nabla R) \bar{R} e^{2Y_N} dx &= \operatorname{Im} \int_{\mathbb{R}^2} (\bar{R} e^{2Y_N} \Delta R + \bar{R} \nabla R \cdot \nabla [e^{2Y_N}]) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^2} (-\nabla R \cdot \nabla [\bar{R} e^{2Y_N}] + \bar{R} \nabla R \cdot \nabla [e^{2Y_N}]) dx \\
&= 0.
\end{aligned}$$

Let $\kappa \in (0, 1)$ such that $\delta \in (0, \frac{2\kappa}{3})$. We have by duality:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |R|^2 e^{2Y_N} &\leq 2|\nabla(Y_N - Y_M)|_{\mathcal{W}_x^{-\kappa, \infty}} |\bar{R} \nabla v^M e^{2Y_N}|_{\mathcal{W}_x^{\kappa, 1}} \\
&\quad + |x(Y_N - Y_M)|_{\mathcal{W}_x^{-\kappa, \infty}} |x \bar{R} v^M e^{2Y_N}|_{\mathcal{W}_x^{\kappa, 1}} \\
&\quad + |:\nabla Y_N|^2 - |\nabla Y_M|^2|_{\mathcal{W}_x^{-\kappa, \infty}} |\bar{R} v^M e^{2Y_N}|_{\mathcal{W}_x^{\kappa, 1}}.
\end{aligned}$$

By Lemma 2.7 and interpolation, we have

$$\begin{aligned}
|\bar{R} \nabla v^M e^{2Y_N}|_{\mathcal{W}_x^{\kappa, 1}} &\lesssim |R|_{\mathcal{W}_x^{\kappa, 2}} |v^M|_{\mathcal{W}_x^{1+\kappa, 2}} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} \\
&\lesssim (|v^N|_{\mathbb{L}_t^\infty \Sigma} + |v^M|_{\mathbb{L}_t^\infty \Sigma}) |v^M|_{\mathbb{L}_t^\infty \Sigma}^{1-\kappa} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} |v^M|_{\mathbb{L}_t^\infty \Sigma}^\kappa \mathcal{W}_x^{2, 2}
\end{aligned}$$

and

$$|x \bar{R} v^M e^{2Y_N}|_{\mathcal{W}_x^{\kappa, 1}} \lesssim (|v^N|_{\mathbb{L}_t^\infty \Sigma} + |v^M|_{\mathbb{L}_t^\infty \Sigma}) |v^M|_{\mathbb{L}_t^\infty \Sigma}^{1-\kappa} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} |v^M|_{\mathbb{L}_t^\infty \Sigma}^\kappa \mathcal{W}_x^{2, 2}$$

and

$$|\bar{R} v^M e^{2Y_N}|_{\mathcal{W}_x^{\kappa, 1}} \lesssim (|v^N|_{\mathbb{L}_t^\infty \Sigma} + |v^M|_{\mathbb{L}_t^\infty \Sigma}) |v^M|_{\mathbb{L}_t^\infty \Sigma} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}}.$$

Integrating in time, we obtain a bound uniform in $t \in [a, b]$ (with a constant depending on $b - a$):

$$\begin{aligned}
|R|_{\mathbb{L}_t^\infty \mathbb{L}_x^2}^2 &\lesssim |e^{-2Y_N}|_{\mathbb{L}_x^\infty} \left[|\nabla(Y_N - Y_M)|_{\mathcal{W}_x^{-\kappa, \infty}} |v^M|_{\mathbb{L}_t^\infty \Sigma}^{1-\kappa} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} |v^M|_{\mathbb{L}_t^\infty \Sigma}^\kappa \mathcal{W}_x^{2, 2} \right. \\
&\quad + |x(Y_N - Y_M)|_{\mathcal{W}_x^{-\kappa, \infty}} |v^M|_{\mathbb{L}_t^\infty \Sigma}^{1-\kappa} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} |v^M|_{\mathbb{L}_t^\infty \Sigma}^\kappa \mathcal{W}_x^{2, 2} \\
&\quad \left. + |:\nabla Y_N|^2 - |\nabla Y_M|^2|_{\mathcal{W}_x^{-\kappa, \infty}} |v^M|_{\mathbb{L}_t^\infty \Sigma} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} \right] (|v^N|_{\mathbb{L}_t^\infty \Sigma} + |v^M|_{\mathbb{L}_t^\infty \Sigma}).
\end{aligned}$$

Let $p < p' < p_0$ and $r > 1$ such that $\frac{2}{p} > \frac{2}{p'} + \frac{1}{r}$, by Hölder's inequality and Corollary 3.9 we have

$$\begin{aligned} |R|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathbb{L}_x^2}^2 &\lesssim |\nabla(Y_N - Y_M)|_{\mathbb{L}_\omega \mathcal{W}_x^{-\kappa, \infty}} \left(|v^N|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \Sigma} + |v^M|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \Sigma} \right) |v^M|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \Sigma}^{1-\kappa} |v^M|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \mathcal{W}_x^{2,2}}^\kappa \\ &\quad + |x(Y_N - Y_M)|_{\mathbb{L}_\omega \mathcal{W}_x^{-\kappa, \infty}} \left(|v^N|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \Sigma} + |v^M|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \Sigma} \right) |v^M|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \Sigma}^{1-\kappa} |v^M|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \mathcal{W}_x^{2,2}}^\kappa \\ &\quad + |:\nabla Y_N|^2: - :|\nabla Y_M|^2:|_{\mathbb{L}_\omega \mathcal{W}_x^{-\kappa, \infty}} \left(|v^N|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \Sigma} + |v^M|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \Sigma} \right) |v^M|_{\mathbb{L}_\omega^{p'} \mathbb{L}_t^\infty \Sigma}. \end{aligned}$$

Let $\kappa' \in (0, 1)$, using Propositions 3.24 and 4.1, we obtain

$$\begin{aligned} |R|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathbb{L}_x^2}^2 &\lesssim \left(|\nabla(Y_N - Y_M)|_{\mathbb{L}_\omega \mathcal{W}_x^{-\kappa, \infty}} + |x(Y_N - Y_M)|_{\mathbb{L}_\omega \mathcal{W}_x^{-\kappa, \infty}} \right) |v_0|_{\mathbb{L}_\omega^{p_0} \Sigma}^{2-\kappa} |v_0|_{\mathbb{L}_\omega^{p_0} \mathcal{W}_x^{2,2}}^\kappa \lambda_M^{\kappa \kappa'} \\ &\quad + |:\nabla Y_N|^2: - :|\nabla Y_M|^2:|_{\mathbb{L}_\omega \mathcal{W}_x^{-\kappa, \infty}} |v_0|_{\mathbb{L}_\omega^{p_0} \Sigma}^2. \end{aligned}$$

Let $\kappa'' \in (\delta, \kappa)$, using Corollary 3.5, we obtain

$$|R|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathbb{L}_x^2}^2 \lesssim |v_0|_{\mathbb{L}_\omega^{p_0} \Sigma}^{2-\kappa} |v_0|_{\mathbb{L}_\omega^{p_0} \mathcal{W}_x^{2,2}}^\kappa \lambda_M^{\kappa \kappa' - \kappa''} + |:\nabla Y_N|^2: - :|\nabla Y_M|^2:|_{\mathbb{L}_\omega \mathcal{W}_x^{-\kappa, \infty}} |v_0|_{\mathbb{L}_\omega^{p_0} \Sigma}^2.$$

Using Corollary 3.18 (recall that $\delta \in (0, \frac{2\kappa}{3})$), we have

$$|R|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathbb{L}_x^2}^2 \lesssim |v_0|_{\mathbb{L}_\omega^{p_0} \mathcal{W}_x^{2,2}}^2 \left(\lambda_M^{\kappa \kappa' - \kappa''} + \lambda_M^{-\delta} \right).$$

We conclude by choosing $\kappa' = \frac{\kappa'' - \delta}{\kappa} \in (0, 1)$. □

We can now prove our main result in the linear case.

Proof. of Theorem 1.1.

Let $\kappa' \in (0, 1)$ and $\delta \in (0, \frac{2}{3})$. By interpolation between Propositions 4.1 and 4.2 we get

$$|v^N - v^M|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}} \lesssim |v^N - v^M|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathcal{W}_x^{2, 2}}^{\frac{\sigma}{2}} |v^N - v^M|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathbb{L}_x^2}^{1-\frac{\sigma}{2}} \lesssim \lambda_N^{\frac{\kappa' \sigma}{2}} \lambda_M^{-(1-\frac{\sigma}{2})\frac{\delta}{2}} |v_0|_{\mathbb{L}_\omega^{p_0} \mathcal{W}_x^{2, 2}}.$$

Let $n \in \mathbb{N}$, $N = 2^{n+1}$ and $M = 2^n$. We have

$$\left| v^{2^{n+1}} - v^{2^n} \right|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}} \lesssim 2^{n(\kappa' \sigma - (1-\frac{\sigma}{2})\delta)/4} |v_0|_{\mathbb{L}_\omega^{p_0} \mathcal{W}_x^{2, 2}}.$$

Remark that:

$$\frac{\kappa' \sigma}{4} - \left(1 - \frac{\sigma}{2}\right) \frac{\delta}{4} < 0 \iff \kappa' < \frac{2 - \sigma}{2\sigma} \delta.$$

Thus, for fixed σ , it is possible to choose $\delta \in (0, \frac{1}{4})$ and $\kappa' \in (0, \frac{2-\sigma}{2\sigma} \delta)$ so that the sequence $\left(\left| v^{2^{n+1}} - v^{2^n} \right|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}} \right)_{n \in \mathbb{N}}$ is summable. Hence, $(v^{2^n})_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathbb{L}^p(\Omega, \mathcal{C}([a, b], \mathcal{W}^{\sigma, 2}))$. Denote by v its limit in this space. Using Lemmas 2.8 and 3.10 and Corollaries 3.5 and 3.18, it is not difficult to show that $-i\partial_t v^{2^n}$ converge in $\mathcal{C}([a, b], \mathcal{W}^{\sigma-2, 2})$ to

$$Hv + 2\nabla v \cdot \nabla Y + xv \cdot xY + :|\nabla Y|^2: v$$

so that $v \in \mathcal{C}^1([a, b], \mathcal{W}^{\sigma-2,2}) \cap \mathcal{C}([a, b], \mathcal{W}^{\sigma,2})$ a.s. and is solution of equation (7).

