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A model for stochastic dependence implied by failures among deteriorating components

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Abstract

A system of n components is here considered, with component deterioration modeled by non decreasing time-scaled Lévy processes. When a component fails, a sudden change in the time-scaling functions of the surviving components is induced, which makes the components stochastically dependent. We compute the reliability function of coherent systems under this new dependence model. We next study the distribution of the ordered failure times, and establish some positive dependence properties. We also provide stochastic comparison results in the usual multivariate stochastic order between failure times of two dependence models with different parameters. Finally, some numerical experiments illustrate the theoretical results.

Keywords: reliability; Lévy process; inverse Gaussian process; functional dependence; stochastic comparison; aging property

1 Introduction

Safety and dependability is a crucial issue in many industries, which has lead to the development of a huge literature devoted to the so-called reliability

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theory. Traditionally, only lifetime data were available, from where appropriate lifetime models were inferred. See, e.g., [2] for a pioneer reference work on the subject. Based on the on-going development of on-line monitoring which allows the effective measurement of a system deterioration, numerous papers nowadays model the degradation in itself, which allows for a better assessment of the system health and a better prediction of its future failure time. The deterioration level of a system is usually considered to be accumulating over time and it is often modeled by non decreasing (or with increasing trend) continuous-time stochastic processes. Most common models are Lévy processes such as gamma processes [1, 5, 19], Wiener processes with trend [6, 22, 26] and inverse Gaussian processes [21, 23] (see also [8, 20] for more references).

In this paper, we consider a system composed of n components, with component deterioration modeled by non decreasing time-scaled Lévy processes and component failures triggered by the reaching of a given known threshold. In the oldest literature, components were most of the time assumed to be independent, which highly facilitated the modeling and assessment of reliability indicators, using for instance structure functions (see, e.g. [2]). Since then, many different kinds of dependence have been considered, see, e.g. [10], where the authors review condition-based maintenance policies for systems with components subject to economic, structural, stochastic and resource dependence. Following their vocabulary, we here envision some "stochastic dependence" between components, which "means that the deterioration or failure processes of components are (partially) dependent" [10]. To be more specific, we here assume that a failure among components has an impact on the future deterioration of the surviving components and we call it stochastic failure dependence. This kind of dependence has been envisioned in different papers such as [4, 11, 12, 13], where the authors model the impact of failures on the lifetimes of the surviving components (including possible cascading failures), see also [10] for many other models and references. However, the papers involving Lévy deteriorating components together with stochastic failure dependence are fewer. One can quote [24], where the authors consider that a component failure "acts as a shock to each surviving component by causing a random magnitude of damage which is additive to the degradation level of the survivor". See also [25] for a similar dependence model. One can also quote [7], where the system health is described by two correlated performance characteristics (or component deterioration levels) modeled by a bivariate Wiener process at a first stage. Once one of the two characteristics has reached its threshold (or once one of the two components has failed), the system operates defectively and enters its second stage, where the surviving characteristic evolves according to a new univariate Wiener process. Within

this setting, the authors compute the system reliability and availability based on different inspection and maintenance policies.

In a similar spirit as [7], we here suggest that each failure among the n components entails an increase in the deterioration rate of the surviving components. As a typical example of such stochastic dependence, one can think about the case where the "multiple components share the total system load. If a component fails, the system keeps operating but the remaining components structurally need to work harder to realize the same output level. The failure of a component thus increases the load on the working components, which will hence deteriorate faster. In practice, this applies for example to a set of pumps that are used to distribute a certain amount of gas." [10]. As another example, one can also think of a parallel-connected power supply system for a server farm, where each server is powered by its own power supply unit (PSU). When one PSU fails, the remaining PSUs need to compensate for the lost power output to meet the power demands of the servers, resulting in faster degradation of those surviving PSUs. One can also think of an electric vehicle powered by a lithium-ion battery pack composed of n identical battery cells. Each cell contributes to storing energy for the vehicle's operation. When a battery cell fails, the overall performance of the pack is disrupted and the operation of the remaining cells must be adjusted, putting additional strain on them and leading to an increased degradation rate. In the same way, the failure of one wind turbine within a wind farm can lead to adjustments in the operation of the remaining turbines, potentially accelerating the overall deterioration of the wind farm's mechanical and electrical components.

In this paper, remembering that the deterioration of a component is modeled by a non decreasing time-scaled Lévy process, we suggest to model the impact of a component failure on the surviving components by some increase in the time-scaling functions of the corresponding Lévy processes. Note that up to our knowledge, such a stochastic failure dependence model with impact on the time-scaling functions of the surviving non decreasing Lévy-deteriorating components has not yet been considered in the literature.

Considering this dependence model, we compute the reliability of a coherent system, as well as the marginal and joint distributions of the component lifetimes and corresponding ordered statistics. These results are used to derive some dependence and stochastic comparison properties, which allow to better understand the impact of the model parameters. Numerical experiments are next given considering inverse Gaussian processes as baseline Lévy processes. These experiments show good agreements between the theoretical results and Monte-Carlo simulations (which require much longer computing times however). These experiments also illustrate the dependence and

stochastic comparison results of the paper and highlight the requirement of several assumptions given for their derivation.

The paper is organized as follows: Section 2 provides a quick summary about time-scaled Lévy processes. The model for dependence is next introduced in Section 3. The reliability function of a coherent system with dependent components is studied in Section 4. The marginal and joint distributions of the component lifetimes are computed in Section 5, as well as the joint distribution of the corresponding ordered statistics. Section 6 is dedicated to dependence and stochastic comparison properties. The numerical experiments are given in Section 7. Section 8 ends the paper with some conclusion and perspectives for future works.

Please note that in all the paper, the terms "increasing" and "decreasing" mean "non-decreasing" and "non-increasing", respectively. Also, a function $\phi : I \subseteq \mathbb{R}^n \rightarrow J \subseteq \mathbb{R}$ is said to be increasing if $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$ whenever $\mathbf{x} \leq \mathbf{y}$ component-wise, for all $\mathbf{x}, \mathbf{y} \in I$.

2 Time-scaled Lévy processes

We here recall some basic facts about time-scaled Lévy processes and start with some definitions.

Definition 1 *A function $\Lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be a time-scaling function if Λ is differentiable and strictly increasing, such that $\Lambda(0) = 0$ and $\lim_{t \rightarrow +\infty} \Lambda(t) = \infty$. Its derivative function is denoted by $\lambda = \Lambda'$.*

Definition 2 *Let $\mathbf{Y} = (Y(t))_{t \geq 0}$ be a càdlàg process, that is a right-continuous process with left-side limits. Then \mathbf{Y} is a Lévy process if:*

- $Y(0) = 0$ almost surely,
- \mathbf{Y} has independent and homogeneous increments: for all $n \in \mathbb{N}^*$, $t_0 = 0 < t_1 < \dots < t_n$, the increments $\Delta Y(t_{i-1}, t_i) = Y(t_i) - Y(t_{i-1})$, $i = 1, \dots, n$ are independent and $\Delta Y(t_{i-1}, t_i)$ is identically distributed as $Y(t_i - t_{i-1})$ for all $i = 1, \dots, n$,
- \mathbf{Y} is stochastically continuous.

Definition 3 *Let $\mathbf{X} = (X(t))_{t \geq 0}$ be a càdlàg process. Then \mathbf{X} is called a time-scaled Lévy process if there exists a time-scaling function Λ and a baseline Lévy process \mathbf{Y} such that*

$$X(t) = Y(\Lambda(t))$$

for all $t \geq 0$. In that case, \mathbf{X} is said to be a time-scaled Lévy process with parameters (Λ, \mathbf{Y}) .

Now, let us consider a component with deterioration level modeled by a time-scaled *non negative* Lévy process $\mathbf{X} = (X(t))_{t \geq 0}$ with parameters (Λ, \mathbf{Y}) as in the previous definition. Recall that both \mathbf{Y} and \mathbf{X} are increasing (and \mathbf{Y} is a subordinator). Assume the component to be failed as soon as $X(t) \geq L$, where L is a given failure threshold. Then, the component failure time is

$$\tau_{\mathbf{X}} = \inf(t \geq 0 : X(t) \geq L)$$

with survival function

$$\bar{F}_{\tau_{\mathbf{X}}}(t) = \mathbb{P}(\tau_{\mathbf{X}} > t) = \mathbb{P}(X(t) \leq L) = F_{X(t)}(L),$$

where $F_{X(t)}$ is the cumulative distribution function (cdf) of $X(t)$. Using similar notations for \mathbf{Y} , one can see that

$$\bar{F}_{\tau_{\mathbf{X}}}(t) = \mathbb{P}(Y(\Lambda(t)) \leq L) = \mathbb{P}(\tau_{\mathbf{Y}} > \Lambda(t)) = \bar{F}_{\tau_{\mathbf{Y}}}(\Lambda(t)). \quad (1)$$

Assuming $\bar{F}_{\tau_{\mathbf{Y}}}$ to be differentiable, then $\bar{F}_{\tau_{\mathbf{X}}}$ also is differentiable with probability density function (pdf)

$$f_{\tau_{\mathbf{X}}}(t) = \lambda(t) f_{\tau_{\mathbf{Y}}}(\Lambda(t)) \quad (2)$$

for all $t \geq 0$.

3 The model for dependence

Let us consider n identical non-repairable components gathered in a system. Between failures among the n components, each component deteriorates according to a time-scaled Lévy process. A component fails as soon as its deterioration level crosses a given failure threshold L . Each failure entails some modification in the future deterioration of the still operating components, as they become more solicited. This modification is modelled by a change in the time-scaling functions for the surviving components. Between failures, the intrinsic deterioration of the surviving components are assumed to be independent.

In order to define the model more specifically, let us introduce independent time-scaled non negative Lévy processes $\mathbf{X}_{i,j}$, $i, j = 1, \dots, n$, with parameters $(\Lambda_{i,j}, \mathbf{Y}_{i,j})$, $i, j = 1, \dots, n$, where:

- index i refers to the i -th component,

- index j refers to the number of already failed components (among the n components),
- the $\Lambda_{i,j}$ are time-scaling functions which only depend on the number of already failed components (and not on the component), that is:

$$\Lambda_{i,j} = \Lambda_j \text{ for all } i, j = 1, \dots, n, \quad (3)$$

- the baseline Lévy processes $\mathbf{Y}_{i,j}, i, j = 1, \dots, n$ are independent and identically distributed as a given non negative Lévy process \mathbf{Y} :

$$\mathbb{P}(Y_{i,j}(t) \leq x) = \mathbb{P}(Y(t) \leq x) \text{ for all } i, j = 1, \dots, n, \text{ all } t \geq 0, \text{ all } x \geq 0. \quad (4)$$

At the beginning, each component deteriorates according to $\mathbf{X}_{i,1}, i \in I_1 = \{1, \dots, n\}$. Then, after the first failure, the surviving components deteriorate according to $\mathbf{X}_{i,2}, i \in I_2$, where I_2 stands for the set of the $n - 1$ surviving components. More generally, after the j -th failure with $1 \leq j \leq n - 1$, the surviving components deteriorate according to $\mathbf{X}_{i,j+1}, i \in I_{j+1}$, where I_{j+1} stands for the set of the $n - j$ surviving components. The failure times of the components are denoted by $\tau_i, i = 1, \dots, n$, and the successive failures happen at time $\tau^{(1)} \leq \tau^{(2)} \leq \dots \leq \tau^{(n)}$, where $(\tau^{(1)}, \dots, \tau^{(n)})$ is the vector of order statistics of (τ_1, \dots, τ_n) . For $j = 1, \dots, n$, the period between $\tau^{(j-1)}$ and $\tau^{(j)}$ is called the j -th phase (where the 0-th failure is considered to happen at time $\tau^{(0)} = 0$). This is illustrated in Figure 1 in the case of four components, with $\tau^{(1)} = \tau_1 < \tau^{(2)} = \tau_3 < \tau^{(3)} = \tau_2 < \tau^{(4)} = \tau_4$.

