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Ordinal measures of the set of finite multisets

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Abstract

Well-partial orders, and the ordinal invariants used to measure them, are relevant in set theory, program verification, proof theory and many other areas of computer science and mathematics. In this article we focus on a common data structure in programming, finite multisets of some well partial order. There are two natural orders one can define on the set of finite multisets of a partial order: the multiset embedding and the multiset ordering. Though the maximal order type of these orders is already known, other ordinal invariants remain mostly unknown. Our main contributions are expressions to compute compositionally the width of the multiset embedding and the height of the multiset ordering. Furthermore, we provide a new ordinal invariant useful for characterizing the width of the multiset ordering.

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Introduction

Measuring partial orders is useful in many domains, from set theory to proof theory, including infinitary combinatorics, program verification, rewriting theory, proof automation and many more.

There are intuitive notions of measure for a partial order when it is finite: its cardinal obviously, but also its height (the length of a maximal chain) or its width (the length of a maximal antichain). Similar notions exist for infinite partial orders, as long as they are *well partial orders* (**wpo**), i.e., well-founded partial orders with no infinite antichains [10, 12]. Two such notions are the *ordinal height*, which is the order type of a maximal chain, and the *maximal order type* (**mot**), which is the order type of a maximal linearisation, a notion introduced by De Jongh and Parikh in order to measure hierarchies of functions [6]. These are transfinite measures, hence we call them *ordinal invariants*. Kříž and Thomas introduced alternative characterizations for **mot** and ordinal height, which naturally led to the definition of a third ordinal invariant, *ordinal width* [11]. Less studied than its counterparts, the width of a **wpo** relates to its antichains, even though it cannot be defined as the order type of a maximal antichain. While exploring techniques for program termination, Blass and Gurevich rediscovered these characterizations with a game-theoretical point of view [4].

Ordinal invariants of **wpos** have also been used to prove complexity bounds. In the last decade there has been a flurry of complexity results for the verification of well-structured transition systems (**wsts**), i.e., transition systems whose set of configurations is a **wpo** and whose transitions respect this ordering [5]. When a **wsts** is based on a **wpo** X of maximal order type ω^α , one can expect the complexity of coverability to be in H_{ω^α} in the Hardy hierarchy, or in F_α in the fast-growing hierarchy [9]. This bound can be refined by looking at controlled antichains instead of controlled bad sequences [14], thus bounding complexity with width instead of maximal order type.



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46 **Computing ordinal invariants compositionally.** Many wpos underlying wsts are
 47 built from classical operations on simpler wpos whose invariants are known. This has spurred
 48 new interest in measuring the ordinal invariants of various well-ordered data structures: De
 49 Jongh and Parikh computed the **mot** of the disjoint sum and the Cartesian product of wpos
 50 [6]. Schmidt then computed the **mot** of word embedding and homeomorphic tree embedding
 51 on a wpo [13]. Abraham and Bonnet pursued this line of study by computing the height of
 52 Cartesian product, but also the width of disjoint sum and lexicographic product [1]. For
 53 a complete survey of these results see [8], where Džamonja et al. computed the ordinal
 54 invariants of the lexicographic product, but also the height of the multiset word and tree
 55 embeddings.

56 **Finite multisets:** In this article, we study the ordinal invariants of the set of finite
 57 multisets. Multisets, also called “bags”, a common data structure in computer science.
 58 Informally, a finite multiset over a set X is a finite subset of X where an element can appear
 59 finitely many times. For instance, $\langle a, a, b \rangle$ denote the multiset where a appears twice and b
 60 once. One can see the set of finite multisets on a wpo as the set of finite words quotiented by
 61 the equivalence relation “equality up to some permutation”. It comes down to describing a
 62 multiset as a word where the order of terms is irrelevant. A finite multiset can be represented
 63 by a function from X to \mathbb{N} with finite support, which associates its multiplicity with each
 64 element.

65 Two orderings are classically defined on the finite multisets of any ordered set. The first
 66 one is the *multiset ordering*, which often appears in rewriting theory and automation of
 67 termination proofs [7]. The other, less-known, ordering is the *multiset embedding*, or term
 68 ordering as it is called in [15]. It was presented by Aschenbrenner and Pong as a natural
 69 extension of the embedding order over finite words [3].

70 Some invariants of these two orderings have already been measured: Van der Meeren,
 71 Rathjen, and Weiermann [15] built on [17] to compute the **mot** of the set of finite multisets
 72 on a wpo X ordered with the multiset ordering, and provided a new proof for the expression
 73 of the **mot** of the multiset embedding computed in [18]. Džamonja et al. [8] proved that
 74 the height of the multiset embedding is equal to the height of the set of finite words ordered
 75 with word embedding. It is noteworthy that these three results give expressions that are
 76 functional in (i.e., can be expressed as a function of) the **mot** and height of X . However, the
 77 height of the multiset ordering still needs to be determined, and the width remains unstudied
 78 for both orderings.

79 **Our contributions:** In this article, we provide functional expressions for the width of
 80 the multiset embedding (Theorem 2.1) and the height of the multiset ordering (Theorem 3.1).

81 We further show that the width of the multiset ordering cannot be expressed as a function
 82 of the three ordinal invariants (Example 3.2). Nonetheless, we get around this issue by
 83 introducing a fourth ordinal invariant, the *friendly order type* (Definition 3.3), in which the
 84 width of the multiset ordering is functional (Theorem 3.4). We then proceed to investigate
 85 and compute this new ordinal invariant.

86 **1** Definitions and state of the art

87 **1.1** Width, height and maximal order type

88 A sequence x_1, \dots, x_n, \dots on a partial order (X, \leq_X) is *good* when there exist $i < j$ such
 89 that $x_i \leq_X x_j$, otherwise it is a *bad* sequence. An *antichain* is a sequence whose elements
 90 are pairwise incomparable.

91 A *well partial order* (**wpo**) is a partial order that has no infinite bad sequences. Equival-
 92 ently, a **wpo** is a partial order that is both well-founded (i.e. no infinite strictly decreasing
 93 sequences) and has no infinite antichains.

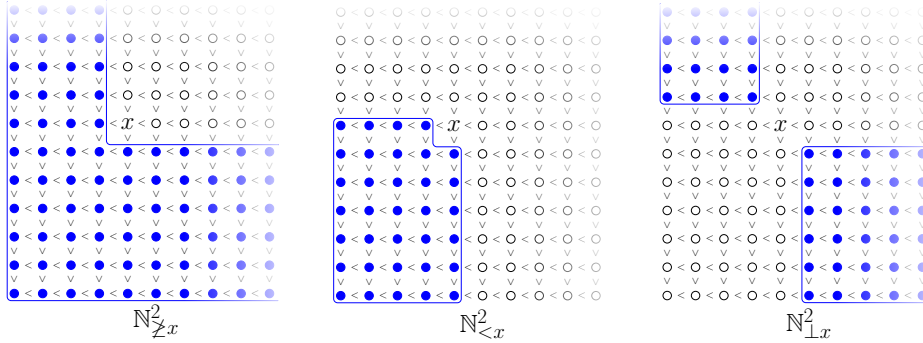
94 Let (X, \leq_X) be a **wpo**. We often write just X when \leq_X is understood. The trees $Bad(X)$,
 95 $Dec(X)$ and $Ant(X)$ are defined as the sets of bad sequences, strictly decreasing sequences,
 96 and antichains of X , respectively, ordered by inverse prefix order (a sequence is smaller than
 97 its prefixes) ([11, 8]). The finiteness of bad sequences, strictly decreasing sequences and
 98 antichains in a **wpo** implies that these trees are well-founded. Therefore, one can define a
 99 notion of rank on these trees: a sequence has rank 0 when it cannot be extended; otherwise
 100 its rank is the smallest ordinal strictly larger than the ranks of its extensions. The rank of a
 101 tree is the rank of the empty sequence (which is the root of the tree).

