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# A model of growth with living capital

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## Abstract

Acknowledging the economic role of all living things can trigger an endogenous production shift, which not only preserves but makes thrive all living capital without assuming that preferences depend on the living nor that production could degrade the living.

In a planned economy, we find that the optimal sequence of total capital is always monotonic. Depending on the productivity of total capital (defined as the sum of physical and living capital), three different regimes hold: bounded growth; asymptotically balanced unbounded growth or unbalanced unbounded growth.

Only at the very early stages of development the economy devotes all its investment effort to increase the stock of physical capital and only physical capital. As the economy develops, it will start using living capital in production. In the first regime, the total capital converges to a steady state with a positive stock of living capital, which is larger than the one without living capital. In the second regime, growth becomes unbounded, and consumption grows at a constant rate. Total capital grows at the same rate but only asymptotically. In the third case, living capital is used increasingly at the beginning. Once the economy is sufficiently rich, physical capital starts growing faster than living capital.

At the end of the paper, we consider a market economy with externalities from the living. In this case, if the government levies taxes to finance the accumulation of living capital and implements exactly the optimal sequence of living capital as in the planner's program, then the equilibrium market prices decentralize exactly the planner's solution.

**Keywords:** Ramsey model, physical capital, living capital, bounded and unbounded growth, global analysis.

**JEL codes:** C61, C62, O44.

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# 1 Introduction

Mankind is currently facing tremendous challenges directly or indirectly caused by economic activities. Among these challenges, we find climate change, the exhaustion of natural resources, the rapid loss in biodiversity, and the diffusion of new lethal diseases. Obviously, one feels compelled to immediately reduce production and consumption, hoping that the virtues of degrowth would re-establish a healthier (sustainable) equilibrium as praised by Georgescu-Roegen (1979). Tradition would suggest to consider human, social and natural capital as productive inputs and as utils, and then to model how production damages them. While this approach is extremely important and informative, we propose a more comprehensive measure for all living things: living capital. Living capital is a broad measure to encompass all living things and their bonds, tangible and intangible, to acknowledge that all living things form a unity. The fact of acknowledging the economic role of all living things can change investment behaviors, privilege their preservation, and bring degrowth even without introducing any negative feedback from production to living capital.

Living capital is defined as a comprehensive measure of all economic inputs that are alive, and which are not raw labor. Hence, living capital includes education, health, social interactions, social networks and social capital, and also nature characterized by environmental quality, climate, biodiversity and natural resources. Besides, living capital includes all bonds and feedbacks among all its components, and this is exactly what distinguishes living capital from natural and social capital, and even their sum. Obviously, there are positive and negative feedbacks, but we shall later assume that the net feedback effect is positive making living capital resilient. The three essential characteristics of living capital are that, first, it is not essential; second, it is irreversible; and third, it depreciates at a slower pace than physical capital. Living capital is not essential because economies can produce without any living capital, using physical capital, labor and some available technology. And still, we prove that, once the role of living capital is accounted for, agents, policy makers and the market will invest and secure its growth. Living capital is irreversible in the sense that physical capital can be converted to living capital, but the opposite is not true. Finally, regarding the third assumption: Why does living capital depreciate slower than physical capital? It cannot be otherwise since living capital is resilient because of the connections among all its components. Hence it is as if we were assuming that the positive feedbacks outperform the negative.

From a historical modelling perspective, we have observed the sequential introduction of various types of productive capitals in economic models. The first growth models of Ramsey (1928) and Solow (1956) focused on the role of physical capital in human development. In the Sixties, the new growth theories pioneered by Arrow (1962) and Uzawa (1965) stress instead the roles of knowledge and human capital. The notion of Natural capital was sequentially introduced in different disciplines in the early Seventies.<sup>1</sup> According to Holdren

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<sup>1</sup>See for instance Holdren and Ehrlich (1974), Solow (1974), Hartwick (1977) and the

and Ehrlich (1974) natural capital is the sum of nature's services which range from food production, to biodiversity warrant, conversion of waste and disease control. For Arrow et al. (2004) natural capital includes all contributions of nature to present and future utility. Contributions can be direct and indirect as preserving biodiversity, flood control, water purification, etc. Only since 2012, international organizations like the United Nations and the European Union started collecting data to account for natural resources and nature services. Although evidence is still scarce, there seems to be a strong will to render nature, namely in terms of renewable resources and ecosystemic services, its complete importance, including its economic role. The most recent productive capital to enter the economic modelling scene is social capital. Coleman (1990) defines social capital as the institutional relations between people understood widely to include both as formal and informal relations. Soon after, Putnam (1993) tested the concept using evidence on the Seventies Italian reform of the regional administration, to underline in particular the connection between social capital and the performance of institutions.

In an otherwise standard Ramsey setup in discrete time, we obtain the optimal trajectories for both physical and living capital assuming that the relative "price" between the two capitals is constant. Following Bruno et al. (2009), Le Van et al. (2010) and Le Van et al. (2018), we characterize the dynamics of total capital defined as the sum of the value of physical and living capital. Depending on the productivity of total capital, three different regimes can arise: (1) bounded growth; (2) asymptotically balanced unbounded growth; and (3) unbalanced unbounded growth. Let us briefly describe each of them:

(1) If the sum of the productivity parameters of physical and living capital is lower than one, then total capital converges to a steady state with a positive stock of living capital.

(2) When this sum is equal to one, growth becomes unbounded and consumption grows at a constant rate. Interestingly, total capital grows at the same rate as consumption but only asymptotically. In fact, the economy will optimally follow an Asymptotically Balanced Growth Path (ABGP).

(3) Finally, when the sum is greater than one, growth is unbounded for both physical and living capital. When total capital trespasses a given threshold, the economy starts accumulating living capital. The ratio of physical to living capital first decreases and then it grows to infinity, after total capital grows beyond a critical threshold.

We can read these results as both an alternative to degrowth theories and a challenge to the literature supporting the Environmental Kuznets Curve (EKC hereafter). Indeed, we show in cases (2) and (3) that an economy which recognizes the productive role of living capital will preserve and promote it along the optimal path. Only at the very early stages of development, the economy will devote all its investment effort to increase the stock of physical capital and only physical capital. As the economy develops, it will start using living capital in production in an increasing manner, relatively reducing investment in physical

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numerous references there on.

capital. Once the economy is sufficiently rich and the stock of living capital sufficiently large, but only then, physical capital will start growing faster than living capital. Hence, our model describes an endogenous production shift that preserves and makes thrive all living capital without assuming that preferences depend on the living nor that production degrades the living.

We also mentioned that our model can challenge the EKC because, in case (3), it actually describes a U-shaped relationship between economic development and physical capital. Here, physical capital does not decrease in the long term to privilege living capital. On the contrary, both capitals will increase together. The premise being that the economy reached a critical development stage in which living capital was sufficiently abundant. Among the early works that unearthed the possibility of an inverted U-shaped relationship between production and environmental quality when environmental damage affected negatively welfare, let us mention John and Pecchenino (1994), Selden and Song (1995), Stokey (1998), Jaeger (1998), Andreoni and Levinson (2001) and Bosi and Desmarchelier (2018), among others. Regarding empirical support, see Grossman and Krueger (1995), Harbaoug et al. (2002) or Haberl et al. (2020). Our results could even be related to the imbalance effect revealed in Ouattara et al. (2016) in the context of tourism-based economies. Like us, they find that economies should first build a sufficiently large stock of capital and infrastructure that will ensure the optimal economic trajectory, before devoting resources to nature preservation.

