Recovering the velocity in a 1-d non-local transport equation

Sylvain Ervedoza, Jiacheng Zhang, Zhiqiang Wang

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Sylvain Ervedoza† Zhiqiang Wang‡ Jiacheng Zhang§

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Abstract

In this article, we consider an inverse problem for the non-local system $\partial_t\rho+\lambda(W(t))\partial_x\rho=0$, in which $W(t)=\displaystyle\int_0^1\rho(x,t)dx$ is the total mass of the system. We propose an algorithm and derive a formula to reconstruct the velocity function $\lambda(\cdot)$, assumed to be strictly positive, in an interval $[W_-,W_+]$ which contains the initial total mass $W(0)$, by suitably choosing the influx condition $u(t)=\lambda(W(t))\rho(0,t)$ and measuring the outflux $y(t)=\lambda(W(t))\rho(1,t)$. Some numerical experiments are provided to illustrate the performance of our method.

Keywords Transport equation, inverse problem, non-local velocity.

AMS subject classifications 35Q49, 35R30, 93B52

1 Introduction

1.1 Main results

Our work is motivated by problems arising in the control of semiconductor manufacturing systems which are characterized by their highly re-entrant character, see [1]. Such model is described by the following partial differential equation, corresponding to a 1-d non-local transport equation:

$$\begin{cases}\partial_t\rho+\lambda(W(t))\partial_x\rho=0, & (x,t)\in\ (0,1)\times(0,T),\\ W(t)=\displaystyle\int_0^1\rho(x,t)dx, & t\in\ (0,T),\\ u(t)=\lambda(W(t))\rho(0,t), & t\in\ (0,T),\\ y(t)=\lambda(W(t))\rho(1,t), & t\in\ (0,T),\\ \rho(x,0)=\rho_0(x), & x\in\ (0,1),\end{cases}\tag{1.1}$$

Here, $T>0$, $\lambda:\mathbb{R}\mapsto(0,\infty)$ is a C^1 continuous function, ρ_0 is the initial data which belongs to $L^p(0,1)$(for some $p>1$) and $\rho(\cdot,\cdot)$ is the corresponding weak solution of (1.1) which is defined in Section 2. In the manufacturing system the natural control input is the influx $u(\cdot)$, and the output is the outflux $y(\cdot)$.

The well-posedness of the open-loop system (1.1) with known influx u is considered in [6], see also [12] for a generalization to the case of a velocity depending on x and $W(t)$ of the form $\lambda(x,W(t))$, and [8] for global well-posedness for closed loop versions (that is with u given in terms of y) of (1.1).

In our work, we assume that λ is not known (but still satisfies the a priori condition $\lambda:\mathbb{R}\mapsto(0,\infty)$ and $\lambda\in C^1(\mathbb{R})$), and our goal is to reconstruct it. In order to do that, we assume the following:

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†Institut de Mathématiques de Bordeaux UMR 5251, Université de Bordeaux and CNRS, F-33405 Talence, France. E-mail: sylvain.ervedoza@math.u-bordeaux.fr

‡School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, China. E-mail: wzq@fudan.edu.cn

§School of Mathematical Sciences, Fudan University, Shanghai 200433, China. Institut de Mathématiques de Bordeaux UMR 5251, Université de Bordeaux, F-33405 Talence, France. E-mail: zhangjc18@fudan.edu.cn

Assumption 1.1. *The initial mass $W(0) = W_0$ is known.*

Assumption 1.2. *There exists $p > 1$ such that $\rho_0 \in L^p(0,1)$ and there exists a positive constant m such that ρ_0 satisfies*

$$\rho_0(x) \geqslant m, \quad a.e\ x \in (0,1). \tag{1.2}$$

Assumption 1.3. *We can measure the output y, and design the input u in terms of y, so that u at time t depends only on $y(t)$.*

The main result of this paper is the following:

Theorem 1.1. *Assume that $\lambda : \mathbb{R} \mapsto (0,\infty)$ is a $C^1(\mathbb{R})$ continuous function and that Assumptions 1.1-1.2–1.3 hold. Let $W_1 > W_0$ and $\alpha : [0,\infty) \to [1,\infty)$ be a continuous non-decreasing function which is larger than 1 when $t > 0$.*

Let us consider the following algorithm:

Step 1. While $t > 0$ is such that

$$\int_0^t y(s)\,ds < W_0,$$

we impose

$$u(t) = y(t) \tag{1.3}$$

in the equation (1.1).

We call T_0 the first time $t > 0$ in which

$$\int_0^t y(s)\,ds = W_0.$$

Step 2. For $t > T_0$, we impose

$$u(t) = \alpha(t - T_0)y(t), \tag{1.4}$$

in the equation (1.1), that we solve up to the time T_1 defined by

$$\int_{T_0}^{T_1} (\alpha(s - T_0) - 1)y(s)ds = W_1 - W_0. \tag{1.5}$$

This algorithm enjoys the following properties:

1. *The times T_0 and T_1 are finite.*
2. *The function $t \mapsto W(t)$ is known on the whole time interval $[0, T_1]$ and is given by*

$$W(t) = W_0 + \int_0^t (u(s) - y(s))ds, \quad t \in [0, T_1]. \tag{1.6}$$

Besides, for all $t \in [0, T_1]$, $W(t) \in [W_0, W_1]$ and W is surjective on $[W_0, W_1]$. The function W is constant on $[0, T_0]$ and increasing on $[T_0, T_1]$.

3. *For every $t \in [T_0, T_1]$, there exists a unique $f(t) \in [0, t)$ such that*

$$\int_{f(t)}^t y(s)ds = W(f(t)). \tag{1.7}$$

This function f is C^1 and increasing on $[T_0, T_1]$.

4. *The function $t \mapsto \lambda(W(t))$ on $[T_0, T_1]$ satisfies*

$$\begin{cases} \lambda(W_0) = \dfrac{1}{T_0}, \\ \lambda(W(t)) = f'(t)\lambda(W(f(t))), \quad t \in [T_0, T_1]. \end{cases} \tag{1.8}$$

Accordingly, we can recover the velocity λ on the whole interval $[W_0, W_1]$ from the information given by the measurement y on the whole time interval $[0, T_1]$.

Remark 1.1. *In Assumption 1.2, the condition (1.2) with $m > 0$ can be replaced by $\rho_0(x) \geqslant 0, a.e. x \in (0,1)$, see Section 4.1 by allowing to choose u under the form $u(t) = y(t) + v(t)$ for some time, where $v \in C^0[0,\infty) \cap L^1(0,\infty)$ is a decreasing positive function.*

Remark 1.2. *If one wants to reconstruct the velocity λ in an interval I of the form $I = [W_-, W_+]$ with $0 < W_- < W_0 < W_+$, we can put ourselves in the situation of Theorem 1.1 by doing as follows. For $k \in (0,1)$, we impose $u(t) = ky(t)$ for $t \in (0, T_*)$ in (1.1), where T_* is the first time such that*

$$W_0 + \int_0^t (u(s) - y(s))\, ds = W_-.$$

It is easy to check that, if $\lambda : \mathbb{R} \mapsto (0,\infty)$ is a $C^1(\mathbb{R})$ function and Assumptions 1.1-1.2–1.3 hold, then T_* is finite, $W(T_*) = W_-$, and assumptions 1.1-1.2–1.3 still hold at time T_*. Therefore, shifting $t = T_*$ to $t = 0$, we can apply Theorem 1.1 to reconstruct λ on $[W_-, W_+]$.