102 The *maximal order type* (or **mot**) of X , denoted by $\mathbf{o}(X)$, is defined as the rank of $Bad(X)$.
 103 Similarly, the *height* $\mathbf{h}(X)$ and the *width* $\mathbf{w}(X)$ of X are defined as the ranks of $Dec(X)$ and
 104 $Ant(X)$, respectively. Together, $\mathbf{o}(X)$, $\mathbf{h}(X)$ and $\mathbf{w}(X)$ are called the *ordinal invariants* of
 105 X .

106 For any **wpo** X , $Dec(X)$ and $Ant(X)$ are subtrees of $Bad(X)$. Thus $\mathbf{h}(X) \leq \mathbf{o}(X)$ and
 107 $\mathbf{w}(X) \leq \mathbf{o}(X)$.

108 Let $x \perp y$ denote that x and y are incomparable. For a relation $*$ among $\{ \not\leq, <, \perp \}$, we
 109 define the residual X_{*x} as $\{ y \in X : y * x \}$. This definition can be extended to subsets
 110 $S \subseteq X$: $X_{*S} \stackrel{\text{def}}{=} \{ y \in X : \forall x \in S, y * x \}$.

111 ► **Example 1.1.** In Figure 1, you can see the residuals at $x = (4, 6)$ of $\mathbb{N} \times \mathbb{N}$ ordered
 112 component-wise.



■ **Figure 1** Residuals of \mathbb{N}^2 at $(4, 6)$.

113 Since the rank of the empty sequence is the smallest ordinal strictly larger than the ranks
 114 of the sequences of length 1, the definitions of **mot**, height and width can be reformulated
 115 inductively through the following *residual equations*:

116
$$\mathbf{o}(X) = \sup_{x \in X} (\mathbf{o}(X_{\geq x}) + 1) \tag{Res-o}$$

117
$$\mathbf{h}(X) = \sup_{x \in X} (\mathbf{h}(X_{<x}) + 1) \tag{Res-h}$$

118
$$\mathbf{w}(X) = \sup_{x \in X} (\mathbf{w}(X_{\perp x}) + 1) \tag{Res-w}$$

120 With these equations we can compute easily the ordinal invariants of \mathbb{N}^2 . For instance,
 121 observe that $(\mathbb{N}^2_{<x})$ is finite for any $x \in \mathbb{N}^2$, so its height is finite but can be arbitrarily big.
 122 Hence $\mathbf{h}(\mathbb{N}^2) = \omega$.

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1.2 Ordinal arithmetic

We suppose well-known the notions of sum, product, subtraction, natural sum, natural product on ordinals, denoted with $+$, \cdot , $-$, \oplus , \otimes [2]. However, let us recall some definitions and notations that might be less familiar to the reader.

An ordinal α is *indecomposable* iff for any $\delta, \gamma < \alpha$, we have $\delta \oplus \gamma < \alpha$. Equivalently, α is indecomposable when there is an ordinal β such that $\alpha = \omega^\beta$. α is an ϵ -number when $\alpha = \omega^\alpha$.

The *Hessenberg-based product* $\alpha \odot \beta$ is defined inductively as follows [1]:

$$\alpha \odot 0 = 0, \quad \alpha \odot (\beta + 1) = (\alpha \odot \beta) \oplus \alpha, \quad \alpha \odot \beta = \sup\{\alpha \odot \gamma : \gamma < \beta\} \text{ for limit } \beta.$$

This definition ensures that $\alpha \cdot \beta \leq \alpha \odot \beta \leq \alpha \otimes \beta$.

For any ordinal $\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_n}$, let $\hat{\alpha} \stackrel{\text{def}}{=} \omega^{\alpha'_1} + \dots + \omega^{\alpha'_n}$, where α'_i is $\alpha_i + 1$ when α_i is the sum of an ϵ -number and a finite ordinal, otherwise $\alpha'_i = \alpha_i$.

For any ordinals α, β , let $\alpha \hat{\oplus} \beta \stackrel{\text{def}}{=} \sup\{\alpha' \oplus \beta' : \alpha' < \alpha, \beta' < \beta\}$.

1.3 Ordinal invariants of basic data structures

For any wpos P, Q , the *disjoint sum* $P \sqcup Q$ is the disjoint union of P and Q ordered such that elements of P and Q cannot be compared together, whereas the *direct sum* $P + Q$ is the disjoint union of P and Q ordered such that for all $p \in P, q \in Q, p \leq q$. For a family of wpos $(A_i)_{i < \alpha}$, let $\Sigma_{i < \alpha} A_i$ denote the direct sum of the A_i s along the ordinal α .

The *Cartesian product* $P \times Q$ is the set of pairs $(p, q) \in P \times Q$ where elements are compared component-wise. The *lexicographic product* of P along Q , written $P \cdot Q$, has the same support as $P \times Q$, with a different ordering: $(p, q) \leq_{P \cdot Q} (p', q')$ iff $q <_Q q'$, or $q = q'$ and $p \leq_P p'$.

Sums and products are the most basic operations on wpos one can find. Their ordinal invariants are easy to compute compositionally (see Table 1), with the notable exception of the width of the Cartesian product which cannot be expressed as a function of the ordinal invariants its factors [16].

■ **Table 1** How to compute ordinal invariants compositionally, [8]. See Section 1.2 for definitions of $\hat{\oplus}$ and \odot

Space X	M.O.T. $\mathfrak{o}(X)$	Height $\mathfrak{h}(X)$	Width $\mathfrak{w}(X)$
$A \sqcup B$	$\mathfrak{o}(A) \oplus \mathfrak{o}(B)$	$\max(\mathfrak{h}(A), \mathfrak{h}(B))$	$\mathfrak{w}(A) \oplus \mathfrak{w}(B)$
$A + B$	$\mathfrak{o}(A) + \mathfrak{o}(B)$	$\mathfrak{h}(A) + \mathfrak{h}(B)$	$\max(\mathfrak{w}(A), \mathfrak{w}(B))$
$A \times B$	$\mathfrak{o}(A) \otimes \mathfrak{o}(B)$	$\mathfrak{h}(A) \hat{\oplus} \mathfrak{h}(B)$	<i>(Not functional)</i>
$A \cdot B$	$\mathfrak{o}(A) \cdot \mathfrak{o}(B)$	$\mathfrak{h}(A) \cdot \mathfrak{h}(B)$	$\mathfrak{w}(A) \odot \mathfrak{w}(B)$

1.4 Comparing wpos

A widely-used and intuitive relation between wpos is the *reflection* relation. A mapping between wpos $f: (A, \leq_A) \rightarrow (B, \leq_B)$ is a *reflection* if $f(x) \leq_B f(y)$ implies $x \leq_A y$, i.e. it is

152 a morphism from (A, \leq_A) to (B, \leq_B) Let $A \hookrightarrow B$ denote that there is a reflection from A to
 153 B .

154 However, in this article, we prefer to use the stronger notions of augmentations and
 155 substructures.

156 ► **Definition 1.2** (Substructure, augmentation). *A wpo (A, \leq_A) is a substructure of a wpo
 157 (B, \leq_B) whenever $A \subseteq B$ and \leq_A is the restriction of \leq_B to A . This relation is written
 158 $A \leq_{st} B$. Similarly (A, \leq_A) is an augmentation of (B, \leq_B) whenever $A = B$ and $\leq_B \subseteq \leq_A$.
 159 We write this relation $A \geq_{aug} B$.*

160 Obviously, $A \leq_{st} B$ or $A \geq_{aug} B$ imply $A \hookrightarrow B$.

161 We often abuse these notations and write $A \leq_{st} B$ (resp. $B \leq_{aug} A$) to mean that A is
 162 isomorphic to a substructure (resp. an augmentation) of B .

163 We denote by $A \equiv B$ that (A, \leq_A) is isomorphic to (B, \leq_B) .

164 In this article, when we consider a subset Y of a wpo X , it is understood that $Y \leq_{st} X$,
 165 i.e. Y is ordered with \leq_X restricted to the subset.