Let us close our brief review of the related literature by mentioning some complementary works, which have adopted different perspectives. As mentioned, ours is a standard Ramsey framework and it does not deal with the problem of sustainability. We could have claimed to do so by trivially summarizing some of the findings in the literature in social welfare considering a sufficiently small discount factor, or a scrap function in the objective function to signal that the policy maker cared about the final state of living capital.<sup>2</sup> However, sustainability deserves a more careful treatment, possible only once living capital is fully understood as a productive factor. Regarding agents' predator behavior, Van der Ploeg (2010) shows that cooperation among agents is crucial for environmental preservation when natural capital can be converted into physical capital. Institutions are also pointed out as key warrants of natural preservation in Veeman and Politylo (2003) and Arrow et al. (2004) (among others). We also understand from the later that we should rest cautious about technological progress since it can increase production without bounds and without accounting for all its consequences. As Arrow et al. (2004), Ehrlich et al. (2012) also highlight the effects that uncontrolled population growth could have on the environment. While all these channels remain key, we have intentionally excluded them from our analysis to study how economic recognition alone can secure living capital.

Finally, let us mention that from a strict technical point of view, and regard-

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<sup>2</sup>See for an introduction to the subject Chichilnisky (1977, 1982, 1996, 1997), Fleurbaey (2009), Fleurbaey and Zuber (2013) or Cairns et al. (2019), and all the references therein and thereafter. Obviously, our reference list is not exhaustive.

ing the dynamic behavior of the optimal trajectories, our results are in line with Le Van et al. (2010). It is shown that if human capital and new technologies are sufficiently efficient, then investment in human capital and new technologies turn out to be larger than investment in physical capital. Conversely, in Le Van et al. (2018), investment in physical capital remains larger than investment in social capital in the long run.

After studying the social planner problem, we consider the market economy with living externalities. In this case, we show that if the government levies taxes to finance the accumulation of living capital and implements exactly the optimal sequence of living capital obtained in the planner's program, then the equilibrium market prices exactly decentralize the planner's solution. Note that we are able to provide the explicit sequences of these market prices.

The rest of the paper is organized as follows. In section 2, we solve the social planner's problem by proving the existence of an optimal path and providing a global analysis of economic trajectories. These results are complemented by numerical simulations. In section 3, we consider a market economy with living externalities and we compute the competitive equilibrium. Section 4 concludes. All proofs are gathered in the Appendix.

## 2 Social planner

Let us begin with the policy maker's problem. The policy maker needs to decide on both the sequences of physical and living capital that will maximize overall social welfare.

### 2.1 Fundamentals

Our economy is endowed with three factors to produce a unique final good, which is used for all purposes. These three productive factors are labor, physical capital and living capital. Denoting by  $K_t$  and  $L_t$  physical and living capital, let us assume that their evolution in time is described as follows:

$$K_{t+1} = I_t^K + (1 - \delta_K) K_t \quad (1)$$

$$L_{t+1} = I_t^L + (1 - \delta_L) L_t \quad (2)$$

where  $I_t^K$  and  $I_t^L$  stand for investment in each of the capitals at time  $t$ , and  $\delta_K$  and  $\delta_L$  are their depreciation rates. Plausibly, we can suppose that living capital depreciates less than physical capital. Let us also assume that output and physical capital are the same good.

Let  $q$  be the constant "price" of living capital in terms of output. Then, total capital at time  $t$  is given by:

$$X_t \equiv K_t + qL_t$$

Note that there is not an *a priori* reason why the growth processes of physical and living capital could not follow different trajectories and trends. Furthermore, physical capital could experience degrowth while total capital grows. This

is exactly the situation we analyze and characterize here, and we will delve with it in detail once the model is fully presented and completely solved analytically.

The final good sector utilizes both capitals together with labor, blending their specific services. However, as we will see next, each input has a very different role in production. Let us denote by  $Y_t$  total output at time  $t$ . Then,

$$Y_t = A(1 + \lambda L_t)^\theta K_t^\alpha N_t^{1-\alpha} \quad (3)$$

Our approach departs from the literature on climate change and, instead of modelling environmental damages from production, the production function in (3) reveals how living capital enhances productivity.<sup>3</sup>

Let us introduce some plausible parameter restrictions on technology and capital depreciation, which is assumed to hold throughout the paper, even if not mentioned explicitly. Our first assumption ensures that positive production is possible without any living capital, while production is zero without physical capital.

**Assumption 1**  $0 < \theta < 1$ ,  $0 < \alpha < 1$ ,  $\lambda > 0$ ,  $A > 0$  and  $0 < \delta_L \leq \delta_K \leq 1$ .

Regarding labor, let us assume an inelastic labor supply:  $N_t = \bar{N} = 1$  for any  $t$ . Then we can write  $Y_t$  as  $Y_t = A(1 + \lambda L_t)^\theta K_t^\alpha$ .

There exists a planner who maximizes a classic time discounted intertemporal utility function  $\sum_{t=0}^{\infty} \beta^t \ln c_t$ , where  $\beta$  is the time discount parameter with  $0 < \beta < 1$ . The policy maker needs to take into account a sequence of constraints:

$$c_t + I_t^K + qI_t^L \leq A(1 + \lambda L_t)^\theta K_t^\alpha$$

for all  $t \geq 0$ , given  $K_0, L_0 > 0$ . Note that investment in natural capital needs to be multiplied by  $q$ , the transformation price. Using (1) and (2), we can write the overall resource constraint of the economy at time  $t$  as

$$c_t + K_{t+1} - (1 - \delta_K) K_t + q[L_{t+1} - (1 - \delta_L) L_t] \leq A(1 + \lambda L_t)^\theta K_t^\alpha$$

**Remark 1** Recall that investment in living capital is irreversible, and that the consumption good and physical capital are the same good. As a result, physical capital can actually be consumed, that is  $I_t^K \equiv K_{t+1} - (1 - \delta_K) K_t < 0$  at some  $t$ . We have also assumed that living capital cannot be transformed into the consumption good, that is  $I_t^L \equiv L_{t+1} - (1 - \delta_L) L_t \geq 0$  for all  $t$ . Note that this assumption is compatible with degrowth in living capital at some period  $t$ , that is  $L_{t+1} < L_t$ , provided that  $I_t^L < \delta_L L_t$ . However, the irreversibility assumption becomes superfluous, because as we will see, the economy always experiences growth in living capital along the optimal trajectory.