The overall deterioration level of the i -th component is denoted by $\mathbf{Z}_i = (Z_i(t))_{t \geq 0}, i = 1, \dots, n$ and we set D to be the down state for a failed component. In the specific case where $\tau_1 < \tau_2 < \dots < \tau_n$ (for sake of simplicity) and setting $\tau_0 = 0$, we have:

$$Z_i(t) = \begin{cases} X_{i,1}(t) = \Delta X_{i,1}(\tau_0, t) & \text{if } t < \tau_1, \\ \Delta X_{i,1}(\tau_0, \tau_1) + \Delta X_{i,2}(\tau_1, t) & \text{if } \tau_1 \leq t < \tau_2, \\ \vdots & \vdots \\ \sum_{j=1}^{k-1} \Delta X_{i,j}(\tau_{j-1}, \tau_j) + \Delta X_{i,k}(\tau_{k-1}, t) & \text{if } \tau_{k-1} \leq t < \tau_k, \\ \vdots & \vdots \\ \sum_{j=1}^{i-1} \Delta X_{i,j}(\tau_{j-1}, \tau_j) + \Delta X_{i,i}(\tau_{i-1}, t) & \text{if } \tau_{i-1} \leq t < \tau_i, \\ D & \text{if } \tau_i \leq t \end{cases} \quad (5)$$

for all $i = 1, \dots, n$ and $t \geq 0$, where $\Delta X_{i,j}(u, v) = X_{i,j}(v) - X_{i,j}(u)$ for all $0 \leq u \leq v$ and all $i, j = 1, \dots, n$.

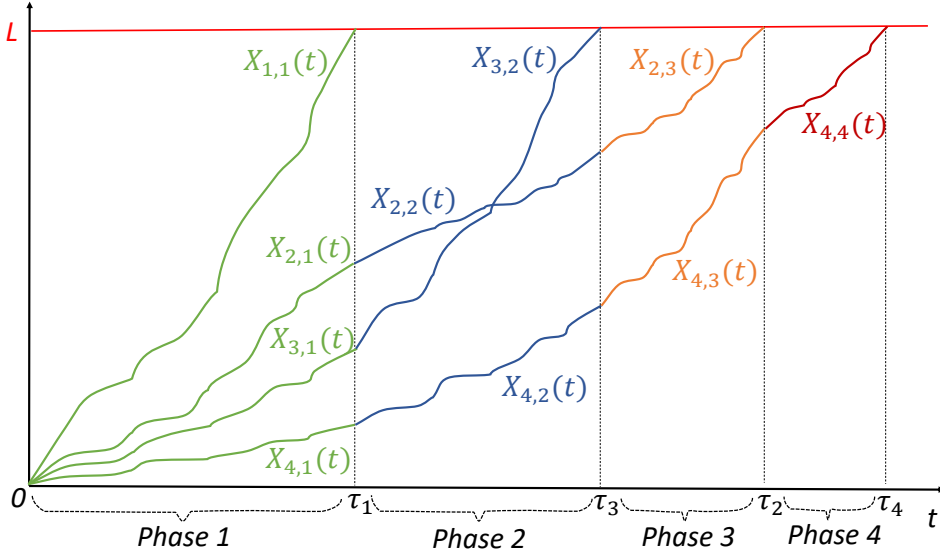


Figure 1: Example of trajectories for the component deterioration levels up to failure, case $n = 4$ and $\tau_1 < \tau_3 < \tau_2 < \tau_4$.

Remark 4 Note that the parameters for this dependence model are the collection of time-scaling functions $\{\Lambda_j, j = 1, \dots, n\}$ and the distribution of the baseline non negative Lévy process \mathbf{Y} .

Technical assumptions 5 In all the paper, we make the following technical assumptions:

- For all $t \geq 0$, the random variable $Y(t)$ admits a pdf with respect to Lebesgue measure,
- The function $F_{Y(t)}(L)$ is differentiable with respect to t .

We end this section with some notations and a few direct consequences of the model definition, that will be used repeatedly in the remaining of the paper.

We set

$$\Delta\Lambda_j(s, t) = \Lambda_j(t) - \Lambda_j(s)$$

for all $0 \leq s \leq t$ and $j = 1, \dots, n$, with $\Delta\Lambda_j(0, t) = \Lambda_j(t)$ when $s = 0$.

Based on (3), (4) and on the homogeneity of \mathbf{Y} , it is easy to check that

$$\begin{aligned} \mathbb{P}(\Delta X_{i,j}(s, t) \leq x) &= \mathbb{P}(Y(\Lambda_j(t)) - Y(\Lambda_j(s)) \leq x) \\ &= \mathbb{P}(Y(\Lambda_j(t) - \Lambda_j(s)) \leq x) \\ &= \mathbb{P}(Y(\Delta\Lambda_j(s, t)) \leq x) \end{aligned}$$

for all $0 \leq s < t$ and all $x \geq 0$.

When x is equal to the failure threshold L , we also have

$$\mathbb{P}(\Delta X_{i,j}(s,t) \leq L) = \mathbb{P}(\tau_{\mathbf{Y}} > \Delta \Lambda_j(s,t)) = \bar{F}_{\tau_{\mathbf{Y}}}(\Delta \Lambda_j(s,t)).$$

Finally, in a similar way as in (1) and (2), we have

$$\bar{F}_{\tau_{\mathbf{X}_{i,j}}}(t) = \bar{F}_{\tau_{\mathbf{Y}}}(\Lambda_j(t)), \quad (6)$$

$$f_{\tau_{\mathbf{X}_{i,j}}}(t) = \lambda_j(t) f_{\tau_{\mathbf{Y}}}(\Lambda_j(t)), \quad (7)$$

for all $t > 0$, $i, j = 1, \dots, n$.

4 Reliability of a coherent system

Considering n identical components with dependence modelled as in the previous section, we now assume that they are gathered in a coherent system and the point is to compute its reliability. To begin with, let us consider a specific coherent system, that is a k -out-of- n system (with $1 \leq k \leq n$). In that case, the system is up as long as at least k components are operating and the system fails at the time of the $n - k + 1$ -th failure among components. The lifetime of the system hence is the order statistics $\tau^{(n-k+1)}$ and the reliability function is

$$R_{k,n}(t) = \mathbb{P}(\tau^{(n-k+1)} > t),$$

for all $t \geq 0$.

Theorem 6 *The reliability function of the k -out-of- n system is given by*

$$R_{k,n}(t) = \sum_{m=0}^{n-k} R_m(t),$$

for all $t \geq 0$, with

$$R_0(t) = [\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda_1(t))]^n \quad (8)$$

and

$$R_m(t) = \frac{n!}{(n-m)!} \int_{\mathbb{R}_+^m} \mathbf{1}_{\{t_1 < t_2 < \dots < t_m < t\}} a_m(t_1, \dots, t_m) \times [\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda_{m+1}^*(t_1, \dots, t_m, t))]^{n-m} dt_1 \dots dt_m \quad (9)$$

for all $t \geq 0$ and all $m = 1, \dots, n-1$, where

$$\Lambda_m^*(t_1, \dots, t_m) = \sum_{j=1}^m \Delta \Lambda_j(t_{j-1}, t_j), \quad (10)$$

$$a_m(t_1, \dots, t_m) = \prod_{i=1}^m \lambda_i(t_i) f_{\tau_Y}(\Lambda_i^*(t_1, \dots, t_i)), \quad (11)$$

for all $m = 1, \dots, n$ and all $t_0 = 0 < t_1 < t_2 < \dots < t_m$.

Proof. Let us first divide the reliability function $R_{k,n}(t)$, according to the number of already failed components at time t :

$$R_{k,n}(t) = \sum_{m=0}^{n-k} R_m(t), \quad (12)$$

where

$$R_m(t) = \mathbb{P}(\tau^{(m)} < t < \tau^{(m+1)}) \quad (13)$$

stands for the probability that exactly m components are failed at time t , with $0 \leq m \leq n-k$ (and $\tau^{(0)} = 0$).

Let us first consider the case $m = 0$, where no components have failed before t :

$$R_0(t) = \mathbb{P}(\tau^{(1)} > t) = \mathbb{P}\left(\min_{1 \leq i \leq n} (\tau_i) > t\right) = \mathbb{P}\left(\bigcap_{i=1}^n \{\tau_i > t\}\right).$$

When no components have failed at time t , all components are in their first phase (see Figure 1) and they all deteriorate according to independent processes $\mathbf{X}_{i,1}$, $i = 1, \dots, n$ up to time t . Hence

$$\begin{aligned} R_0(t) &= \mathbb{P}\left(\bigcap_{i=1}^n \{X_{i,1}(t) < L\}\right) \\ &= \prod_{i=1}^n \mathbb{P}(X_{i,1}(t) < L) \\ &= [\bar{F}_{\tau_Y}(\Lambda_1(t))]^n. \end{aligned}$$

Now let us consider the generic term $R_m(t)$, with $1 \leq m \leq n-k$ and let $\mathcal{A}_{n,m}$ be the set of sequences (i_1, \dots, i_m) of m elements of $\{1, \dots, n\}$ without repetition, with

$$\text{card}(\mathcal{A}_{n,m}) = \frac{n!}{(n-m)!}.$$

Then, the reliability can be written as

$$R_m(t) = \sum_{(i_1, \dots, i_m) \in \mathcal{A}_{n,m}} \mathbb{P} \left(\tau_{i_1} < \tau_{i_2} < \dots < \tau_{i_m} < t < \min_{j \notin \{i_1, \dots, i_m\}} \tau_j \right) \quad (14)$$

Noting that the probability in the right side of (14) does not depend on (i_1, \dots, i_m) (because all components are identical), we obtain

$$R_m(t) = \frac{n!}{(n-m)!} S_m(t) \quad (15)$$

with

$$S_m(t) = \mathbb{P} \left(\tau_1 < \tau_2 < \dots < \tau_m < t < \min_{m+1 \leq i \leq n} \tau_i \right).$$

Conditioning by τ_1, \dots, τ_m , we get

$$S_m(t) = \mathbb{E} \left(\mathbf{1}_{\{\tau_1 < \tau_2 < \dots < \tau_m < t\}} \Phi(\tau_1, \dots, \tau_m) \right) \quad (16)$$

with

$$\begin{aligned} \Phi(t_1, \dots, t_m) &= \mathbb{P} \left(t < \min_{m+1 \leq i \leq n} \tau_i \mid \tau_1 = t_1, \dots, \tau_m = t_m \right) \\ &= \mathbb{P} \left(\bigcap_{m+1 \leq i \leq n} \{t < \tau_i\} \mid \tau_1 = t_1, \dots, \tau_m = t_m \right) \\ &= \mathbb{P} \left(\bigcap_{m+1 \leq i \leq n} \{Z_i(t) < L\} \mid \tau_1 = t_1, \dots, \tau_m = t_m \right) \end{aligned}$$

for $t_1 < t_2 < \dots < t_m < t$.

Now, given that $\tau_1 = t_1 < \dots < \tau_m = t_m$, all $n-m$ still operating components are in their $m+1$ -th phase, and remembering (5), their deterioration level is given by

$$\begin{aligned} Z_i(t) &= \sum_{j=1}^m \Delta X_{i,j}(t_{j-1}, t_j) + \Delta X_{i,m+1}(t_m, t) \\ &=_{st} Y \left(\sum_{j=1}^m \Delta \Lambda_j(t_{j-1}, t_j) + \Delta \Lambda_{m+1}(t_m, t) \right) \\ &=_{st} Y \left(\Lambda_{m+1}^*(t_1, \dots, t_m, t) \right) \end{aligned} \quad (17)$$

for all

$$t_m \leq t < \tau^{(m+1)} = \min_{m+1 \leq k \leq n} \tau_k$$

and all $i = m + 1, \dots, n$, where $=_{st}$ means equality in distribution. The second line is a direct consequence of the model assumptions described in Subsection 3. Noting that, given $\tau_1 = t_1 < \dots < \tau_m = t_m$, the processes $(Z_i(t))_{t \geq t_m}$, $i = m + 1, \dots, n$ are conditionally independent up to the next failure at time $\tau^{(m+1)}$, we obtain

$$\begin{aligned}\Phi(t_1, \dots, t_m) &= \prod_{m+1 \leq i \leq n} \mathbb{P}(Y(\Lambda_{m+1}^*(t_1, \dots, t_m, t)) < L) \\ &= [\bar{F}_{\tau_Y}(\Lambda_{m+1}^*(t_1, \dots, t_m, t))]^{n-m}.\end{aligned}$$

Starting again from (16), we now have

$$\begin{aligned}S_m(t) &= \int_{\mathbb{R}_+^m} \mathbf{1}_{\{t_1 < t_2 < \dots < t_m < t\}} [\bar{F}_{\tau_Y}(\Lambda_{m+1}^*(t_1, \dots, t_m, t))]^{n-m} \\ &\quad \times f_{(\tau_1, \dots, \tau_m)}(t_1, \dots, t_m) dt_1 \dots dt_m \\ &= \int_{\mathbb{R}_+^m} \mathbf{1}_{\{t_1 < t_2 < \dots < t_m < t\}} [\bar{F}_{\tau_Y}(\Lambda_{m+1}^*(t_1, \dots, t_m, t))]^{n-m} f_{\tau_1}(t_1) \\ &\quad \times \prod_{i=2}^m f_{\tau_i | \tau_1, \dots, \tau_{i-1}}(t_i | t_1, \dots, t_{i-1}) dt_1 \dots dt_m\end{aligned}\quad (18)$$

with

$$f_{\tau_1}(t_1) = \lambda_1(t_1) f_{\tau_Y}(\Lambda_1(t)) = \lambda_1(t_1) f_{\tau_Y}(\Lambda_1^*(t))$$

(see (7)) and where $f_{\tau_i | \tau_1, \dots, \tau_{i-1}}(t_i | t_1, \dots, t_{i-1})$ stands for the conditional pdf of τ_i given $\tau_1 = t_1, \dots, \tau_{i-1} = t_{i-1}$.