166 These notions of augmentations and substructures allow us to compare the ordinal
 167 invariants of wpos.

168 ► **Lemma 1.3.** *Let A and B be wpos.*

169 *If $A \leq_{st} B$ then $\mathbf{i}(A) \leq \mathbf{i}(B)$ for $\mathbf{i} \in \{ \mathbf{o}, \mathbf{h}, \mathbf{w} \}$.*

170 *If $A \geq_{aug} B$ then $\mathbf{o}(A) \leq \mathbf{o}(B)$ and $\mathbf{w}(A) \leq \mathbf{w}(B)$. However $\mathbf{h}(A) \geq \mathbf{h}(B)$.*

171 The substructure and augmentation relations are monotonous through most operations
 172 on wpos. For instance, if $A \leq_{st} A'$, then $A \times B \leq_{st} A' \times B$.

173 An ordinal, as defined by Von Neumann, is the linear wpo that contains all smaller
 174 ordinals. Thus augmentations and substructures relations can also be used to compare
 175 directly ordinals to wpos. The following result is well-known:

176 ► **Proposition 1.4.** *For any wpo X , $\mathbf{h}(X)$ and $\mathbf{o}(X)$ are the largest ordinals such that
 177 $\mathbf{h}(X) \leq_{st} X$ and $\mathbf{o}(X) \geq_{aug} X$.*

178 1.5 Orderings on the set of finite multisets

179 We assume familiarity with finite multisets and the associated operations as used in [17]:
 180 union, intersection and subtraction, denoted by \cup, \cap and \setminus , respectively. Let $\langle x_1, \dots, x_n \rangle$
 181 denote the finite multiset that contains the elements x_1, \dots, x_n (they do not have to be
 182 distinct). For any $k \in \mathbb{N}$, $m \times k$ means the union of k copies of m . Let $|m|$ denote the number
 183 of elements of a multiset m .

184 There are two main orderings classically defined on the set of finite multisets $M(X)$ of a
 185 partial order X :

186 ► **Definition 1.5** (Multiset embedding [18]). *The multiset embedding on $M(X)$, also known
 187 as the term ordering, is defined as:*

188 $m \leq_{\diamond} m'$ iff there exists $f : m \rightarrow m'$ injective such that for any $x \in m$, $x \leq f(x)$.

► **Definition 1.6** (Multiset ordering [17]). *The multiset ordering on $M(X)$ is defined as:*

$$m \leq_r m' \iff m = m' \text{ or } \forall x \in m \setminus (m \cap m'), \exists y \in m' \setminus (m \cap m'), x < y.$$

189 We write $M^{\diamond}(X)$ for $(M(X), \leq_{\diamond})$ and $M^r(X)$ for $(M(X), \leq_r)$.

190 The multiset ordering and the multiset embedding are both augmentations of the word
 191 embedding on X^* the set of finite words on X . Therefore, according to Higman's lemma [10],

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192 $M^\diamond(X)$ and $M^r(X)$ are wpos when X is. Moreover $M^\diamond(X) \leq_{\text{aug}} M^r(X)$, as was observed
 193 by Aschenbrenner and Pong [3].

194 Observe that if X is a linear ordering, then $M^r(X)$ is linear, while $M^\diamond(X)$ is not as long
 195 as X has more than two elements.

196 ► **Proposition 1.7** (Transformation equations). *For any wpos A and B ,*

$$197 \quad M^*(A \sqcup B) \equiv M^*(A) \times M^*(B) \text{ for } * \in \{\diamond, r\}, \quad (\text{Trans-1})$$

$$198 \quad M^r(A + B) \equiv M^r(A) \cdot M^r(B), \quad (\text{Trans-2})$$

$$199 \quad M^\diamond(A + B) \leq_{\text{aug}} M^\diamond(A) \cdot M^\diamond(B). \quad (\text{Trans-3})$$

201 ► **Lemma 1.8** (Width of $M(X)$ on Γ_k). *For any $k < \omega$, we denote by Γ_k the wpo that
 202 contains k incomparable elements. Then $\mathbf{w}(M^\diamond(\Gamma_k)) = \mathbf{w}(M^r(\Gamma_k)) = \omega^{k-1}$.*

203 **Proof.** Since $M^\diamond(\Gamma_1) \equiv M^r(\Gamma_1) \equiv \omega$, Equation (Trans-1) tells us that $M^\diamond(\Gamma_k)$ and $M^r(\Gamma_k)$
 204 are both isomorphic to the k -fold Cartesian product $\omega \times \cdots \times \omega$. This special case of the
 205 width of a Cartesian product is known [16]: $\mathbf{w}(\omega \times \cdots \times \omega) = \omega^{k-1}$. ◀

206 The augmentation and substructure relations are monotone with respect to the multiset
 207 ordering and multiset embedding:

208 ► **Proposition 1.9.** *Let A, B be two wpos. Then $A \leq_{\text{st}} B$ implies $M^\diamond(A) \leq_{\text{st}} M^\diamond(B)$ and
 209 $M^r(A) \leq_{\text{st}} M^r(B)$. Moreover, $A \geq_{\text{aug}} B$ implies that $M^\diamond(A) \geq_{\text{aug}} M^\diamond(B)$ and $M^r(A) \geq_{\text{aug}}$
 210 $M^r(B)$.*

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212 Van der Meeren, Rathjen and Weiermann computed the mot of $M^\diamond(X)$ and $M^r(X)$.

213 ► **Theorem 1.10** (Mot of multiset embedding [15, 18]). *For any wpo X , $\mathbf{o}(M^\diamond(X)) = \omega^{\widehat{\mathbf{o}(X)}}$.*

214 ► **Theorem 1.11** (Mot of multiset ordering [15, 17]). *For any wpo X , $\mathbf{o}(M^r(X)) = \omega^{\mathbf{o}(X)}$.*

215 Observe that $\omega^{\mathbf{o}(X)} \leq \omega^{\widehat{\mathbf{o}(X)}}$, as one would expect since $M^r(X) \geq_{\text{aug}} M^\diamond(X)$. Further-
 216 more, we expect that $\mathbf{w}(M^r(X)) \leq \mathbf{w}(M^\diamond(X))$, while $\mathbf{h}(M^r(X)) \geq \mathbf{h}(M^\diamond(X))$.

217 ► **Theorem 1.12** (Height of the multiset embedding [8]). *Let X be a wpo.*

218 *Then $\mathbf{h}(M^\diamond(X)) = \mathbf{h}^*(X)$, where*

$$219 \quad \mathbf{h}^*(X) \stackrel{\text{def}}{=} \begin{cases} \mathbf{h}(X) & \text{if } \mathbf{h}(X) \text{ is infinite and indecomposable,} \\ \mathbf{h}(X) \cdot \omega & \text{otherwise.} \end{cases}$$

220 1.6 A tool to compute the width: Quasi-incomparable subsets

221 Of all three ordinal invariants, the width is the less studied, since it has been introduced
 222 more recently, and also the hardest invariant to study for lack of tools.

223 A powerful tool to analyse the width of a wpo is the notion of *quasi-incomparable* subsets
 224 of a wpo, which was first introduced in [16] for the Cartesian product of several ordinals.

225 For any subsets Y, Z of X , let $Y \perp Z$ denote that for every $y \in Y, z \in Z, y \perp z$.

226 ► **Definition 1.13.** *Let A be a wpo, and A_1, \dots, A_n be n subsets of A . Then $(A_i)_{1 \leq i \leq n}$ is a
 227 quasi-incomparable family of subsets of A iff for any $i < n$, for any finite $Y \subseteq A_1 \cup \cdots \cup A_i$,
 228 there exists $A'_{i+1} \subseteq A_{i+1}$ such that $A'_{i+1} \perp Y$ and $A'_{i+1} \equiv A_{i+1}$.*

229 This definition is slightly more restrictive than the one in [16], which only required that
230 $\mathbf{w}(A'_{i+1}) = \mathbf{w}(A_{i+1})$.