1.2 Related references

There is a wide range of literature on inverse problems for partial differential equations. We refer for instance to the textbooks [2, 10, 11] for an overview of the domain and presentation of some relevant results for inverse problems for hyperbolic equations, and in particular the wave equation.

In the context of the recovery of the velocity in a transport equation, we refer in particular to [4] and [5]. In [4], the authors consider the inverse problems of determining the velocity function c in the quasi-linear transport equation

$$u_t + c(u)u_x = 0. \tag{1.9}$$

Under some assumptions on the monotonicity of the initial and boundary data, and the knowledge of the output boundary conditions, the inverse problem is proved to have a unique solution, which is well-posed or ill-posed depending on which side the additional measurement is performed. We also point out the discussion on the determination of source terms in quasilinear transport equations in [4, 5].

We are not aware of many results in the same spirit, and we believe that Theorem 1.1 is the first one to address the recovery of the velocity in a non-local transport equation of the form (1.1).

In fact, so far, equation (1.1) has been mainly discussed from the controllability and stabilization points of view. Regarding controllability properties, we refer in particular to [6, 12] and in the context of networks, to [9].

The output feedback stabilization (i.e. with a closed loop control) for the system (1.1) under a suitable feedback law is another natural and interesting control problem, which has been studied in [7], where it has been proved that if $\overline{\rho}$ is the target constant states around which we want to stabilize, for $k \in (-1,1)$, the proportional controller

$$u(t) - \bar{\rho}\lambda(\bar{\rho}) = k(y(t) - \bar{\rho}\lambda(\bar{\rho})), \quad t \in (0,\infty), \tag{1.10}$$

stabilizes the closed-loop system (1.1) if and only if

$$d := \frac{\overline{\rho}\lambda'(\overline{\rho})}{\lambda(\overline{\rho})} \text{ satisfies } d > -1. \tag{1.11}$$

We also refer to [7] for the spectral analysis of the linearized model, to [3] for the global exponential stabilization result in $L^2(0,1)$ for some classes of velocity functions, or to [8] for a time-varying feedback control to achieve a semi-global stabilization for (1.1) under the assumption that $s \mapsto s\lambda(s)$ is injective (which, locally around $\overline{\rho}$, only means $d \neq -1$).

Outline. The rest of the paper is organized as follows. In Section 2, we recall the well-posedness results for the problem (1.1). The proof of Theorem 1.1 is given in Section 3. Some extensions of Theorem 1.1 are discussed in Section 4. Finally in Section 5, we give some numerical experiments to illustrate the performances of the algorithm proposed in Theorem 1.1.

2 Preliminaries

Before considering the reconstructing problem, we need to recall the usual definition of a weak solution for (1.1) and give the corresponding well-posedness result for our closed-loop system.

Definition 2.1. *Let $p > 1$ and ρ_0 be a non-negative function in $L^p(0,1)$. Let $T > 0$, α be a positive function in $L^\infty(0,T)$ and v be a non-negative function in $L^p(0,T)$. A weak solution of the Cauchy problem* (1.1) *with boundary condition*

$$u(t) = \alpha(t)y(t) + v(t), \quad t \in (0,T) \tag{2.1}$$

is a function $\rho \in C^0([0,T]; L^p(0,1))$ such that for every $s \in (0,T]$ and every $\varphi \in C^1([0,1] \times [0,s])$ satisfying

$$\varphi(x,s) = 0, \quad \forall x \in [0,1],$$

one has:

$$\int_0^s \int_0^1 \rho(x,t)(\partial_t \varphi(x,t) + \lambda(W(t))\partial_x \varphi(x,t))\, dxdt + \int_0^s (\alpha(t)y(t) + v(t))\varphi(0,t)\, dt - \int_0^s y(t)\varphi(1,t)\, dt + \int_0^1 \rho_0(x)\varphi(x,0)\, dx = 0. \tag{2.2}$$

Applying a fixed point argument as in [8, Theorem 2.1], we can get the following well-posedness result:

Theorem 2.1. *Assume that $\lambda : \mathbb{R} \mapsto \mathbb{R}_+^*$ is a C^1 function. Let $p > 1$ and ρ_0 be a non-negative function in $L^p(0,1)$. Let α be a positive function $L^\infty_{loc}(0,\infty)$ and v be a non-negative function in $L^p_{loc}(0,\infty)$. There exists a unique time T^* and a unique weak solution $\rho \in C^0([0,T^*); L^p(0,1))$ of system* (1.1)–(2.1). *Moreover, we have*

$$W \in W^{1,p}_{loc}([0,T^*)), \quad y \in L^p_{loc}(0,T^*). \tag{2.3}$$

In this case, T^* is finite if and only if $\lim_{T \to T^*} \|\rho(\cdot,t)\|_{L^p(0,1)} = \infty$.

Remark 2.1. *In our setting, the initial data ρ_0 and the boundary data v are non-negative. Consequently, the solution ρ is non-negative in $(0,L) \times (0,T^*)$ and u, y are non-negative in $(0,T^*)$.*

Remark 2.2. *In the following, we will check that the maps $t \mapsto \|\rho(\cdot,t)\|_{L^p(0,1)}$ and $t \mapsto W(t)$ do not blow up in the reconstruction process of Theorem 1.1, which, according to Theorem 2.1, entails that the solution of* (1.1) *complemented with* (1.3) *on $(0,T_0)$ and* (1.4) *on (T_0,T_1) exist up to T_1.*

Remark 2.3. *For later use, let us also point out that, for solutions ρ of* (1.1)–(2.1)*, integrating in x the equation $(1.1)_{(1)}$, we easily get*

$$\frac{dW(t)}{dt} = u(t) - y(t), \quad \text{in} \quad \mathcal{D}'(0,T). \tag{2.4}$$

3 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1.

The time interval $[0,T_0)$. We first focus on the time interval $[0,T_0)$. During that time interval, the equations (1.1) are completed with the boundary conditions (1.3), so that from (2.4),

$$\forall t \in [0,T_0), \quad W(t) = W_0,$$

and the velocity $\lambda(W(t))$ is constant equal to $\lambda(W_0)$ for all $t \in [0,T_0)$.

Note that we also have that the norm $\|\rho(\cdot,t)\|_{L^p(0,1)}$ is constant for $t \in [0,T_0)$ since the boundary conditions (1.3) amounts to solve a transport equation at constant speed on a torus, so that the solution is well-defined up to the time T_0 according to Theorem 2.1. In fact, for this argument to be made rigorous, we should set $\tilde{T}_0$ as the supremum of the times $\tau > 0$ such that $\sup_{t \in [0,\tau]} \|\rho(\cdot,t)\|_{L^p(0,1)} < \infty$ and $\displaystyle\int_0^\tau y(s)\, ds \leqslant W_0$, and check that this supremum $\tilde{T}_0$ necessarily satisfies $\tilde{T}_0 = \infty$ or $\tilde{T}_0 < \infty$ and $\displaystyle\int_0^{\tilde{T}_0} y(s)\, ds = W_0$. We will check next that $\tilde{T}_0$ is necessarily finite, so that it coincides with the time T_0 given in Theorem 1.1.