Using a similar argument as in (17), we have

$$\begin{aligned}\bar{F}_{\tau_i | \tau_1, \dots, \tau_{i-1}}(t_i | t_1, \dots, t_{i-1}) &= \mathbb{P}(Z_i(t_i) < L | \tau_1 = t_1, \dots, \tau_{i-1} = t_{i-1}) \\ &= \mathbb{P}(Y(\Lambda_i^*(t_1, \dots, t_i)) < L) \\ &= \bar{F}_{\tau_Y}(\Lambda_i^*(t_1, \dots, t_i))\end{aligned}$$

for all $2 \leq i \leq m$, from where we easily derive that

$$\begin{aligned}f_{\tau_i | \tau_1, \dots, \tau_{i-1}}(t_i | t_1, \dots, t_{i-1}) &= -\frac{\partial}{\partial t_i} \bar{F}_{\tau_Y}(\Lambda_i^*(t_1, \dots, t_i)) \\ &= \lambda_i(t_i) f_{\tau_Y}(\Lambda_i^*(t_1, \dots, t_i)).\end{aligned}\quad (19)$$

This allows to conclude, based on (12), (15) and (18). ■

Remark 7 In the specific case where $\Lambda_i(t) = \Lambda(t)$ for all $t \geq 0$ and all $i = 1, \dots, n$, all components are independent. In that case, Formula (9) provides

$$\begin{aligned}
R_m(t) &= \frac{n!}{(n-m)!} \int_{\mathbb{R}_+^n} \mathbf{1}_{\{t_1 < t_2 < \dots < t_m < t\}} (\bar{F}_{\tau_Y}(\Lambda(t)))^{n-m} \\
&\quad \times \prod_{i=1}^m \lambda(t_i) f_{\tau_Y}(\Lambda(t_i)) dt_1 \dots dt_m \\
&= \frac{n!}{(n-m)!} (\bar{F}_{\tau_Y}(\Lambda(t)))^{n-m} \frac{1}{m!} \int_{[0,t]^m} \prod_{i=1}^m \lambda(t_i) f_{\tau_Y}(\Lambda(t_i)) dt_1 \dots dt_m \\
&= \frac{n!}{(n-m)!} (\bar{F}_{\tau_Y}(\Lambda(t)))^{n-m} \frac{1}{m!} \left(\int_0^t \lambda(t_1) f_{\tau_Y}(\Lambda(t_1)) dt_1 \right)^m \\
&= \binom{n}{m} (\bar{F}_{\tau_Y}(\Lambda(t)))^{n-m} (F_{\tau_Y}(\Lambda(t)))^m,
\end{aligned}$$

which is coherent, as the distribution for the number of failed components at time t is binomial $\mathcal{B}(n, F_{\tau_Y}(\Lambda(t)))$.

Remark 8 Remembering that series and parallel systems are specific k -out-of- n systems with $k = n$ and $k = 1$, respectively, we easily derive that their reliability functions are

$$R_{n,n}(t) = R_0(t)$$

for the series case and

$$R_{1,n}(t) = \sum_{m=0}^{n-1} R_m(t)$$

for the parallel case, using the notations of Theorem 6.

Let us now assume that the components are gathered in a general *coherent system*, that is in a system such that all components are relevant and with a non decreasing structure function, see e.g. [17] for more details. Noting that the n components are exchangeable, we can observe from [16] that the structure of the system is characterized by its signature $\mathbf{p} = (p_1, \dots, p_n)$, where p_i is the probability that the failure of the system is induced by the i -th failure among components, that is

$$p_i = \mathbb{P}(T = \tau^{(i)})$$

for all $i = 1, \dots, n$, where T stands for the failure time of the system. Also, the signature \mathbf{p} is known to be independent on the joint distribution

of (τ_1, \dots, τ_n) (please see [16] for more details). Finally, as simultaneous failures are not possible in our context, \mathbf{p} is a probability vector, that is $\sum_{i=1}^n p_i = 1$.

We can now derive the system reliability as a direct application of [16, Lemma 1].

Proposition 9 *The reliability of the coherent system with signature $\mathbf{p} = (p_1, \dots, p_n)$ is given by:*

$$R(t) = \sum_{k=1}^n p_k R_{k,n}(t),$$

for all $t \geq 0$.

5 Distributions

In this section we compute the marginal and joint distributions of the component lifetimes, as well as the joint distribution of the corresponding ordered statistics.

5.1 Marginal distributions

Proposition 10 *For each $i = 1, \dots, n$, the survival function of τ_i is given by*

$$\mathbb{P}(\tau_i > t) = \sum_{m=0}^{n-1} \frac{n-m}{n} R_m(t)$$

for all $t > 0$ (independent on i), where $R_m(t)$ has been introduced in Theorem 6 (see also (13)).

Proof. We may write

$$\mathbb{P}(\tau_i > t) = \sum_{m=0}^{n-1} \mathbb{P}(\tau_i > t | \tau^{(m)} < t < \tau^{(m+1)}) R_m(t) = \sum_{m=0}^{n-1} \frac{n-m}{n} R_m(t),$$

for all $t > 0$, where the last equality comes from the fact that the probability that component i is one of the $n-m$ surviving components is $(n-m)/n$ (as all components are identical). ■

5.2 Joint distributions

We first compute the joint probability density function of (τ_1, \dots, τ_n) .

Theorem 11 *The probability density function of (τ_1, \dots, τ_n) is given by*

$$\begin{aligned} f_{(\tau_1, \dots, \tau_n)}(t_1, \dots, t_n) &= \prod_{i=1}^n \lambda_i(t^{(i)}) f_{\tau_Y}(\Lambda_i^*(t^{(1)}, \dots, t^{(i)})) \\ &= a_n(t^{(1)}, \dots, t^{(n)}) \end{aligned}$$

for all distinct $t_1, t_2, \dots, t_n \in \mathbb{R}_+$ (0 otherwise), where $(t^{(1)}, \dots, t^{(n)})$ is the sequence (t_1, \dots, t_n) arranged in increasing order and where a_n is defined in (11).

Proof. Let us first compute the multivariate survival function of (τ_1, \dots, τ_n) . Using a similar method as for the computation of $R_m(t)$ in the proof of Theorem 6, we distinguish the order in which failures occur, that is we write

$$\mathbb{P}(\tau_1 > t_1 \dots \tau_n > t_n) = \sum_{(k_1, \dots, k_n) \in \mathcal{A}_n} \mathbb{P}(\tau_1 > t_1 \dots \tau_n > t_n, \tau_{k_1} < \dots < \tau_{k_n})$$

for all $t_1, t_2, \dots, t_n \in \mathbb{R}_+$, where $\mathcal{A}_n := \mathcal{A}_{n,n}$ is the set of all permutations of $\{1, \dots, n\}$.

Noting that the set $\{\tau_1 > t_1 \dots \tau_n > t_n\}$ does not depend on the ordering of the τ_i 's, we now have

$$\begin{aligned} &\mathbb{P}(\tau_1 > t_1 \dots \tau_n > t_n) \\ &= \sum_{(k_1, \dots, k_n) \in \mathcal{A}_n} \mathbb{P}(\tau_{k_1} > t_{k_1}, \tau_{k_2} > t_{k_2}, \dots, \tau_{k_n} > t_{k_n}, \tau_{k_1} < \dots < \tau_{k_n}) \\ &= \sum_{(k_1, \dots, k_n) \in \mathcal{A}_n} \mathbb{P}(\tau_{k_1} > t_{k^{(1)}}, \tau_{k_2} > t_{k^{(2)}}, \dots, \tau_{k_n} > t_{k^{(n)}}, \tau_{k_1} < \dots < \tau_{k_n}) \quad (20) \end{aligned}$$

where $t_{k^{(i)}} := t_{k_1} \vee \dots \vee t_{k_i}$, $i = 1, \dots, n$.

Next, we know that

$$f_{(\tau_1, \dots, \tau_n)}(t_1, \dots, t_n) = (-1)^n \frac{\partial^n}{\partial t_1 \dots \partial t_n} \mathbb{P}(\tau_1 > t_1 \dots \tau_n > t_n).$$

To compute this derivative, note that the sequence $(t_{k^{(1)}}, t_{k^{(2)}}, \dots, t_{k^{(n)}})$ will contain repeated terms unless $t_{k_1} < \dots < t_{k_n}$, that is unless t_i 's are all distinct and $t^{(i)} = t_{k_i}$, $i = 1, 2, \dots, n$. As a consequence, the only term in (20) which will not vanish after differentiation is the one in which $(t_{k_1}, \dots, t_{k_n}) = (t^{(1)}, \dots, t^{(n)})$.

Hence we have

$$\begin{aligned}
& f_{(\tau_1, \dots, \tau_n)}(t_1, \dots, t_n) \\
&= (-1)^n \frac{\partial^n}{\partial t_1 \dots \partial t_n} \mathbb{P}(\tau_{k_1} > t^{(1)}, \tau_{k_2} > t^{(2)}, \dots, \tau_{k_n} > t_{k^{(n)}}, \tau_{k_1} < \dots < \tau_{k_n}) \\
&= (-1)^n \frac{\partial^n}{\partial t_1 \dots \partial t_n} \mathbb{P}(\tau_1 > t^{(1)}, \tau_2 > t^{(2)}, \dots, \tau_n > t_n, \tau_1 < \dots < \tau_n)
\end{aligned}$$

by symmetry. We next compute this expression using similar arguments as in (18) and (19), which provides

$$\begin{aligned}
f_{(\tau_1, \dots, \tau_n)}(t^{(1)}, \dots, t^{(n)}) &= \prod_{i=1}^n f_{\tau_{k_i} | \tau_{k_1}, \dots, \tau_{k_{i-1}}} (t^{(i)} | t^{(1)}, \dots, t^{(i-1)}) \\
&= \prod_{i=1}^n \lambda_i (t^{(i)}) f_{\tau_{\mathbf{Y}}} (\Lambda_i^* (t^{(1)}, \dots, t^{(i)})), \quad (21)
\end{aligned}$$

and gives the result. ■

We next derive the pdf of $(\tau^{(1)}, \dots, \tau^{(n)})$ in the following corollary.

Corollary 12 *The pdf of $(\tau^{(1)}, \dots, \tau^{(n)})$ is given by*

$$f_{(\tau^{(1)}, \dots, \tau^{(n)})}(t_1, \dots, t_n) = n! \prod_{i=1}^n \lambda_i (t_i) f_{\tau_{\mathbf{Y}}} (\Lambda_i^* (t_1, \dots, t_i)) = n! a_n (t_1, \dots, t_n)$$

for $t_1 < t_2 < \dots < t_n$ (0 otherwise).

Proof. The result is a direct consequence of Theorem 11, considering the $n!$ possible orderings among the failures. ■

Finally, we derive the probability density function of $(\tau^{(1)}, \dots, \tau^{(k)})$ for $1 \leq k \leq n$, which is next used to derive the conditional distributions of the ordered failure times.