We can define gross production as

$$F(K_t, L_t) \equiv A(1 + \lambda L_t)^\theta K_t^\alpha + (1 - \delta_K) K_t + q(1 - \delta_L) L_t \quad (4)$$

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<sup>3</sup>Alternatively, living capital could also be interpreted as a part of the Total Factor Productivity (TFP) of physical capital and labor. A higher stock of living capital would make both physical capital and labor more productive. Our results do not depend on this specific interpretation.

so that gross savings at time  $t$  coincides with  $X_{t+1}$ , that is

$$X_{t+1} \equiv K_{t+1} + qL_{t+1}$$

Accordingly, the resource constraints at  $t$  can be rewritten as:

$$c_t + X_{t+1} \leq F(K_t, L_t)$$

Hence, the social planner's problem becomes:

$$\max \sum_{t=0}^{\infty} \beta^t \ln c_t$$

subject to

$$c_t + X_{t+1} \leq \mathcal{F}(X_t) \tag{5}$$

given the initial condition  $X_0 > 0$ , where

$$\mathcal{F}(X_t) \equiv \max \{F(K_t, L_t) : K_t + qL_t \leq X_t\} \tag{6}$$

is the maximal gross output given the aggregate capital  $X_t$ .

Our analysis begins by proving the existence of a (unique) critical level of total capital,  $\bar{X}$ , with the following characteristic: if the economy's stock of total capital lies below  $\bar{X}$ , then production does not use any living capital, that is  $X_t = K_t$ . However, beyond  $\bar{X}$ , production will always include some living capital. We will come back to this result and its implications after we actually prove it. For now, let us start by proving that this critical value  $\bar{X}$  is the unique solution to equation

$$f(X) = q(\delta_K - \delta_L) \geq 0 \tag{7}$$

where  $f(X) \equiv (\alpha q - \lambda \theta X) A X^{\alpha-1}$ . Note that function  $f$  is continuous for any  $X > 0$ , strictly decreasing from  $\infty$  to 0 in  $(0, \alpha q / (\lambda \theta)]$  and negative for any  $X > \alpha q / (\lambda \theta)$ . Therefore, equation (7) has a unique solution  $\bar{X}$  and, clearly, this solution satisfies

$$0 < \bar{X} \leq \frac{q}{\lambda} \frac{\alpha}{\theta} \tag{8}$$

Before presenting our first results, we would like to underline that the model's dynamics are driven by  $\alpha$  and  $\theta$ , the productivity parameters for physical and living capital. Furthermore, their sum,

$$\gamma \equiv \alpha + \theta > 0$$

captures the productivity of total capital, and the ratio  $\alpha/\theta$ , the relative productivity of physical to living capital. As we will see, the magnitude of  $\gamma$ , less or greater than one, does determine whether the economy will experience bounded or unbounded growth.



**Lemma 2 (maximal gross output)** *Function  $\mathcal{F}$  has the following properties:*

(1) *If  $X_t \leq \bar{X}$ , then  $L^*(X_t) = 0$  and  $K^*(X_t) = X_t$ , and  $\mathcal{F}$  is given by*

$$\mathcal{F}(X_t) = AX_t^\alpha + (1 - \delta_K) X_t \quad (9)$$

(2) *If  $X_t > \bar{X}$ , then  $L_t = L^*(X_t) \in (0, X_t/q)$  is the solution to  $\mathcal{L}(L_t) = \mathcal{R}(L_t)$  with*

$$\mathcal{L}(L_t) \equiv \lambda\theta A(1 + \lambda L_t)^{\theta-1} (X_t - qL_t)^\alpha + q\delta_K \quad (10)$$

$$\mathcal{R}(L_t) \equiv \alpha q A(1 + \lambda L_t)^\theta (X_t - qL_t)^{\alpha-1} + q\delta_L \quad (11)$$

and  $K_t = K^*(X_t) = X_t - qL^*(X_t) \in (0, X_t)$ .  $\mathcal{F}$  is given by

$$\begin{aligned} \mathcal{F}(X_t) &= F(K^*(X_t), L^*(X_t)) \\ &= A[1 + \lambda L^*(X_t)]^\theta [X_t - qL^*(X_t)]^\alpha \\ &\quad + (1 - \delta_K)[X_t - qL^*(X_t)] + (1 - \delta_L)qL^*(X_t) \end{aligned} \quad (12)$$

(3) *Function  $\mathcal{F}$  is continuous, strictly increasing and differentiable for any  $X_t \geq 0$ , and so it is at  $X_t = \bar{X}$ .*

(4) *Functions  $K^*(X_t)$  and  $L^*(X_t)$  are strictly increasing for  $X_t > \bar{X}$ .*

(5)  *$\mathcal{F}$  is strictly concave if  $X_t < \bar{X}$ . If  $X_t > \bar{X}$ ,  $\mathcal{F}$  is strictly concave if and only if  $\gamma < 1$ .*

Lemma 2 shows that poor economies, those whose total capital lies below the threshold, do not consider living capital as a production factor. Lemma 2 proves that economies below the threshold will specialize in "heavy" industries, relying uniquely on physical capital. This result could be even more consequential if industrial pollution and its damage was introduced in the model. Indeed, most probably we would see a "living poverty trap" emerge in which poor countries would not contemplate living capital as linked to production, and could never do so since they would induce irreversible damages to living and physical capital that would retain them below the threshold, in the trap.

In order to gain further understanding on the structure of the economy, we define the ratio of physical to living capital used in the following lemma as

$$\zeta(X_t) \equiv \frac{K^*(X_t)}{qL^*(X_t)}$$

As proven in Lemma 2, this ratio is only defined on  $(\bar{X}, \infty)$  since  $L^*(X_t) = 0$  for values of  $X_t$  below  $\bar{X}$ .

**Lemma 3 (asymptotic properties of capitals)** *Physical and living capital have the following asymptotic properties:*

(1) *When aggregate capital is unbounded, its components are unbounded as well:*

$$\begin{aligned} \lim_{X_t \rightarrow \infty} L^*(X_t) &= \infty \\ \lim_{X_t \rightarrow \infty} K^*(X_t) &= \infty \end{aligned}$$

(2) If  $\gamma < 1$ , then

$$\lim_{X_t \rightarrow \infty} \frac{\mathcal{F}(X_t)}{X_t} \leq 1 - \delta_L < 1$$

(3) If  $\gamma = 1$ , there exists  $\mu \in (0, \alpha/\theta]$  such that, for any sequence  $(X_n)$  with  $\lim_{n \rightarrow \infty} X_n = \infty$ , we have

$$\mu \leq \liminf_{n \rightarrow \infty} \zeta(X_n) \leq \limsup_{n \rightarrow \infty} \zeta(X_n) \leq \frac{\alpha}{\theta}$$

(4) If  $\gamma > 1$ , function  $\zeta$  has the following properties:

$$\begin{aligned} \lim_{X_t \rightarrow \bar{X}^-} \zeta(X_t) &= \infty \\ \lim_{X_t \rightarrow \infty} \zeta(X_t) &= \frac{\alpha}{\theta} \end{aligned}$$

Moreover, if  $\delta_K > \delta_L$ ,  $\zeta(X_t) < \alpha/\theta$  for any  $X_t$  large enough.

Some comments are in order. Although (1) could seem *a priori* a trivial result, it is absolutely not. (1) proves that if total capital increases to infinite, then both physical and living capital necessarily grow boundless. That is to say, if the optimal solution implies that total capital will grow forever, it will be impossible to observe a trajectory along which, for instance, physical capital will grow while living capital will decrease to reach zero. Both capitals need to grow in order to sustain the infinite accumulation of total capital.

Points (2), (3) and (4) study the limit behavior of some ratios depending on the productivity of total capital. Note that these limits are taken with respect to total capital or time. If total capital productivity is smaller than 1, then case (2) shows that production per unit of total capital will be bounded by  $1 - \delta_L$ , that is by the persistence of living capital.