Using the characteristic method and Assumption 1.2, we easily get that ρ is bounded from below by m for all $(x,t) \in [0,1] \times [0,\tilde{T}_0)$, and thus that

$$\forall t \in [0,\tilde{T}_0), \quad y(t) = \lambda(W(t))\rho(1,t) = \lambda(W_0)\rho(1,t) \geqslant m\lambda(W_0) > 0.$$

Accordingly, the map $t \mapsto \displaystyle\int_0^t y(s)\,ds$ is strictly increasing and is bounded from below by $t \mapsto m\lambda(W_0)t$. This implies that $\tilde{T}_0 = T_0$ is finite and

$$\int_0^{T_0} y(s)\,ds = W_0. \tag{3.1}$$

The time interval $[T_0, T_1)$. We start by noticing that $\rho(\cdot, T_0)$ is still bounded from below by m and belongs to $L^p(0,1)$ with the same p as in Assumption 1.2.

We then introduce $\tilde{T}_1$ as the supremum of the times $\tau > T_0$ such that $\sup_{t\in[T_0,\tau]} \|\rho(\cdot,t)\|_{L^p(0,1)} < \infty$ and $\displaystyle\int_{T_0}^{\tau} (\alpha(s-T_0)-1)y(s)\,ds \leqslant W_1 - W_0$, and check that this supremum $\tilde{T}_1$ necessarily satisfies $\tilde{T}_1 = \infty$ or $\tilde{T}_1 < \infty$ and $\displaystyle\int_{T_0}^{\tilde{T}_1} (\alpha(s-T_0)-1)y(s)\,ds = W_1 - W_0$.

First, it is easy to check that for $t \in (T_0, \tilde{T}_1)$, $\rho(\cdot,t)$ stays bounded from below by m, y stays positive and, thus, according to (2.4), W is increasing. Together with (2.4), the choice (1.5) imposes that W is necessarily smaller than W_1 on the time interval $[T_0, \tilde{T}_1)$. Therefore, for all $t \in [T_0, \tilde{T}_1)$, $W(t) \in [W_0, W_1]$.

Since the velocity λ is bounded from below by $\inf_{s\in[W_0,W_1]}\{\lambda(s)\}$ and from above by $\sup_{s\in[W_0,W_1]}\{\lambda(s)\}$ on the time interval $[T_0, \tilde{T}_1)$, the map $t \mapsto \|\rho(\cdot,t)\|_{L^p(0,1)}$ is bounded in any time interval of the form $[T_0,\tau]$ for τ finite and smaller than $\tilde{T}_1$, thus guaranteeing that $\tilde{T}_1 = \infty$ or $\tilde{T}_1 < \infty$ and can be alternatively defined by (1.5).

But, since $\rho(\cdot,t)$ stays bounded from below by m and W stays in the interval $[W_0, W_1]$,

$$\forall t \in [T_0, \tilde{T}_1), \quad y(t) \geqslant m \inf_{s\in[W_0,W_1]}\{\lambda(s)\} > 0. \tag{3.2}$$

Accordingly, $t \mapsto \displaystyle\int_{T_0}^t (\alpha(s-T_0)-1)y(s)\,ds$ is strictly increasing for $t \in [T_0, \tilde{T}_1]$ and, for all $\epsilon > 0$, for all $t \in [T_0+\epsilon, \tilde{T}_1)$,

$$\int_{T_0}^t (\alpha(s-T_0)-1)y(s)\,ds \geqslant (\alpha(\epsilon)-1)(t-(T_0+\epsilon))m \inf_{s\in[W_0,W_1]}\{\lambda(s)\}.$$

Since $\alpha(\epsilon) > 1$, we conclude that necessarily $\tilde{T}_1 = T_1$ is finite and can be alternatively defined by (1.5).

Properties of W. Note that, in the two above paragraphs, we also proved that W is non-decreasing, that it is surjective on $[W_0, W_1]$ and takes value in $[W_0, W_1]$ on $[0, T_1]$. The formula (1.6) directly comes from (2.4). The fact that then W is known at all times then comes from the fact that u is explicitly given in terms of y in the time interval $[T_0, T_1]$. In particular, we have

$$W(t) = \begin{cases} W_0, & \text{for } t \in [0,T_0], \\ W_0 + \displaystyle\int_{T_0}^t (\alpha(s-T_0)-1)y(s)\,ds, & \text{for } t \in [T_0,T_1]. \end{cases} \tag{3.3}$$

In fact, using (3.2), we also have that W is strictly increasing in $[T_0, T_1]$.

Existence and uniqueness of $f(t)$ *for* $t \in [T_0, T_1]$ *satisfying* (1.7). We introduce the function

$$F(t,\tau) = \int_\tau^t y(s)\,ds - W(\tau), \quad (t,\tau) \in [0,T_1]^2. \tag{3.4}$$

Note that W is non-decreasing on $[0, T_1]$, always larger than W_0, and y is positive on $[0, T_1]$. Recalling the property (3.1), we easily get that for $t \in [T_0, T_1]$, $F(t,0) \geqslant 0 > F(t,t)$. Also, since $W \in W^{1,p}(0,T_1)$ (recall Theorem 2.1) and is non-decreasing on $[0,T_1]$, and y is strictly positive on $[0,T_1]$, for $t \in [0,T_1]$, $\tau \mapsto F(t,\tau)$

is continuous and strictly decreasing with respect to $\tau \in [0,t]$. By continuity and strict monotonicity, we conclude that for any fixed $t \in [T_0, T_1]$, there exists a unique $f(t) \in [0,t)$ such that $F(t, f(t)) = 0$, i.e., (1.7) holds.

Now we investigate the continuity and regularity of the function f on $[T_0, T_1]$. In order to do that, we rewrite F as

$$F(t,\tau) = h(t) - g(\tau), \quad \text{with } h(t) = \int_0^t y(s)\,ds \ \text{ and } \ g(\tau) = \int_0^\tau y(s)\,ds + W(\tau). \tag{3.5}$$

The above arguments show that g is continuous on $[0, T_1]$, and strictly increasing. Consequently, g has a continuous strictly increasing inverse g^{-1} defined on $[g(0), g(T_1)]$. Noting that $h(t) \in [g(0), g(T_1)]$, $t \in [T_0, T_1]$, f is simply given by

$$f(t) = g^{-1}(h(t)), \quad t \in [T_0, T_1].$$

Since $h \in W^{1,p}(0,T)$, it is continuous, so that f is continuous on $[T_0, T_1]$. Besides, since h is strictly increasing on $[0, T_1]$, f is also strictly increasing on $[T_0, T_1]$.