Proposition 13 *For each $1 \leq k \leq n$, the pdf of $(\tau^{(1)}, \dots, \tau^{(k)})$ is given by*

$$\begin{aligned}
& f_{(\tau^{(1)}, \dots, \tau^{(k)})}(t_1, \dots, t_k) \\
&= \frac{n!}{(n-k)!} (\bar{F}_{\tau_{\mathbf{Y}}} (\Lambda_k^* (t_1, \dots, t_k)))^{n-k} a_k (t_1, \dots, t_k) \quad (22) \\
&= \frac{n!}{(n-k)!} (\bar{F}_{\tau_{\mathbf{Y}}} (\Lambda_k^* (t_1, \dots, t_k)))^{n-k} \prod_{i=1}^k \lambda_i (t_i) f_{\tau_{\mathbf{Y}}} (\Lambda_i^* (t_1, \dots, t_i)),
\end{aligned}$$

for $t_1 < t_2 < \dots < t_k$ (0 otherwise).

Proof. Note that (22) can be obtained by noticing that the first part is $k!$ times the probability that $n - k$ components have survived to time t_k and the second part is the joint pdf of the first k failure times, divided by $k!$. For a more specific proof, we start from

$$\begin{aligned} & f_{(\tau^{(1)}, \dots, \tau^{(k)})}(t_1, \dots, t_k) \\ &= \int_{t_k}^{\infty} \int_{u_{k+1}}^{\infty} \cdots \int_{u_{n-2}}^{\infty} \int_{u_{n-1}}^{\infty} f_{(\tau^{(1)}, \dots, \tau^{(n)})}(t_1, \dots, t_k, u_{k+1}, \dots, u_n) du_{k+1} \dots du_n \end{aligned} \quad (23)$$

Taking into account Corollary 12, the last integral is equal to

$$\begin{aligned} & \int_{u_{n-1}}^{\infty} f_{(\tau^{(1)}, \dots, \tau^{(n)})}(t_1, \dots, t_k, u_{k+1}, \dots, u_n) du_n \\ &= n! a_{n-1}(t_1, \dots, t_k, u_{k+1}, \dots, u_{n-1}) \int_{u_{n-1}}^{\infty} \lambda_n(t_n) f_{\tau_{\mathbf{Y}}}(\Lambda_n^*(t_1, \dots, t_k, u_{k+1}, \dots, u_n)) du_n \\ &= n! a_{n-1}(t_1, \dots, t_k, u_{k+1}, \dots, u_{n-1}) \bar{F}_{\tau_{\mathbf{Y}}}(\Lambda_{n-1}^*(t_1, \dots, t_k, u_{k+1}, \dots, u_{n-1})). \end{aligned}$$

Substituting this expression into (23), we get

$$\begin{aligned} & f_{(\tau^{(1)}, \dots, \tau^{(k)})}(t_1, \dots, t_k) \\ &= n! \int_{t_k}^{\infty} \int_{u_{k+1}}^{\infty} \cdots \int_{u_{n-2}}^{\infty} a_{n-1}(t_1, \dots, t_k, u_{k+1}, \dots, u_{n-1}) \\ & \quad \times \bar{F}_{\tau_{\mathbf{Y}}}(\Lambda_{n-1}^*(t_1, \dots, t_k, u_{k+1}, \dots, u_{n-1})) du_{k+1} \dots du_{n-1} \\ &= n! \int_{t_k}^{\infty} \int_{u_{k+1}}^{\infty} \cdots \int_{u_{n-3}}^{\infty} a_{n-2}(t_1, \dots, t_k, u_{k+1}, \dots, u_{n-2}) \\ & \quad \times \frac{1}{2!} (\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda_{n-1}^*(t_1, \dots, t_k, u_{k+1}, \dots, u_{n-2})))^2 du_{k+1} \dots du_{n-2} \end{aligned}$$

by integrating the last integral. The conclusion follows integrating repeatedly the last expression. ■

Proposition 14 *The conditional survival function of $\tau^{(i)}$ given $(\tau^{(1)}, \dots, \tau^{(i-1)})$ is provided by*

$$\mathbb{P}(\tau^{(i)} > t | \tau^{(1)} = t_1, \dots, \tau^{(i-1)} = t_{i-1}) = \left(\frac{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda_i^*(t_1, \dots, t_{i-1}, t))}{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda_{i-1}^*(t_1, \dots, t_{i-1}))} \right)^{n-i+1}$$

for all $0 < t_1 < \dots < t_{i-1}$, all $t > t_{i-1}$ and all $i = 2, \dots, n$.

Proof. Taking into account Proposition 13, we have

$$\begin{aligned}
& f_{\tau^{(i)}|\tau^{(1)},\dots,\tau^{(i-1)}}(t_i|t_1,\dots,t_{i-1}) \\
&= \frac{f_{(\tau^{(1)},\dots,\tau^{(i)})}(t_1,\dots,t_i)}{f_{(\tau^{(1)},\dots,\tau^{(i-1)})}(t_1,\dots,t_{i-1})} \\
&= \frac{(n-i+1) \left(\bar{F}_{\tau_Y}(\Lambda_i^*(t_1,\dots,t_i))\right)^{n-i} \lambda_i(t_i) f_{\tau_Y}(\Lambda_i^*(t_1,\dots,t_i))}{\left(\bar{F}_{\tau_Y}(\Lambda_{i-1}^*(t_1,\dots,t_{i-1}))\right)^{n-i+1}}
\end{aligned}$$

for all $t_i > t_{i-1}$. As a consequence of the previous equality, we easily get

$$\begin{aligned}
\mathbb{P}(\tau^{(i)} > t | \tau^{(1)} = t_1, \dots, \tau^{(i-1)} = t_{i-1}) &= \int_t^\infty f_{\tau^{(i)}|\tau^{(1)},\dots,\tau^{(i-1)}}(t_i|t_1,\dots,t_{i-1}) dt_i \\
&= \frac{\left(\bar{F}_{\tau_Y}(\Lambda_i^*(t_1,\dots,t_{i-1},t))\right)^{n-i+1}}{\left(\bar{F}_{\tau_Y}(\Lambda_{i-1}^*(t_1,\dots,t_{i-1}))\right)^{n-i+1}},
\end{aligned}$$

from which the conclusion follows. ■

6 Dependence and stochastic comparison properties

6.1 Stochastic orders and aging properties

We here recall the definitions of the stochastic orders and aging properties that are used further in this section. We begin with univariate stochastic orders, please see [15, Chapter 1] for a more detailed account on these notions.

Definition 15 *Let X and Y be non-negative random variables, with cdfs F and G , and survival functions \bar{F} and \bar{G} , respectively. X is said to be smaller than Y in the*

- (a) *Usual stochastic order ($X \leq_{st} Y$) if $\bar{G}(x) \geq \bar{F}(x)$ for all $x \in \mathbb{R}$.*
- (b) *Hazard rate order ($X \leq_{rh} Y$) if $\frac{\bar{G}(x)}{\bar{F}(x)}$ is increasing in x (or equivalently $\bar{G}(y)\bar{F}(x) - \bar{G}(x)\bar{F}(y) \geq 0$ for all $x \leq y$, to avoid divisions by 0).*
- (c) *Reversed hazard rate order ($X \leq_{rh} Y$) if $\frac{G(x)}{F(x)}$ is increasing in x (or $G(y)F(x) - G(x)F(y) \geq 0$ for all $x \leq y$).*

(c) *Likelihood ratio order* ($X \leq_{lr} Y$) if X and Y are absolutely continuous with respective pdfs f and g such that $\frac{g(x)}{f(x)}$ is increasing in x (or $g(y)f(x) - g(x)f(y) \geq 0$ for all $x \leq y$).

Remark 16 *We recall that the likelihood ratio order implies both the rh and hr orders and that both rh and hr orders imply the usual stochastic order.*

In Section 6.3, we shall also use the multivariate stochastic order as defined in the next definition. Please see [15, Chapter 3] for more details.

Definition 17 *Let $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ be two random vectors. We say that \mathbf{X} is less or equal than \mathbf{Y} in the multivariate stochastic order (written $\mathbf{X} \leq_{st} \mathbf{Y}$) if $E[\phi(\mathbf{X})] \leq E[\phi(\mathbf{Y})]$ for all bounded increasing function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$.*

Remark 18 *It is well known that, for $n = 1$ (univariate case), Definitions 15 (a) and 17 are equivalent (see [15, Thm. 1.2.13, p. 6]). However, in the multivariate case, even though $\mathbf{X} \leq_{st} \mathbf{Y}$ implies that the survival functions or cdfs of both vectors are ordered (which correspond to the upper and lower orthant orders, respectively), these orders are not equivalent any more.*

In Section 6.3, we will study dependence properties between the failure times of a system. Many dependence concepts can be used to this aim (see [15, Section 3.10]). Here, we will use the concept of Conditionally Increasing In Sequence property, which we define below and where we use the following notation: given two random vectors \mathbf{X} and \mathbf{Y} , we write $\mathbf{X} \uparrow \mathbf{Y}$ if \mathbf{X} is stochastically increasing in \mathbf{Y} , namely if the conditional distribution of \mathbf{X} given $\mathbf{Y} = \mathbf{y}$ is increasing for all $\mathbf{y} \in \mathbb{R}^n$.

Definition 19 *Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random vector. It is said to be Conditionally Increasing in Sequence (CIS) if $X_i \uparrow (X_1, \dots, X_{i-1})$, for all $i = 1, \dots, n$.*

Remark 20 *The concept of Conditionally Increasing in Sequence describes a positive dependence property between the random variables under consideration. Among these concepts, it is not the strongest (for instance, the Multivariate Totally Positive of order 2 (MTP2) property is stronger), but it implies many other common dependence concepts such as positive upper/lower orthant dependence or association, for instance. See [15, p. 146] for a summary of the relationships between these concepts of positive dependence).*

We finally come to the Increasing Failure Rate property, which is used in reliability to describe the aging of a system which is deteriorating over time (see e.g. [14, p.104] for more details).

Definition 21 *Let X be a non negative random variable with F and \bar{F} the corresponding cdf and survival functions. The random variable X (or the cdf F) is said to be Increasing failure rate (IFR) if $\mathbb{P}(X > t + h | X > t)$ is decreasing in t for all $h > 0$ and all t such that the previous expression makes sense, that is $\bar{F}(t + h)/\bar{F}(t)$ is decreasing in t for all $h > 0$.*

Remark 22 *A function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is said to be log-concave if $\log \phi$ is a concave function (where finite). It is well known that ϕ is log-concave if and only if (see e.g. [14, p.696])*

$$\phi(x)\phi(y + h) \leq \phi(y)\phi(x + h), \quad \text{for all } y \geq x \geq 0, \quad h > 0. \quad (24)$$

Thus, the IFR property for X is equivalent to the log-concavity of the survival function \bar{F} [14, p.105].

6.2 Dependence properties

The aim of this section is to study dependence properties of the ordered failure times $(\tau^{(1)}, \dots, \tau^{(n)})$. Based on the fact that the failure times are ordered, we specifically investigate positive dependence properties. With that aim, we will need the following assumption for the derivatives $\lambda_j, j = 1, \dots, n$ of the time-scaling functions $\Lambda_j, j = 1, \dots, n$:

$$\lambda_i \leq \lambda_j \text{ for all } i < j, \quad (25)$$

which mostly means that the deterioration rate of the surviving components increases with the number of already failed components, and implies that the larger the number of observed failures, the shorter the waiting time for the next failure. Under this condition, and assuming also the Increasing Failure Rate (IFR) property for $\tau_{\mathbf{Y}}$, we show that $(\tau^{(1)}, \dots, \tau^{(n)})$ is conditionally increasing in sequence (CIS). More specifically, we have the following result.

Proposition 23 *Assume $\tau_{\mathbf{Y}}$ to be IFR. Under Assumption (25), the ordered vector $(\tau^{(1)}, \dots, \tau^{(n)})$ is conditionally increasing in sequence, that is, for $i = 2, \dots, n$,*

$$[\tau^{(i)} | \tau^{(1)} = t_1, \dots, \tau^{(i-1)} = t_{i-1}] \leq_{st} [\tau^{(i)} | \tau^{(1)} = z_1, \dots, \tau^{(i-1)} = z_{i-1}],$$

for all $(t_1, \dots, t_i) \leq (z_1, \dots, z_i)$, where both vectors are arranged in increasing order.