According to (3), if  $\gamma = 1$ , that is, if  $\theta = 1 - \alpha$  and production can be understood as a Cobb-Douglas production function in physical and living capital, then the ratio of physical capital to living capital will remain with time between  $\mu$  and  $\alpha/\theta = \alpha/(1 - \alpha)$ . The upper limit of the ratio of physical to living capital is the ratio between their productivity parameters. That is to say, the ratio of the capitals does not necessarily converge towards the ratio of their productivities, it could remain "stuck" below. If on the contrary the production function exhibits increasing returns to scale, i.e.  $\gamma > 1$  then we have that first, the ratio of capitals does converge to  $\alpha/\theta$  when total capital tends to infinite. Second, if total capital converges (from above) towards  $\bar{X}$ , then the ratio of physical to living capital will tend to infinite (as living capital will tend to zero as shown in Lemma 2).

On the technical side, we observe that the maximization in (6) takes place at any period, included  $t = 0$ . Thus, given  $(K_0, L_0)$  and, then,  $X_0 = K_0 + qL_0$ , the initial artificial and natural capitals available for production are possibly different and equal to  $(K^*(X_0), L^*(X_0)) = (X_0, 0)$  if  $X_0 \leq \bar{X}$ , and to  $(K^*(X_0), L^*(X_0)) = (X_0 - qL^*(X_0), L^*(X_0))$  with  $L^*(X_0) \in (0, X_0/q)$  if  $X_0 > \bar{X}$ .

## 2.2 Existence of an optimal path

The following two lemmas prove that, first, the utility function either converges to a finite real number or diverges to  $-\infty$ , and, second, most importantly, they also show that the utility function has a maximum.

As a preliminary step, we need to characterize the steady state of an economy where living capital does not play any role. Then we tackle the more general case.

When living capital does not have any role in production, that is, when  $\lambda = 0$ , then

$$\bar{X} = \left( \frac{\alpha A}{\delta_K - \delta_L} \right)^{\frac{1}{1-\alpha}} \quad (13)$$

and, if, additionally, we assume that both capitals depreciate at the same rate, that is, if  $\delta_K = \delta_L$ , then  $X_t \leq \bar{X} = \infty$ ,  $L^*(X_t) = 0$  and  $K^*(X_t) = X_t$ . The maximum of production is  $\mathcal{F}(X_t) = AX_t^\alpha + (1 - \delta_K)X_t$ . Since there is only one capital in the model now, we naturally recover the traditional Ramsey model. Then, recall that the Modified Golden Rule is the Euler equation at the steady state, that is:  $\alpha A \bar{X}^{\alpha-1} + 1 - \delta_K = 1/\beta$ . As a result, the steady state without living capital is given by

$$\tilde{X} \equiv \left[ \frac{\alpha \beta A}{1 - \beta(1 - \delta_K)} \right]^{\frac{1}{1-\alpha}} \quad (14)$$

Let us deal next with the general case in which  $\lambda \neq 0$ . We shall assume in the following that the threshold of total capital is below the Modified Golden Rule:

**Assumption 2**  $\bar{X} < \tilde{X}$ .

Assumption 2 is satisfied when living capital is very efficient, that is, when both of  $L_t$ 's productivity parameters  $\lambda$  and  $\theta$  are high, or when the "relative price" of living capital  $q$  is low. Indeed, when  $\lambda$  and  $\theta$  are large enough or  $q$  is sufficiently small, we have that

$$\bar{X} \leq \frac{\alpha q}{\lambda \theta} < \tilde{X}$$

In order to prove the following lemma, we introduce a third critical value for total capital  $\hat{X}$  defined as:

$$\hat{X} \equiv \left( \frac{A}{\delta_K} \right)^{\frac{1}{1-\alpha}} \quad (15)$$

and compare the three critical values for total capital defined in (13), (14) and (15). Since  $\alpha, \beta \in (0, 1)$ , we have  $\tilde{X} < \hat{X}$  and, because of Assumption 2,

$$\bar{X} < \tilde{X} < \hat{X} \quad (16)$$

**Lemma 4 (convergence of intertemporal utility)** *Let  $0 < \gamma < 1/\beta$ . Under Assumption 2, the objective function of the policy maker is finite, that is*

$\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$  for any  $X_0 \geq 0$ . Moreover, the limit of the utility function along any feasible path exists. It converges to a finite real number or to  $-\infty$ :  $\lim_{T \rightarrow \infty} \sum_{t=0}^T \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}] \in \mathbb{R} \cup \{-\infty\}$ .

Using (5), we can re-write the policy maker's objective as  $\sum_{t=0}^{\infty} \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}]$ . Note that the objective function is now an infinite sum which depends exclusively on the sequence of total capital. Let  $\Pi(X_0)$  denote the set of sequences of composite capital stocks  $\mathbf{X} \equiv (X_t)_{t=0}^{\infty}$ , feasible from  $X_0$ , that is the set of  $\mathbf{X} \equiv (X_t)_{t=0}^{\infty}$  such that  $0 \leq X_{t+1} \leq \mathcal{F}(X_t)$  for any  $t \geq 0$ . We can prove that there exists an optimal solution to the planner's problem.

**Proposition 5 (existence of an optimal path)** *Let  $0 < \gamma < 1/\beta$ . Under Assumption 2, the function  $U(\mathbf{X}) \equiv \sum_{t=0}^{\infty} \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}]$  is upper semi-continuous in  $\Pi(X_0)$  with respect to the product topology and it attains a maximum in  $\Pi(X_0)$ .*

## 2.3 Dynamics

Having proved the existence of a solution to the social planner problem, and having characterized some of the asymptotic properties of feasible trajectories for  $X_t$ , let us study next the problem of the convergence of the optimal solution. Although we have made an effort to keep our setting as simple as possible on the technical side, production involves two different types of capital which differ not only in their definition but also in their role in production. Hence, our problem is essentially complex.

The following lemma provides us with interesting and extremely useful results: the optimal path of total capital is monotonic. Furthermore, it also shows that the stock of living capital is positive from a critical date on.

**Lemma 6 (monotonicity and threshold)** *Let  $X_0 > 0$ . The optimal sequence  $(X_t)_{t=0}^{\infty}$  is monotonic. Furthermore, the optimal solution does not converge to zero when  $t \rightarrow \infty$ . Moreover, there exists  $T$  such that, for any  $t \geq T$ , the sequence remains above the threshold, that is, total capital is always larger than  $\bar{X}$ :  $X_t > \bar{X}$ , and the optimal natural capital is positive,  $L_t > 0$ , for any  $t \geq T$ .*

In particular, the previous lemma implies that if the economy is sufficiently well endowed at  $t = 0$ , that is, if  $X_0 > \bar{X}$ , then  $T = 0$ , implying that living capital is always strictly positive along the optimal solution.

**Remark 7** *Our lemma can also be read as follows. If  $X_0 < \bar{X}$ , then the optimal sequence is monotonically increasing and there will be a time  $T$  at which total capital will be equal to  $\bar{X}$ , when the economy will start employing living capital in production. After  $T$ , total capital will continue growing so that living capital will always be positive after  $T$ . Otherwise, if  $X_0 > \bar{X}$ , then the optimal sequence of capital could be either monotonically increasing or decreasing. What we know for sure is that the sequence remains above the threshold for all  $t$ , and that the economy always involves living capital in production.*

Whether the optimal sequence reaches a steady state or whether it increases forever is the subject of the next series of three propositions. These propositions distinguish three cases depending on whether the productivity parameter of total capital is smaller, equal or larger than one.