Since equation (7) is linear, it is sufficient, in order to prove uniqueness, to prove that any solution with initial data 0 is constant to 0. Let v be a solution in $\mathbb{L}^p(\Omega, \mathcal{C}([a, b], \mathcal{W}^{\sigma,2}))$ of equation (7) with initial data 0. Then the Lions-Magenes lemma (see [24]) yields:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |v|^2 e^{2Y_N} dx &= \langle \partial_t v, v e^{2Y_N} \rangle_{\mathcal{W}_x^{\sigma-2,2}, \mathcal{W}_x^{2-\sigma,2}} \\ &= \langle i(Hv + 2\nabla v \cdot \nabla Y + xv \cdot xY + :|\nabla Y|^2 : v), v e^{2Y_N} \rangle_{\mathcal{W}_x^{\sigma-2,2}, \mathcal{W}_x^{2-\sigma,2}}. \end{aligned}$$

Recall that

$$\langle i(Hv + 2\nabla v \cdot \nabla Y_N + xv \cdot xY_N + :|\nabla Y_N|^2 : v), v e^{2Y_N} \rangle_{\mathcal{W}_x^{\sigma-2,2}, \mathcal{W}_x^{2-\sigma,2}} = 0.$$

It follows

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |v|^2 e^{2Y_N} dx &= \langle i(2\nabla v \cdot \nabla(Y - Y_N) + xv \cdot x(Y - Y_N)v), v e^{2Y_N} \rangle_{\mathcal{W}_x^{\sigma-2,2}, \mathcal{W}_x^{2-\sigma,2}} \\ &\quad + \langle i(:|\nabla Y|^2 : - :|\nabla Y_N|^2 :) v, v e^{2Y_N} \rangle_{\mathcal{W}_x^{\sigma-2,2}, \mathcal{W}_x^{2-\sigma,2}}. \end{aligned}$$

Let $\kappa \in (0, \sigma - 1)$, we have a.s.

$$\begin{aligned} \left| \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |v|^2 e^{2Y_N} dx \right| &\lesssim \left(|\nabla(Y - Y_N)|_{\mathcal{W}_x^{-\kappa, \infty}} |v|_{\mathcal{W}_x^{1+\kappa, 2}} + |x(Y - Y_N)|_{\mathcal{W}_x^{-\kappa, \infty}} |v|_{\mathcal{W}_x^{1+\kappa, 2}} \right. \\ &\quad \left. + :|\nabla Y|^2 : - :|\nabla Y_N|^2 : \Big|_{\mathcal{W}_x^{-\kappa, \infty}} |v|_{\mathcal{W}_x^{\kappa, 2}} \right) |v|_{\mathcal{W}_x^{\kappa, 2}} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} \\ &\lesssim \left(|\nabla(Y - Y_N)|_{\mathcal{W}_x^{-\kappa, \infty}} + |x(Y - Y_N)|_{\mathcal{W}_x^{-\kappa, \infty}} \right) |v|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}^2 |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} \\ &\quad + :|\nabla Y|^2 : - :|\nabla Y_N|^2 : \Big|_{\mathcal{W}_x^{-\kappa, \infty}} |v|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}^2 |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}}. \end{aligned}$$

Integrating in time, taking expectation and using Lemma 3.10 and Corollaries 3.5 and 3.18 we obtain

$$\sup_{[a, b]} \int_{\mathbb{R}^2} |v|^2 e^{2Y} = 0 \text{ a.s.}$$

which concludes the uniqueness.

To prove now that the whole sequence converges to v , let $N \geq 2$ and $n = \lfloor \log_2(N) \rfloor$. Note $M = 2^n$. Then, interpolating between Propositions 4.1 and 4.2, we have

$$\begin{aligned} |v^N - v|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}} &\lesssim |v^N - v^{2^n}|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathcal{W}_x^{2, 2}}^{\frac{\sigma}{2}} |v^N - v^{2^n}|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathbb{L}_x^2}^{1-\frac{\sigma}{2}} + |v^{2^n} - v|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}} \\ &\lesssim \lambda_N^{\frac{\kappa'\sigma}{2}} \lambda_M^{-(1-\frac{\sigma}{2})\frac{\delta}{2}} + |v^{2^n} - v|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}} \\ &\lesssim 2^{\log_2(N) \frac{\kappa'\sigma}{4} - n(1-\frac{\sigma}{2})\frac{\delta}{4}} + |v^{2^n} - v|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}, \end{aligned}$$

which goes to 0 when N goes to $+\infty$ if, for example, we choose $\delta \in (0, \frac{1}{4})$, $\delta' \in (0, (1 - \frac{\sigma}{2}) \frac{\delta}{4})$ and $\kappa' \in \left(0, \min\left(1, \frac{(2-\sigma)\delta - 8\delta'}{4\sigma}\right)\right)$ in order to have

$$\frac{\log_2(N)}{\lfloor \log_2(N) \rfloor} \frac{\kappa'\sigma}{4} - \left(1 - \frac{\sigma}{2}\right) \frac{\delta}{4} < -\delta'$$

uniformly in $N \geq 2$.

It remains to prove conservation laws. The conservation of the transformed mass follows from the convergence of v^N in $\mathbb{L}^2(\mathbb{R}^2)$ and Lemma 3.10 and the conservation of the transformed energy follows from convergence of v^N in Σ and Lemmas 2.7, 2.8 and 3.10. This concludes the proof. \square

5 The nonlinear equation

First we prove that there exists at most one solution to equation (4).

Proposition 5.1. *Let $\lambda \in \mathbb{R}$. Almost surely, for every $a < 0 < b$, every $\sigma \in (1, 2)$ and every $v_0 \in \mathcal{W}^{\sigma, 2}$, there exists at most one solution to equation (4) in $\mathcal{C}([a, b], \mathcal{W}^{\sigma, 2})$ and it belongs to $\mathcal{C}([a, b], \mathcal{W}^{\sigma, 2}) \cap \mathcal{C}^1([a, b], \mathcal{W}^{\sigma-2, 2})$.*

Proof. Using the continuous injection of $\mathcal{C}([a', b'], \mathcal{W}^{\sigma', 2})$ in $\mathcal{C}([a, b], \mathcal{W}^{\sigma, 2})$ for $a' < a$, $b' > b$ and $\sigma' > \sigma$, it is sufficient to prove the claim almost surely for fixed a, b and σ . So let $a < 0 < b$ and $\sigma \in (1, 2)$.

Let $\kappa \in (0, \sigma - 1)$ and $\delta \in (0, \frac{2\kappa}{3})$. Define

$$\Omega_0 = \left\{ e^{-2Y} \in \mathbb{L}^\infty(\mathbb{R}^2), e^{2Y_N} \xrightarrow[N \rightarrow +\infty]{\mathbb{L}^\infty(\mathbb{R}^2)} e^{2Y}, \sup_{N \in \mathbb{N}} |e^{2Y_N} - 1|_{\mathcal{W}_x^{1-\kappa, \infty}} < +\infty \right\} \\ \cap \left\{ \sup_{N \in \mathbb{N}} \lambda_N^\delta \max \left(|\nabla(Y - Y_N)|_{\mathcal{W}_x^{-\kappa, \infty}}, |x(Y - Y_N)|_{\mathcal{W}_x^{-\kappa, \infty}}, |:\nabla Y|^2: - :|\nabla Y_N|^2: |_{\mathcal{W}_x^{-\kappa, \infty}} \right) < +\infty \right\},$$

using Lemma 3.10 and Corollaries 3.5 and 3.18, it follows that $\mathbb{P}(\Omega_0) = 1$.

Let $v_0 \in \mathcal{W}^{\sigma, 2}$, v and w be two solutions of equation (4) in $\mathcal{C}([a, b], \mathcal{W}^{\sigma, 2})$ with initial data v_0 and set $r = v - w$. Then $r \in \mathcal{C}([a, b], \mathcal{W}^{\sigma, 2})$ and verifies

$$\begin{cases} i\partial_t r + Hr + 2\nabla Y \cdot \nabla r + xY \cdot xr + :|\nabla Y|^2:r + \lambda(|v|^2v - |w|^2w) e^{2Y} \\ r(0) = 0. \end{cases} \quad (16)$$

Moreover, using Lemma 2.8, one can show that $v, w \in \mathcal{C}([a, b], \mathcal{W}^{\sigma, 2}) \cap \mathcal{C}^1([a, b], \mathcal{W}^{\sigma-2, 2})$ so that $r \in \mathcal{C}([a, b], \mathcal{W}^{\sigma, 2}) \cap \mathcal{C}^1([a, b], \mathcal{W}^{\sigma-2, 2})$. Using Lemma 3.10, on Ω_0 , we have

$$\sup_{t \in [a, b]} \|r(t)\|_{\mathbb{L}_x^2}^2 \leq |e^{-2Y}|_{\mathbb{L}_x^\infty} \sup_{t \in [a, b]} \int_{\mathbb{R}^2} |r|^2 e^{2Y} dx \leq |e^{-2Y}|_{\mathbb{L}_x^\infty} \lim_{N \rightarrow +\infty} \sup_{t \in [a, b]} \int_{\mathbb{R}^2} |r|^2 e^{2Y_N} dx. \quad (17)$$

Recall we chose $\kappa \in (0, \sigma - 1)$. As in the linear case, by taking the time derivative, the Lions-Magenes

lemma yields

$$\begin{aligned}
\left| \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r|^2 e^{2Y_N} dx \right| &= \left| \langle i(2\nabla r \cdot \nabla(Y - Y_N) + xr \cdot x(Y - Y_N) + (|\nabla Y|^2 - |\nabla Y_N|^2)) r \right. \\
&\quad \left. + \lambda(|v|^2 v - |w|^2 w) e^{2Y}, re^{2Y_N} \rangle_{\mathcal{W}_x^{-\kappa, 2}, \mathcal{W}_x^{\kappa, 2}} \right| \\
&\leq |2\nabla r \cdot \nabla(Y - Y_N) + xr \cdot x(Y - Y_N)|_{\mathcal{W}_x^{-\kappa, 2}} |re^{2Y_N}|_{\mathcal{W}_x^{\kappa, 2}} \\
&\quad + (|\nabla Y|^2 - |\nabla Y_N|^2) r|_{\mathcal{W}_x^{-\kappa, 2}} |re^{2Y_N}|_{\mathcal{W}_x^{\kappa, 2}} \\
&\quad + \left| \lambda \int_{\mathbb{R}^2} (|v|^2 v - |w|^2 w) e^{2Y} \bar{r} e^{2Y_N} dx \right|.
\end{aligned}$$

Using Lemmas 2.7, 2.8 and 3.11 and Corollary 2.5, it gives

$$\begin{aligned}
\left| \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |r|^2 e^{2Y_N} dx \right| &\lesssim \left(|\nabla(Y - Y_N)|_{\mathcal{W}_x^{-\kappa, \infty}} + |x(Y - Y_N)|_{\mathcal{W}_x^{-\kappa, \infty}} \right) |r|_{\mathcal{W}_x^{1+\kappa, 2}} |r|_{\mathcal{W}_x^{\kappa, 2}} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} \\
&\quad + (|\nabla Y|^2 - |\nabla Y_N|^2)|_{\mathcal{W}_x^{-\kappa, \infty}} |r|_{\mathcal{W}_x^{\kappa, 2}} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} \\
&\quad + \left(|v|_{\mathbb{L}_x^\infty}^2 + |w|_{\mathbb{L}_x^\infty}^2 \right) |e^{2Y}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |r|^2 e^{2Y_N} dx \\
&\lesssim \left(|\nabla(Y - Y_N)|_{\mathcal{W}_x^{-\kappa, \infty}} + |x(Y - Y_N)|_{\mathcal{W}_x^{-\kappa, \infty}} \right) |r|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} \\
&\quad + (|\nabla Y|^2 - |\nabla Y_N|^2)|_{\mathcal{W}_x^{-\kappa, \infty}} |r|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} \\
&\quad + \left(|v|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}^2 + |w|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}^2 \right) |e^{2Y}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |r|^2 e^{2Y_N} dx.
\end{aligned}$$

Recall that $\delta \in (0, \frac{2\kappa}{3})$, so there exists a positive random variable C finite on Ω_0 (by definition of Ω_0) such that for any $N \in \mathbb{N}$, it holds

$$\frac{d}{dt} \int_{\mathbb{R}^2} |r|^2 e^{2Y_N} dx \leq C \left(|r|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}^2 \lambda_N^{-\delta} + \left(|v|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}^2 + |w|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}^2 \right) |e^{2Y}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |r|^2 e^{2Y_N} dx \right).$$

Then by Gronwall's lemma, we obtain

$$\forall N \in \mathbb{N}, \quad \sup_{t \in [a, b]} \int_{\mathbb{R}^2} |r|^2 e^{2Y_N} dx \leq C |r|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}^2 \lambda_N^{-\delta} e^{(b-a)C} \left(|v|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}^2 + |w|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}^2 \right) |e^{2Y}|_{\mathbb{L}_x^\infty}$$

which in addition to equation (17) implies that $v = w$ almost surely. \square

We will only prove the existence of a solution in the defocusing case (i.e. Theorem 1.2). In view of Proposition 3.27, analogous proofs will hold on the conditional probability space $(\overline{\Omega}_L, \mathcal{A}_{|\overline{\Omega}_L}, \mathbb{P}_{|\overline{\Omega}_L})$ in the focusing case if we restrict ourselves to initial data $v_0 \in \mathcal{W}^{2,2}$ satisfying $|v_0|_{\mathbb{L}_x^2} \leq L$. So, assume $\lambda \leq 0$. For $N \in \mathbb{N}$, denote by v^N the unique global solution to equation (10). First as in the linear case, we obtain a diverging bound but this time on $|v^N|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}$ for any $\sigma \in (\frac{3}{2}, 2)$. In order to obtain this bound, we prove the following generalized Brezis-Gallouet inequality (for the classical one see [6]).