Differentiability of f. To get that f is in fact differentiable, it is convenient to characterize f as follows: for all $t \in [T_0, T_1]$, $f(t)$ coincides with $\tilde{f}(t)$ defined as the unique element of $[0, T_1]$ such that

$$\int_{\tilde{f}(t)}^t \lambda(W(s))\,ds = 1. \tag{3.6}$$

Indeed, since $s \mapsto \lambda(W(s))$ is continuous and strictly positive, for all $t \in [T_0, T_1]$, there exists at most one element $\tilde{f}(t)$ such that (3.6) holds. Now, by the characteristic method, for $t \in [T_0, T_1]$, setting $\tilde{t} \in [0, T_1]$ such that

$$\int_{f(t)}^{\tilde{t}} \lambda(W(s))\,ds = 1,$$

we have, if $\tilde{t} > t$, i.e. $\displaystyle\int_{f(t)}^t \lambda(W(s))\,ds < 1$,

$$\begin{aligned} W(f(t)) = \int_{f(t)}^t y(s)\,ds &= \int_{f(t)}^t \lambda(W(s))\rho(1,s)\,ds \\ &= \int_{f(t)}^t \lambda(W(s))\rho\left(1 - \int_{f(t)}^s \lambda(W(\theta))d\theta, f(t)\right) ds \\ &= \int_{1-\int_{f(t)}^t \lambda(W(\theta))d\theta}^1 \rho(x, f(t))\,dx, \end{aligned}$$

and, with similar computations, if $\tilde{t} < t$, i.e. $\displaystyle\int_{f(t)}^t \lambda(W(s))\,ds > 1$,

$$\begin{aligned} W(f(t)) = \int_{f(t)}^t y(s)\,ds &= \int_{f(t)}^{\tilde{t}} \lambda(W(s))\rho(1,s)ds + \int_{\tilde{t}}^t \lambda(W(s))\rho(1,s)\,ds \\ &= \int_0^1 \rho(x, f(t))\,dx + \int_{\tilde{t}}^t \lambda(W(s))\rho(1,s)\,ds. \end{aligned}$$

Since, for $t \in [T_0, T_1]$,

$$W(f(t)) = \int_0^1 \rho(x, f(t))\,dx,$$

by definition of W, and ρ and $\lambda(W)$ are strictly positive, we necessarily have that $\tilde{t} = t$, that is, from the definition of $\tilde{t}$,

$$\int_{f(t)}^t \lambda(W(s))\,ds = 1.$$

Therefore, for all $t \in [T_0, T_1]$, we have $f(t) = \tilde{f}(t)$, where $\tilde{f}(t)$ is defined by (3.6).

It follows that, equivalently to (1.7), f can be defined by the implicit relation

$$k(t, f(t)) = 0, \quad \text{where } k(t,\tau) = \int_\tau^t \lambda(W(s))\, ds - 1, \text{ for } t \in [T_0, T_1] \text{ and } \tau \in [0, T_1]. \tag{3.7}$$

Since W is continuous on $[0, T_1]$, k is C^1 on $[T_0, T_1] \times [0, T_1]$. Since we also have that $\partial_\tau k(t,\tau) = -\lambda(W(\tau))$ does not vanish for $(t,\tau) \in [T_0, T_1] \times [0, T_1]$, f is thus C^1 on $[T_0, T_1]$ by the implicit function theorem and satisfies the equation

$$\lambda(W(t)) - \lambda(W(f(t)))f'(t) = 0, \quad \text{for } t \in [T_0, T_1].$$

This proves items 3 and 4 of Theorem 1.1, except for the relation $\lambda(W_0) = 1/T_0$ that we now prove.

Indeed, we easily check that $f(T_0) = 0$, since W_0 satisfies (3.1). Accordingly, we have, from (3.7), that

$$\int_0^{T_0} \lambda(W(s))\, ds = 1.$$

But, for all $s \in [0, T_0]$, $W(s) = W_0$ and $\lambda(W(s)) = \lambda(W_0)$. Therefore, we obtain from the previous relation that $T_0\lambda(W_0) = 1$, as announced. This concludes the proof of Theorem 1.1.

4 Extensions of Theorem 1.1

In this section, we give two extensions of Theorem 1.1. The first one provides a relaxation of the condition (1.2) in Assumption 1.2. The second one discusses the choice of the function α in the algorithm of Theorem 1.1.

4.1 Relaxation of Assumption 1.2

In Assumption 1.2, the condition $\rho_0(x) \geqslant m > 0, a.e.\ x \in (0,1)$ can be replaced by $\rho_0(x) \geqslant 0, a.e.\ x \in (0,1)$ according to the following proposition.

Proposition 4.1. *Assume $p > 1$, $\lambda : \mathbb{R} \mapsto (0,\infty)$ is a $C^1(\mathbb{R})$ function and that the initial data $\rho_0 \in L^p(0,1)$ is non-negative. Let*

$$u(t) = y(t) + v(t), \quad t \in (0,\infty), \tag{4.1}$$

where v is a positive function in $C^0([0,\infty)) \cap L^1(0,\infty)$. Then define T^* by

$$T^* = \sup\left\{\tau \geqslant 0 \text{ such that } \int_0^\tau y(s)ds = W_0\right\} \tag{4.2}$$

where y is the outflux corresponding to the closed-loop system (1.1)-(4.1). Moreover, for any $T > T^*$, there exists a positive constant m_T depending on T such that

$$\rho(x,T) \geqslant m_T, \quad a.e\ x \in (0,1), \quad \text{and} \quad W(T) = W_0 + \int_0^T v(t)\, dt.$$

Remark 4.1. *A simple example of function v in $C^0([0,\infty)) \cap L^1(0,\infty)$ satisfying the assumption of Proposition 4.1 is given by $v(t) = \frac{1}{1+t^2}$ for $t \in [0,\infty)$.*

Proof. Below, we will not discuss the fact that the solution of (1.1)–(4.1) is well-defined for all times. The proof of this result can be done easily, using Theorem 2.1 and proving that the L^p norm of the solution cannot blow up in finite time, and its proof is left to the reader. The fact that the solution ρ stays non-negative everywhere is also important and is easy to check.