**Proposition 8 (bounded growth)** *If  $\gamma < 1$ , then the optimal sequence  $(X_t)_{t=0}^{\infty}$  converges to a steady state  $X^*$ . This steady state is larger than the golden rule steady state value for  $X_t$  without living capital:  $X^* > \bar{X}$ .*

We know from Lemma 6 that any optimal sequence includes some living capital from a time  $T$  onwards, so that the limit value of total capital does include a positive amount of living capital. What Proposition 8 further offers is the proof that taking into account the services of nature and human capital as part of the production function leads the economy towards a higher level of total capital. However, since the maintenance of living capital requires some of the final good, the increase in living capital could come in detriment of physical capital. Although this increase could also come from the extra production generated, or from an extraordinary effort in savings, truth is we do not have the answer this far and the comparison established with the standard Ramsey model could just stop at total capital. The following lemma sheds light on the asymptotic composition of total capital.

**Lemma 9** *Let  $\gamma < 1$ . If  $\delta_K = \delta_L$ , then*

$$\lim_{t \rightarrow \infty} \zeta_t = \frac{\alpha}{\theta} \left( 1 + \frac{1}{\lambda L^*} \right) > \frac{\alpha}{\theta}$$

*where  $L^*$  is the stationary stock of living capital.*

Then, when  $\delta_K = \delta_L$ , living capital does enhance the accumulation of physical capital in the long run since the ratio physical to living capital is strictly larger than the ratio of their productivities. In other words, physical capital is accumulated beyond its "relative productivity".

When the parameter of total productivity of both capitals is equal to one, we obtain a balanced growth path as one would have expected in light of similar results in the standard Ramsey model. The following proposition characterizes the optimal trajectories and growth rates for consumption and each of the capitals.

**Proposition 10 (asymptotically balanced unbounded growth)** *If  $\gamma = 1$ ,*

then the optimal paths are given by

$$\begin{aligned} c_t &= \bar{c}_0 [\beta \mathcal{F}'(\bar{X})]^t \\ X_{t+1} &= \bar{c}_0 [\beta \mathcal{F}'(\bar{X})]^t \frac{\beta}{1-\beta} - z \\ L_{t+1} &= \left( \bar{c}_0 [\beta \mathcal{F}'(\bar{X})]^t \frac{\beta}{1-\beta} - z - \bar{X} \right) \frac{1}{q + \lambda \bar{X}} \\ K_{t+1} &= \left( \bar{c}_0 [\beta \mathcal{F}'(\bar{X})]^t \frac{\beta}{1-\beta} - z + \frac{q}{\lambda} \right) \frac{\lambda \bar{X}}{q + \lambda \bar{X}} \end{aligned}$$

where

$$z \equiv \frac{q}{\lambda} \left[ 1 + \frac{\delta_L}{\mathcal{F}'(\bar{X}) - 1} \right] \quad (17)$$

and  $\bar{c}_0$  is the value of initial consumption which maximizes  $V(c_0) = \sum_{t=0}^{\infty} \beta^t \ln(c_0 [\beta \mathcal{F}'(\bar{X})]^t)$ . The maximum is attained at

$$\bar{c}_0 \equiv (1 - \beta) \mathcal{F}'(\bar{X}) \left( X_0 + \frac{q}{\lambda} \left[ 1 + \frac{\delta_L}{\mathcal{F}'(\bar{X}) - 1} \right] \right)$$

Therefore, growth is unbounded

$$\lim_{t \rightarrow \infty} K_t = \lim_{t \rightarrow \infty} L_t = \infty$$

with

$$\lim_{t \rightarrow \infty} \frac{K_t}{qL_t} = \bar{X} \frac{\lambda}{q} \leq \frac{\alpha}{\theta}$$

but asymptotically balanced

$$\frac{L_{t+1}}{L_t} \sim \frac{K_{t+1}}{K_t} \sim \frac{X_{t+1}}{X_t} \sim \frac{c_{t+1}}{c_t} = \beta \mathcal{F}'(\bar{X}) > 1 \quad (18)$$

along the ABGP (Asymptotically Balanced Growth Path).

**Proposition 11 (unbalanced unbounded growth)** Let  $1 < \gamma < 1/\beta$ . The optimal sequence  $(X_t)_{t=0}^{\infty}$  converges to infinity with  $\lim_{t \rightarrow \infty} L_t = \lim_{t \rightarrow \infty} K_t = \infty$  and

$$\lim_{t \rightarrow \infty} \zeta_t = \lim_{t \rightarrow \infty} \frac{K_t}{qL_t} = \frac{\alpha}{\theta}$$

(1) If  $\delta_K = \delta_L$ , the capital ratio always stays above its limit, that is, the ratio physical to living capital converges towards its limit value from above, i.e. for any  $t$ :

$$\zeta_t \equiv \frac{K_t}{qL_t} = \frac{\alpha}{\lambda \theta} \frac{1}{L_t} + \frac{\alpha}{\theta} > \frac{\alpha}{\theta}$$

(2) If  $\delta_K > \delta_L$ , then the sequence of capital ratios  $(\zeta_t)$  is uniformly bounded away from zero and from  $\infty$ . Moreover, when  $X_t$  is sufficiently close to  $\bar{X}$  from the right,  $\zeta'(X_t) < 0$ , and, when  $X_t$  is sufficiently large,  $\zeta'(X_t) > 0$ . Moreover, if  $\delta_K > \delta_L$ , for  $X_t$  large enough,  $\zeta(X_t) < \alpha/\theta$ .

We observe that, when total capital is very efficient ( $1 \leq \gamma < 1/\beta$ ), growth is unbounded for both physical and living capital ( $\lim_{t \rightarrow \infty} K_t^* = \lim_{t \rightarrow \infty} L_t^* = \infty$ ) and so it is for total capital ( $\lim_{X \rightarrow \infty} X_t^* = \infty$ ). Moreover, if  $1 < \gamma < 1/\beta$  and if  $\theta > \alpha$ , so that living capital is more efficient, or if it has a more important role in production, then the economy ends up accumulating relatively more living capital in the long run:

$$\lim_{t \rightarrow \infty} \frac{K_t}{qL_t} < 1$$

Before closing this section, we would like to come back to Proposition 11. The proposition clearly shows that when physical capital and living capital wear down at the same rate, so that capitals are indistinguishable from this point of view, then  $\zeta_t > \alpha/\theta$  for all  $t$ . As mentioned, the economy will optimally accumulate more physical capital in relative terms than its relative productivity. The more productive physical capital, the more the economy will favor it. However, the most interesting result is presented in point (2). Since  $1 < \gamma < 1/\beta$ , we know that total capital will be monotonically increasing. What point (2) brings anew is that, when physical capital deteriorates faster than living capital, the composition of total capital will vary with time. When total capital is low, close to  $\bar{X}$  and that the economy has recently started incorporating nature and human capital services into account for production purposes, the economy privileges investment in living capital so that the ratio  $\zeta_t$  decreases. Then, when total capital becomes large enough, then its relative composition changes letting physical capital dominate the mix from that moment onwards. In this respect, stressing the role of the natural component in the living capital, we can say that there our model generates a reverse environmental Kuznets curve.