Proposition 5.2. *Generalized Brezis-Gallouet inequality*

Let $\sigma > 1$, there exists a constant $C > 0$ such that

$$\forall v \in \mathcal{W}^{\sigma,2}, |v|_{\mathbb{L}_x^\infty} \leq C \left(1 + |v|_\Sigma \sqrt{1 + \ln(1 + |v|_{\mathcal{W}_x^{\sigma,2}})} \right)$$

Proof. By continuity of both sides and density of $\mathcal{S}(\mathbb{R}^2)$ in $\mathcal{W}^{\sigma,2}$, it is sufficient to obtain the claim for $v \in \mathcal{S}(\mathbb{R}^2)$. Then, we have for any $R > 0$:

$$\begin{aligned} |v|_{\mathbb{L}_{\mathbb{R}^2}^\infty} &\lesssim \int_{|\eta| < R} \langle \eta \rangle \mathcal{F}v(\eta) \frac{d\eta}{\langle \eta \rangle} + \int_{|\eta| \geq R} \langle \eta \rangle^\sigma \mathcal{F}v(\eta) \frac{d\eta}{\langle \eta \rangle^\sigma} \\ &\lesssim |v|_\Sigma \sqrt{\ln(1 + R)} + \frac{|v|_{\mathcal{W}^{\sigma,2}}}{(1 + R^2)^{\frac{\sigma-1}{2}}} \\ &\lesssim |v|_\Sigma \sqrt{\ln(1 + R)} + \frac{|v|_{\mathcal{W}^{\sigma,2}}}{1 + R^{\sigma-1}} \end{aligned}$$

using Cauchy-Schwartz inequality. We conclude choosing $R = \left(\frac{|v|_{\mathcal{W}^{\sigma,2}}}{1 + |v|_\Sigma} \right)^{\frac{1}{\sigma-1}} \leq |v|_{\mathcal{W}^{\sigma,2}}^{\frac{1}{\sigma-1}}$ and using properties of the logarithm. \square

Using this inequality, we obtain the following diverging bound which grows polynomially in N .

Proposition 5.3. *Almost surely, for all $v_0 \in \mathcal{W}^{2,2}$, for all $a < 0 < b$, for all $\sigma \in (1, 2)$ and all $\kappa \in (0, 1)$, there exists $C > 0$ such that*

$$\forall N \in \mathbb{N}, |v^N|_{\mathbb{L}_{[a,b]}^\infty \mathcal{W}^{\sigma,2}} \leq C \lambda_N^\kappa.$$

Then, as in the linear case, we obtain a Cauchy estimate in $\mathbb{L}^2(\mathbb{R}^2)$.

Proposition 5.4. *Almost surely, for all $v_0 \in \mathcal{W}^{2,2}$, for all $a < 0 < b$, for all $\kappa \in (0, 1)$ and all $\delta \in (0, \frac{1}{3})$, there exists $C > 0$ such that*

$$\forall N > M \geq 0, |v^N - v^M|_{\mathbb{L}_{[a,b]}^\infty \mathbb{L}_{\mathbb{R}^2}^2} \leq C \lambda_N^\kappa \lambda_M^{-\delta}.$$

The proofs of both Propositions 5.3 and 5.4 are given in appendix C. We can now prove the theorem.

Proof. of Theorem 1.2.

Let $\overline{\Omega} = \Omega' \cap \Omega'' \cap \Omega_0$ where Ω' , Ω'' and Ω_0 are the full probability event of Propositions 5.1, 5.3 and 5.4 respectively. Then $\mathbb{P}(\overline{\Omega}) = 1$ and for any $N \in \mathbb{N}$ and any $v_0 \in \mathcal{W}^{2,2}$ and any $\omega \in \overline{\Omega}$, $v^N(\omega)$ is global.

Let $\omega \in \overline{\Omega}$, $v_0 \in \mathcal{W}^{2,2}$, $a < 0 < b$ and $\sigma, \sigma' \in (1, 2)$ such that $\sigma' > \sigma$, $\delta \in (0, \frac{1}{3})$ and let $\kappa = (1 - \frac{\sigma}{\sigma'}) \frac{\delta}{2} \in (0, 1)$. Then by interpolation, for any $N > M \geq 0$, it holds

$$|v^N(\omega) - v^M(\omega)|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma,2}} \leq |v^N(\omega) - v^M(\omega)|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma',2}}^{\frac{\sigma}{\sigma'}} |v^N(\omega) - v^M(\omega)|_{\mathbb{L}_t^\infty \mathbb{L}_x^2}^{1 - \frac{\sigma}{\sigma'}}.$$

By Propositions 5.3 and 5.4, there exists $C(\omega, v_0) \in (0, +\infty)$ (which is independent of N and M , and may vary from line to line) such that

$$\forall N > M \geq 0, |v^N(\omega) - v^M(\omega)|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma,2}} \leq C(\omega, v_0) \lambda_N^\kappa \lambda_M^{-(1 - \frac{\sigma}{\sigma'})\delta}. \quad (18)$$

Let $n \in \mathbb{N}$, $M = 2^n$ and $N = 2^{n+1}$. Then equation (18) gives

$$\left| v^{2^{n+1}}(\omega) - v^{2^n}(\omega) \right|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma,2}} \leq C(\omega, v_0) 2^{\frac{n}{2}(\kappa - (1 - \frac{\sigma}{\sigma'})\delta)} = C(\omega, v_0) 2^{-n(1 - \frac{\sigma}{\sigma'})\frac{\delta}{4}},$$

which is summable. Hence the sequence $(v^{2^n}(\omega))_{n \in \mathbb{N}}$ is Cauchy in the Banach space $\mathcal{C}([a, b], \mathcal{W}^{\sigma,2})$ and thus converges to an element $v(\omega)$. As in the linear case, using

$$\left| v^N(\omega) - v(\omega) \right|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma,2}} \leq \left| v^N(\omega) - v^{2^{\lfloor \log_2 N \rfloor}}(\omega) \right|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma,2}} + \left| v^{2^{\lfloor \log_2 N \rfloor}}(\omega) - v(\omega) \right|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma,2}}$$

in addition to equation (18) with a well-chosen κ shows the convergence of $v^N(\omega)$ to $v(\omega)$ in $\mathcal{C}([a, b], \mathcal{W}^{\sigma,2})$. The function v is measurable as an almost sure limit of measurable functions. Using Lemmas 2.8 and 3.10 and Corollaries 3.5 and 3.18, it is not difficult to show that $-i\partial_t v^N(\omega)$ converges in $\mathcal{C}([a, b], \mathcal{W}^{\sigma-2,2})$ to

$$Hv(\omega) + 2\nabla v(\omega) \cdot \nabla Y(\omega) + xv(\omega) \cdot xY(\omega) + :|\nabla Y|^2:(\omega)v(\omega) + \lambda|v(\omega)|^2v(\omega)e^{2Y(\omega)}$$

so that $v \in \mathcal{C}^1([a, b], \mathcal{W}^{\sigma-2,2}) \cap \mathcal{C}([a, b], \mathcal{W}^{\sigma,2})$ a.s. and is solution of equation (4). Moreover, as in the linear case, the regularity of v is sufficient to pass to the limit conservation laws. \square

Remark 5.5.

- In the case of subcubic nonlinearities, $|u|^{2\gamma}u$ with $\gamma \in (\frac{1}{2}, 1)$, analogous of Propositions 3.26, 5.1, 5.3 and 5.4 and Theorem 1.2 holds and there exists almost surely, for all initial data in $\mathcal{W}^{2,2}$, a unique solution which lives in $\mathcal{C}(\mathbb{R}, \mathcal{W}^{2-,2})$.
- If instead of working with deterministic initial data in the focusing case, one choose a random initial datum $v_0 \in \mathcal{W}^{2,2}$ a.s. such that

$$\lambda|v_0|_{\mathbb{L}_x^2}^2 \left| e^{-2Y} \right|_{\mathbb{L}_x^\infty}^2 \left| e^{2Y} \right|_{\mathbb{L}_x^\infty} \left| e^{4Y} \right|_{\mathbb{L}_x^\infty} < 4$$

holds almost surely, then by a slight modification of previous arguments, one can show that there exists almost a unique global solution.

A Proofs of results from Section 2

A.1 Hermite-Sobolev spaces

Proof. of Proposition 2.4

It is known from [4] that, for $k \in \mathbb{N}$, an equivalent norm on $\mathcal{W}^{k,p}$ is given by

$$|u|_{k,p} = |u|_{\mathbb{L}_x^p} + \sum_{m=1}^k \sum_{1 \leq |i_1|, \dots, |i_m| \leq 2} |A_{i_1} \cdots A_{i_m} u|_{\mathbb{L}_x^p}.$$

Then, for $u \in \mathcal{W}^{k,p}$ with $k \in \mathbb{N}^*$ and $i \in \{\pm 1, \pm 2\}$, we have

$$|A_i u|_{k-1,p} = |A_i u|_{\mathbb{L}_x^p} + \sum_{m=1}^{k-1} \sum_{1 \leq |i_1|, \dots, |i_m| \leq 2} |A_{i_1} \cdots A_{i_m} A_i u|_{\mathbb{L}_x^p} \leq |u|_{k,p},$$

so $A_i \in \mathcal{L}(\mathcal{W}^{k,p}, \mathcal{W}^{k-1,p})$. The case $k \in -\mathbb{N}$ is shown by duality and the general case $s \in \mathbb{R}$ is obtained by interpolation. \square

Proof. of Proposition 2.6.

The first point follows from usual Sobolev embedding of $W^{s-\sigma,p}$ in $\mathbb{L}^q(\mathbb{R}^2)$ and continuous embedding of $\mathcal{W}^{s-\sigma,p}$ in $W^{s-\sigma,p}$. When the condition is sharp, one can use the compact embedding of $\mathcal{W}^{\varepsilon,p}$ in $\mathbb{L}^p(\mathbb{R}^2)$ (for $\varepsilon > 0$ and $p \in (1, +\infty)$) to obtain compactness of the embedding.

It remains to prove continuous embedding in the case $p < q = +\infty$. Assume first that $\sigma < 0$ and let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. As $\frac{1}{p} - \frac{s}{2} < \frac{1}{q} - \frac{\sigma}{2}$, it follows $1 < \frac{1}{p'} + \frac{s-\sigma}{2}$. For any $k \in \mathbb{N}$, $\mathcal{W}^{k,1}$ is continuously embedded in $W^{k,1}$ which is continuously embedded in $W^{k+\sigma-s,p'}$ by classical Sobolev embeddings. By interpolation, $\mathcal{W}^{-\sigma,1}$ is continuously embedded in $W^{-s,p'}$. By duality and continuous embedding of $\mathcal{W}^{s,p}$ in $W^{s,p}$ we obtain the claim for $\sigma < 0$.