Firstly, we prove that T^* given by (4.2) is finite. Integrating (2.4) and using (4.1) yield that

$$W(t) = W_0 + \int_0^t v(s)ds, \quad t \in [0,\infty).$$

Thus, for all $t \geqslant 0$,

$$0 \leqslant W(t) \leqslant M \triangleq W_0 + \|v\|_{L^1(0,\infty)}, \quad t \in [0,\infty). \tag{4.3}$$

Then $\lambda(W(t))$ is bounded below by $\inf_{s\in[0,M]}\{\lambda(s)\} > 0$. Therefore the characteristic curve starting from the point $(0,0)$ reaches the right boundary $x=1$ in finite time T_*, i.e.,

$$\int_0^{T_*} \lambda(W(s))ds = 1. \tag{4.4}$$

Using the characteristic formula, we immediately get

$$\int_0^{T_*} y(s)ds = \int_0^{T_*} \lambda(W(s))\rho(1,s)ds = \int_0^1 \rho_0(x)dx = W_0. \tag{4.5}$$

For any $T > T_*$, for almost every $x \in (0,1)$, the characteristic curve passing through the point (x,T) and going backwards in time intersects the left boundary $x=0$ at a positive time $\tau(x,T) > 0$ (characterized by $\int_{\tau(x,T)}^T \lambda(W(s))ds = x$). Additionally,

$$\rho(x,T) = \rho(0,\tau(x,T)).$$

By the choice of u in (4.1) and (4.3), and since ρ is non-negative everywhere, we deduce that

$$\rho(x,T) = \rho(0,\tau(x,T)) = \frac{u(\tau(x,T)))}{\lambda(W(\tau(x,T))))} \geqslant \frac{\inf_{s\in[\tau(1,T),T]}\{v(s)\}}{\sup_{s\in[0,M]}\{\lambda(s)\}} > 0.$$

This also implies that for all $t > T_*$, $y(t) > 0$, and thus, for all $T > T_*$,

$$\int_0^T y(s)\,ds > W_0.$$

Accordingly, T_* coincides with T^* given by (4.2). This concludes the proof of Proposition 4.1. □

4.2 Discussion on the choice of the function α in the reconstruction algorithm

The algorithm proposes in Theorem 1.1 provides a formula (1.8) to recover the velocity λ, which requires, to obtain the value of λ at $W(t)$, to know the value of λ at an early time $W(f(t))$, where $f(t)$ is computed by (1.7).

The discussion below aims at indicating how the choice of the function α in Theorem 1.1 can guarantee that

$$\forall t \in [T_0,T_1], \quad f(t) \in [0,T_0]. \tag{4.6}$$

Indeed, in such case, for all $t \in [T_0,T_1]$, $W(f(t))$ is simply given by W_0 and the computations of $f(t)$ in (1.7) and of $\lambda(W(t))$ in (1.8) become easier; at least from the numerical point, they will generate less numerical errors (see Section 5 for some examples).

Discussion 1. The case of a constant α. In this case, we get the following result:

Proposition 4.2. *Under the assumptions of Theorem 1.1.*
If α is a constant function, condition (4.6) *holds if and only if*

$$\alpha \geqslant \frac{W_1}{W_0}. \tag{4.7}$$

Proof. Recall from Theorem 1.1 that f is increasing on $[T_0,T_1]$. Therefore, condition (4.6) holds if and only if

$$f(T_1) \leqslant T_0. \tag{4.8}$$

Recalling the definition (1.5) of T_1, we get

$$\int_{T_0}^{T_1} y(s)ds = \frac{W_1 - W_0}{\alpha - 1}$$

On the other hand, since W is constant equal to W_0 on $[0, T_0]$ and strictly increasing on $[T_0, T_1]$, y is strictly positive, and

$$\int_{f(T_1)}^{T_1} y(s)ds = W(f(T_1)),$$

the condition (4.8) holds if and only if

$$\int_{T_0}^{T_1} y(s)ds \leqslant W_0.$$

Accordingly, condition (4.6) is equivalent to the condition (4.7). □

Discussion 2. When α is not constant. We find the following sufficient condition to (4.6).

Proposition 4.3. *Under the assumptions of Theorem* (1.1)*, with $\alpha : [0, \infty) \to [1, \infty)$ being a continuous non-decreasing function which is strictly larger than* 1 *when $t > 0$.*

If the function α is such that there exists $\tau > 0$ such that

$$\int_{T_0}^{T_0+\tau} (\alpha(s - T_0) - 1)y(s)ds \leqslant W_1 - W_0, \tag{4.9}$$

and

$$\tau \|y\|_{L^\infty(T_0, T_0+\tau)} + \frac{1}{\alpha(\tau) - 1}(W_1 - W_0) \leqslant W_0, \tag{4.10}$$

then condition (4.6) *holds.*

Proof. First, since $(\alpha(s - T_0) - 1)y(s) > 0$ for all $s > T_0$, it follows from (4.9) and (1.5), that

$$\tau \leqslant T_1 - T_0.$$

Moreover, by the monotonicity of α and the definition of T_1 in (1.5), we have

$$\begin{aligned} W_1 - W_0 &= \int_{T_0}^{T_1} (\alpha(s - T_0) - 1)y(s)ds \\ &\geqslant (\alpha(\tau) - 1) \int_{T_0+\tau}^{T_1} y(s)ds \\ &\geqslant (\alpha(\tau) - 1)\Big(\int_{T_0}^{T_1} y(s)ds - \tau \|y\|_{L^\infty(T_0, T_0+\tau)} \Big). \end{aligned}$$

Therefore, if (4.10) holds, we have

$$\int_{T_0}^{T_1} y(s)ds \leqslant W_0,$$

Since for all $t \in [0, T_1]$, $W(t) \in [W_0, W_1]$ by Theorem 1.1, we necessarily have

$$\int_{f(T_1)}^{T_1} y(s)ds = W(f(T_1)) \geqslant W_0 \geqslant \int_{T_0}^{T_1} y(s)ds.$$

Since y is positive, we thus necessarily have $f(T_1) \leqslant T_0$, which entails (4.6) since f is non-decreasing. This concludes the proof of Proposition 4.3. □

In practice, it is easy to construct α satisfying the conditions of Proposition 4.3 if we additionally have rough estimates on $\|\rho_0\|_{L^\infty(0,1)}$ and $\|\lambda\|_{C^0([W_0,W_1])}$ of the form $\|\rho_0\|_{L^\infty(0,1)} \leqslant M_0$ and $\|\lambda\|_{C^0([W_0,W_1])} \leqslant L$. Indeed, we can choose $\tau > 0$ sufficiently small such that

$$\tau M_0 L \leqslant \frac{W_0}{2} \quad \text{and} \quad \tau L \leqslant 1. \tag{4.11}$$

The second condition guarantees, that for $t \in [T_0, T_0 + \tau]$, the characteristic going backwards in time and starting from $(1, t)$ reaches $x = 0$ for a time $t \leqslant T_0$, and then, since $u(s) = y(s)$ for $s \in [0, T_0]$, we easily get

$$\rho(1, t) \leqslant M_0, \quad t \in [T_0, T_0 + \tau].$$

In particular, we get

$$\tau\|y\|_{L^\infty(T_0,T_0+\tau)} \leqslant \frac{W_0}{2}.$$

Choose then the value of $\alpha(\tau)$ such that

$$\frac{2W_1}{W_0} - 1 \leqslant \alpha(\tau) \leqslant \frac{W_1 - W_0}{\tau M_0 L} + 1,$$

Then (4.10) and (4.9) are both satisfied. Note that such choice is always possible due to the first condition in (4.11).

Here is an example of a continuous non-decreasing function α strictly larger than 1:

$$\alpha_{c,\tau}(t) = 1 + (c-1)\frac{t}{\tau}, \text{ for } t > 0, \quad \text{with } c \in \left[\frac{2W_1}{W_0} - 1, \frac{W_1 - W_0}{\tau M_0 L} + 1\right].$$

It is then easy to check that for

$$\beta \geqslant \frac{2L(W_1 - W_0)}{W_0 \min\{1, W_0/(2M_0)\}},$$

the function α_β given by

$$\alpha_\beta(t) = 1 + \beta t, \text{ for } t > 0, \tag{4.12}$$

coincides with $\alpha_{c,\tau}$ for $c = 2W_1/W_0 - 1$ and $\tau = 2(W_1 - W_0)/(W_0\beta)$ satisfying (4.11).