231 The idea behind these quasi-incomparable subsets is that sometimes one can slice a wpo
232 A into simpler subsets A_1, \dots, A_n whose width is known, such that $\text{Ant}(A_n) + \dots + \text{Ant}(A_1)$
233 is embedded in $\text{Ant}(A)$. Intuitively, it means that one can combine antichains of A_1, \dots, A_n
234 into one antichain of A .

235 This entails a practical relation between the widths of A and its subsets:

236 ► **Lemma 1.14** ([16]). *Let $(A_i)_{i \leq n}$ be a quasi-incomparable family of subsets of A . Then*
237 $\mathbf{w}(A) \geq \mathbf{w}(A_n) + \dots + \mathbf{w}(A_1)$.

238 **2 Ordinal width of the multiset embedding**

239 In this section we compute the width of $M^\diamond(X)$ for any wpo X , which happens to be
240 functional in the width of X :

241 ► **Theorem 2.1** (Width of the multiset embedding). *For any wpo X , $\mathbf{w}(M^\diamond(X)) = \omega^{\widehat{\mathbf{o}(X)}-1}$.*
242 *(See Section 1.2 for the definition of $\widehat{\alpha}$.)*

243 It is already known that, in some cases, the width of the multiset embedding reaches its
244 mot.

245 ► **Lemma 2.2** ([8]). *If $\mathbf{o}(X)$ is infinite and indecomposable, $\mathbf{w}(M^\diamond(X)) = \mathbf{o}(M^\diamond(X))$.*

246 We focus for now on the set of finite multisets on a linear wpo, i.e., an ordinal. Let us
247 treat first the case of successor ordinals.

248 ► **Lemma 2.3.** *For any successor ordinal $\alpha = \beta + 1$, $\mathbf{w}(M^\diamond(\alpha)) \geq \mathbf{w}(M^\diamond(\beta)) \cdot \omega$.*

249 **Proof.** We denote with $M_{>k}^\diamond(X)$ the subset $\{ m \in M^\diamond(X) : |m| > k \}$ for any $k \in \mathbb{N}$ of
250 $M^\diamond(X)$ for any wpo X , for any $k < \omega$.

251 Let $m_n \stackrel{\text{def}}{=} \langle \beta \rangle \times n$ for any $n \in \mathbb{N}$. According to Equation (Res-w),

$$252 \quad \mathbf{w}(M^\diamond(\alpha)) = \sup \{ \mathbf{w}(M^\diamond(\alpha)_{\perp m}) + 1 : m \in M^\diamond(\alpha) \}$$

$$253 \quad \geq \sup \{ \mathbf{w}(M^\diamond(\alpha)_{\perp m_n}) + 1 : n \in \mathbb{N} \}.$$

255 Let $M_k \stackrel{\text{def}}{=} \{ \langle \beta \rangle \times (n - k) \cup m : m \in M_{>k}^\diamond(\beta) \}$ for $k \in [1, n]$. These subsets of $M^\diamond(\alpha)$ are
256 actually subsets of $M^\diamond(\alpha)_{\perp m_n}$: for all $m \in M_k$, $m \perp m_n$ since $|m| > |m_n|$. Observe also that
257 for any $k \in [1, n]$, $M_k \equiv M^\diamond(\beta)$.

258 Moreover, $(M_k)_{k \in [1, n]}$ is a quasi-incomparable family of subsets of $M^\diamond(\alpha)_{\perp m_n}$: for any
259 $i < n$, for any finite $Y \subset M_1 \cup \dots \cup M_i$, let $s(Y) = \max\{|m|, m \in Y\}$. Observe that M_{i+1}
260 contains $M_{i+1} \cap M_{>s(Y)}^\diamond(\beta)$ which is incomparable to Y , and isomorphic to M_{i+1} .

261 Therefore, $\mathbf{w}(M^\diamond(\alpha)_{\perp m_n}) \geq \mathbf{w}(M_n) + \dots + \mathbf{w}(M_1) = \mathbf{w}(M^\diamond(\beta)) \cdot n$ according to
262 Lemma 1.14. Thus $\mathbf{w}(M^\diamond(\alpha)) \geq \sup \{ \mathbf{w}(M^\diamond(\beta)) \cdot n + 1 : n \in \mathbb{N} \} = \mathbf{w}(M^\diamond(\beta)) \cdot \omega$. ◀

263 ► **Lemma 2.4.** *For any infinite ordinal α , $\mathbf{w}(M^\diamond(\alpha)) = \mathbf{o}(M^\diamond(\alpha))$.*

264 **Proof.** We already know that $\mathbf{w}(M^\diamond(\alpha)) \leq \mathbf{o}(M^\diamond(\alpha))$. We prove the lower bound by
265 induction on α :

266 ■ If α is indecomposable, see Lemma 2.2.

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267 ■ If $\alpha = \beta + 1$, then according to Lemma 2.3,

$$\begin{aligned}
 268 \quad \mathbf{w}(M^\diamond(\alpha)) &\geq \mathbf{w}(M^\diamond(\beta)) \cdot \omega \\
 269 \quad &= \mathbf{o}(M^\diamond(\beta)) \cdot \omega && \text{by induction hypothesis,} \\
 270 \quad &= \omega^{\widehat{\beta+1}} = \omega^{\widehat{\beta+1}} = \mathbf{o}(M^\diamond(\alpha)) && \text{according to Theorem 1.10.} \\
 271
 \end{aligned}$$

272 ■ If $\alpha = \beta + \omega^\rho$ with $\beta, \omega^\rho < \alpha$ and $\rho > 0$, then according to the transformation equation
273 Trans-3, $M^\diamond(\alpha) \leq_{\text{aug}} M^\diamond(\beta) \cdot M^\diamond(\omega^\rho)$. Hence according to Lemma 1.3 and Table 1,

$$\begin{aligned}
 274 \quad \mathbf{w}(M^\diamond(\alpha)) &\geq \mathbf{w}(M^\diamond(\beta)) \odot \mathbf{w}(M^\diamond(\omega^\rho)) \\
 275 \quad &= \mathbf{o}(M^\diamond(\beta)) \odot \mathbf{o}(M^\diamond(\omega^\rho)) && \text{by induction hypothesis,} \\
 276 \quad &= \omega^{\widehat{\beta}} \odot \omega^{\widehat{\omega^\rho}} = \omega^{\widehat{\alpha}} \\
 277 \quad &= \mathbf{o}(M^\diamond(\alpha)) && \text{according to Theorem 1.10.} \blacktriangleleft \\
 278
 \end{aligned}$$

279 We can now prove that Lemma 2.4 generalizes to non-linear wpos.

280 ► **Lemma 2.5.** *If $\mathbf{o}(X)$ is infinite then $\mathbf{w}(M^\diamond(X)) = \mathbf{o}(M^\diamond(X))$.*

Proof. Let $\alpha = \mathbf{o}(X)$. Then $X \leq_{\text{aug}} \alpha$ from Proposition 1.4, hence $M^\diamond(X) \leq_{\text{aug}} M^\diamond(\alpha)$ according to Lemma 1.3 and Proposition 1.9. Thus

$$\mathbf{w}(M^\diamond(\alpha)) \leq \mathbf{w}(M^\diamond(X)) \leq \mathbf{o}(M^\diamond(X)).$$

281 Now $\mathbf{o}(M^\diamond(X)) = \omega^{\widehat{\alpha}} = \mathbf{o}(M^\diamond(\alpha))$ according to Theorem 1.10. Now with Lemma 2.4
282 $\mathbf{w}(M^\diamond(\alpha)) = \mathbf{o}(M^\diamond(\alpha))$, hence $\mathbf{w}(M^\diamond(X)) = \mathbf{o}(M^\diamond(X))$. ◀