Now, for $\sigma = 2k \in 2\mathbb{N}$, the claim follows from usual continuous embeddings of $W^{s,p}$ in $W^{2k,\infty}$ for the Sobolev part and using the continuous injection of $\mathcal{W}^{s-2k,p}$ in $\mathbb{L}^\infty(\mathbb{R}^2)$ combined with Corollary 2.5 to obtain that $\langle x \rangle^{2k} : \mathcal{W}^{s,p} \rightarrow \mathbb{L}^\infty(\mathbb{R}^2)$ is continuous. The general case $\sigma \geq 0$ follows by interpolation. \square

A.2 The smooth truncation

Proof. of Lemma 2.12.

As S_N commute with $(-H)^{\frac{\alpha}{2}}$, it is sufficient to prove the case $\alpha = 0$. By interpolation, it is sufficient to only prove the case $s = 2n$ for $n \in \mathbb{N}$. Let $N \in \mathbb{N}$ and $\phi \in \mathbb{L}^p(\mathbb{R}^2)$, write it $\phi = \sum_{k \in \mathbb{N}^2} \phi_k h_k$, then we have

$$|S_N \phi|_{\mathcal{W}_x^{2n,p}}^p = \int_{\mathbb{R}^2} \left| \sum_{k \in \mathbb{N}^2} \chi_{k,N} \lambda_k^{2n} \phi_k \right|^p dx = \lambda_N^{2np} \int_{\mathbb{R}^2} \left| \sum_{k \in \mathbb{N}^2} \chi_{k,N} \left(\frac{\lambda_k^2}{\lambda_N^2} \right)^n \phi_k h_k(x) \right|^p dx.$$

Define $\psi_n(\lambda) = \lambda^n \chi(\lambda)$, then $\psi \in \mathcal{S}(\mathbb{R})$, so by Lemma 2.10, it follows

$$|S_N \phi|_{\mathcal{W}_x^{2n,p}}^p = \lambda_N^{2np} \left| \psi_n \left(\frac{-H}{\lambda_N^2} \right) \phi \right|_{\mathbb{L}_x^p}^p \lesssim \lambda_N^{2np} |\phi|_{\mathbb{L}_x^p}^p.$$

\square

Proof. of Lemma 2.13.

Let $\phi = \sum_{k \in \mathbb{N}^2} \phi_k h_k \in \mathcal{W}^{\alpha+s,2}$. It holds

$$|\phi - S_N \phi|_{\mathcal{W}_x^{\alpha,2}}^2 = \sum_{k \in \mathbb{N}^2} (1 - \chi_{k,N}^2) \lambda_k^{2\alpha} |\phi_k|^2 \leq \lambda_{\lfloor \frac{N}{2} \rfloor}^{-2s} \sum_{k \in \mathbb{N}^2} \lambda_k^{2\alpha+2s} |\phi_k|^2 \leq \lambda_{\lfloor \frac{N}{2} \rfloor}^{-2s} |\phi|_{\mathcal{W}_x^{\alpha+s,2}}^2.$$

\square

B Proofs of results from Section 3

B.1 Noise regularity

Proof. of Lemma 3.3.

Let $\beta < \alpha$ and $p > \max\left(1, \frac{2}{\alpha-\beta}\right)$. By Markov's inequality, we have

$$\mathbb{P}\left(\lambda_N^\beta |X_N| > 1\right) \leq \lambda_N^{-p(\alpha-\beta)} \left(\sup_{N \in \mathbb{N}} \lambda_N^\alpha |X_N|_{\mathbb{L}^p}\right)^p.$$

Let $A_N = \left\{\lambda_N^\beta |X_N| > 1\right\}$. The previous inequality shows $\sum_{N \in \mathbb{N}} \mathbb{P}(A_N) < +\infty$, thus, by the Borel-Cantelli lemma $\mathbb{P}\left(\limsup_{N \rightarrow +\infty} A_N\right) = 0$. Hence, almost surely, there exists $n \in \mathbb{N}$ such that for any $N \geq n$, $\lambda_N^\beta |X_N| \leq 1$. This implies that almost surely $\sup_{N \in \mathbb{N}} \lambda_N^\beta |X_N| < +\infty$. \square

Proof. of Lemma 3.4.

By Hölder's inequality, it is sufficient to prove the case $p \geq q$. By Minkowski's inequality and gaussianity, we have

$$\mathbb{E}\left[|Y - Y_N|_{\mathcal{W}_x^{1-s,q}}^p\right]^{\frac{q}{p}} \lesssim \int_{\mathbb{R}^2} \mathbb{E}\left[\left|\sum_{k \in \mathbb{N}^2} (1 - \chi_{k,N}) \lambda_k^{-(1+s)} h_k(x) \xi_k\right|^2\right]^{\frac{q}{2}} dx \lesssim \lambda_N^{-q(s-s')} |K|_{\mathbb{L}^q \mathcal{W}_y^{1-s',2}}^q.$$

The assumption $s'q > 2$ implies $1 - s' < 1 - \frac{2}{q}$ and we conclude using Proposition 2.2. \square

Proof. of Corollary 3.5.

Let $\epsilon = \sqrt{\frac{\kappa'}{\kappa}}$, $s = \epsilon\kappa$, $s' = (1 - \epsilon)s$ and $q > \max\left(\frac{2}{s'}, \frac{2}{\kappa-s}\right)$. By Sobolev embeddings and Corollary 2.5, there exists $c > 0$ such that

$$\max\left(|Y - Y_N|_{\mathcal{W}_x^{1-\kappa,\infty}}, |\nabla Y - \nabla Y_N|_{\mathcal{W}_x^{-\kappa,\infty}}, |xY - xY_N|_{\mathcal{W}_x^{-\kappa,\infty}}\right) \leq c|Y - Y_N|_{\mathcal{W}_x^{1-s,q}}.$$

Denote by $C_N = c|Y - Y_N|_{\mathcal{W}_x^{1-s,q}} \lambda_N^{s-s'}$. As $s - s' = \kappa'$, we obtain

$$\max\left(|Y - Y_N|_{\mathcal{W}_x^{1-\kappa,\infty}}, |\nabla Y - \nabla Y_N|_{\mathcal{W}_x^{-\kappa,\infty}}, |xY - xY_N|_{\mathcal{W}_x^{-\kappa,\infty}}\right) \leq C_N \lambda_N^{-\kappa'}.$$

Let $p \in [1, +\infty)$, by Lemma 3.4, $\sup_{N \in \mathbb{N}} |C_N|_{\mathbb{L}^p} < +\infty$. The only point remaining is to show that almost surely $\sup_{N \in \mathbb{N}} C_N < +\infty$. Let $\kappa'' \in (\kappa', \kappa)$, then by taking $s = \sqrt{\kappa\kappa'}$, $s' = \left(1 - \sqrt{\frac{\kappa''}{\kappa}}\right)s$ and $q > \max\left(\frac{2}{s'}, \frac{2}{\kappa-s}\right)$, we can conclude using Lemmas 3.3 and 3.4. \square

Proof. of Corollary 3.6.

Let $q > \frac{4}{\kappa}$ and $s = \frac{\kappa}{2}$, by Sobolev embedding and Corollary 2.5, there exists a constant $c > 0$ such that

$$\max\left(|\nabla Y_N|_{\mathbb{L}_x^\infty}, |xY_N|_{\mathbb{L}_x^\infty}\right) \leq c|Y_N|_{\mathcal{W}_x^{1+s,q}}.$$

Hence, using Lemma 2.12 and Corollary 2.11, there exists $c' > 0$ such that

$$\max\left(|\nabla Y_N|_{\mathbb{L}_x^\infty}, |xY_N|_{\mathbb{L}_x^\infty}\right) \leq c'|Y|_{\mathcal{W}_x^{1-s,q}} \lambda_N^\kappa,$$

where we used that $\kappa = 2s$. \square

Proof. of Lemma 3.8.

Due to the condition on s , $\mathcal{W}^{1-s,q}$ is continuously embedded in $\mathbb{L}^\infty(\mathbb{R}^2)$. Thus, $\mathcal{W}^{1-s,q}$ is an algebra and there exists a constant $c \geq 1$ such that for all $u, v \in \mathcal{W}^{s,q}$ we have

$$|uv|_{\mathcal{W}_x^{1-s,q}} \leq c|u|_{\mathcal{W}_x^{1-s,q}}|v|_{\mathcal{W}_x^{1-s,q}}. \quad (19)$$

Using Minkowski's inequality, we obtain

$$\forall N \in \mathbb{N}, \mathbb{E} \left[|e^{aY_N} - 1|_{\mathcal{W}_x^{1-s,q}}^p \right] \leq \mathbb{E} \left[e^{cp|a||Y_N|_{\mathcal{W}_x^{1-s,q}}} \right] \text{ and } \mathbb{E} \left[|e^{aY} - 1|_{\mathcal{W}_x^{1-s,q}}^p \right] \leq \mathbb{E} \left[e^{cp|a||Y|_{\mathcal{W}_x^{1-s,q}}} \right].$$

By Corollary 2.11, there exists $c' \geq 1$ such that, for any $N \in \mathbb{N}$, we have $|Y_N|_{\mathcal{W}_x^{1-s,q}} \leq c'|Y|_{\mathcal{W}_x^{1-s,q}}$.

By Fernique's theorem (see [16]), there exists $\beta > 0$ such that $\mathbb{E} \left[e^{\beta|Y|_{\mathcal{W}_x^{1-s,q}}^2} \right] < +\infty$. So, let

$c'' = \frac{(cc'p|a|)^2}{\beta}$, then Young's inequality implies,

$$\forall N \in \mathbb{N}, \mathbb{E} \left[|e^{aY_N} - 1|_{\mathcal{W}_x^{1-s,q}}^p \right] \leq e^{c''} \mathbb{E} \left[e^{\beta|Y|_{\mathcal{W}_x^{1-s,q}}^2} \right]$$

and

$$\mathbb{E} \left[|e^{aY} - 1|_{\mathcal{W}_x^{1-s,q}}^p \right] \leq e^{c''} \mathbb{E} \left[e^{\beta|Y|_{\mathcal{W}_x^{1-s,q}}^2} \right].$$

□

Proof. of Lemma 3.10.

The second claim follows easily from the first one using Lemma 3.3. In order to prove the first claim, let $q > \frac{4}{\kappa - \kappa'}$ and $s \in \left(\kappa' + \frac{2}{q}, \kappa - \frac{2}{q} \right)$. Then $\mathcal{W}^{1-s,q}$ is continuously embedded in $\mathcal{W}^{1-\kappa,\infty}$ and is an algebra, thus equation (19) is verified. Denote by C the constant of Sobolev embedding of $\mathcal{W}^{1-s,q}$ in $\mathcal{W}^{1-\kappa,\infty}$. Using the algebra property of $\mathcal{W}^{1-s,q}$ norms, we obtain

$$|e^{aY_N} - e^{aY}|_{\mathcal{W}_x^{1-\kappa,\infty}} \leq cC |e^{aY}|_{\mathcal{W}_x^{1-s,q}} |e^{a(Y_N - Y)} - 1|_{\mathcal{W}_x^{1-s,q}}.$$

By Minkowski's inequality, we have

$$|e^{a(Y_N - Y)} - 1|_{\mathcal{W}_x^{1-s,q}} \leq \sum_{n=1}^{+\infty} \frac{c^{n-1} |a|^n |Y - Y_N|_{\mathcal{W}_x^{1-s,q}}^n}{n!} \leq |a| |Y - Y_N|_{\mathcal{W}_x^{1-s,q}} e^{c|a||Y - Y_N|_{\mathcal{W}_x^{1-s,q}}}.$$

Using Hölder's inequality, Corollary 2.11 and Lemmas 3.4 and 3.8 with $s' = s - \kappa'$ we obtain the first claim. □

B.2 Renormalization

Proof. of Lemma 3.17.