## 2.4 Simulations

We close this section providing some numerical illustrations to hopefully offer with some answers to the remaining open questions. Note that while illustrating the cases when  $\gamma \leq 1$  has little interest since all trajectories are explicit, it is extremely informative to obtain more information on the transition dynamics in the more challenging case of unbalanced and unbounded growth when  $1 < \gamma < 1/\beta$ . Here, we define adequate ratios for each case that converge in the long-run. Then, from that converging sequence we will be able to recover (numerically this time) some further information on the original capitals or ratios. Let us consider the following subcases: (1)  $\delta_K = \delta_L$  and (2)  $\delta_K > \delta_L$ .

When the depreciation rates are equal, it is possible to write the dynamics in variable  $\zeta_t$ , which, we know, converges to a limit  $\alpha/\theta$  according to Proposition 11.

**Proposition 12 (simulations with equal depreciation rates)** *Let  $1 < \gamma < 1/\beta$  and  $\delta_K = \delta_L$ . The optimal dynamics are driven by the Euler equation*

$$\frac{\psi(\zeta_{t+1}, \zeta_{t+2})}{\psi(\zeta_t, \zeta_{t+1})} = \beta \mathcal{F}'(X(\zeta_{t+1})) \quad (19)$$

where

$$\psi(\zeta_t, \zeta_{t+1}) \equiv \mathcal{F}(X(\zeta_t)) - X(\zeta_{t+1}) = c_t \quad (20)$$

represents the resource constraint and

$$X_t = X(\zeta_t) \equiv \frac{q \alpha \zeta_t + \alpha}{\lambda \theta \zeta_t - \alpha}$$

total capital.

The optimal trajectory for  $\zeta_t$  is determined by (19) together with the transversality condition:

$$\lim_{t \rightarrow \infty} \zeta_t = \frac{\alpha}{\theta}$$

Using (67) and (20), we will obtain the sequences of total capital  $(X_t)_{t=0}^{\infty}$  and consumption  $(c_t)_{t=0}^{\infty}$ .

When the depreciation rates are different, it is possible to write the optimal dynamics in terms of a new variable  $\varphi_t$ , which also converges to a limit  $\lambda\theta/(\alpha q)$ .

**Proposition 13 (simulations with different depreciation rates)** *Let  $1 < \gamma < 1/\beta$  and  $\delta_K > \delta_L$ . Economic dynamics are driven by a recursive transition function*

$$\varphi_{t+2} = \mathcal{T}(\varphi_t, \varphi_{t+1}) \quad (21)$$

where

$$\varphi_t = \varphi(X_t) \equiv \frac{1 + \lambda L^*(X_t)}{X_t - q L^*(X_t)} \quad (22)$$

and

$$\mathcal{T}(\varphi_t, \varphi_{t+1}) \equiv \varphi(\mathcal{F}(\varphi^{-1}(\varphi_{t+1})) - \beta \mathcal{F}'(\varphi^{-1}(\varphi_{t+1})) [\mathcal{F}(\varphi^{-1}(\varphi_t)) - \varphi^{-1}(\varphi_{t+1})])$$

The trajectory is determined by (21) jointly with the transversality condition:

$$\lim_{t \rightarrow \infty} \varphi_t = \frac{\lambda\theta}{\alpha q}$$

We close this paper by using the theoretical results presented in the previous subsection in some numerical illustrations that will complete the long-term description of the dynamics of our model.

In the first simulation, we assume that both physical and living capital depreciate at the same rate:  $\delta_K = \delta_L = 0.04$ . The calibration is summarized in the following table.

$A$	TFP	0.5
$\alpha$	physical capital productivity parameter	1/3
$\theta$	living capital productivity parameter	2/3 + 0.01
$\beta$	time discount	0.99
$\delta_K$	physical capital depreciation rate	0.04
$\delta_L$	living capital depreciation rate	0.04
$q$	relative price of living capital	1/2
$\lambda$	living capital parameter	1/2



[angle=0,width=11cm]fig3<sub>9</sub>23.png[angle = 0, width = 11cm]fig4<sub>9</sub>23.png

[angle=0,width=11cm]fig5<sub>9</sub>23.png[angle = 0, width = 11cm]fig6<sub>9</sub>23.png

Figures 1 and 2 present the corresponding results and illustrate well Proposition 12.

*itbpFU2.0116in1.5126in0inFig.11.pngitbpFU2.0116in1.5126in0inFig.22.png*

Our program uses the fact that the ratio  $\zeta$  converges towards  $\alpha/\theta$  and recovers backwards the trajectory for  $\zeta$  (Figure 1). Using this trajectory, we are able to compute the optimal trajectories for  $X$ ,  $K$  and  $L$ . Our results show that under our calibration, when  $1 < \gamma < 1/\beta$ , all original variables grow with time. Besides, natural capital grows faster than artificial capital until  $\zeta$  approaches the steady state. At that point in time, both capitals grow at the same rate forever (Figure 2).

The second set of numerical exercises illustrate Proposition 13. We assume this time different depreciation rates for physical and living capital, that is  $\delta_K = 0.04 > \delta_L = 0.039$  and  $q = 1/4$ , all other parameters keep the same values as in the previous exercise. Let us assume that  $X_0 = 0.9983$ , just slightly larger than  $\bar{X} = 0.9982$ . The following exercises show that the relative values of physical and living capital at time 0 do matter for the transitional dynamics. Figures 3 and 4 present results when  $K_0 = 0.75X_0 = 0.7487$  and  $L_0 = (X_0 - K_0)/q = 0.9983$ .

As predicted by Proposition 11 when  $X_0$  is close to  $\bar{X}$ ,  $\zeta$  decreases with time and converges towards a constant. Both physical and living capital grow forever.

Next, Figures 5 and 6 present results when  $K_0 = 0.5X_0 = 0.4991$  and  $L_0 = (X_0 - K_0)/q = 1.99$ . This example depicts an example in which living capital is initially "too" abundant with respect to physical capital. Note that the economy readjusts in one period. After readjustment, and although both  $K$  and  $L$  increase monotonically, the ratio  $\zeta$  decreases as expected.

### 3 Optimal growth and competitive equilibrium with living externalities

To complete our analysis, we consider in this section alternative problems where the trajectory for living capital is known to the decision maker, and it is equal to the optimal trajectory from the planner's program (P0). We will say that there are externalities from the living or "living externalities". In the first subsection, we compare (P0) to the problem of a policy maker identical to that of the previous section except that, as mentioned, she takes living capital to be equal to its optimal trajectory in (P0). We prove that both problems do have the same optimal solution for physical capital and hence consumption. In the final

subsection, we consider instead a market economy. There, the policy maker will also take living capital as given and will levy taxes to finance investment in living capital. We prove in our last theorem that there exist prices that decentralize the policy maker's problem and ensure that the economy will follow the optimal solution for (P0).