283 We can also compute the width of $M^\diamond(X)$ when X is a finite wpo:

284 ► **Lemma 2.6.** *If $\mathbf{o}(X)$ is finite, then $\mathbf{w}(M^\diamond(X)) = \omega^{\mathbf{o}(X)-1}$.*

285 **Proof.** Let $k = \mathbf{o}(X)$. Then $\Gamma_k \leq_{\text{aug}} X \leq_{\text{aug}} k$, hence $\mathbf{w}(M^\diamond(\Gamma_k)) \geq \mathbf{w}(M^\diamond(X)) \geq$
286 $\mathbf{w}(M^\diamond(k))$ thanks to Lemma 1.3. According to Lemma 1.8, $\mathbf{w}(M^\diamond(\Gamma_k)) = \omega^{k-1}$, and
287 according to Lemma 2.3 applied $(k-1)$ times, $\mathbf{w}(M^\diamond(k)) \geq \mathbf{w}(M^\diamond(1)) \cdot \omega^{k-1} = \omega^{k-1}$.
288 Therefore $\mathbf{w}(M^\diamond(X)) = \omega^{k-1} = \omega^{\mathbf{o}(X)-1}$. ◀

289 This section's main result follows directly from Lemmas 2.5 and 2.6.

290 **Proof of Theorem 2.1.** If $\mathbf{o}(X)$ is finite, then $\widehat{\mathbf{o}(X)} - 1 = \mathbf{o}(X) - 1$. On the other hand, if
291 $\mathbf{o}(X)$ is infinite, then $\widehat{\mathbf{o}(X)} - 1 = \widehat{\mathbf{o}(X)}$. ◀

292 **3 Ordinal height and width of the multiset ordering**

293 For the height of $M^r(X)$, we obtain a result similar to Theorem 1.11.

294 ► **Theorem 3.1** (Height of the multiset ordering). *Let X be a wpo.*

295 *Then $\mathbf{h}(M^r(X)) = \omega^{\mathbf{h}(X)}$.*

296 **Proof.** Observe that the multiset ordering of any linear ordering is also linear. Thus, for
297 any ordinal α , $M^r(\alpha)$ is isomorphic to ω^α (the function $\langle x_1, \dots, x_n \rangle \mapsto \omega^{x_1} \oplus \dots \oplus \omega^{x_n}$ is
298 an isomorphism).

299 According to Proposition 1.4, $X \geq_{\text{st}} \mathbf{h}(X)$, and thus $M^r(X) \geq_{\text{st}} M^r(\mathbf{h}(X)) \equiv \omega^{\mathbf{h}(X)}$
300 (Proposition 1.9). Therefore $\mathbf{h}(M^r(X)) \geq \omega^{\mathbf{h}(X)}$ according to Lemma 1.3. See the proof of
301 the upper bound in Appendix A. ◀

302 The width of the multiset ordering is harder to compute, as $\mathbf{w}(M^r(X))$ is not functional
 303 in the ordinal invariants of X . The following example exhibits two wpos X_1 and X_2 , with
 304 identical ordinal invariants, such that $\mathbf{w}(M^r(X_1)) \neq \mathbf{w}(M^r(X_2))$.

305 ▶ **Example 3.2.** Let $H \stackrel{\text{def}}{=} \Sigma_{n < \omega} \Gamma_n$. An interesting property of H is that $\mathbf{w}(H) = \mathbf{h}(H) =$
 306 $\mathbf{o}(H) = \omega$. Since $M^r(H) \geq_{\text{st}} M^r(\Gamma_n)$, then $\omega^{n-1} \leq \mathbf{w}(M^r(H)) \leq \mathbf{o}(M^r(H)) = \omega^\omega$ for all
 307 $n < \omega$ according to Lemma 1.8 and Theorem 1.11. Hence $\mathbf{w}(M^r(H)) = \omega^\omega$.

308 Consider $X_1 = H + H$ and $X_2 = H + \omega$, two wpos with the same ordinal invariants:
 309 $\mathbf{o}(X_i) = \mathbf{h}(X_i) = \omega \cdot 2$ and $\mathbf{w}(X_i) = \omega$ for $i \in \{1, 2\}$. According to Equation (Trans-2)
 310 and Table 1, $\mathbf{w}(M^r(X_1)) = \mathbf{w}(M^r(H)) \odot \mathbf{w}(M^r(H)) = \omega^\omega \odot \omega^\omega = \omega^{\omega \cdot 2}$ and $\mathbf{w}(M^r(X_2)) =$
 311 $\mathbf{w}(M^r(H)) \odot \mathbf{w}(M^r(\omega)) = \omega^\omega \odot 1 = \omega^\omega$.

312 Fortunately, we uncovered a new ordinal invariant, defined similarly to the usual invariants,
 313 in which the width of the multiset ordering is functional.

314 ▶ **Definition 3.3** (Friendly order type). A bad sequence is open-ended if it is empty or of the
 315 form sx where s is an open-ended sequence and x has a “friend”¹ in the residual $X_{\not\geq s}$, i.e.,
 316 an element incomparable to x . For any wpo X , let $\text{Bad}_\perp(X)$ be the subtree of $\text{Bad}(X)$ which
 317 contains all open-ended bad sequences. As $\text{Bad}_\perp(X)$ is a substructure of $\text{Bad}(X)$, it has a
 318 rank that we denote by $\mathbf{o}_\perp(X)$ the friendly order type of X (or fot).

319 This definition can be expressed as the following residual equation:

$$320 \quad \mathbf{o}_\perp(X) = \sup_{x \in X, X_{\perp x} \neq \emptyset} (\mathbf{o}_\perp(X_{\not\geq x}) + 1) \quad (\text{Res-f})$$

322 ▶ **Theorem 3.4.** For any wpo X , $\mathbf{w}(M^r(X)) = \omega^{\mathbf{o}_\perp(X)}$

323 **Proof.** See Appendix B. The proof of Theorem 3.4 is quite technical, and relies on the notion
 324 of quasi-incomparable subsets. ◀

325 4 Computing the friendly order type

326 Like the usual ordinal invariants, the fot can be computed compositionally for some basic
 327 operations on wpos:

328 ▶ **Proposition 4.1.** For any non empty wpo A, B ,

- 329 1. $\mathbf{o}_\perp(A + B) = \mathbf{o}_\perp(A) + \mathbf{o}_\perp(B)$,
- 330 2. $\mathbf{o}_\perp(A \sqcup B) = 1 + (\mathbf{o}(A) - 1) \oplus (\mathbf{o}(B) - 1)$,

331 **Proof.** 1. For any sequences s_A, s_B in $\text{Bad}_\perp(A), \text{Bad}_\perp(B)$, the concatenation $s_B s_A$ is a
 332 sequence of $\text{Bad}_\perp(A + B)$. Furthermore, any sequence of $\text{Bad}_\perp(A + B)$ is of this form.
 333 2. For any two sequences s_1, s_2 , let $s_1 \sqcup s_2$ denote the set of sequences obtained through
 334 shuffling s_1, s_2 together (e.g. $abcd \in aba \sqcup cd$). Let x_A, x_B be two minimal elements of A
 335 and B . For any sequences s_A, s_B in $\text{Bad}(A \setminus \{x_A\}), \text{Bad}(B \setminus \{x_B\})$, for any $s \in s_A \sqcup s_B$, we
 336 know that s and $s x_A$ and $s x_B$ are in $\text{Bad}_\perp(A \sqcup B)$. Reciprocally, from any $s \in \text{Bad}_\perp(A \sqcup B)$,
 337 there is a partition $s_A \in \text{Bad}(A), s_B \in \text{Bad}(B)$ such that $s \in s_A \sqcup s_B$. Furthermore,
 338 the natural sum of the ranks of s_A in $\text{Bad}(A)$ and s_B in $\text{Bad}(B)$ is strictly positive.
 339 Suppose for contradiction sake that s_A and s_B have rank 0 in $\text{Bad}(A)$ and $\text{Bad}(B)$. Let
 340 $s = s'x$. Then $(A \sqcup B)_{\not\geq s} = \emptyset$ and in particular x has no friend in $(A \sqcup B)_{\not\geq s'}$. Thus
 341 $s \notin \text{Bad}_\perp(A \sqcup B)$, contradiction. ◀

¹ Can one be friend with one’s superior or inferior? No. Your true friends are those you cannot (and do not have to) compare yourselves with.