Proof. Let us assume that both sequences $\mathbf{t} = (t_1, \dots, t_{i-1})$ and $\mathbf{z} = (z_1, \dots, z_{i-1})$ are arranged in increasing order (in order that the conditional probability makes sense). Let

$$A_{\mathbf{t}}^i = \{\tau^{(1)} = t_1, \dots, \tau^{(i-1)} = t_{i-1}\}; A_{\mathbf{z}}^i = \{\tau^{(1)} = z_1, \dots, \tau^{(i-1)} = z_{i-1}\}. \quad (26)$$

We have to show that

$$\mathbb{P}(\tau^{(i)} > t | A_{\mathbf{t}}^i) \leq \mathbb{P}(\tau^{(i)} > t | A_{\mathbf{z}}^i) \quad (27)$$

for all $t \geq 0$.

Let us first note that if $t \leq z_{i-1}$, then $\mathbb{P}(\tau^{(i)} > t | A_{\mathbf{z}}^i) = 1$ so that (27) is true. Hence we can restrict the study to $t > z_{i-1}$.

Let us set

$$\Lambda^1(t) = \sum_{k=1}^i \Lambda_k^*(t_1, \dots, t_{k-1}, t) \mathbf{1}_{[t_{k-1}, t_k)}(t), \quad (28)$$

$$\Lambda^2(t) = \sum_{k=1}^i \Lambda_k^*(z_1, \dots, z_{k-1}, t) \mathbf{1}_{[z_{k-1}, z_k)}(t) \quad (29)$$

for all $t \geq 0$ with $t_i = z_i = \infty$.

Noticing that

$$\begin{aligned} \Lambda^1(t) &= \Lambda_i^*(t_1, \dots, t_{i-1}, t) \\ &= \Lambda^1(t_{i-1}) + \Delta\Lambda_i(t_{i-1}, t) \\ &= \Lambda^1(t_{i-1}) + \Delta\Lambda_i(t_{i-1}, z_{i-1}) + \Delta\Lambda_i(z_{i-1}, t) \\ &= \Lambda^1(z_{i-1}) + \Delta\Lambda_i(z_{i-1}, t) \end{aligned}$$

for all $t > z_{i-1} \geq t_{i-1}$, we get from Proposition 17 that

$$\mathbb{P}(\tau^{(i)} > t | A_{\mathbf{z}}^i) = \left(\frac{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^2(z_{i-1}) + \Delta\Lambda_i(z_{i-1}, t))}{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^2(z_{i-1}))} \right)^{n-i+1}, \quad (30)$$

$$\begin{aligned} \mathbb{P}(\tau^{(i)} > t | A_{\mathbf{t}}^i) &= \left(\frac{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(z_{i-1}) + \Delta\Lambda_i(z_{i-1}, t))}{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(t_{i-1}))} \right)^{n-i+1} \\ &\leq \left(\frac{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(z_{i-1}) + \Delta\Lambda_i(z_{i-1}, t))}{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(z_{i-1}))} \right)^{n-i+1} \end{aligned} \quad (31)$$

due to $\Lambda^1(z_{i-1}) \geq \Lambda^1(t_{i-1})$ for the last inequality.

Now, let us show that

$$\Lambda^1(t) \geq \Lambda^2(t) \quad (32)$$

for all $t > z_{i-1}$. Let λ^1 and λ^2 be the derivative of Λ^1 and Λ^2 on \mathbb{R}_+ , respectively. At first, note that for each $u \geq 0$, there exists j_1 such that $t_{j_1-1} \leq u < t_{j_1}$ and j_2 such that $z_{j_2-1} \leq u < z_{j_2}$. Also $j_1 \geq j_2$ due to $(t_1, \dots, t_i) \leq (z_1, \dots, z_i)$. This implies

$$\lambda_{j_1}(u) = \lambda^1(u) \geq \lambda_{j_2}(u) = \lambda^2(u)$$

based on Assumption (25). Hence $\lambda^1(u) \geq \lambda^2(u)$ for all $u \geq 0$, which implies (32) for all $t > z_{i-1}$.

Remembering that $\tau_{\mathbf{Y}}$ is increasing failure rate, we know that

$$\frac{\bar{F}_{\tau_{\mathbf{Y}}}(y+h)}{\bar{F}_{\tau_{\mathbf{Y}}}(y)} \leq \frac{\bar{F}_{\tau_{\mathbf{Y}}}(x+h)}{\bar{F}_{\tau_{\mathbf{Y}}}(x)}, \quad \text{for all } y \geq x \geq 0, \quad h > 0. \quad (33)$$

Based on (32), we derive that

$$\frac{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(z_{i-1}) + \Delta\Lambda_i(z_{i-1}, t))}{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(z_{i-1}))} \leq \frac{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^2(z_{i-1}) + \Delta\Lambda_i(z_{i-1}, t))}{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^2(z_{i-1}))},$$

which taking into account (30) and (31) shows (27), and therefore the result. ■

Remark 24 *Note that it is not possible to use similar arguments as in the previous proof to derive a conditionally decreasing in sequence property, because the reversed inequality of (27) would not be valid, as for $t_{i-1} < t < z_{i-1}$ the second term in (27) is 1, whereas the first term can be less than 1.*

Finally, let us mention that the IFR property for $\tau_{\mathbf{Y}}$ is quite a natural condition fulfilled by most usual Lévy processes, which can easily derived from the following Lemma. As an example, it is checked for the inverse Gaussian process (as we will see in the next section). It is also valid for the Gamma process, based on similar arguments.

Lemma 25 *If the baseline Lévy process \mathbf{Y} is such that $Y(t)$ increases with t in the reversed hazard rate ordering, the crossing time $\tau_{\mathbf{Y}}$ of level L is Increasing Failure Rate (IFR).*

Proof. Our aim is to use Proposition 3.1 from [18]. At first, notice that the constant threshold L can be seen as an IFR random variable and that \mathbf{Y} is a non negative Lévy process, and hence in the class \mathcal{IPII} (independent positive increasing increments) in the sense of [18]. As $\mathbb{P}(Y_t = L) = 0$, we have now checked all conditions from [18, Prop. 3.1 (a)], which allows to derive that $\tau_{\mathbf{Y}}$ is IFR. ■

6.3 Stochastic comparisons

In this section, we provide conditions to compare the failure times in two different deterioration models, with, possibly, different baseline processes and time-scaling functions. We begin with a simple result comparing the times of the first failures among components, or equivalently, comparing the failure times of series systems.

Proposition 26 *Let us consider two deterioration models described by the pair of baseline Lévy processes \mathbf{Y} and \mathbf{Z} , with $\{\Lambda_j^m, j = 1, \dots, n\}$ the corresponding time-scaling functions. Assume that $Y(t) \geq_{st} Z(t)$ for all $t \geq 0$ and $\Lambda_1^1 \geq \Lambda_1^2$. Then, setting $\tau^{(1)}$ and $\hat{\tau}^{(1)}$ to be the first failure times in the first and second models, respectively, we have*

$$\tau^{(1)} \leq_{st} \hat{\tau}^{(1)} \quad (34)$$

Proof. Observe that $\Lambda_1^1(t) \geq \Lambda_1^2(t)$ implies that

$$\mathbb{P}(\tau^{(1)} > t) = \mathbb{P}(Y(\Lambda_1^1(t)) \leq L)^n \leq \mathbb{P}(Y(\Lambda_1^2(t)) \leq L)^n. \quad (35)$$

Now $Y(t) \geq_{st} Z(t)$ implies that

$$\mathbb{P}(Y(\Lambda_1^2(t)) \leq L)^n \leq \mathbb{P}(Z(\Lambda_1^2(t)) \leq L)^n = \mathbb{P}(\hat{\tau}^{(1)} > t) \quad (36)$$

for all $t > 0$. Gathering (35) and (36) shows (34). ■

Remark 27 *As a specific case, assume that the two deterioration models in Proposition 26 share the same baseline Lévy processes \mathbf{Y} . As $Y(t) \leq_{st} \hat{Y}(t)$ for all $t \geq 0$, if $\Lambda_1^1 \geq \Lambda_1^2$, we get that $\tau^{(1)} \leq_{st} \hat{\tau}^{(1)}$. Considering $\Lambda_2^m(t) = t$ for all $t \geq 0$ and all m (which means that the second deterioration process is the baseline process), we can see that if $\Lambda_1^1(t) \geq (\leq)t$ for all t , then the time of the first failure in the deterioration model is stochastically smaller (larger) than the time of the first failure in the baseline model. Equivalently, this provides comparison results between the lifetimes $\tau_{\mathbf{X}}$ (deterioration model) and $\tau_{\mathbf{Y}}$ (baseline model) in case of a common series structure.*

We next study the multivariate stochastic ordering property of the ordered failure times. As previous steps, we prove two technical lemmas, which have some interest in themselves.

Lemma 28 *Let \mathbf{Y} and \mathbf{Z} be two baseline Lévy processes such that $Y(t) \geq_{rh} Z(t)$ for all $t \geq 0$. Then $\tau_{\mathbf{Y}} \leq_{hr} \tau_{\mathbf{Z}}$.*

Proof. The assumption $Y(t) \geq_{rh} Z(t)$ for all $t \geq 0$ means that

$$\frac{F_{Z(t)}(x)}{F_{Z(t)}(x+h)} \geq \frac{F_{Y(t)}(x)}{F_{Y(t)}(x+h)} \quad (37)$$

for all $h, t \geq 0$ and $x > -h$, where $F_{Y(t)}$ and $F_{Z(t)}$ stand for the cumulative distribution functions of $Y(t)$ and $Z(t)$, respectively. The point is to show that $\tau_{\mathbf{Y}} \leq_{hr} \tau_{\mathbf{Z}}$, that is

$$\frac{\mathbb{P}(\tau_{\mathbf{Y}} > s+t)}{\mathbb{P}(\tau_{\mathbf{Y}} > t)} \leq \frac{\mathbb{P}(\tau_{\mathbf{Z}} > s+t)}{\mathbb{P}(\tau_{\mathbf{Z}} > t)} \quad (38)$$

for all $s, t \geq 0$. Note that, based on the independent and homogenous increment property of \mathbf{Y} , we have

$$Y(s+t) =_{st} Y(t) + Y^*(s),$$

where \mathbf{Y}^* is an independent copy of \mathbf{Y} . Hence

$$\begin{aligned} \mathbb{P}(\tau_{\mathbf{Y}} > s+t) &= \mathbb{P}(Y(s+t) \leq L) \\ &= \mathbb{P}(Y(t) \leq L - Y^*(s)) \\ &= \mathbb{E}[F_{Y(t)}(L - Y^*(s))] \\ &= \mathbb{E}[F_{Y(t)}(L - Y(s))] \end{aligned}$$

We derive

$$\begin{aligned} \frac{\mathbb{P}(\tau_{\mathbf{Y}} > s+t)}{\mathbb{P}(\tau_{\mathbf{Y}} > t)} &= \mathbb{E}\left(\frac{F_{Y(t)}(L - Y(s))}{F_{Y(t)}(L)}\right) \\ &\leq \mathbb{E}\left(\frac{F_{Z(t)}(L - Y(s))}{F_{Z(t)}(L)}\right), \end{aligned} \quad (39)$$

due to (37) for the last inequality, taking $x = L - Y(s)$ and $h = Y(s)$.