By Minkowski's inequality and Proposition 3.15, we have

$$|: |\nabla Y_N|^2 :|_{\mathbb{L}_x^{q'} \mathcal{W}_x^{\alpha,q}} \leq \int_{\mathbb{R}^2} \mathbb{E} \left[|(-H)^{\frac{\alpha}{2}} : |\nabla Y_N|^2 : (x)|^{q'} \right]^{\frac{q}{q'}} dx \leq (q-1) |(-H)^{\frac{\alpha}{2}} : |\nabla Y_N|^2 :|_{\mathbb{L}_x^q \mathbb{L}_x^2}.$$

Using one more time Minkowski's and Hölder's inequalities, we obtain the claim. □

Proof. of Corollary 3.18.

First, we show the moment estimates in the case $q < +\infty$. Recall that Proposition 3.12 implies

$$\forall N \geq M, \left| |\nabla Y_N|^2 - |\nabla Y_M|^2 \right|_{\mathbb{L}_\omega^q \mathcal{W}_x^{-s,q}} \lesssim \lambda_M^{-\delta}.$$

Nelson's estimate and Hölder's inequality then implies

$$\forall N \geq M, \left| |\nabla Y_N|^2 - |\nabla Y_M|^2 \right|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}} \lesssim \lambda_M^{-\delta}.$$

In the case $q = +\infty$, let $q' > \max\left(\frac{5}{s}, \frac{10}{2s-3\delta}\right)$ and define $s' = s - \frac{3}{q'}$. By definition of q' , we have $s' > \frac{2}{q'}$ and the Sobolev embedding of $\mathcal{W}^{-s',q'}$ in $\mathcal{W}^{-s,q}$. Moreover, $\delta < \frac{2s'}{3} - \frac{4}{3q'}$. Hence the previous estimate leads to

$$\forall N \geq M, \left| |\nabla Y_N|^2 - |\nabla Y_M|^2 \right|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}} \lesssim \left| |\nabla Y_N|^2 - |\nabla Y_M|^2 \right|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s',q'}} \lesssim \lambda_M^{-\delta}.$$

Let $M \in \mathbb{N}$, by triangular inequality, we have

$$\left| |\nabla Y_M|^2 \right|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}} \leq \left| |\nabla Y_M|^2 - |\nabla Y|^2 \right|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}} + \left| |\nabla Y|^2 \right|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}}.$$

Lemma 3.17 shows that $\left| |\nabla Y|^2 \right|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}} < +\infty$ and the previous point shows that (by sending N to $+\infty$)

$$\forall M \in \mathbb{N}, \left| |\nabla Y_M|^2 - |\nabla Y|^2 \right|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}} \lesssim \lambda_M^{-\delta} \lesssim 1. \quad (20)$$

Hence

$$\max\left(\left| |\nabla Y|^2 \right|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}}, \sup_{N \in \mathbb{N}} \left| |\nabla Y_N|^2 \right|_{\mathbb{L}_\omega^p \mathcal{W}_x^{-s,q}}\right) < +\infty$$

and we conclude using Lemma 3.3 and equation (20). \square

Proof. of Lemma 3.20.

By Fubini's theorem and Nelson's estimate, we have

$$\mathbb{E} \left[\left| |\nabla Y_N|^2 \right|_{\mathbb{L}_x^q}^q \right] = \int_{\mathbb{R}^2} \mathbb{E} \left[\left| |\nabla Y_N|^2(x) \right|^q \right] dx \lesssim \int_{\mathbb{R}^2} \mathbb{E} \left[\left| |\nabla Y_N|^2(x) \right|^2 \right]^{\frac{q}{2}} dx.$$

Using $\left| |\nabla Y_N|^2(x) \right| = \left| \nabla Y_N(x) \right|^2 - \mathbb{E} \left[\left| \nabla Y_N(x) \right|^2 \right]$, one has

$$\mathbb{E} \left[\left| |\nabla Y_N|^2(x) \right|^2 \right] = \text{Var} \left(\left| \nabla Y_N(x) \right|^2 \right) \leq \mathbb{E} \left[\left| \nabla Y_N(x) \right|^4 \right].$$

Using Jensen's inequality, we obtain

$$\mathbb{E} \left[\left| |\nabla Y_N|^2 \right|_{\mathbb{L}_x^q}^q \right] \lesssim \int_{\mathbb{R}^2} \mathbb{E} \left[\left| \nabla Y_N(x) \right|^4 \right]^{\frac{q}{2}} dx \lesssim \int_{\mathbb{R}^2} \mathbb{E} \left[\left(\left| \nabla Y_N(x) \right|^2 \right)^q \right] dx \lesssim \left| \nabla Y_N \right|_{\mathbb{L}_\omega^q \mathbb{L}_x^q}^q,$$

and we conclude using Hölder's inequality. \square

Proof. of Corollary 3.21.

Using Hölder's inequality for $p < q$ and Lemma 3.17 for $p > q$, it is sufficient to prove the case $p = q$. Using Lemmas 2.12 and 3.20, we have

$$\left| |\nabla Y_N|^2 \right|_{\mathbb{L}_\omega^q \mathbb{L}_x^q} \lesssim \left| \nabla Y_N \right|_{\mathbb{L}_\omega^{2q} \mathbb{L}_x^{2q}}^2 \lesssim \lambda_N^s \left| \nabla Y_N \right|_{\mathbb{L}_\omega^{2q} \mathcal{W}_x^{-\frac{s}{2}, 2q}}^2$$

and we conclude using Lemma 3.4. \square

B.3 The regularized equation

Proof. of Proposition 3.24

First using definition of the transformed energy, we obtain

$$\begin{aligned} \frac{1}{2} |v^N(t)|_{\Sigma}^2 &= \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla v^N(t)|^2 + |xv^N(t)|^2) e^{2Y_N} e^{-2Y_N} dx \\ &\leq |e^{-2Y_N}|_{\mathbb{L}_x^\infty} \left(\tilde{\mathcal{E}}_N(v^N(t)) + \int_{\mathbb{R}^2} |xv^N(t)|^2 Y_N e^{2Y_N} dx + \int_{\mathbb{R}^2} |v^N(t)|^2 : |\nabla Y_N|^2 : e^{2Y_N} dx \right). \end{aligned}$$

Conservation of the transformed energy implies

$$\begin{aligned} \frac{1}{2} |v^N(t)|_{\Sigma}^2 &\leq |e^{-2Y_N}|_{\mathbb{L}_x^\infty} \tilde{\mathcal{E}}_N(v_0) + |e^{-2Y_N}|_{\mathbb{L}_x^\infty} |e^{2Y_N}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |xv^N(t)|^2 |Y_N| dx \\ &\quad + \left| |e^{-2Y_N}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |v^N(t)|^2 : |\nabla Y_N|^2 : e^{2Y_N} dx \right|. \end{aligned} \quad (21)$$

We treat the two integral terms separately.

- Let $q > 2$, $s > 0$ such that $qs > 2$ and $q' > 1$ verifying $\frac{1}{q} + \frac{1}{q'} = 1$. Using Hölder's inequality, Sobolev embeddings and interpolation, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |xv^N(t)|^2 |Y_N| dx &\lesssim \left\| |x|^{\frac{1+s}{2}} v^N(t) \right\|_{\mathbb{L}_x^{2q'}}^2 \left\| |x|^{1-s} Y_N \right\|_{\mathbb{L}_x^q} \\ &\lesssim |v^N(t)|_{\mathcal{W}_x^{\frac{1+s}{2}, 2q'}}^2 |Y_N|_{\mathcal{W}_x^{1-s, q}} \\ &\lesssim |v^N(t)|_{\mathbb{L}_x^2}^{1-\frac{2}{q}-s} |v^N(t)|_{\Sigma}^{1+\frac{2}{q}+s} |Y_N|_{\mathcal{W}_x^{1-s, q}}. \end{aligned}$$

Let $q_1 > 1$ such that $q_1 \left(1 + \frac{2}{q} + s\right) = 2$ and p_1 such that $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Using Young's inequality and mass conservation, there exists a constant $c_1 > 0$ such that

$$|e^{-2Y_N}|_{\mathbb{L}_x^\infty} |e^{2Y_N}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |xv^N|^2 |Y_N| dx \leq \frac{1}{8} |v^N|_{\Sigma}^2 + c_1 |e^{-2Y_N}|_{\mathbb{L}_x^\infty}^{2+p_1} |e^{2Y_N}|_{\mathbb{L}_x^\infty}^{1+p_1} |Y_N|_{\mathcal{W}_x^{1-s, q}}^{p_1} |v_0|_{\mathbb{L}_x^2}^2.$$

Using Lemmas 3.3 and 3.4 for the $\mathcal{W}_x^{1-s, q}$ norm and Lemma 3.10 and Corollary 3.9 for the terms involving exponentials of Y_N , we obtain that $C_{1, N} = c_1 |e^{-2Y_N}|_{\mathbb{L}_x^\infty}^{2+p_1} |e^{2Y_N}|_{\mathbb{L}_x^\infty}^{1+p_1} |Y_N|_{\mathcal{W}_x^{1-s, q}}^{p_1}$ verifies Property (*) and

$$|e^{-2Y_N}|_{\mathbb{L}_x^\infty} |e^{2Y_N}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |xv^N(t)|^2 |Y_N| dx \leq \frac{1}{8} |v^N(t)|_{\Sigma}^2 + C_{1, N} |v_0|_{\mathbb{L}_x^2}^2. \quad (22)$$

- Let $\kappa \in (0, 1)$, using duality and Lemma 2.7 we have

$$\begin{aligned} \left| \int_{\mathbb{R}^2} |v^N(t)|^2 : |\nabla Y_N|^2 : e^{2Y_N} dx \right| &\leq | : |\nabla Y_N|^2 : |_{\mathcal{W}_x^{-\kappa, \infty}} | |v^N(t)|^2 e^{2Y_N} |_{\mathcal{W}_x^{\kappa, 1}} \\ &\lesssim | : |\nabla Y_N|^2 : |_{\mathcal{W}_x^{-\kappa, \infty}} |v^N(t)|_{\mathcal{W}_x^{\kappa, 2}}^2 |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} \\ &\lesssim | : |\nabla Y_N|^2 : |_{\mathcal{W}_x^{-\kappa, \infty}} |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa, \infty}} |v^N(t)|_{\mathbb{L}_x^2}^{2-2\kappa} |v^N(t)|_{\Sigma}^{2\kappa}. \end{aligned}$$

Let $\tilde{C}_{2,N} = |e^{-2Y_N}|_{\mathbb{L}^\infty} : |\nabla Y_N|^2 : |_{\mathcal{W}^{-\kappa,\infty}} |e^{2Y_N}|_{1+\mathcal{W}^{\kappa,\infty}}$, by Young's inequality an mass conservation, there exists a constant $c_2 > 0$ such that

$$|e^{-2Y_N}|_{\mathbb{L}^\infty} \left| \int_{\mathbb{R}^2} |v^N(t)|^2 : |\nabla Y_N|^2 : e^{2Y_N} dx \right| \leq \frac{1}{8} |v^N(t)|_\Sigma^2 + c_2 \tilde{C}_{2,N}^{\frac{1}{1-\kappa}} |e^{-2Y_N}|_{\mathbb{L}^\infty} |e^{2Y_N}|_{\mathbb{L}^\infty} |v_0|_{\mathbb{L}_x^2}^2.$$

Let $C_{2,N} = c_2 \tilde{C}_{2,N}^{\frac{1}{1-\kappa}} |e^{2Y_N}|_{\mathbb{L}^\infty} |e^{2Y_N}|_{\mathbb{L}^\infty}$, using Lemma 3.10 and Corollary 3.9 for the terms involving exponentials of Y_N and Sobolev embeddings, Corollary 3.18 and Lemma 3.3 for the renormalized term, it follows $C_{2,N}$ verifies Property (*) and

$$|e^{-2Y_N}|_{\mathbb{L}^\infty} \left| \int_{\mathbb{R}^2} |v^N(t)|^2 : |\nabla Y_N|^2 : e^{2Y_N} dx \right| \leq \frac{1}{8} |v^N(t)|_\Sigma^2 + C_{2,N} |v_0|_{\mathbb{L}_x^2}^2. \quad (23)$$