### 3.1 Optimal growth with living externalities

Reconsider gross output (4) and the maximal output (6) with total capital  $X_t \equiv K_t + qL_t$  given. A solution  $(K^*(X_t), L^*(X_t))_{t=0}^{\infty}$  satisfies

$$\mathcal{F}(X_t) = A[1 + \lambda L^*(X_t)]^\theta K^*(X_t)^\alpha + (1 - \delta_K) K^*(X_t) + q(1 - \delta_L) L^*(X_t)$$

Let us redefine the gross output as a function of total capital and living capital:

$$G(X_t, L_t) \equiv A(1 + \lambda L_t)^\theta (X_t - qL_t)^\alpha + (1 - \delta_K)(X_t - qL_t) + q(1 - \delta_L)L_t$$

and denote the first derivatives by  $G_X$  and  $G_L$ . We know that  $L^*(X_t)$  is solution to  $G_L(X_t, L_t) = 0$ . Invoking the Envelope Theorem, we have

$$\mathcal{F}'(X_t) = G_X(X_t, L^*(X_t)) + G_L(X_t, L^*(X_t)) L^{*'}(X_t) = G_X(X_t, L^*(X_t))$$

when  $X_t > \bar{X}$  (that is  $L^* > 0$ ). Hence, if  $X_t > \bar{X}$ ,

$$\mathcal{F}'(X_t) = G_X(X_t, L^*(X_t)) = \alpha A[1 + \lambda L^*(X_t)]^\theta [X_t - qL^*(X_t)]^{\alpha-1} + 1 - \delta_K$$

We observe that  $\mathcal{F}'(X_t) = F_K(K_t, L^*(X_t)) > 0$ .

In the following subsections, we compare the initial planner's program (P0) to the intertemporal utility maximization under the sequence of resource constraints, with the program (say (P1)) where the planner chooses the optimal sequence  $(K_t^*)_{t=0}^{\infty}$  taking as given the sequence  $(L_t^*)_{t=0}^{\infty}$  solution to (P0). In program (P1)  $(L_t^*)_{t=0}^{\infty}$  is just a particular sequence of externalities from the living. From now on, for simplicity, we denote a sequence  $(z_t)_{t=0}^{\infty}$  by  $(z_t)$ .

#### 3.1.1 Optimal growth

Let us begin by recalling the initial program (P0) and the properties of its optimal solution.

**(P0) Optimal growth.**

$$\max_{(K_t, L_t)} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$c_t + K_{t+1} + qL_{t+1} \leq F(K_t, L_t)$$

for any  $t \geq 0$ .

Let  $(K_t^*, L_t^*)_{t=0}^{\infty}$  be the optimal path of program (P0) with  $(K_0^*, L_0^*) = (K_0, L_0)$ . Assuming  $u(c) \equiv \ln c$ , we have obtained the following properties of

the optimal sequence  $(K_t^*, L_t^*)$  depending on the "total productivity" parameter  $\gamma \equiv \alpha + \theta$ .

(1) If  $\gamma < 1$ , then  $(K_t^*, L_t^*)$  converges to the steady state  $(K^*, L^*)$  with

$$\lim_{t \rightarrow \infty} \frac{K_t^*}{qL_t^*} = \frac{K^*}{qL^*} > 0 \quad (23)$$

(2) If  $\gamma = 1$ , growth is unbounded and asymptotically balanced with

$$\lim_{t \rightarrow \infty} \frac{K_t^*}{qL_t^*} = \bar{X} \frac{\lambda}{q} \leq \frac{\alpha}{\theta} \quad (24)$$

(3) If  $1 < \gamma < 1/\beta$ , growth is unbounded and unbalanced with

$$\lim_{t \rightarrow \infty} \frac{K_t^*}{qL_t^*} = \frac{\alpha}{\theta} \quad (25)$$

### 3.1.2 Optimal growth with living externalities

Let us assume that the planner takes as given the optimal sequence of living externalities  $(L_t^*)$ , which is the optimal solution to (P0) and that she maximizes her objective with respect to the sequence of physical capital levels  $(K_t)$ . This is the problem denoted by (P1), which is defined and solved below.

**(P1) Optimal growth with living externalities.**

$$\begin{aligned} & \max_{(K_t)} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & c_t + K_{t+1} \leq M(K_t, L_t^*, L_{t+1}^*) \end{aligned}$$

for any  $t \geq 0$ , where  $M(K_t, L_t^*, L_{t+1}^*) \equiv F(K_t, L_t^*) - qL_{t+1}^*$ .

There exists an optimal solution  $(\tilde{K}_t)$  and, because of the strict concavity of  $F(K, L)$  in  $K$  and the strict concavity of  $\ln c$  in  $c$ , this solution is unique.

One can prove that actually the optimal solutions to (P1) and (P0) coincide, that is that  $(\tilde{K}_t) = (K_t^*)$ , where  $(K_t^*)$  is the optimal solution to (P0). Indeed, maximizing the Lagrangian function  $\sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t [F(K_t, L_t^*) - c_t - K_{t+1} - qL_{t+1}^*]$  we obtain the first-order conditions  $\lambda_t = \beta^t u'(c_t)$  and  $\lambda_t = \lambda_{t+1} F_K(K_{t+1}, L_{t+1}^*)$  for any  $t$ . As usual, the first order conditions can be combined to attain the Euler equation

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta F_K(K_{t+1}, L_{t+1}^*) = \beta \mathcal{F}'(X_{t+1})$$

which coincides with the Euler equation of program (P0).

### 3.2 Competitive equilibrium with living externalities

Now, consider a market economy where the sequence of living capital  $(L_t)$  is given. In particular, let us assume that the market economy takes as given the optimal sequence  $(L_t^*)$ , which is solution to (P0). In other terms, it is as if the government chooses  $(L_t) = (L_t^*)$ , which is attainable by levying an opportune sequence of real taxes  $(\tau_t^*)$  to finance the living capital accumulation, that is  $\tau_t^* = q [L_{t+1}^* - (1 - \delta_L) L_t^*]$ .

The firm chooses the sequence of physical capital demands  $(K_t)$  that maximizes its intertemporal profit:

$$\pi^* = \max_{(K_t)} \left( \sum_{t=0}^{\infty} p_t^* \left[ A (1 + \lambda L_t^*)^\theta K_t^\alpha + (1 - \delta_K) K_t - K_{t+1} - \tau_t^* \right] - r^* K_0 \right)$$

At the same time, the representative agent maximizes her intertemporal utility function under the intertemporal budget constraint:  $\sum_{t=0}^{\infty} p_t^* c_t = \pi^* + r^* K_0$ . The associated competitive equilibrium (P2) is defined as follows:

**(P2) Competitive equilibrium  $(p_t^*, q^*, K_t^*)$  with  $(L_t^*)$  as externality.**

(P2) comprehends the firm's and the representative consumer's problems and the clearing of all markets:

(1) The firm chooses the sequence of physical capital demands  $(K_t)$  to maximize the profit:

$$\pi^* = \max_{(K_t)} \left( \sum_{t=0}^{\infty} p_t^* [M(K_t, L_t^*, L_{t+1}^*) - K_{t+1}] - r^* K_0 \right)$$

Profit maximization gives

$$p_t^* = p_{t+1}^* M_K(K_{t+1}, L_{t+1}^*, L_{t+2}^*) = p_{t+1}^* F_K(K_{t+1}, L_{t+1}^*)$$

for any  $t \geq 0$ .