Now, as $Y(s) \geq_{rh} Z(s)$, we also have $Y(s) \geq_{st} Z(s)$, which entails that

$$\mathbb{E}\left(\frac{F_{Z(t)}(L - Y(s))}{F_{Z(t)}(L)}\right) \leq \mathbb{E}\left(\frac{F_{Z(t)}(L - Z(s))}{F_{Z(t)}(L)}\right) = \frac{\mathbb{P}(\tau_{\mathbf{Z}} > s+t)}{\mathbb{P}(\tau_{\mathbf{Z}} > t)} \quad (40)$$

and consequently (38) is true, gathering (39) and (40). ■

Lemma 29 *Let Λ^1 and Λ^2 be two time-scaling functions such that $\Lambda^1 \geq \Lambda^2$. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ be a log-concave decreasing function. Then, for all $0 \leq s < t$ such that $\Delta\Lambda^1(s, t) \geq \Delta\Lambda^2(s, t)$ we have*

$$\phi(\Lambda^1(t)) \phi(\Lambda^2(s)) \leq \phi(\Lambda^2(t)) \phi(\Lambda^1(s)). \quad (41)$$

Proof. By assumption, we know that $\Lambda^1(s) \geq \Lambda^2(s)$. Then, using (24) for the second line and the decreasing property for the third one, we have

$$\begin{aligned} \phi(\Lambda^1(t)) \phi(\Lambda^2(s)) &= \phi(\Lambda^1(s) + \Delta\Lambda^1(s, t)) \phi(\Lambda^2(s)) \\ &\leq \phi(\Lambda^2(s) + \Delta\Lambda^1(s, t)) \phi(\Lambda^1(s)) \\ &\leq \phi(\Lambda^2(s) + \Delta\Lambda^2(s, t)) \phi(\Lambda^1(s)) \\ &= \phi(\Lambda^2(t)) \phi(\Lambda^1(s)), \end{aligned}$$

and the conclusion follows. ■

Proposition 30 *Let us consider two deterioration models described by the pair of baseline Lévy processes \mathbf{Y} and \mathbf{Z} , with $\{\Lambda_j^m, j = 1, \dots, n\}$ and $\{\lambda_j^m, j = 1, \dots, n\}$, $m = 1, 2$, as corresponding time-scaling functions and derivatives, respectively. Assume the following conditions to hold:*

- (a) $Y(t) \geq_{rh} Z(t)$ for all $t \geq 0$ and $\Lambda_1^1 \geq \Lambda_1^2$.
- (b) $\lambda_j^1 \geq \lambda_j^2$, for all $j = 2, \dots, n$.
- (c) *One of the deterioration models satisfies conditions in Proposition 23 (that is, Condition (25) for the time-scaling functions, as well as the IFR property for the failure time in the baseline Lévy process).*

Then, setting $(\tau^{(1)}, \dots, \tau^{(n)})$ and $(\hat{\tau}^{(1)}, \dots, \hat{\tau}^{(n)})$ to be the ordered failure times in the first and second models, respectively, we have

$$(\tau^{(1)}, \dots, \tau^{(n)}) \leq_{st} (\hat{\tau}^{(1)}, \dots, \hat{\tau}^{(n)}).$$

Proof. Taking into account condition (a) and that the fact that the hazard rate order implies the usual stochastic order, we know from Proposition 26 that $\tau^{(1)} \leq_{st} \hat{\tau}^{(1)}$. As a second step, let us use the notation introduced in (26), and let \hat{A}_t^i and \hat{A}_z^i be the corresponding overcomes for the second model. With these notations, we want to show that

$$[\tau^{(i)} | A_t^i] \leq_{st} [\hat{\tau}^{(i)} | \hat{A}_z^i] \quad (42)$$

for all $(t_1, \dots, t_n) \leq (z_1, \dots, z_n)$ and all $2 \leq i \leq n$.

Assume first that Condition (c) is satisfied for the first model. Based on Proposition 23, we then have $[\tau^{(i)} | A_t^i] \leq_{st} [\tau^{(i)} | A_z^i]$, so that (42) will follow if we show that

$$[\tau^{(i)} | A_z^i] \leq_{st} [\hat{\tau}^{(i)} | \hat{A}_z^i]. \quad (43)$$

Let us set

$$\Lambda^l(t) = \sum_{k=1}^i \Lambda_k^{*l}(z_1, \dots, z_{k-1}, t) \mathbf{1}_{[z_{k-1}, z_k)}(t),$$

with $z_i = \infty$ for all $t \geq 0$ and $l = 1, 2$, where Λ_k^{*l} , $l = 1, 2$ is defined as Λ_k^* in (10) for $1 \leq k \leq i$.

Then, due to (30), we have

$$\begin{aligned} \mathbb{P}(\tau^{(i)} > t | A_{\mathbf{z}}^i) &= \left(\frac{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(t))}{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(z_{i-1}))} \right)^{n-i+1}, \\ \mathbb{P}(\hat{\tau}^{(i)} > t | \hat{A}_{\mathbf{z}}^i) &= \left(\frac{\bar{F}_{\tau_{\mathbf{Z}}}(\Lambda^2(t))}{\bar{F}_{\tau_{\mathbf{Z}}}(\Lambda^2(z_{i-1}))} \right)^{n-i+1} \end{aligned} \quad (44)$$

for all $t > z_{i-1}$.

Condition (a), together with the condition $\lambda_j^1 \geq \lambda_j^2$, $j \geq 2$ obviously implies that $\Lambda^1(z_{i-1}) \geq \Lambda^2(z_{i-1})$. Moreover, $\Delta\Lambda^1(z_{i-1}, t) \geq \Delta\Lambda^2(z_{i-1}, t)$ (because $\lambda_j^1 \geq \lambda_j^2$, $j \geq 2$). As $\tau_{\mathbf{Y}}$ is increasing failure rate, conditions in Lemma 29 are satisfied and we have

$$\frac{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(t))}{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(z_{i-1}))} \leq \frac{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^2(t))}{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^2(z_{i-1}))} \quad (45)$$

Now based on $Y(t) \geq_{rh} Z(t)$ for all $t \geq 0$, we can derive from Lemma 28 that $\tau_{\mathbf{Y}} \leq_{hr} \tau_{\mathbf{Z}}$, and consequently (38) is true. This provides

$$\frac{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^2(t))}{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^2(z_{i-1}))} \leq \frac{\bar{F}_{\tau_{\mathbf{Z}}}(\Lambda^2(t))}{\bar{F}_{\tau_{\mathbf{Z}}}(\Lambda^2(z_{i-1}))} \quad (46)$$

for all $t > z_{i-1}$. Then, from (45) and (46), we conclude that

$$\frac{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(t))}{\bar{F}_{\tau_{\mathbf{Y}}}(\Lambda^1(z_{i-1}))} \leq \frac{\bar{F}_{\tau_{\mathbf{Z}}}(\Lambda^2(t))}{\bar{F}_{\tau_{\mathbf{Z}}}(\Lambda^2(z_{i-1}))},$$

and this last inequality together with (44) show (43). Finally, the result follows by (34) and (42) (see [15, Thm. 3.3.7, p. 93]). In the case where Condition (b) is satisfied for the second model, it is easy to check that similar arguments remain valid. ■

Remark 31 Let $R_{k,n}(t)$ and $\hat{R}_{k,n}(t)$ be the reliability functions of a k -out-of- n system for two deterioration models as defined in Proposition 30. Proposition 26 gives conditions for the case $k = 1$ (series system). Moreover, as the multivariate stochastic order implies the univariate stochastic order of the

marginals (see [15, Thm.3.3.10, p.94]), we immediately deduce from Proposition 30 that $R_{k,n}(t) \leq \hat{R}_{k,n}(t)$, $k = 2, \dots, n$. In particular, this includes stochastic comparisons for parallel systems ($k = n$). Moreover, if $R(t)$ and $\hat{R}(t)$ are the reliability functions of a specific coherent system for both models, we have by Proposition 9 that $R(t) \leq \hat{R}(t)$.

Remark 32 Let us assume that the two deterioration models in Proposition 30 share the same baseline Lévy process \mathbf{Y} and that $\Lambda_j^2(t) = t$ for all $t \geq 0$ and all $j = 1, \dots, n$. Then Condition (25) is true for the second deterioration process (which is the baseline Lévy process). Assume further that $\tau_{\mathbf{Y}}$ is IFR and that $\lambda_j^1(t) \geq (\leq) 1$, for all $j = 1, \dots, n$. Then, as $Y(t) \geq_{rh} Y(t)$ for all $t \geq 0$, we know from Proposition 30 that

$$(\tau^{(1)}, \dots, \tau^{(n)}) \leq_{st} (\geq_{st}) (\hat{\tau}^{(1)}, \dots, \hat{\tau}^{(n)})$$

where $\hat{\tau}^{(1)}, \dots, \hat{\tau}^{(n)}$ stand for the ordered failure times of the baseline Lévy process model. Proposition 30 hence provides us with conditions under which the ordered failure times of the deterioration model are stochastically smaller or larger than those of the baseline process. Based on Remark 31, the reliability functions of a specific coherent system for both deterioration and baseline models are also comparable under the same conditions.

7 Illustration in the case of inverse Gaussian processes

We here consider n identical and dependent components as in Section 3, with an inverse Gaussian process as underlying deterioration process \mathbf{Y} . As a first step, we recall some well-known facts about inverse Gaussian processes.

7.1 The inverse Gaussian process

Following the parametrization of [21], we recall that an inverse Gaussian process \mathbf{Y} is a Lévy process such that $Y(t) \sim IG(t, bt^2)$ with pdf and cdf

$$f_t(y) = t \sqrt{\frac{b}{2\pi y^3}} \exp\left[-\frac{b(y-t)^2}{2y}\right],$$

$$F_t(y) = \Phi\left[(y-t)\sqrt{\frac{b}{y}}\right] + e^{2bt} \Phi\left[-(y+t)\sqrt{\frac{b}{y}}\right]$$

for all $t, y > 0$, respectively, where $b > 0$ and Φ is the cdf of the standard normal distribution.

The technical assumptions from Section 3 are clearly true and remembering that $\tau_{\mathbf{Y}}$ stands for the crossing time of level L , then

$$\bar{F}_{\tau_{\mathbf{Y}}}(t) = F_t(L)$$

and

$$\begin{aligned} f_{\tau_{\mathbf{Y}}}(t) &= -\frac{\partial F_t(L)}{\partial t} \\ &= \sqrt{\frac{b}{L}}\phi\left[(L-t)\sqrt{\frac{b}{L}}\right] + e^{2bt}\sqrt{\frac{b}{L}}\phi\left[-(L+t)\sqrt{\frac{b}{L}}\right] \\ &\quad - 2be^{2bt}\Phi\left[-(L+t)\sqrt{\frac{b}{L}}\right], \end{aligned}$$

where ϕ is the pdf of the standard normal distribution.

Finally, based on [14, Section 13.A.e page 458], it is known that $Y(t)$ increases with t in the likelihood hazard rate ordering, and consequently, it also increases in the reversed hazard rate ordering. Based on Lemma 25, we hence derive that the crossing time $\tau_{\mathbf{Y}}$ of level L is Increasing Failure Rate. Finally

As a consequence, we now have all the material to illustrate the previous results considering an inverse Gaussian process as baseline Lévy process \mathbf{Y} . Note however that $IG(t, bt^2)$ is not monotonous in the reversed hazard rate ordering with respect to b , so that for Proposition 30, we have to consider the same inverse gaussian process as baseline Lévy process \mathbf{Y} .

7.2 Numerical experiments

We assume in all this section that \mathbf{Y} is an inverse Gaussian process such that $Y(t) \sim IG(t, t^2)$ (that is $b = 1$) and $L = 1$. The collection of time-scaling functions is denoted by $\{\Lambda_j, j = 1, \dots, n\}$, as in Section 3.

As a first step, the results provided by Theorem 6 are compared to those obtained by Monte-Carlo simulations on an example.

7.2.1 Comparison with Monte-Carlo simulations

We here consider a 2-out-of-3 system. Then Theorem 6 provides

$$R_{2,3}(t) = (F_{\Lambda_1(t)}(L))^3 + 3 \int_0^t (F_{\Lambda_1(t_1)+\Delta\Lambda_2(t_1,t)}(L))^2 \lambda_1(t_1) f_{\tau_{\mathbf{Y}}}[\Lambda_1(t_1)] dt_1 \quad (47)$$

for all $t \geq 0$.

We take

$$\begin{aligned}\Lambda_1(t) &= e^t - 1, \\ \Lambda_2(t) &= e^{2t} - 1, \\ \Lambda_3(t) &= e^{3t} - 1\end{aligned}$$

for all $t \geq 0$ and compute the reliability on $[0, 1.75]$ through (47) and also through (MC) Monte-Carlo simulations, using Matlab software. As for the MC simulations, 10^4 trajectories of $(\mathbf{Y}(t))_{0 \leq t \leq 2.5}$ are simulated through random walk approximation with a small time-step ($h = 0.001$), in order to get mostly reliable crossing times for level L . This provides an empirical estimate of $R_{2,3}(t)$ denoted by $\hat{R}_{2,3}(t)$. The results are provided in Figure 2, where we can see that $\hat{R}_{2,3}(t)$ and $R_{2,3}(t)$ are mostly superimposed (left plot), with a difference less than 5×10^{-3} (right plot). Hence, both methods provide similar results. However, the computation takes 242 cpu time on a standard laptop computer for the MC simulation and around 0.5 cpu time for the computation through (47). Even though the code might surely be bettered for the MC simulation, there clearly is a huge gap between the computing times through MC simulations and through (47). In the following, all computations are hence made through the theoretical results from the previous sections.