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342 Observe how friendly order type behaves similarly to mot . It is not unusual to have fot
 343 coincides with mot , for instance $\mathbf{o}_\perp(\omega \sqcup \omega) = \mathbf{o}(\omega \sqcup \omega)$ (Proposition 4.1).

344 To bring this new ordinal invariant closer to familiar grounds, we bound the fot of a wpo
 345 X with the mot of a special subset of X , the *stripped* subset.

► **Definition 4.2** (Stripped subset). *The stripped subset of a wpo X , denoted by $\text{str}(X)$, is X without its friendless elements:*

$$\text{str}(X) \stackrel{\text{def}}{=} \{ x \in X : X_{\perp x} \neq \emptyset \}.$$

346 Since $\text{Bad}_\perp(X)$ is a subtree of $\text{Bad}(\text{str}(X))$, we know that $\mathbf{o}_\perp(X) \leq \mathbf{o}(\text{str}(X))$. Here is
 347 an example where this inequality is strict:

348 ► **Example 4.3.** Let $X = \omega \sqcup \{\clubsuit\}$. Here $\text{str}(X) = X$, so $\mathbf{o}(\text{str}(X)) = \omega + 1$. However,
 349 in $\text{Bad}_\perp(X)$, the singleton \clubsuit has rank 0, and the singleton n for any $n \in \omega$ has rank n .
 350 Therefore $\mathbf{o}_\perp(X) = \omega < \mathbf{o}(\text{str}(X))$.

351 Let us show that $\mathbf{o}(\text{str}(X))$ also appears in a lower bound on $\mathbf{o}_\perp(X)$, by introducing an
 352 alternative characterisation of fot as the mot of a specific subset.

353 A *maximal linearisation* is a monotonic function from a wpo X onto $\mathbf{o}(X)$.

354 ► **Definition 4.4** (Friendly subset). *A subset X' of X is friendly if there exist a maximal
 355 linearisation $\ell : X' \rightarrow \mathbf{o}(X')$ such that for any bad sequence $s = x_1, \dots, x_n$ in X' verifying
 356 $\ell(x_1) > \dots > \ell(x_n)$, s is open-ended. We say that ℓ witnesses the friendly condition.*

357 Observe that every friendly subset of X is a substructure of $\text{str}(X)$.
 For any ordinal α , let

$$\delta(\alpha) \stackrel{\text{def}}{=} \begin{cases} \alpha & \text{if } \alpha \text{ is limit,} \\ \gamma + \lfloor n/2 \rfloor & \text{if } \alpha = \gamma + n \text{ with } \gamma \text{ limit and } n < \omega. \end{cases}$$

358 ► **Theorem 4.5** (Alternative characterisation of $\mathbf{o}_\perp(X)$). *Let X be a wpo . There exists
 359 a friendly subset X' of X which maximizes $\mathbf{o}(X')$, and $\mathbf{o}_\perp(X) = \mathbf{o}(X')$. Furthermore,
 360 $\delta(\mathbf{o}(\text{str}(X))) \leq \mathbf{o}_\perp(X) \leq \mathbf{o}(\text{str}(X))$.*

361 **Proof.** See proof in Appendix C. ◀

362 ► **Example 4.6** (Following on Example 3.2). Remember that $H \stackrel{\text{def}}{=} \sum_{n < \omega} \Gamma_n$. Thus $\text{str}(H) =$
 363 $\sum_{2 \leq n < \omega} \Gamma_n$, and $\mathbf{o}(\text{str}(H)) = \mathbf{o}(H) = \omega$. Consider $X_1 = H + H$ and $X_2 = H + \omega$. Observe that
 364 $\text{str}(X_1) = \text{str}(H) + \text{str}(H)$ whereas $\text{str}(X_2) = \text{str}(H)$. Therefore, according to Theorem 4.5,
 365 $\mathbf{o}_\perp(X_1) = \omega \cdot 2$ and $\mathbf{o}_\perp(X_2) = \omega$.

366 ► **Corollary 4.7.** *For any wpo X , if $\mathbf{o}(X)$ is limit and $\mathbf{o}(\text{str}(X)) = \mathbf{o}(X)$, then $\mathbf{o}_\perp(X) =$
 367 $\mathbf{o}(X)$.*

368 The conditions in Corollary 4.7 are often satisfied:

369 ► **Proposition 4.8.** *For any wpo non-empty X , $\mathbf{o}_\perp(M^\circ(X)) = \mathbf{o}(M^\circ(X))$.*

370 **Proof.** Observe that $M^\circ(X) = M^\circ(X) \setminus \{\emptyset\}$. Thus $\mathbf{o}(\text{str}(M^\circ(X))) = \mathbf{o}(M^\circ(X)) - 1 =$
 371 $\mathbf{o}(M^\circ(X))$ (Theorem 1.10). We conclude with Corollary 4.7. ◀

■ **Table 2** Ordinal invariants of the set of finite multisets.

Invariants	Multiset embedding of X	Multiset ordering of X
Mot \mathbf{o}	$\omega^{\widehat{\mathbf{o}(X)}}$	$\omega^{\mathbf{o}(X)}$
Height \mathbf{h}	$\mathbf{h}^*(X)$	$\omega^{\mathbf{h}(X)}$
Width \mathbf{w}	$\omega^{\widehat{\mathbf{o}(X)}-1}$	$\omega^{\mathbf{o}_\perp(X)}$

Conclusion

372

373 Table 2 sums up this article’s contributions (in the gray cases) amidst the former state of
374 the art.

375 These results are part of a more general research program (see [8, 16]) aimed at measuring
376 more precisely and more effectively the complexity of wpos used in well-structured systems,
377 termination proofs, and other algorithmic applications.

378 Investigating the friendly order type is a subject for further research: How does it relate
379 to other concepts? Can it be computed compositionally for more operations? Can we define
380 a class of wpos where friendly order type always coincides with mot?

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381

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434 **A Proof of Theorem 3.1**

435 We write $m \overset{\varnothing}{\leq}_r m'$ (resp. $m \overset{\varnothing}{<} m'$, $m \overset{\varnothing}{\perp} m'$) when $m \cap m' \neq \emptyset$ and $m \leq_r m'$ (resp $m < m'$,
 436 $m \perp m'$). With these new notations, the multiset ordering can be reformulated as follows

437 ► **Definition A.1** (Multiset ordering (reformulated)). $M^r(X) = (M(X), \leq_r)$ is ordered with the
 438 multiset ordering: $m \leq_r m'$ iff there exists m_1, m'_1, m_2 such that $m = m_1 \cup m_2$, $m' = m'_1 \cup m_2$,
 439 and $m_1 \overset{\varnothing}{<} m'_1$.

► **Lemma A.2.** Let $A = \cup_{i \leq n} A_i$ a set partitioned in n subsets, for some $n \in \mathbb{N}$. Let \leq_A a well-partial ordering on A , and \leq_{A_i} the same ordering restricted to the subset A_i for $i \leq n$. Then

$$h(A, \leq_A) \leq \bigoplus_{i \leq n} h(A_i, \leq_{A_i}).$$

440 **Proof.** From any decreasing sequence s on A , one can extract a decreasing sequence s_i by
 441 restricting s to A_i for any $i \leq n$. By induction on the rank of s in $Dec(A)$, one shows that
 442 $rk(s) \leq \bigoplus_{i \leq n} rk(s_i)$. ◀

443 **Proof of Theorem 3.1.** We prove the upper bound by induction on $h(X)$.