Of course, this suggests that, when we do not know any a priori guess on $\|\rho_0\|_{L^\infty(0,1)}$ and $\|\lambda\|_{C^0([W_0,W_1])}$, we might simply consider choosing $\alpha = \alpha_\beta$ in (4.12) for sufficiently large β.

5 Numerics

5.1 Numerical implementations

The numerical scheme is implemented as follows.

For $N \in \mathbb{N}^*$, the space interval $(0,1)$ is divided into N parts, corresponding to a mesh size $\Delta x = 1/N$. The corresponding time discretization parameter is $\Delta t = \Delta x/\lambda_M$, where $\lambda_M \geqslant \max\{\lambda(s), s \in \mathbb{R}\}$, or at least $\lambda_M \geqslant \max\{\lambda(s), s \in [W_0, W_1]\}$, in which $[W_0, W_1]$ is the interval in which we want to reconstruct λ (note that this requires an a priori information on the L^∞ bound of λ on the interval of interest).

In the following, we assume that the initial condition ρ^0 satisfies the assumption 1.2 and is continuous, so that we can approximate it by a discrete function in a reasonable sense, and that

$$W_0 = \Delta x \sum_{k=1}^{N} \rho^0(k\Delta x),$$

and we set $\rho_k^0 = \rho^0(k\Delta x)$ for all $k \in \{0, \cdots, N\}$.

The function ρ is then approximated by a sequence of vectors $(\vec{\rho}^{\,n})_{n\in\mathbb{N}} = (\rho_0^n, \cdots, \rho_N^n)_{n\in\mathbb{N}}$, in the sense that ρ_m^n denotes the discrete approximation of the value of ρ at (t_n, x_m) where $t_n = n\Delta t$ and $x_m = m\Delta x$. Similarly, $(W^n)_{n\in\mathbb{N}}$ is a sequence of real number approximating W at time t_n and given by

$$W^n = \Delta x \sum_{k=1}^{N} \rho_k^n, \qquad (n \in \mathbb{N}).$$

Corresponding to the algorithm presented in Theorem 1.1, we also introduce a continuous non-decreasing function $\alpha : (0,\infty) \mapsto (1,\infty)$.

We then use the following explicit solver, given for $n \in \mathbb{N}$ by

$$\begin{cases} \rho_m^{n+1} = \rho_m^n - \dfrac{\Delta t \lambda(W^n)}{\Delta x}(\rho_m^n - \rho_{m-1}^n), \qquad m = 1, \cdots, N, \\ \rho_0^{n+1} = \begin{cases} \rho_N^{n+1}, & \text{if } t_{n+1} \leqslant T_0, \\ \alpha(t_{n+1} - T_0)\rho_N^{n+1} & \text{if } t_{n+1} > T_0, \end{cases} \\ \rho_m^0 = \rho_0(m\Delta x), \qquad \text{for } m = 0, \cdots, N, \\ \text{with } W^n = \Delta x \displaystyle\sum_{k=1}^{N} \rho_k^n, \end{cases} \tag{5.1}$$

where T_0 is given by

$$T_0 = t_{n_0} + \frac{W_0 - \Delta t \sum_{n=0}^{n_0-1} y(t_n)}{y(t_{n_0})},$$

and where t_{n_0} is such that

$$\Delta t \sum_{n=0}^{n_0-1} y(t_n) < W_0 \leqslant \Delta t \sum_{n=0}^{n_0} y(t_n).$$

Note that this amounts to approximate the observation y by the step function which is equal to $\lambda(W(t_n))\rho_{N+1}^n$ on $[t_n, t_{n+1})$.

We stop the algorithm (5.1) when

$$W^n > W_1,$$

which guarantees that $t_n > T_1$ in some sense (recall that, in the continuous setting, $W(T_1) = W_1$ and that W is strictly increasing on $[T_0, T_1]$).

Note that, since ρ_0 satisfies Assumption 1.2 and is continuous, it is easy to check that $(\rho_m^n)_{n\in\mathbb{N},m\in\{0,\cdots,N\}}$ solving (5.1) stays positive. This is crucial to guarantee the fact that T_0 and T_1 are finite, and the existence and uniqueness of the various quantities defined below and obtained from the direct simulation of (5.1) (which can be proved in the same lines as the proof of Theorem 1.1).

From now on, we assume that we have computed the solution (5.1) and that only $(y(t_n))_{n\in\mathbb{N}}$ and W^0 are now known, where $y(t_n) = \lambda(W^n)\rho_m^n$.

Note that W^n can be computed immediately by the formula

$$W^n = \begin{cases} W^0 & \text{if } t_n < T_0, \\ W^0 + \Delta t \displaystyle\sum_{k=n_0}^{n-1} (\alpha(t_k - T_0) - 1) y(t_k) & \text{if } t_n \geqslant T_0. \end{cases}$$

and is non-decreasing.

In the following, since the natural class for y is rather $L^p(0, T_1)$ and for W is rather $W^{1,p}(0, T)$, we do our computations by thinking that y is a piecewise constant function taking value $y(t_k)$ on the interval $[t_k, t_{k+1})$, and W as a continuous piecewise linear function which coincides with $t \mapsto W^k + (t - t_k)(W^{k+1} - W^k)/\Delta t$ on the interval $[t_k, t_{k+1}]$.

For any mesh point $t_k \in [T_0, T_1]$, we then define $f(t_k)$ as follows. We introduce n_k as the unique positive integer satisfying

$$W^{n_k} \leqslant \Delta t \sum_{n=n_k}^{k} y(t_n), \quad \text{and} \quad \Delta t \sum_{n=n_k+1}^{k} y(t_n) \leqslant W^{n_k+1},$$

where the uniqueness is deduced from the positivity of y and the monotonicity of W, and we then set $f(t_k)$ as the unique solution of

$$W^{n_k} + (f(t_k) - t_{n_k}) \left(\frac{W^{n_k+1} - W^{n_k}}{\Delta t} \right) = \Delta t \sum_{n=n_k}^{k} y(t_n) - (f(t_k) - t_{n_k}) y(t_{n_k}).$$

This yields

$$f(t_k) = t_{n_k} + \Delta t \left(\frac{\Delta t \sum_{n=n_k}^{t_k} y(t_n) - W^{n_k}}{(\Delta t) y(t_{n_k}) + W^{n_k+1} - W^{n_k}} \right). \tag{5.2}$$

Finally, considering the approximation of $\lambda(W)$ by a linear piecewise continuous function, we approximate $\lambda(W(f(t_k)))$ as follows

$$\lambda(W(f(t_k))) := \lambda(W^{n_k}) + (f(t_k) - t_{n_k}) \left(\frac{\lambda(W^{n_k+1}) - \lambda(W^{n_k})}{\Delta t} \right). \tag{5.3}$$

Formula (1.8) is then approximated as follows: For any mesh point $t_k \in [T_0, T_1]$,

$$\lambda(W^k) = \begin{cases} \lambda(W_0) & \text{if } t_k = T_0, \\ \lambda(W_0) \dfrac{f(t_k) - 0}{t_k - T_0} & \text{if } t_{k-1} \leqslant T_0 < t_k, \\ \lambda(W(f(t_k))) \left(\dfrac{f(t_k) - f(t_{k-1})}{\Delta t} \right) & \text{if } t_{k-1} > T_0. \end{cases} \tag{5.4}$$

5.2 Numerical tests

In this part, we will use Matlab to test numerically the algorithm of Theorem 1.1 for several choices of λ.