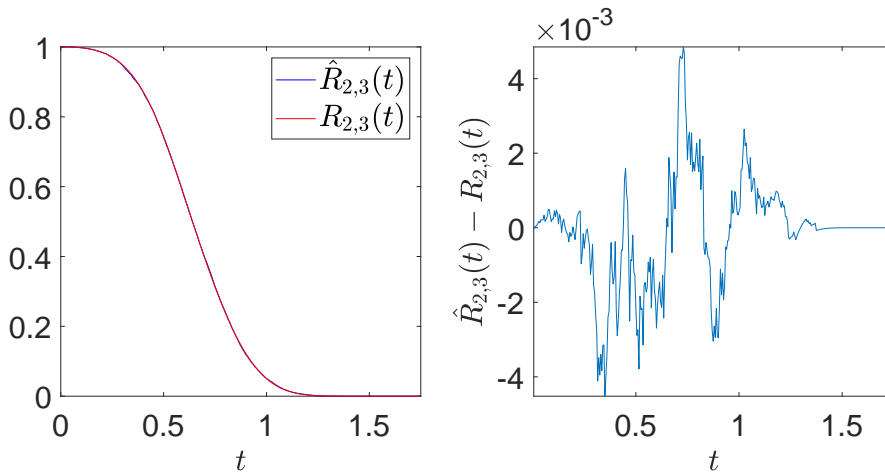


Figure 2: $\hat{R}_{2,3}(t)$ and $R_{2,3}(t)$ (left plot) and $\hat{R}_{2,3}(t) - R_{2,3}(t)$ (right plot)

7.2.2 Conditionally Increasing in Sequence property

The point here is to illustrate the results from Proposition 23. For each $2 \leq i \leq 9$, $\mathbf{t} = (t_j)_{1 \leq j \leq i-1}$ and $\mathbf{z} = (z_j)_{1 \leq j \leq i-1} \geq \mathbf{t}$, we set

$$\begin{aligned} G_{\mathbf{t}}(u) &= \mathbb{P}(\tau^{(i)} > u | \tau^{(1)} = t_1, \dots, \tau^{(i-1)} = t_{i-1}) \\ &= \left(\frac{\bar{F}_{\tau_{\mathbf{y}}} \left(\sum_{j=1}^{i-1} \Delta \Lambda_j(t_{j-1}, t_j) + \Delta \Lambda_i(t_{i-1}, u) \right)}{\bar{F}_{\tau_{\mathbf{y}}} \left(\sum_{j=1}^{i-1} \Delta \Lambda_j(t_{j-1}, t_j) \right)} \right)^{n-i+1} \end{aligned}$$

for all $u > t_{i-1}$ and a similar expression for $G_{\mathbf{z}}(u)$, with \mathbf{t} substituted by \mathbf{z} .

Considering $n = 10$, $k = 6$ and $t_j = 0.1 j$ for $1 \leq j \leq 5$, we envision two different sets of scaling functions and two different \mathbf{z} . As a first case, we take

$$\Lambda_j(t) = 0.1 (e^{t\alpha_j} - 1)$$

for all $t \geq 0$, with $\alpha_j = 0.1 j$ for $1 \leq j \leq 10$, so that Assumption (25) is fulfilled. We set $\mathbf{z} = 4\mathbf{t} \geq \mathbf{t}$. The functions $G_{\mathbf{t}}(u)$ and $G_{\mathbf{z}}(u)$ are plotted in Figure 3 (left plot). As expected, we can see that $G_{\mathbf{t}}(u) \leq G_{\mathbf{z}}(u)$ for all $u > z_{i-1}$, which is coherent with the fact that $(\tau^{(1)}, \dots, \tau^{(n)})$ is conditionally increasing in sequence (CIS property), as shown in Proposition 23.

As a second case, we take $\Lambda_j(t) = 0.1 t^{\alpha_j}$ with $\alpha_j = 0.2(11 - j)$ for $1 \leq j \leq 10$ and $\mathbf{z} = 10\mathbf{t}$. Note that Assumption (25) is not fulfilled any more. The functions $G_{\mathbf{t}}(u)$ and $G_{\mathbf{z}}(u)$ are plotted in Figure 3 (right plot). It can be seen that the functions $G_{\mathbf{t}}(u)$ and $G_{\mathbf{z}}(u)$ are crossing, which shows that the CIS property does not hold any more for $(\tau^{(1)}, \dots, \tau^{(n)})$.

Hence, Assumption (25) appears to be a necessary assumption, in order to guaranty that $(\tau^{(1)}, \dots, \tau^{(n)})$ is conditionally increasing in sequence.

7.2.3 Stochastic comparisons

The point here is to illustrate the results from Proposition 30. We here consider n components and look at the survival and cumulative distribution functions of $(\tau^{(1)}, \tau^{(2)})$. Starting from the joint pdf of $(\tau^{(1)}, \tau^{(2)})$ provided in

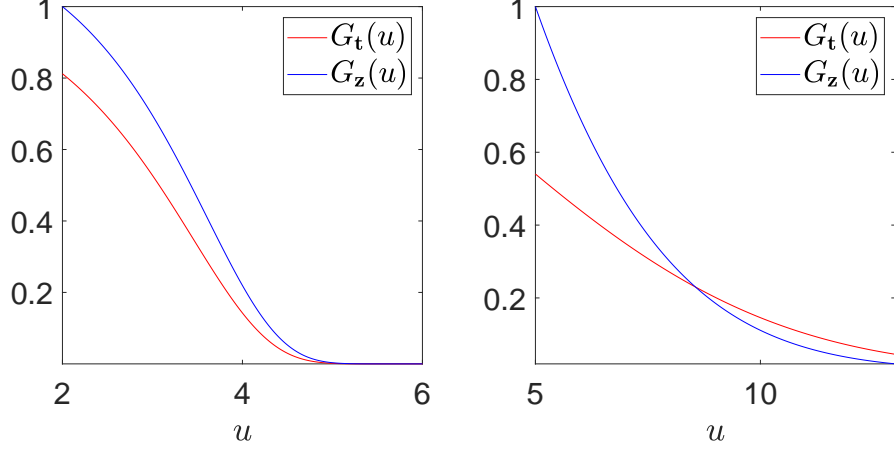


Figure 3: The functions $G_t(u)$ and $G_z(u)$. Left plot: first case; right plot: second case.

Proposition 13, we obtain that

$$\begin{aligned}
& \bar{F}_{(\tau^{(1)}, \tau^{(2)})}(t_1, t_2) \\
&= \mathbb{P}(\tau^{(1)} > t_1, \tau^{(2)} > t_2) \\
&= n \int_{t_1}^{\infty} \lambda_1(u_1) f_{\tau_Y}(\Lambda_1^*(u_1)) \\
&\times \left(\int_{t_2}^{\infty} \mathbf{1}_{\{u_1 < u_2\}} (n-1) \lambda_2(u_2) f_{\tau_Y}(\Lambda_2^*(u_1, u_2)) [\bar{F}_{\tau_Y}(\Lambda_2^*(u_1, u_2))]^{n-2} du_2 \right) du_1 \\
&= n \int_{t_1}^{\infty} \lambda_1(u_1) f_{\tau_Y}(\Lambda_1(u_1)) [\bar{F}_{\tau_Y}(\Lambda_2^*(u_1, \max(t_2, u_1)))]^{n-1} du_1 \\
&= [\bar{F}_{\tau_Y}(\Lambda_1(t^{(2)}))]^n \\
&+ \mathbf{1}_{\{t_1 < t_2\}} n \int_{t_1}^{t_2} \lambda_1(u_1) f_{\tau_Y}(\Lambda_1(u_1)) [\bar{F}_{\tau_Y}(\Lambda_1(u_1) + \Delta\Lambda_2(u_1, t_2))]^{n-1} du_1
\end{aligned} \tag{48}$$

for all $t_1, t_2 \geq 0$, where we distinguish the two cases $t_1 < t_2$ and $t_2 \leq t_1$ to derive the last line, remembering that $t^{(2)} = \max(t_1, t_2)$.

Similar computations provide

$$\begin{aligned}
& F_{(\tau^{(1)}, \tau^{(2)})}(t_1, t_2) \\
&= n \int_0^{t^{(1)}} \lambda_1(u_1) f_{\tau_Y}(\Lambda_1^*(u_1)) \\
&\quad \times \left(\int_{u_1}^{t_2} (n-1) \lambda_2(u_2) f_{\tau_Y}(\Lambda_2^*(u_1, u_2)) [\bar{F}_{\tau_Y}(\Lambda_2^*(u_1, u_2))]^{n-2} du_2 \right) du_1 \\
&= n \int_0^{t^{(1)}} \lambda_1(u_1) f_{\tau_Y}(\Lambda_1(u_1)) \left([\bar{F}_{\tau_Y}(\Lambda_1(u_1))]^{n-1} - [\bar{F}_{\tau_Y}(\Lambda_2^*(u_1, t_2))]^{n-1} \right) du_1 \\
&= 1 - [\bar{F}_{\tau_Y}(\Lambda_1(t^{(1)}))]^n - n \int_0^{t^{(1)}} \lambda_1(u_1) f_{\tau_Y}(\Lambda_1(u_1)) [\bar{F}_{\tau_Y}(\Lambda_2^*(u_1, t_2))]^{n-1} du_1
\end{aligned} \tag{49}$$

for all $t_1, t_2 \geq 0$, where we remind that $t^{(1)} = \min(t_1, t_2)$.

Using the notations of Proposition 30, we take $Y(t) = Z(t) \sim IG(t, t^2)$ (based on the remark at the end of Section 7.1),

$$\Lambda_j^m(t) = \beta_j^m (e^{t\alpha_j^m} - 1)$$

for all $t \geq 0$ and $j = 1, \dots, n$, $m = 1, 2$, with

$$\alpha_1^1 = 1.25, \beta_1^1 = 1, \alpha_2^1 = 1.25, \beta_2^1 = 1.5, \alpha_1^2 = 1, \beta_1^2 = 1, \alpha_2^2 = 1, \beta_2^2 = 0.5.$$

Then it is easy to check that Assumption (25) is true for $\{\Lambda_1^1, \Lambda_2^1\}$, but not for $\{\Lambda_1^2, \Lambda_2^2\}$. Also $\lambda_j^1 \geq \lambda_j^2$ for $j = 1, \dots, n$.

Let $\bar{F}_m(t_1, t_2)$ and $F_m(t_1, t_2)$, $m = 1, 2$ be the survival and cumulative distribution functions of $(\tau^{(1)}, \tau^{(2)})$ and $(\hat{\tau}^{(1)}, \hat{\tau}^{(2)})$, respectively, which are computed through (48) and (49). The differences $(\bar{F}_2 - \bar{F}_1)(t_1, t_2)$ and $(F_1 - F_2)(t_1, t_2)$ are plotted in Figure 4. As expected from Proposition 30, we can see that $\bar{F}_2(t_1, t_2) \geq \bar{F}_1(t_1, t_2)$ and $F_1(t_1, t_2) \geq F_2(t_1, t_2)$ for all $t_1, t_2 \geq 0$, which is coherent with the fact that

$$(\tau^{(1)}, \tau^{(2)}) \leq_{st} (\hat{\tau}^{(1)}, \hat{\tau}^{(2)})$$

(recall Remark 18).