Putting equations (22) and (23) back in equation (21), we obtain

$$|v^N(t)|_\Sigma^2 \leq 8 |e^{-2Y_N}|_{\mathbb{L}^\infty} \tilde{\mathcal{E}}_N(v_0) + 8 (C_{1,N} + C_{2,N}) |v_0|_{\mathbb{L}_x^2}^2.$$

As the right hand side is independent of t , for all $N \in \mathbb{N}$, v^N is global and we have

$$|v^N|_{\mathbb{L}_t^\infty \Sigma}^2 \leq 8 |e^{-2Y_N}|_{\mathbb{L}^\infty} \tilde{\mathcal{E}}_N(v_0) + 8 (C_{1,N} + C_{2,N}) |v_0|_{\mathbb{L}_x^2}^2. \quad (24)$$

Now, recall the definition of the transform energy,

$$\tilde{\mathcal{E}}_N(v_0) = \frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla v_0|^2 + |xv_0|^2 (1 - Y_N) - : |\nabla Y_N|^2 : |v_0|^2 \right) e^{2Y_N} dx \quad (25)$$

so that using Lemmas 2.7 and 2.8, Sobolev embeddings and interpolation, for some arbitrary $\kappa \in (0, 1)$, we obtain

$$2\tilde{\mathcal{E}}_N(v_0) \leq \left(1 + |Y_N|_{\mathbb{L}^\infty} + : |\nabla Y_N|^2 : |_{\mathcal{W}_x^{-\kappa,\infty}} \right) |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa,\infty}} |v_0|_\Sigma^2. \quad (26)$$

Define $C_N = 8(C_{1,N} + C_{2,N}) + 4 |e^{-2Y_N}|_{\mathbb{L}^\infty} \left(1 + |Y_N|_{\mathbb{L}^\infty} + : |\nabla Y_N|^2 : |_{\mathcal{W}_x^{-\kappa,\infty}} \right) |e^{2Y_N}|_{1+\mathcal{W}_x^{\kappa,\infty}}$, then as previously, C_N verifies Property (*) and using equations (24) and (26), we obtain

$$|v^N|_{\mathbb{L}_t^\infty \Sigma}^2 \leq C_N |v_0|_\Sigma^2.$$

Taking the square root and \mathbb{L}_ω^p norm of both side and using Hölder's inequality, we finally obtain

$$\sup_{N \in \mathbb{N}} |v^N|_{\mathbb{L}_\omega^p \mathbb{L}_t^\infty \Sigma} \leq C |v_0|_{\mathbb{L}_\omega^0 \Sigma}$$

using that all moments of C_N are bounded uniformly in N by Property (*). \square

Proof. of Proposition 3.26.

As in the proof of Proposition 3.24, as $\lambda \leq 0$, for every $N \in \mathbb{N}$, v^N is global and equation (24) holds. It remains to bound the term $\tilde{\mathcal{E}}_N(v_0)$. Now recall that

$$\tilde{\mathcal{E}}_N(v_0) = \tilde{\mathcal{E}}_N^{lin}(v_0) - \frac{\lambda}{4} \int_{\mathbb{R}^2} |v|^4 e^{4Y_N} dx$$

where $\tilde{\mathcal{E}}_N^{lin}(v_0)$ is defined in equation (25) and verifies equation (26). Using the same notation as in the proof of Proposition 3.24, we obtain

$$|v^N|_{\mathbb{L}_x^\infty \Sigma}^2 \leq C_N |v_0|_\Sigma^2 - 2\lambda |e^{-2Y_N}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |v_0|^4 e^{4Y_N} dx. \quad (27)$$

Using Lemma 3.25, we have

$$|v^N|_{\mathbb{L}_x^\infty \Sigma}^2 \leq C_N |v_0|_\Sigma^2 - \lambda |e^{-2Y_N}|_{\mathbb{L}_x^\infty} |e^{4Y_N}|_{\mathbb{L}_x^\infty} |v_0|_\Sigma^4. \quad (28)$$

As C_N verifies Property (*), the claim follows from equation (28) and Lemma 3.10. \square

Proof. of Proposition 3.27.

In the case where $\lambda > 0$, equation (21) does not hold, but instead, we have

$$\begin{aligned} \frac{1}{2} |v^N(t)|_\Sigma^2 &\leq |e^{-2Y_N}|_{\mathbb{L}_x^\infty} \tilde{\mathcal{E}}_N(v_0) + |e^{-2Y_N}|_{\mathbb{L}_x^\infty} |e^{2Y_N}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |xv^N(t)|^2 |Y_N| dx \\ &+ \left| |e^{-2Y_N}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |v^N(t)|^2 : |\nabla Y_N|^2 : e^{2Y_N} dx \right| + \frac{\lambda |e^{-2Y_N}|_{\mathbb{L}_x^\infty}}{4} \int_{\mathbb{R}^2} |v^N(t)|^4 e^{4Y_N} dx. \end{aligned} \quad (29)$$

Using Lemma 3.25 and the mass conservation, we obtain

$$\frac{\lambda |e^{-2Y_N}|_{\mathbb{L}_x^\infty}}{4} \int_{\mathbb{R}^2} |v^N(t)|^4 e^{4Y_N} dx \leq \frac{\lambda |e^{-2Y_N}|_{\mathbb{L}_x^\infty}^2 |e^{4Y_N}|_{\mathbb{L}_x^\infty}}{8} \tilde{M}_N(v_0) |v^N(t)|_\Sigma^2. \quad (30)$$

For $L > 0$, let $\bar{\Omega}_L = \left\{ \lambda L^2 |e^{-2Y}|_{\mathbb{L}_x^\infty}^2 |e^{2Y}|_{\mathbb{L}_x^\infty} |e^{4Y}|_{\mathbb{L}_x^\infty} < 4 \right\}$. Then for any $\omega \in \bar{\Omega}_L$, there exists $N_0(\omega) \in \mathbb{N}$ and $\varepsilon \in (0, \frac{1}{2})$ such that for any $v_0 \in \Sigma$ verifying $|v_0|_{\mathbb{L}_x^2} \leq L$, it holds

$$\forall N \geq N_0(\omega), \quad \frac{\lambda |e^{-2Y_N}|_{\mathbb{L}_x^\infty}}{4} \int_{\mathbb{R}^2} |v^N(t)|^4 e^{4Y_N} dx \leq \frac{\lambda |e^{-2Y_N}|_{\mathbb{L}_x^\infty}^2 |e^{2Y_N}|_{\mathbb{L}_x^\infty} |e^{4Y_N}|_{\mathbb{L}_x^\infty}}{8} |v_0|_{\mathbb{L}_x^2}^2 \leq \frac{1}{2} - \varepsilon.$$

Then, equation (29) gives on $\bar{\Omega}_L$,

$$\begin{aligned} \forall N \geq N_0, \quad \varepsilon |v^N(t)|_\Sigma^2 &\leq |e^{-2Y_N}|_{\mathbb{L}_x^\infty} \tilde{\mathcal{E}}_N(v_0) + |e^{-2Y_N}|_{\mathbb{L}_x^\infty} |e^{2Y_N}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |xv^N(t)|^2 |Y_N| dx \\ &+ \left| |e^{-2Y_N}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |v^N(t)|^2 : |\nabla Y_N|^2 : e^{2Y_N} dx \right|. \end{aligned} \quad (31)$$

A slight modification of the proof of Proposition 3.24, shows there exists $\tilde{C}_{1,N}$ and $\tilde{C}_{2,N}$ verifying Property (*) and

$$|e^{-2Y_N}|_{\mathbb{L}_x^\infty} |e^{2Y_N}|_{\mathbb{L}_x^\infty} \int_{\mathbb{R}^2} |xv^N(t)|^2 |Y_N| dx \leq \frac{\varepsilon}{4} |v^N(t)|_\Sigma^2 + \tilde{C}_{1,N} |v_0|_{\mathbb{L}_x^2}^2, \quad (32)$$

$$|e^{-2Y_N}|_{\mathbb{L}_x^\infty} \left| \int_{\mathbb{R}^2} |v^N(t)|^2 : |\nabla Y_N|^2 : e^{2Y_N} dx \right| \leq \frac{\varepsilon}{4} |v^N(t)|_\Sigma^2 + \tilde{C}_{2,N} |v_0|_{\mathbb{L}_x^2}^2. \quad (33)$$

Moreover, in that case

$$\tilde{\mathcal{E}}_N(v_0) = \tilde{\mathcal{E}}_N^{lin}(v_0) - \frac{\lambda}{4} \int_{\mathbb{R}^2} |v|^4 e^{4Y_N} dx \leq \tilde{\mathcal{E}}_N^{lin}(v_0)$$

where $\tilde{\mathcal{E}}_N^{lin}(v_0)$ is defined in equation (25) and verifies equation (26). Define the random variable $\tilde{C}_N = \frac{2}{\varepsilon} \left(\left(1 + |Y_N|_{\mathbb{L}^\infty} + |\cdot|_{\mathbb{W}_x^{-\kappa, \infty}} : |\nabla Y_N|^2 : |\mathbb{W}_x^{-\kappa, \infty}\right) |e^{2Y_N}|_{1+\mathbb{W}_x^{\kappa, \infty}} + \tilde{C}_{1,N} + \tilde{C}_{2,N} \right)$ for some arbitrary $\kappa \in (0, 1)$, it verifies Property (*) as previously. Then, putting equations (26), (32) and (33) in equation (29), we obtain for every $\omega \in \overline{\Omega}_L$ and every $N \geq N_0(\omega)$,

$$\forall t \in \mathbb{R}, |v^N(\omega, t)|_\Sigma^2 \leq \tilde{C}_N(\omega) |v_0|_\Sigma^2.$$

The claim follows as usual. \square

C Proofs of Proposition 5.3 and 5.4

Lemma C.1. *Let $a < 0 < b$, $\lambda \leq 0$, $\sigma \in (\frac{3}{2}, 2)$. For any $v_0 \in \mathcal{W}^{2,2}$, there exists a positive random variable $C(v_0)$ almost surely finite such that for any $\kappa \in (0, 1]$ there exists a positive random variable $C'(\kappa, v_0)$ almost surely finite, such that*

$$\forall N \in \mathbb{N}, |v^N|_{\mathbb{L}_t^\infty \mathbb{W}_x^{\sigma, 2}} \leq C'(\kappa, v_0) \lambda_N^{C(v_0)\kappa}.$$

Moreover, C and $C'(\kappa, \cdot)$ are nondecreasing functions of $|v_0|_{\mathbb{W}_x^{2,2}}$.

Proof. of Lemma C.1.

In what follows, we always assume conclusions of Proposition 3.26 to hold. We will show a bound for $|v^N|_{\mathbb{W}_x^{\sigma, 2}}$ uniform on $t \in [0, b]$, then reversing time, we obtain a bound of the same type uniformly in $t \in [a, 0]$. Let $v_0 \in \mathcal{W}^{2,2}$. Set $\mathfrak{C} = C(1 + |v_0|_\Sigma)^2 \geq \sup_{N \in \mathbb{N}} |v^N|_{\mathbb{L}_t^\infty \Sigma}$ the upper bound given by Proposition 3.26 and let $N \in \mathbb{N}$.