We take the initial value ρ_0 given by $\rho_0(x) = \sin(\pi x) + 1$, whose corresponding total mass is $W(0) = \frac{2}{\pi} + 1 \approx 1.6366$. We choose $W_1 = 2W_0$, which means our reconstructing interval is $[W(0), 2W(0)]$, i.e about $[1.6366, 3.2732]$.

We will consider the following choices of λ:

$$\begin{aligned} \lambda_1(s) &= 3 + 2s, \\ \lambda_2(s) &= 2 + \sin(10s), \\ \lambda_3(s) &= 2 + \sin\Big(\frac{10}{2-s}\Big)(2-s), \\ \lambda_4(s) &= ([s] \mod 2) + 2. \end{aligned} \tag{5.5}$$

Note that λ_1, λ_2 belong to C^1 and satisfy the assumptions of Theorem 1.1, while λ_3 and λ_4 do not: indeed, λ_3 presents strong oscillations and λ_4 presents discontinuities.

5.2.1 Example 1: Choosing a constant α in the algorithm of Theorem 1.1

In this part, we choose a constant α in the algorithm in Theorem 1.1: see Figure 1 in the case $\alpha = 3$.

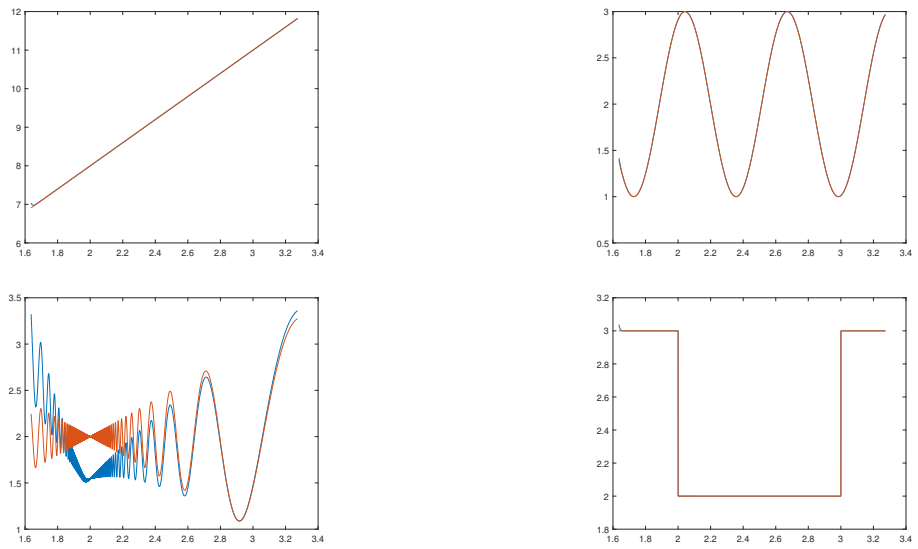

Figure 1: Plots of the velocities reconstructed using the Algorithm presented in Theorem 1.1 discretized as explained in Section 5.1, with the choice $\alpha = 3$, for the velocities λ in (5.5). Top left corresponds to λ_1, top right to λ_2, bottom left to λ_3 and bottom right to λ_4. The reconstructed velocity is plotted in blue, and the true velocity in red.

For λ_1, λ_2 and λ_4, the red curve and the blue curve almost overlap except for a small error close to W_0, probably due to some instability of the formula (5.4) close to T_0.

For λ_3, the algorithm does not manage to catch the strong oscillations near $s = 2$, which results in a significant error close to $s = 2$. It seems a bit surprising however, that the algorithm seems to be able to catch rather correctly the velocity after this singularity.

The case of λ_4 also performs well, even if λ_4 presents discontinuities which are in principle not compatible with the regularity properties needed for the formula (1.8) to make sense. This is probably due to the fact, that since λ_4 is piecewise smooth, the singularities on $f(t)$ should be sparse, and the formula (1.8) makes sense almost everywhere.

We also consider the choice $\alpha = 2$ in the algorithm of Theorem 1.1, see Figure 2 for the recovery of λ_2. Note that $\alpha = 2$ is the limiting case in the formula (4.7).

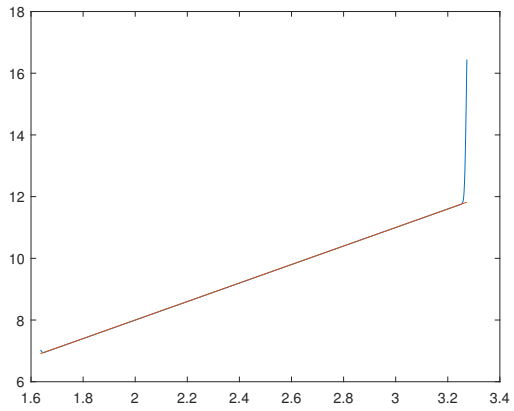

Figure 2: Plots of the velocities reconstructed using the Algorithm presented in Theorem 1.1 discretized as explained in Section 5.1, with the choice $\alpha = 2$, for the velocity λ_2 in (5.5). The reconstructed velocity is plotted in blue, and the true velocity in red.

Figure 2 exhibits a large error in the reconstruction of λ_2 near W_1. A simple explanation could be that the computation of λ_2 at W_1 is done by considering the characteristic of the equation (1.1) started from $(x,t) = (0,T_0)$, which is precisely a point in which the boundary conditions is multiplied by the factor α, and is thus discontinuous. Accordingly, the numerical scheme is very likely quite imprecise there, thus yielding this large error in the reconstruction process.

5.2.2 Example 2: Choosing a linear α in the algorithm of Theorem 1.1

In this section, we choose a linear function α

$$\alpha(t) = 1 + \beta t, \quad t \geqslant 0, \tag{5.6}$$

where β is a positive constant.

Figures 3 and 4 show the numerical reconstructions corresponding to the choice α as in (5.6) for the choices $\beta = 1$ and $\beta = 200$ in the algorithm of Theorem 1.1, which yields rather similar results as the ones plotted in Figure 1.