Remark 33 *Note that we have not been able to find any counter-example such that $\lambda_j^1 \geq \lambda_j^2$ $j = 1, 2$, where neither $\{\Lambda_j^1, j = 2, \dots, n\}$ nor $\{\Lambda_j^2, j = 2, \dots, n\}$ fulfills Assumption (25), and such that $(\tau^{(1)}, \tau^{(2)})$ is not stochastically smaller than $(\hat{\tau}^{(1)}, \hat{\tau}^{(2)})$. Hence, Assumption (25) may be not required for the stochastic comparison result to hold, but we have not been able to prove it without this assumption. Then the question remains open whether Assumption (25) is required to hold for one of the two models or not.*

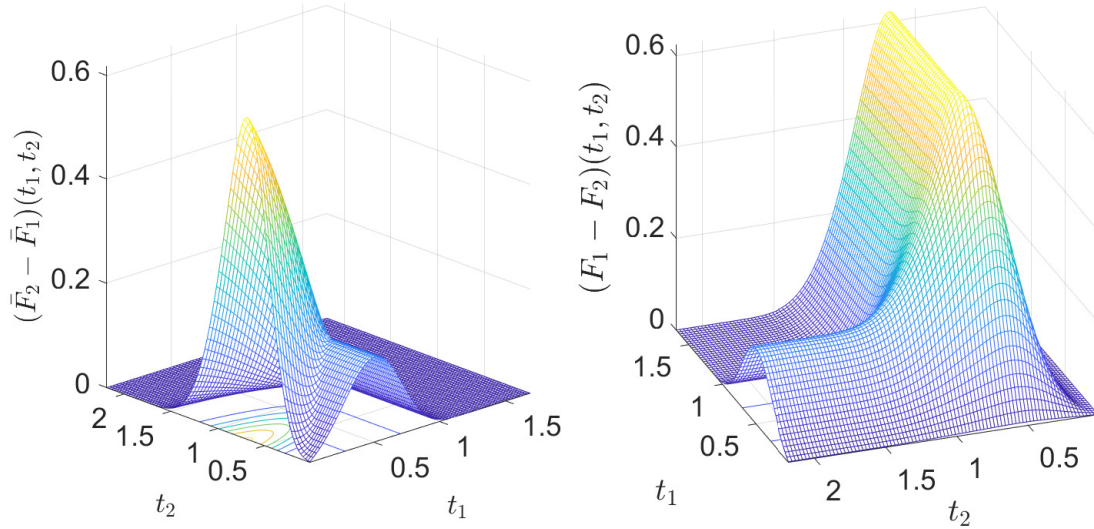


Figure 4: $(\bar{F}_2 - \bar{F}_1)(t_1, t_2)$ (left plot) and $(F_1 - F_2)(t_1, t_2)$ (right plot)

We next explore the results of Proposition 30 and Remark 31, which allow to compare the reliability of systems with identical structures and different impacts of component failures on the time-scaling functions of the surviving components.

Example 1 *As a first example, 2-out-of-5 systems are considered, with*

$$R_{2,5}(t) = \sum_{m=0}^3 R_m(t)$$

	α_1	α_2	α_3	α_4
Set 1 (independent case)	1	1	1	1
Set 2	1	1.5	2	2.5
Set 3	1	2	3	4
Set 4	1	2.5	3.5	4.5

Table 1: The four parameter sets for Figure 5

and

$$R_0(t) = (F_{\Lambda_1(t)}(L))^5 \quad (50)$$

$$R_1(t) = 5 \int_0^t (\bar{F}_{\tau\mathbf{Y}}(\Lambda_1(t_1) + \Delta\Lambda_2(t_1, t)))^4 \lambda_1(t_1) f_{\tau\mathbf{Y}}[\Lambda_1(t_1)] dt_1, \quad (51)$$

$$R_2(t) = 20 \int_{[0,t]^2} \mathbf{1}_{\{t_1 < t_2\}} (\bar{F}_{\tau\mathbf{Y}}(\Lambda_1(t_1) + \Delta\Lambda_2(t_1, t_2) + \Delta\Lambda_3(t_2, t)))^3 \\ \times \lambda_1(t_1) f_{\tau\mathbf{Y}}[\Lambda_1(t_1)] \lambda_2(t_2) f_{\tau\mathbf{Y}}[\Lambda_1(t_1) + \Delta\Lambda_2(t_1, t_2)] dt_1 dt_2, \quad (52)$$

$$R_3(t) = 60 \int_{[0,t]^3} \mathbf{1}_{\{t_1 < t_2 < t_3\}} (\bar{F}_{\tau\mathbf{Y}}(\Lambda_1(t_1) + \Delta\Lambda_2(t_1, t_2) + \Delta\Lambda_3(t_2, t_3) + \Delta\Lambda_4(t_3, t)))^2 \\ \times \lambda_1(t_1) f_{\tau\mathbf{Y}}[\Lambda_1(t_1)] \lambda_2(t_2) f_{\tau\mathbf{Y}}[\Lambda_1(t_1) + \Delta\Lambda_2(t_1, t_2)] \\ \times \lambda_3(t_3) f_{\tau\mathbf{Y}}[\Lambda_1(t_1) + \Delta\Lambda_2(t_1, t_2) + \Delta\Lambda_3(t_2, t_3)] dt_1 dt_2 dt_3 \quad (53)$$

(please see Theorem 6).

For illustration purpose, we take $\Lambda_j(t) = e^{\alpha_j t} - 1$ for $j = 1, 2, 3, 4$ (note that Λ_5 is not involved in $R_{2,5}(t)$) and consider four different sets of parameters α_j 's, which are described in Table 1.

As can be seen, α_j (and hence λ_j) increases with the set number, and for each parameter set, Assumption (25) is true. Based on the fact that $R_{2,5}(t) = \mathbb{P}(\tau^{(4)} > t)$, we hence expect $R_{2,5}(t)$ to decrease with the set number due to Proposition 30. This is illustrated in Figure 5, where $R_{2,5}(t)$ is plotted for the four parameter sets (including the independence case which corresponds to the first set of parameters), where we can see that the results are coherent with Proposition 30 and Remark 31.

Example 2 We next consider a 5-component system with a bridge structure, as shown in Figure 6.

The corresponding signature is given by $\mathbf{p} = (0, 1/5, 3/5, 1/5, 0)$, please see Example 4.1 page 50 in [17]. We hence derive from Proposition 9 that

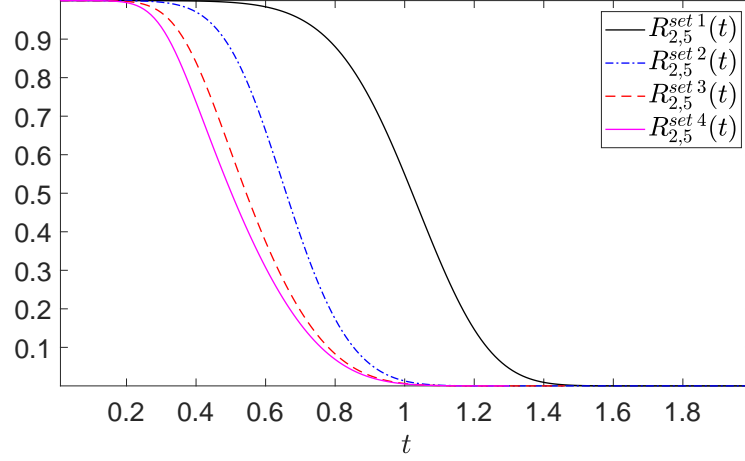


Figure 5: The reliability function $R_{2,5}(t)$ for the different parameter sets of Example 1 (2-out-of-5 system)

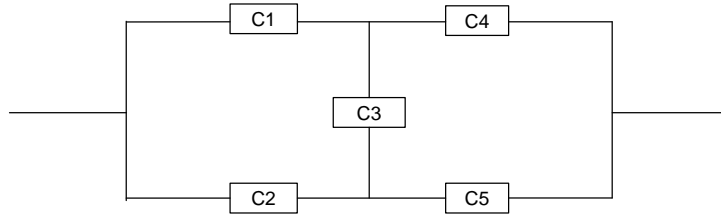


Figure 6: The bridge structure

the reliability function is

$$R(t) = \frac{1}{5}R_{2,5}(t) + \frac{3}{5}R_{3,5}(t) + \frac{1}{5}R_{4,5}(t)$$

where

$$R_{k,5}(t) = \sum_{m=0}^{5-k} R_m(t)$$

for $k \in \{2, 3, 4\}$ and the $R_m(t)$'s are given in (50)–(53). This easily provides

$$\begin{aligned} R(t) &= \frac{1}{5} \sum_{m=0}^3 R_m(t) + \frac{3}{5} \sum_{m=0}^2 R_m(t) + \frac{1}{5} \sum_{m=0}^1 R_m(t) \\ &= R_0(t) + R_1(t) + \frac{4}{5}R_2(t) + \frac{1}{5}R_3(t) \end{aligned}$$

We take again $\Lambda_j(t) = e^{\alpha_j t} - 1$ for $j = 1, 2, 3, 4$ and consider $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_4 = 4$ as a first parameter set and $\alpha_1 = 1, \alpha_2 = 1.75, \alpha_3 = 5, \alpha_4 = 5$ as a second parameter set.

We can see that the corresponding time-scaling functions both fulfill Assumption (25). However we do not have $\lambda_j^1 \leq \lambda_j^2$ for all j , nor $\lambda_j^1 \geq \lambda_j^2$ for all j so that we cannot apply the results of Proposition 30 and Remark 31. The corresponding reliability functions are plotted in the left plot of Figure 7 together with their difference in the right plot. We can see that the two reliability functions are crossing, showing that they are not comparable.

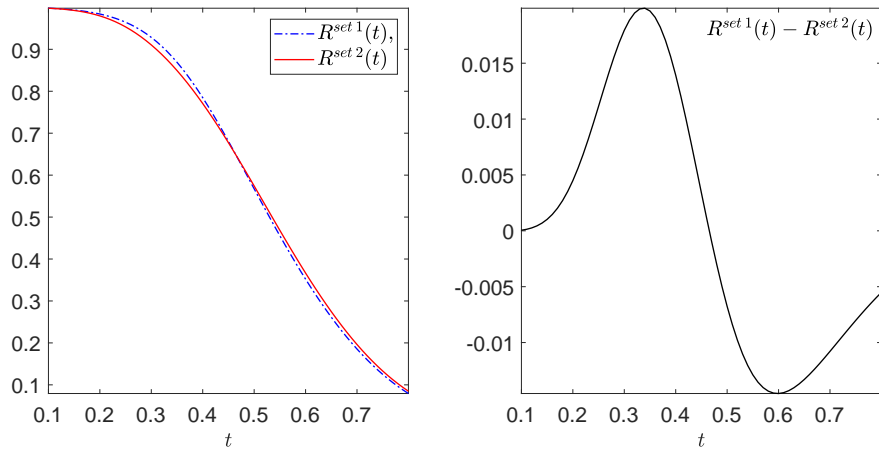


Figure 7: The reliability functions for the two parameter sets of Example 2 (left plot) together with their difference (right plot)

As a conclusion from Example 2, Assumption $\lambda_j^1 \leq \lambda_j^2$ for all j is a necessary condition to derive the stochastic comparison results from Proposition 30 and Remark 31.

8 Conclusion and perspectives

Mimicking some existing lifetime models where failures among components entail an increase in the failure rates of surviving components, we have here suggested to consider Lévy-deteriorating components, with an increase in the time-scaling functions of the Lévy processes for the surviving components. The reliability of a coherent system with components linked by this specific

stochastic dependence has been computed, together with the joint distribution of the component lifetimes. Some dependence and comparison results have been provided and illustrated considering inverse Gaussian processes as baseline Lévy processes.

We remind that for the stochastic comparison result of Proposition 30, we have not been able to conclude about the necessity of Assumption (25) for the result to hold. Hence, the question remains open whether this assumption is necessary, and this requires further research. Also, this paper has focused on the most usual multivariate stochastic order for comparison purpose. It should be possible to consider other types of multivariate stochastic orders and try to derive other results. Beyond that, one could also think about other types of conditions to derive stochastic comparison results, such as conditions on the Lévy measures of the baseline Lévy processes, in a similar spirit as [3].

Furthermore, beyond the few papers quoted in the introduction [7, 24, 25] and the present paper, there remains many work left for adapting previous stochastic dependence models (following the vocabulary of [10]) to the case of Lévy-deteriorating components. At first, we have here chosen to model the impact of a failure as a change in the time-scaling function of the Lévy processes. Other possibilities include the multiplication by a given function of the Lévy process itself or maybe use similar ideas as those developed by [9] for sequential order statistics of lifetimes. Also, we could consider both sudden increase in the component deterioration levels as in [24, 25] and some change in the Lévy process for the surviving components as in the present paper. Other ideas include trying to adapt existing models for cascading failures, load-sharing or spare-part-sharing systems, starting for instance from the literature provided in [10].

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