444 If $h(X) = 0$ then $X = \emptyset$ and $h(M^r(\emptyset)) = 1 = \omega^0$.

445 Suppose that X is not empty. For any non-empty multiset $m \in M^r(X)$, the residual
446 $M^r(X)_{<m}$ can be partitioned as follows:

$$447 \quad M^r(X)_{<m} = \bigcup_{m_1+m_2=m, m_1 \neq \emptyset} \{ m' + m_2 : m' \not\prec m_1 \}.$$

449 Note that this union is a partition of the support of $M^r(X)_{<m}$, it does not say anything on
450 the order between the elements of the subsets in the union.

451 For any non-empty multiset m , we define $S_m \stackrel{\text{def}}{=} (\cap_{x \in m} X_{\not\prec x}) \cap (\cup_{x \in m} X_{<x})$ a subset of
452 X . Thus for any multiset m' in $M^r(X)$, $m' \not\prec m$ iff $m' \in M^r(S_m)$. Therefore:

$$453 \quad M^r(X)_{<m} = \bigcup_{m_1+m_2=m, m_1 \neq \emptyset} \{ m' + m_2 : m' \in M^r(S_{m_1}) \}.$$

Observe that $\mathbf{h}(S_{m_1}) < \mathbf{h}(X)$ by definition of S_{m_1} . Hence by induction hypothesis
454 $\mathbf{h}(M^r(S_{m_1})) \leq \omega^{\mathbf{h}(S_{m_1})} < \omega^{\mathbf{h}(X)}$. Moreover, $\omega^{\mathbf{h}(X)}$ is indecomposable. Hence according to
Lemma A.2:

$$\mathbf{h}(M^r(X)_{<m}) \leq \bigoplus_{m_1+m_2=m, m_1 \neq \emptyset} \mathbf{h}(M^r(\cup_{x \in m_1} X_{<x})) < \omega^{\mathbf{h}(X)}.$$

455 Therefore $\mathbf{h}(M^r(X)) \leq \omega^{\mathbf{h}(X)}$ according to Equation (Res-h). ◀

456 B Proof of Theorem 3.4

457 First we prove intermediary lower and upper bounds on the width of the multiset ordering.

► **Lemma B.1.** *Let X be a wpo. Then*

$$\mathbf{w}(M^r(X)) \geq \sup_{x \in X, n < \omega} \mathbf{w}(M^r(X)_{\perp \langle x \rangle}) \cdot n + 1$$

458 **Proof.** This proof follows the same structure as the proof of Lemma 2.3: We study the
459 residual of $M^r(X)$ which contains every element incomparable to some multiset of the form
460 $\langle x \rangle \times n$, and slice this residual into a family of quasi-incomparable subsets.

461 According to Equation (Res-w),

$$462 \quad \mathbf{w}(M^r(X)) = \sup_{m \in M^r(X)} \mathbf{w}(M^r(X)_{\perp m}) + 1$$

$$463 \quad \geq \sup_{x \in X, n < \omega} \mathbf{w}(M^r(X)_{\perp \langle x \rangle \times n}) + 1.$$

465 For all $k \in [1, n]$, let $M_k = \{ \langle x \rangle \times (n - k) \cup m : m \in M^r(X)_{\perp \langle x \rangle} \}$. Observe that
466 $M_k \equiv M^r(X)_{\perp \langle x \rangle}$ for any $k \in [1, n]$, and for all $m \in M_k$, $m \perp \langle x \rangle \times n$. We claim that
467 $(M_k)_{k \in [1, n]}$ is a quasi-incomparable family of subsets of $M^r(X)_{\perp \langle x \rangle \times n}$: Let $i < n$ and Y a
468 finite subset of $M_1 \cup \dots \cup M_i$. We define m_Y and M'_{i+1} as

$$469 \quad m_Y \stackrel{\text{def}}{=} \bigcup_{j \leq i} \bigcup_{m \in (M_j \cap Y)} (m \setminus (\langle x \rangle \times (n - j))),$$

$$470 \quad M'_{i+1} \stackrel{\text{def}}{=} \{ \langle x \rangle \times (n - i - 1) \cup m_Y \cup m : m \in M^r(X)_{\perp \langle x \rangle} \}.$$

472 Observe that M'_{i+1} is an isomorphic subset of M_{i+1} , and $Y \perp M'_{i+1}$.

473 Therefore according to Lemma 1.14, $\mathbf{w}(M^r(X)_{\perp \langle x \rangle \times n}) \geq \mathbf{w}(M^r(X)_{\perp \langle x \rangle}) \cdot n$. ◀

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► **Lemma B.2.** *Let X be a wpo. Then*

$$\mathbf{w}(M^r(X)) \leq \sup_{x \in X, n < \omega} \mathbf{w}(M^r(X)_{\perp \langle x \rangle}) \otimes n + 1$$

474 **Proof.** By definition, for any multisets $m, m' \in M^r(X)$, $m \perp m'$ means that $m \neq m'$ and
475 there exists m_1, m'_1, m_2 such that $m = m_1 \cup m_2$, $m' = m'_1 \cup m_2$ and $m_1 \overset{\uparrow}{\perp} m'_1$.

476 Therefore, the residual $M^r(X)_{\perp m}$ can be partitioned as an augmentation of a disjoint
477 union:

$$M^r(X)_{\perp m} \geq_{\text{aug}} \bigsqcup_{m_1 + m_2 = m, m_1 \neq \emptyset} \{ m'_1 + m_2 : m' \in M^r(X), m'_1 \overset{\uparrow}{\perp} m_1 \},$$

478 which can be reformulated into

$$M^r(X)_{\perp m} \geq_{\text{aug}} \bigsqcup_{m_1 \subseteq m, m_1 \neq \emptyset} M^r(X)_{\perp m_1}^{\uparrow}$$

479 where $M^r(X)_{\perp m_1}^{\uparrow}$ is the residual $\{ m' \in M^r(X) : m' \overset{\uparrow}{\perp} m_1 \}$.

480 Let us observe this residual: $m' \overset{\uparrow}{\perp} m_1$ means that m' and m_1 are disjoint and there exists
481 $x \in m_1$ such that for all $y' \in m'$, $x \not\leq y'$, and there exists $x' \in m'$ such that for all $y \in m_1$,
482 $x' \not\leq y$. In particular $x' \not\leq x$. Hence $m' \overset{\uparrow}{\perp} m_1$ implies there exists $x \in m_1$ such that $\langle x \rangle \overset{\uparrow}{\perp} m'$,
483 which is equivalent to $\langle x \rangle \perp m'$. Therefore the support of $M^r(X)_{\perp m_1}^{\uparrow}$ is included in a union
484 on $x \in m_1$ of residuals $M^r(X)_{\perp \langle x \rangle}$. With an augmentation we get a disjoint union:

$$M^r(X)_{\perp m_1}^{\uparrow} \leq_{\text{st}} \geq_{\text{aug}} \bigsqcup_{x \in m_1} M^r(X)_{\perp \langle x \rangle}.$$

$$\text{Hence according to Table 1, } M^r(X)_{\perp m} \leq \bigoplus_{m_1 \subseteq m, m_1 \neq \emptyset} \bigoplus_{x \in m_1} \mathbf{w}(M^r(X)_{\perp \langle x \rangle}).$$

Let $x \in m$ such that $\mathbf{w}(M^r(X)_{\perp \langle x \rangle})$ is maximal. Then $\mathbf{w}(M^r(X)_{\perp m}) \leq \mathbf{w}(M^r(X)_{\perp \langle x \rangle}) \otimes n$ for some $n < \omega$. Hence according to Equation (Res-w),

$$\mathbf{w}(M^r(X)) = \sup_{m \in M^r(X)} \mathbf{w}(M^r(X)_{\perp m}) + 1 \leq \sup_{x \in X, n < \omega} \mathbf{w}(M^r(X)_{\perp \langle x \rangle}) \otimes n + 1. \quad \blacktriangleleft$$

485 The bounds provided in Lemmas B.1 and B.2 actually match. Furthermore, they can be
486 reformulated in such a way that the residual on $M^r(X)$ boils down to a residual on X :

487 ► **Lemma B.3.** *For any non-linear wpo X ,*

$$488 \quad \mathbf{w}(M^r(X)) = \sup \{ \mathbf{w}(M^r(X_{\not\leq x})) \cdot \omega : x \in X, X_{\perp x} \neq \emptyset \}. \quad (\text{W})$$

489 **Proof.** For any ordinal α , $\sup_{n < \omega} (\alpha \cdot n + 1) = \sup_{n < \omega} (\alpha \otimes n + 1) = \alpha \cdot \omega$. Hence according
490 to Lemmas B.1 and B.2, $\mathbf{w}(M^r(X)) = \sup_{x \in X} (\mathbf{w}(M^r(X)_{\perp \langle x \rangle}) \cdot \omega)$.