Define $w^N = \partial_t v^N$, then equation (10) gives

$$\begin{aligned} |v^N|_{\mathbb{W}_x^{\sigma, 2}} &\leq |w^N|_{\mathbb{L}_x^2} + 2|\nabla v^N \cdot \nabla Y_N|_{\mathbb{W}_x^{\sigma-2, 2}} + |xv^N \cdot xY_N|_{\mathbb{W}_x^{\sigma-2, 2}} \\ &\quad + |v^N : |\nabla Y_N|^2 : |_{\mathbb{W}_x^{\sigma-2, 2}} + |\lambda| |v^N|^2 v^N e^{2Y_N}|_{\mathbb{L}_x^2}. \end{aligned}$$

Theorem 3.23 and Proposition 3.26 imply $w^N \in \mathcal{C}(\mathbb{R}, \mathbb{L}^2(\mathbb{R}^2))$, so we can define $W_N = \tilde{M}_N(w^N)$. Then, by Young's inequality, we have

$$|w^N|_{\mathbb{L}_x^2} \leq \frac{W_N}{2} + \frac{|e^{-2Y_N}|_{\mathbb{L}_x^\infty}}{2} \leq \frac{W_N}{2} + \frac{\sup_{N \in \mathbb{N}} |e^{-2Y_N}|_{\mathbb{L}_x^\infty}}{2}.$$

For the nonlinear term, by Sobolev embeddings, there exists a constant $c_0 > 0$ such that

$$||v^N|^2 v^N e^{2Y_N}|_{\mathbb{L}_x^2} \leq c_0 |e^{2Y_N}|_{\mathbb{L}_x^\infty} |v^N|_\Sigma^3 \leq c_0 \mathfrak{C}^3 \sup_{N \in \mathbb{N}} |e^{2Y_N}|_{\mathbb{L}_x^\infty}. \quad (34)$$

Using Lemma 2.8 and Corollary 2.5, there exists a constant $c_1 > 0$ depending only of σ such that:

$$\begin{aligned} |\nabla v^N \cdot \nabla Y_N|_{\mathbb{W}_x^{\sigma-2, 2}} &\leq c_1 |v^N|_{\mathbb{W}_x^{3-\sigma, 2}} |\nabla Y_N|_{\mathbb{W}_x^{\sigma-2, \infty}}, \\ |xv^N \cdot xY_N|_{\mathbb{W}_x^{\sigma-2, 2}} &\leq c_1 |v^N|_{\mathbb{W}_x^{3-\sigma, 2}} |xY_N|_{\mathbb{W}_x^{\sigma-2, \infty}} \end{aligned}$$

and

$$|v^N : |\nabla Y_N|^2 : |_{\mathcal{W}_x^{\sigma-2,2}} \leq c_1 |v^N|_{\mathcal{W}_x^{2-\sigma,2}} : |\nabla Y_N|^2 : |_{\mathcal{W}_x^{\sigma-2,\infty}}.$$

Let $\theta = \frac{2-\sigma}{1-\sigma}$. As $\sigma > \frac{3}{2}$ we have $0 < 2 - \sigma < 3 - \sigma < \sigma$, $\theta \in (0, 1)$ and $3 - \sigma = \sigma\theta + (1 - \theta)$. By interpolation, there exists $c_2 > 0$ such that

$$|\nabla v^N \cdot \nabla Y_N|_{\mathcal{W}_x^{\sigma-2,2}} \leq c_2 \sup_{N \in \mathbb{N}} |\nabla Y_N|_{\mathcal{W}_x^{\sigma-2,\infty}} \mathfrak{E}^{1-\theta} |v^N|_{\mathcal{W}_x^{\sigma,2}}^\theta, \quad (35)$$

$$|xv^N \cdot xY_N|_{\mathcal{W}_x^{\sigma-2,2}} \leq c_2 \sup_{N \in \mathbb{N}} |xY_N|_{\mathcal{W}_x^{\sigma-2,\infty}} \mathfrak{E}^{1-\theta} |v^N|_{\mathcal{W}_x^{\sigma,2}}^\theta, \quad (36)$$

$$|v^N : |\nabla Y_N|^2 : |_{\mathcal{W}_x^{\sigma-2,2}} \leq c_2 \sup_{N \in \mathbb{N}} | : |\nabla Y_N|^2 : |_{\mathcal{W}_x^{\sigma-2,\infty}} \mathfrak{E}^{1-\theta} |v^N|_{\mathcal{W}_x^{\sigma,2}}^\theta. \quad (37)$$

By Corollaries 3.5 and 3.18 and Lemma 3.10, almost surely

$$\sup_{N \in \mathbb{N}} \max \left(|e^{-2Y_N}|_{\mathbb{L}_x^\infty}, |e^{2Y_N}|_{\mathbb{L}_x^\infty}, |\nabla Y_N|_{\mathcal{W}_x^{\sigma-2,\infty}}, |xY_N|_{\mathcal{W}_x^{\sigma-2,\infty}}, | : |\nabla Y_N|^2 : |_{\mathcal{W}_x^{\sigma-2,\infty}} \right) < +\infty. \quad (38)$$

By Young's inequality in equations (35) to (37), and by equation (34), there exists a positive random variable C_0 , almost surely finite, such that

$$|v^N|_{\mathcal{W}_x^{\sigma,2}} \leq \frac{W_N}{2} + \frac{|v^N|_{\mathcal{W}_x^{\sigma,2}}}{2} + \frac{C_0}{2}.$$

Hence, we have

$$|v^N|_{\mathcal{W}_x^{\sigma,2}} \leq W_N + C_0.$$

Using that w^N verify the equation:

$$\begin{cases} i\partial_t w^N + Hw^N + 2\nabla Y_N \cdot \nabla w^N + xY_N \cdot xw^N + : |\nabla Y_N|^2 : w^N \\ \quad + \lambda \left(|v^N|^2 w^N + 2 \operatorname{Re} \left(v^N \overline{w^N} \right) v^N \right) e^{2Y_N} = 0 \\ w^N(0) = i \left(Hv_0 + 2\nabla v_0 \cdot \nabla Y_N + xv_0 \cdot xY_N + : |\nabla Y_N|^2 : v_0 + \lambda |v_0|^2 v_0 e^{2Y_N} \right) \end{cases}$$

we obtain

$$\frac{1}{2} \frac{d}{dt} W_N \leq 2|\lambda| \int_{\mathbb{R}^2} \operatorname{Re} \left(v^N \overline{w^N} \right) \operatorname{Im} \left(v^N \overline{w^N} \right) e^{4Y_N} dx \leq 2|\lambda| |v^N|_{\mathbb{L}_x^\infty}^2 |e^{2Y_N}|_{\mathbb{L}_x^\infty} W_N.$$

Using Proposition 5.2, there exists a positive random variable C_1 , almost surely finite, such that

$$\frac{d}{dt} W_N \leq C_1 \left(1 + \ln \left(1 + |v^N|_{\mathcal{W}_x^{\sigma,2}} \right) \right) W_N \leq C_1 (1 + \ln(1 + W_N + C_0)) W_N.$$

Define $G_N = 1 + \ln(1 + W_N + C_0)$, then it verifies almost surely $G'_N \leq C_1 G_N$. Hence, Gronwall's lemma gives

$$\forall t \in [0, b], G_N(t) \leq G_N(0) e^{tC_1} \leq (1 + \ln(1 + C_0 + W_N(0))) e^{bC_1}$$

which implies

$$|v^N|_{\mathbb{L}_x^\infty \mathcal{W}_x^{\sigma,2}} \leq e^{e^{bC_1}} (1 + C_0 + W_N(0))^{e^{bC_1}}.$$

Now, let $\kappa \in (0, 1)$. Remark that

$$W_N(0) \leq |e^{2Y_N}|_{\mathbb{L}_x^\infty} |w_0^N|_{\mathbb{L}_x^2}^2 \leq \sup_{n \in \mathbb{N}} |e^{2Y_n}|_{\mathbb{L}_x^\infty} \left(|v_0|_{\mathcal{W}_x^{2,2}} + |\lambda| |v_0|_{\mathbb{L}_x^6}^3 \sup_{n \in \mathbb{N}} |e^{2Y_n}|_{\mathbb{L}_x^\infty} \right. \\ \left. + \sup_{n \in \mathbb{N}} [\lambda_n^{-\kappa} \max(|\nabla v_0 \cdot \nabla Y_n|_{\mathbb{L}_x^2}, |xv_0 \cdot xY_n|_{\mathbb{L}_x^2}, |v_0 : |\nabla Y_n|^2 : \mathbb{L}_x^2)] \lambda_N^\kappa \right)^2.$$

By Corollaries 3.7 and 3.22, almost surely

$$\sup_{n \in \mathbb{N}} [\lambda_n^{-\kappa} \max(|\nabla v_0 \cdot \nabla Y_n|_{\mathbb{L}_x^2}, |xv_0 \cdot xY_n|_{\mathbb{L}_x^2}, |v_0 : |\nabla Y_n|^2 : \mathbb{L}_x^2)] < +\infty.$$

Moreover, there exists a positive random variable $C'_0(\kappa)$, depending on v_0 through $|v_0|_{\mathcal{W}_x^{2,2}}$ non-decreasingly, almost surely finite, such that $W_N(0) \leq C'_0(\kappa) (1 + \lambda_N^{2\kappa})$, and as $\lambda_N \geq 1$, we obtain

$$|v^N|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma,2}} \leq (2e(1 + C_0 + C'_0(\kappa)))^{e^{bC_1}} \lambda_N^{2\kappa e^{bC_1}}.$$

Setting $C(v_0) = 2e^{bC_1}$ and $C'(\kappa, v_0) = (2e(1 + C_0 + C'_0(\kappa)))^{e^{bC_1}}$, we obtain the claim. \square

Proof. of Proposition 5.3.

For $n \in \mathbb{N}$, define $\sigma_n = 2 - 2^{-(n+2)}$. Hence, for any $a < 0 < b$ and any $\sigma \in (1, 2)$ there exists $n \in \mathbb{N}$ such that $\mathbb{L}^\infty([-n, n], \mathcal{W}^{\sigma_n, 2})$ is continuously embedded in $\mathbb{L}^\infty([a, b], \mathcal{W}^{\sigma, 2})$. Thus it is sufficient to prove the claim for fixed $a < 0 < b$ and $\sigma \in (\frac{3}{2}, 2)$. Moreover, as $\mathcal{W}^{2,2} = \bigcup_{n \in \mathbb{N}} B_{\mathcal{W}^{2,2}}(0, n)$, it is sufficient to prove the claim uniformly in $v_0 \in B_{\mathcal{W}^{2,2}}(0, L)$ a $\mathcal{W}^{2,2}$ -ball of fixed radius $L \in \mathbb{N}$.