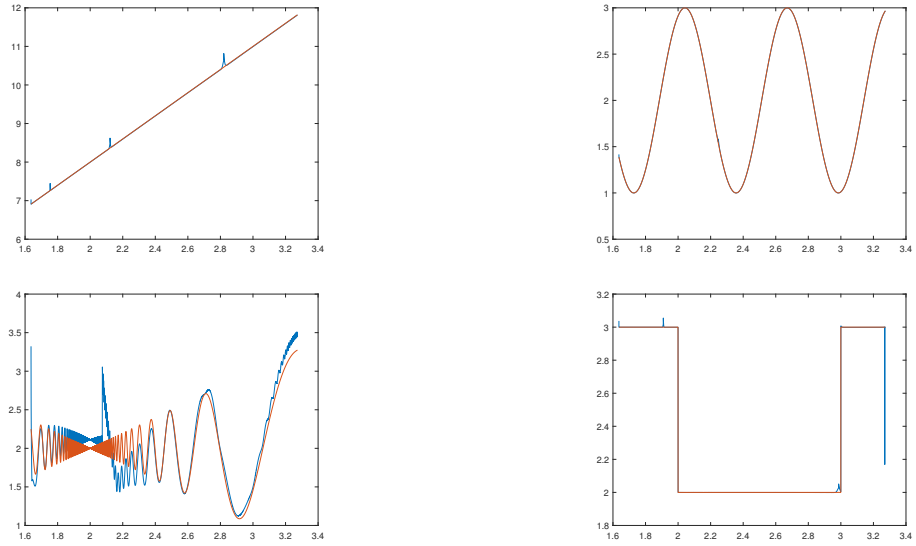

Figure 3: Plots of the velocities reconstructed using the Algorithm presented in Theorem 1.1 discretized as explained in Section 5.1, with the choice $\alpha(t) = 1 + t$, for the velocities λ in (5.5). Top left corresponds to λ_1, top right to λ_2, bottom left to λ_3 and bottom right to λ_4. The reconstructed velocity is plotted in blue, and the true velocity in red.

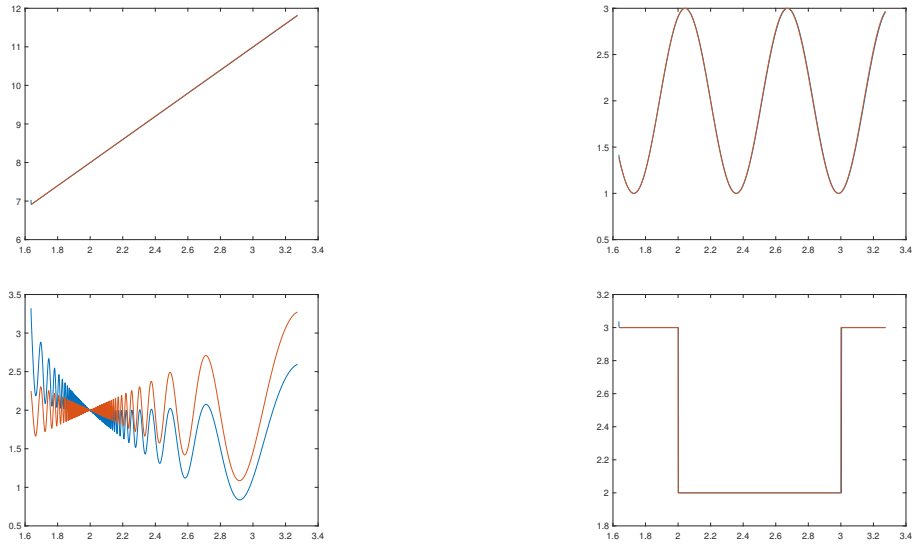

Figure 4: Plots of the velocities reconstructed using the Algorithm presented in Theorem 1.1 discretized as explained in Section 5.1, with the choice $\alpha(t) = 1 + 200t$, for the velocities λ in (5.5). Top left corresponds to λ_1, top right to λ_2, bottom left to λ_3 and bottom right to λ_4. The reconstructed velocity is plotted in blue, and the true velocity in red.

When $\alpha(t) = 1 + t$, we see some kind of regular fluctuations in the plots (for instance on Top Right, in Figure 3). These are due to the error transported by the characteristic line starting from $(x, t) = (0, T_0)$, similar to what we saw in the previous section in Figure 2.

As plotted in Figure 4, choosing β larger in (5.6), for instance $\beta = 200$, this shortcoming can be corrected. This is of course related to the discussion in Section 4.2.

5.2.3 Convergence rates

To conclude our study, we also provide a rough analysis on the convergence of the reconstructed velocities in the aforementioned cases. Firstly, we define the discrete L^2 error between the true velocity λ and the reconstructed one $\widehat{\lambda}$ by the formula

$$\|\lambda - \widehat{\lambda}\|_{\ell^2(W_0,W_1)} = \sqrt{\sum_{i=i_1}^{i_2} (\lambda(W^i) - \widehat{\lambda}(W^i)|_{t=t_i})^2 (W(t_{i+1}) - W(t_i))} \tag{5.7}$$

where $i_1 \in \mathbb{N}$ and $i_2 \in \mathbb{N}$ are the indices such that $\{t_i\}_{i\in\{i_1,\cdots,i_2\}}$ are all the mesh points in $[T_0, T_1]$. Taking λ_2 in (5.5) for example, we can compute the error for different N, which are given in Table 1.

N	100	200	500	1000	2000	5000	10000
error of $\alpha = 3$	0.3558	0.1835	0.0753	0.0383	0.0195	0.0081	0.0042
error of $\alpha = 1 + t$	0.2201	0.1143	0.0482	0.0252	0.0133	0.0059	0.0032
error of $\alpha = 1 + 200t$	1.2027	0.6824	0.2847	0.1435	0.0721	0.0288	0.0144

Table 1: Tables of the errors $\|\lambda - \widehat{\lambda}\|_{\ell^2(W_0,W_1)}$ with different N for the choices $\alpha(t) = 3$, $\alpha(t) = 1 + t$, and $\alpha(t) = 1 + 200t$, in the case $\lambda = \lambda_2$ in (5.5).

Figure 5 displays the plot of the convergence rates of the algorithm in logarithmic scales, underlining the fact that all choices of α have similar convergence rates, all of them being roughly of order 1.

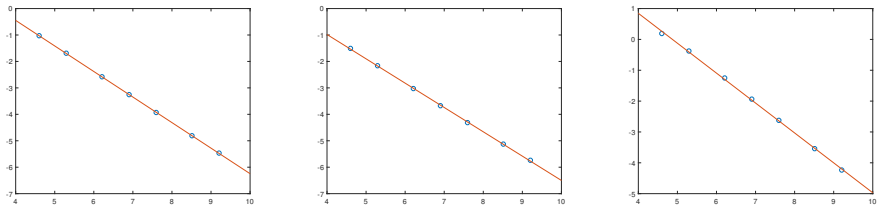

Figure 5: Plot of the errors $\|\lambda - \widehat{\lambda}\|_{\ell^2(W_0,W_1)}$ versus N in logarithmic scales for the choices $\alpha(t) = 3$ (left, the linear regression has slope=-0.9663), $\alpha(t) = 1 + t$ (middle, the linear regression has slope=-0.9199), and $\alpha(t) = 1 + 200t$ (right, the linear regression has slope=-0.9695), in the case $\lambda = \lambda_2$ in (5.5).

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