Let $x \in X$. If $X_{\perp x} = \emptyset$, then $M^r(X)_{\perp \langle x \rangle} = \emptyset$. Otherwise let $y \in X_{\perp x}$. Observe that, for any $m \in M^r(X_{\not\leq x})$, $m \cup \langle y \rangle \perp \langle x \rangle$. Hence

$$\{ \langle y \rangle \cup m : m \in M^r(X_{\not\leq x}) \} \leq_{\text{st}} M^r(X)_{\perp \langle x \rangle} \leq_{\text{st}} M^r(X_{\not\leq x}).$$

491 Therefore $\mathbf{w}(M^r(X)_{\perp \langle x \rangle}) = \mathbf{w}(M^r(X_{\not\leq x}))$ if $X_{\perp x} \neq \emptyset$, otherwise $\mathbf{w}(M^r(X)_{\perp \langle x \rangle}) = 0$. ◀

492 **Proof of Theorem 3.4.** If X is linear, $Bad_{\perp}(X)$ only contains the empty sequence, hence
 493 $\mathbf{o}_{\perp}(X) = 0$ and $\mathbf{w}(Mulr(X)) = 1$. Otherwise, observe that Equation (W) is quite similar
 494 to Equation (Res-f) in its structure. Thus $\mathbf{w}(M^r(X)) = \omega^{\mathbf{o}_{\perp}(X)}$ follows directly from
 495 Equation (W). ◀

496 **C Proof of Theorem 4.5**

497 ▶ **Lemma C.1.** *For any wpo X , for any maximal linearisation $\ell : str(X) \rightarrow \mathbf{o}(str(X))$, there*
 498 *exists a friendly subset X' such that ℓ restricted to X' verifies the friendly condition, and*
 499 *$\mathbf{o}(X') \geq \delta(\mathbf{o}(str(X)))$.*

500 **Proof.** We claim that for any $\beta \leq \mathbf{o}(str(X))$, there exists $X_{\beta} \subseteq \ell^{-1}(\{\gamma : \gamma < \beta\})$ a friendly
 501 subset of X where ℓ restricted to X_{β} verifies the friendly condition, such that $\mathbf{o}(X_{\beta}) \geq \delta(\beta)$.
 502 In this proof, when we say that a subset is friendly, it is always implied that ℓ restricted to
 503 this subset witnesses the friendly condition.

504 We build the subsets $(X_{\beta})_{\beta \leq \mathbf{o}(str(X))}$ as follows:

- 505 ■ $X_0 = \emptyset$,
- 506 ■ For γ limit, $X_{\gamma} = \bigcup_{\beta < \gamma} X_{\beta}$,
- 507 ■ For any β , $X_{\beta+1} = X_{\beta} \cup \ell^{-1}(\beta)$ if friendly, otherwise $X_{\beta+1} = X_{\beta}$.

508 First observe that X_{β} is friendly for any $\beta \leq \mathbf{o}(str(X))$. Indeed, X_0 is friendly, and since
 509 for any $\beta < \beta'$, $X_{\beta} \subseteq X_{\beta'}$, then the union $\bigcup_{\beta < \gamma} X_{\beta}$ for γ limit is friendly by induction.

510 Let us prove the claim $\mathbf{o}(X_{\beta}) \geq \delta(\beta)$, by showing that for any $\beta + 2 \leq \mathbf{o}(str(X))$, we
 511 have $\mathbf{o}(X_{\beta+2}) > \mathbf{o}(X_{\beta})$. Let $x = \ell^{-1}(\beta')$ and $x' = \ell^{-1}(\beta' + 1)$. Assume for the sake of
 512 contradiction that $X_{\beta+2} = X_{\beta}$. This means that neither $X_{\beta} \cup \{x\}$ nor $X_{\beta} \cup \{x'\}$ are friendly.
 513 Hence there exists $y, y' \in X_{\beta}$ such that for any $z \in X$, we have $z \perp y \implies z \geq x$ and
 514 $z \perp y' \implies z \geq x'$. Now because of ℓ we know that $x \not\geq x'$ and $y, y' \not\geq x, x'$. Since
 515 $y, y' \in str(X)$, then $X_{\perp y}$ and $X_{\perp y'}$ are both non-empty, so actually $x \perp y$ and $x' \perp y'$. And
 516 since $x \not\geq x'$, we know $y' < x$. Therefore $x \perp x'$, hence $y < x'$. Which leads to a contradiction
 517 on the relationship between y and y' . ◀

518 For any friendly subset X' , $\mathbf{o}(X') \leq \mathbf{o}(str(X))$, and there exist X' such that $\mathbf{o}(X') \geq$
 519 $\delta(\mathbf{o}(str(X)))$. Therefore there exists a friendly subset X' which maximizes $\mathbf{o}(X')$.

520 **Proof of Theorem 4.5.** We say that a bad sequence x_1, \dots, x_n respects a maximal linear-
 521 isation ℓ when $\ell(x_1) > \dots > \ell(x_n)$. Let X' be a friendly subset of X and ℓ a maximal
 522 linearisation of X' that verifies the friendly condition. Observe that $Bad(X')$ restricted
 523 to sequences that respect ℓ has for rank $\mathbf{o}(X')$, and is embedded in $Bad_{\perp}(X)$. Hence
 524 $\mathbf{o}_{\perp}(X) \geq \mathbf{o}(X')$.

525 We prove the upper bound by induction on $\mathbf{o}_{\perp}(X)$. If $\mathbf{o}_{\perp}(X) = 0$ then the only friendly
 526 subset of X is the empty set. Now suppose that $\mathbf{o}_{\perp}(X) > 0$. For any $x \in str(X)$, by
 527 induction hypothesis on $X_{\not\geq x}$, there exists a friendly subset X' of $X_{\not\geq x}$, with a maximal
 528 linearisation ℓ which verifies the friendly condition, such that $\mathbf{o}(X') \geq \mathbf{o}_{\perp}(X_{\not\geq x})$. We extend
 529 ℓ to the subset $X' \cup \{x\}$ of X , such that $\ell(x) = \mathbf{o}(X')$. Now ℓ is a maximal linearisation
 530 of $X' \cup \{x\}$ which verifies the friendly condition, therefore $\mathbf{o}(X' \cup \{x\})$ is a friendly subset
 531 of X and $\mathbf{o}(X' \cup \{x\}) > \mathbf{o}_{\perp}(X_{\not\geq x})$. Let X' be a friendly subset of X which maximizes
 532 $\mathbf{o}(X')$. Then for any $x \in str(X)$, $\mathbf{o}_{\perp}(X_{\not\geq x}) < \mathbf{o}(X')$. Therefore $\mathbf{o}_{\perp}(X) \leq \mathbf{o}(X')$ according to
 533 Equation (Res-f). ◀