Let $L \in \mathbb{N}$. For $j \in \mathbb{N}$, let $\kappa_j = 2^{-j}$. As $\lambda_N > 1$ for all N , it is sufficient to show conclusion only for $(\kappa_j)_{j \in \mathbb{N}^*}$. Recall the random variable $C(v_0)$ and $C'(\kappa, v_0)$ defined in Lemma C.1. Denote by C_L and $C'_L(\kappa)$ the value of $C(v'_0)$ and $C'(\kappa, v'_0)$ respectively for a fixed $v'_0 \in \mathcal{W}_x^{2,2}$ of $\mathcal{W}_x^{2,2}$ -norm equal to L , recall that C and C' are nondecreasing functions of $|v_0|_{\mathcal{W}_x^{2,2}}$. For $k \in \mathbb{N}$, let $\tilde{\Omega}_k = \{2^k \leq C_L < 2^{k+1}\}$ and let $\tilde{\Omega}_{-1} = \{C_L < 1\}$. By definition of C_L , $\tilde{\Omega} = \bigcup_{k \geq -1} \tilde{\Omega}_k$ is of probability one. Let $\kappa' \in (0, 1)$ and $k \geq -1$, in view of Lemma C.1, on $\tilde{\Omega}_k$, it holds

$$\forall N \in \mathbb{N}, \forall v_0 \in B_{\mathcal{W}^{2,2}}(0, L), |v^N|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma,2}} \leq C'_L(\kappa') \lambda_N^{C_L \kappa'} \leq C'_L(\kappa') \lambda_N^{2^{k+1} \kappa'}.$$

Let $\Omega'_\kappa = \{C'_L(\kappa) < +\infty\}$, from Lemma C.1 it is of probability one for any $\kappa \in (0, 1]$. Thus $\Omega' = \bigcap_{n \in \mathbb{N}} \Omega'_{2^{-n}}$ is of probability one too. On $\Omega' \cap \tilde{\Omega}_k$, it holds

$$\forall N \in \mathbb{N}, \forall v_0 \in B_{\mathcal{W}^{2,2}}(0, L), |v^N|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma,2}} \leq C'_L(2^{-(k+j+1)}) \lambda_N^{\kappa_j}.$$

Hence, on $\tilde{\Omega} \cap \Omega'$, which is of probability one, it holds

$$\forall j \in \mathbb{N}, \forall N \in \mathbb{N}, \forall v_0 \in B_{\mathcal{W}^{2,2}}(0, L), |v^N|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma,2}} \leq C_L(\kappa_j) \lambda_N^{\kappa_j},$$

for a certain positive random variable $C_L(\kappa_j)$ finite on $\tilde{\Omega} \cap \Omega'$, which allows to conclude. \square

Remark C.2. If one tries to apply the proof of Lemma C.1 in the case $\sigma = 2$, it would end with a random variable depending of κ in the exponent, thus the proof of Proposition 5.3 would not hold.

Lemma C.3. *Let $a < 0 < b$ and $\lambda \leq 0$. For any $v_0 \in \mathcal{W}^{2,2}$, there exists a positive random variable $C(v_0)$ almost surely finite such that for any $\kappa \in (0, 1]$ and any $\delta \in (0, \frac{1}{3})$ there exists a positive random variable $C''(\kappa, \delta, v_0)$ almost surely finite, such that*

$$\forall N > M \geq 0, |v^N - v^M|_{\mathbb{L}_t^\infty \mathbb{L}_x^2} \leq C''(\kappa, \delta, v_0) \lambda_N^{C(v_0)\kappa} \lambda_M^{-\delta}.$$

Moreover, C and C'' are nondecreasing functions of $|v_0|_{\mathcal{W}_x^{2,2}}$.

Proof. of Lemma C.3.

Assume in what follows that both conclusions of Propositions 3.26 and 5.3 hold. Let $\delta \in (0, \frac{1}{3})$, $\delta' \in (2\delta, \frac{2}{3})$, $\kappa' \in (\frac{3\delta'}{2}, 1)$ and $\sigma \in (1 + \kappa', 2)$. Define $\theta = \frac{1+\kappa'}{\sigma} \in (0, 1)$. Let $N > M \geq 2$. As in the proof of Lemma C.1, let $v_0 \in \mathcal{W}^{2,2}$ and set $\mathfrak{C} = C(1 + |v_0|_\Sigma)^2 \geq \sup_{n \in \mathbb{N}} |v^n|_{\mathbb{L}_t^\infty \Sigma}$ the upper bound given by Proposition 3.26. Define $R = v^N - v^M$, then it verifies

$$\begin{cases} -i\partial_t R = & HR + 2\nabla Y_N \cdot \nabla R + 2\nabla(Y_N - Y_M) \cdot \nabla v^M + xY_N \cdot xR + :|\nabla Y_N|^2 : R \\ & + \lambda |v^N|^2 R e^{2Y_N} + x(Y_N - Y_M) \cdot xv^M + (:|\nabla Y_N|^2 : - :|\nabla Y_M|^2 :) v^M \\ & + \lambda (|v^N|^2 - |v^M|^2) v^M e^{2Y_N} + \lambda |v^M|^2 v^M (e^{2Y_N} - e^{2Y_M}) \\ R(0) = 0. \end{cases} \quad (39)$$

Now, set $W_N = \int_{\mathbb{R}^2} |R|^2 e^{2Y_N} dx$. Taking the time derivative, we obtain

$$\begin{aligned} \frac{1}{2} W'_N &\leq 2|\nabla(Y_N - Y_M)|_{\mathcal{W}^{-\kappa', \infty}} |\overline{R} \nabla v^M e^{2Y_N}|_{\mathcal{W}^{\kappa', 1}} + |x(Y_N - Y_M)|_{\mathcal{W}^{-\kappa', \infty}} |x \overline{R} v^M e^{2Y_N}|_{\mathcal{W}^{\kappa', 1}} \\ &+ |:|\nabla Y_N|^2 : - :|\nabla Y_M|^2 :|_{\mathcal{W}^{-\kappa', \infty}} |\overline{R} v^M e^{2Y_N}|_{\mathcal{W}^{\kappa', 1}} \\ &+ \left| \lambda \int_{\mathbb{R}^2} (|v^N|^2 - |v^M|^2) v^M \overline{R} e^{4Y_N} dx \right| + \left| \lambda \int_{\mathbb{R}^2} |v^M|^2 v^M (e^{2Y_N} - e^{2Y_M}) \overline{R} e^{2Y_N} dx \right|. \end{aligned} \quad (40)$$

First, by Cauchy-Schwarz inequality and Sobolev embeddings, there exists $c_0 > 0$ such that

$$\left| \lambda \int_{\mathbb{R}^2} |v^M|^2 v^M (e^{2Y_N} - e^{2Y_M}) \overline{R} e^{2Y_N} dx \right| \leq c_0 \mathfrak{C}^4 \sup_{n \in \mathbb{N}} |e^{2Y_n}|_{\mathbb{L}_x^\infty} \sup_{n \in \mathbb{N}} \lambda_n^{\delta'} |e^{2Y_n} - e^{2Y}|_{\mathbb{L}_x^\infty} \lambda_M^{-\delta'}. \quad (41)$$

Using Lemma 2.7 and Corollary 2.5, by interpolation, there exists $c_1 > 0$ such that

$$\max \left(|\overline{R} \nabla v^M e^{2Y_N}|_{\mathcal{W}_x^{\kappa', 1}}, |x \overline{R} v^M e^{2Y_N}|_{\mathcal{W}_x^{\kappa', 1}} \right) \leq c_1 \sup_{n \in \mathbb{N}} |e^{2Y_n}|_{1+\mathcal{W}_x^{\kappa', \infty}} \mathfrak{C}^{2-\theta} |v^M|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}} \quad (42)$$

$$|\overline{R} v^M e^{2Y_N}|_{\mathcal{W}_x^{\kappa', 1}} \leq c_1 \sup_{n \in \mathbb{N}} |e^{2Y_n}|_{1+\mathcal{W}_x^{\kappa', \infty}} \mathfrak{C}^2. \quad (43)$$

Moreover, we have

$$|\nabla(Y_N - Y_M)|_{\mathcal{W}_x^{-\kappa', \infty}} \leq 2 \sup_{n \in \mathbb{N}} \lambda_n^{\delta'} |\nabla(Y_n - Y)|_{\mathcal{W}_x^{-\kappa', \infty}} \lambda_M^{-\delta'} \quad (44)$$

$$|x(Y_N - Y_M)|_{\mathcal{W}_x^{-\kappa', \infty}} \leq 2 \sup_{n \in \mathbb{N}} \lambda_n^{\delta'} |x(Y_n - Y)|_{\mathcal{W}_x^{-\kappa', \infty}} \lambda_M^{-\delta'} \quad (45)$$

$$|:|\nabla Y_N|^2 : - :|\nabla Y_M|^2 :|_{\mathcal{W}_x^{-\kappa', \infty}} \leq 2 \sup_{n \in \mathbb{N}} \lambda_n^{\delta'} |:|\nabla Y_n|^2 : - :|\nabla Y|^2 :|_{\mathcal{W}_x^{-\kappa', \infty}} \lambda_M^{-\delta'} \quad (46)$$

$$\left| \int_{\mathbb{R}^2} (|v^N|^2 - |v^M|^2) v^M \overline{R} e^{4Y_N} dx \right| \leq \sup_{n \in \mathbb{N}} |e^{2Y_n}|_{\mathbb{L}_x^\infty} |v^M|_{\mathbb{L}_x^\infty} \left(|v^M|_{\mathbb{L}_x^\infty} + |v^N|_{\mathbb{L}_x^\infty} \right) W_N. \quad (47)$$

From Corollaries 3.5 and 3.18 and Lemma 3.10, it follows almost surely

$$\sup_{n \in \mathbb{N}} \lambda_n^{\delta'} \max \left(|\nabla(Y_n - Y)|_{\mathcal{W}_x^{-\kappa', \infty}}, |x(Y_n - Y)|_{\mathcal{W}_x^{-\kappa', \infty}}, |:\nabla Y_n|^2 : - :|\nabla Y|^2 : |_{\mathcal{W}_x^{-\kappa', \infty}} \right) < +\infty$$

and

$$\sup_{n \in \mathbb{N}} \max \left(\lambda_n^{\delta'} |e^{2Y_n} - e^{2Y}|_{\mathbb{L}_x^\infty}, |e^{2Y_n}|_{1+\mathcal{W}_x^{\kappa', \infty}} \right) < +\infty.$$

Thanks to Propositions 3.26 and 5.2, there exists a deterministic $c_2 > 0$ such that, almost surely, it holds

$$\forall n \in \mathbb{N}, |v^n|_{\mathbb{L}_x^\infty} \leq c_2 \sqrt{1 + \mathfrak{C} \left(1 + \log \left(1 + |v^n|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}} \right) \right)}.$$

Using Propositions 3.26 and 5.3 and equations (40) to (47), there exists a positive random variable $C_0 = C_0(v_0)$, almost surely finite, which is independent of N , M and t , such that, almost surely, for any $\kappa \in (0, 1)$,

$$W'_N \leq C_0 \left(1 + |v^M|_{\mathbb{L}_t^\infty \mathcal{W}_x^{\sigma, 2}}^\theta \right) \lambda_M^{-\delta'} + C_1 (1 + \ln(1 + C'_\kappa \lambda_N^\kappa)) W_N,$$

where

$$C_1 = 2c_2^2 \sup_{n \in \mathbb{N}} |e^{2Y_n}|_{\mathbb{L}_x^\infty} (1 + \mathfrak{C})$$

and C'_κ , given by Proposition 5.3, are almost surely finite random variables. Moreover, C_0 , C_1 and C'_κ are nondecreasing functions of $|v_0|_{\mathcal{W}_x^{2,2}}$. We now apply Proposition 5.3 with an arbitrary $\kappa'' \in \left(0, \min \left(1, \frac{\delta' - 2\delta}{\theta} \right) \right)$ to obtain

$$W'_N \leq \tilde{C}_\delta \lambda_M^{-2\delta} + C_1 (1 + \ln(1 + C'_\kappa \lambda_N^\kappa)) W_N,$$

where \tilde{C}_δ is a positive random variable, almost surely finite, which is independent of N , M and t , and is a nondecreasing function of $|v_0|_{\mathcal{W}_x^{2,2}}$. Then by Gronwall's lemma and Lemma 3.10, we conclude that there exists a positive random variable $C''(\kappa, \delta, v_0)$ almost surely finite, independent of N and M such that

$$|v^N - v^M|_{\mathbb{L}_t^\infty \mathbb{L}_x^2}^2 \leq C''(\kappa, \delta, v_0) \lambda_N^{C_1 \kappa} \lambda_M^{-\delta}.$$

Moreover, $C'''(\kappa, \delta, v_0)$ is a nondecreasing function of $|v_0|_{\mathcal{W}_x^{2,2}}$. □

Proof. of Proposition 5.4.

The proof is analogous to the proof of Proposition 5.3. □

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