**Remark 14** *We observe that there is only one firm acting as a competitive firm without market power. In other terms, the firm takes prices as given and does not play strategically by exploiting its market power. In the case of constant returns to scale, a zero profit condition holds and maximizing the profits of a large number of firms yields the same solution in terms of optimal aggregate demand for capital than maximizing the profit of an aggregate firm. However, in our case, returns to scale are not constant and we assume that the unique existing firm is a price taker. Even if this is a strong assumption, we can partially justify it by noticing that the capital good and the consumption good are the same and the firms shares the market power with many consumers.*

(2) The representative consumer maximizes her intertemporal utility func-

tion under an intertemporal budget constraints:

$$\begin{aligned} & \max_{(c_t)} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & \sum_{t=0}^{\infty} p_t^* c_t \leq \pi^* + r^* K_0 \end{aligned}$$

where  $\pi^*$  is the profit she earns as (unique) shareholder.  $r^*$  is the price of  $K_0$  at period  $-1$ . Indeed, note that physical capital is bought one period before being used.

(3) All markets clear:

$$c_t + K_{t+1}^* = M(K_t^*, L_t^*, L_{t+1}^*)$$

with  $K_0^* = K_0$ .

Our final theorem provides with the prices that allow to decentralize the policy maker's maker.

**Theorem 15** *Let  $0 < \gamma < 1/\beta$  and let the sequence  $(c_t^*, K_t^*)$  be solution to (P1). Define*

$$\begin{aligned} p_t^* & \equiv \beta^t u'(c_t^*) = \beta^t / c_t^* \\ r^* & \equiv p_0^* M_K(K_0, L_0^*, L_1^*) = \alpha A (1 + \lambda L_0)^\theta K_0^{\alpha-1} + 1 - \delta_K \end{aligned}$$

*Then  $(r^*, (p_t^*, c_t^*, K_t^*)_{t=0}^{\infty})$  is a competitive equilibrium where  $(L_t^*)$ , solution to (P0), is a sequence of externalities.*

Very briefly, note that the price the firm has to pay for  $K_0$  increases with the role of living capital, and in particular with its relative productivity  $\lambda$  and with  $\theta$ , its share in production. It also increases with the initial level of living capital. Regarding the impact of physical capital,  $p_t$  increases with the share of physical capital in production and it decreases with  $K_0$ .

## 4 Conclusion

We have defined living capital as a comprehensive measure of all economic inputs that are alive. The three characteristics of living capital are that: first, it is not essential; second, it is irreversible; third, it depreciates at a slower pace than physical capital. Among these, the one that makes living capital truly different from any other productive capital is its non-essentiality, which means that the economy could very well produce without living capital.

Then, we defined total capital as a sum of physical and living capital in terms of value and we have considered, first, the planner's solution and, then, the market economy.

In the case of the policy maker's problem, we have shown that the optimal sequence of capital is always monotonic. Depending on the productivity of total

capital, three different regimes hold: (1) bounded growth; (2) asymptotically balanced unbounded growth; (3) unbalanced unbounded growth. In case (1), when the sum of the productivity parameters of physical and living capital is lower than one, total capital converges to a steady state with a positive stock of living capital, which is larger than the one without living capital. In the second case, when this sum is equal to one, growth becomes unbounded and the economy experiences an asymptotically balanced growth path. Finally, in case (3), when the sum of the productivity parameters of physical and living capital is greater than one, growth is unbounded for both physical and living capital. Clearly, this implies that total capital grows forever. Beyond a critical point of total capital, living capital starts to grow but the ratio physical to living capital first decreases and then, after a critical date, it grows to infinity.

In the case of a market economy, we show that if the government levies taxes to finance the accumulation of living capital and implements exactly the optimal sequence of living capital as in the planner's program, then the equilibrium market prices decentralize exactly the planner's solution. We provide the explicit sequence of market prices.

## 5 Appendix

### Proof of Lemma 2

We maximize  $F(K_t, L_t) \equiv A(1 + \lambda L_t)^\theta K_t^\alpha + (1 - \delta_K)K_t + (1 - \delta_L)qL_t$  under the constraint  $K_t + qL_t \leq X_t$  with  $L_t, K_t \geq 0$ .

We observe that  $F(X_t - qL_t, L_t)$  is strictly concave in  $L_t$ :

$$\frac{[F(X_t - qL_t, L_t)]''}{A(1 + \lambda L_t)^\theta K_t^\alpha} = -\alpha(1 - \alpha) \frac{q^2}{K_t^2} - \theta(1 - \theta) \frac{\lambda^2}{(1 + \lambda L_t)^2} - 2\alpha\theta \frac{q\lambda}{K_t(1 + \lambda L_t)} < 0$$

that is  $[F(X_t - qL_t, L_t)]'' < 0$ .

Equivalently, we can maximize  $A(1 + \lambda L_t)^\theta (X_t - qL_t)^\alpha + (\delta_K - \delta_L)qL_t$  with respect to  $L_t \geq 0$ .

If  $L_t > 0$ , we obtain the first-order condition:

$$\lambda\theta A(1 + \lambda L_t)^{\theta-1} (X_t - qL_t)^\alpha - \alpha q A(1 + \lambda L_t)^\theta (X_t - qL_t)^{\alpha-1} + q(\delta_K - \delta_L) = 0 \quad (26)$$

or, equivalently,  $\mathcal{L}(L_t) = \mathcal{R}(L_t)$ , where  $\mathcal{L}(L_t)$  and  $\mathcal{R}(L_t)$  are given by (10) and (11).

We observe that, for  $L \in [0, X/q]$ ,  $\mathcal{L}'(L) < 0$  and  $\mathcal{R}'(L) > 0$ .

Moreover,  $\lim_{L_t \rightarrow 0} \mathcal{L}(L_t) = \lambda\theta AX_t^\alpha + q\delta_K$ ,  $\lim_{L_t \rightarrow 0} \mathcal{R}(L_t) = \alpha q AX_t^{\alpha-1} + q\delta_L$ ,  $\lim_{L_t \rightarrow X_t/q} \mathcal{L}(L_t) = q\delta_K$  and  $\lim_{L_t \rightarrow (X_t/q)^-} \mathcal{R}(L_t) = \infty$ .

Thus, there exists a unique value  $L^* \in [0, X_t/q]$  such that  $\mathcal{L}(L^*) = \mathcal{R}(L^*)$  if and only if  $\lambda\theta AX_t^\alpha + q\delta_K \geq \alpha q AX_t^{\alpha-1} + q\delta_L$ .

The critical value  $\bar{X}$  is defined by  $\lambda\theta A\bar{X}^\alpha + q\delta_K = \alpha q A\bar{X}^{\alpha-1} + q\delta_L$ , that is by (7).

When  $X_t = \bar{X}$ , then  $L^* = 0$ ; when  $X_t > \bar{X}$ , then  $L^* > 0$  because  $\lambda\theta AX_t^\alpha + q\delta_K - \alpha q AX_t^{\alpha-1} - q\delta_L > 0$  (indeed, the LHS of this inequality is strictly increasing in  $X_t$ ).

Therefore, we obtain the following.

(1) If  $X_t \leq \bar{X}$ , then  $L^*(X_t) = 0$  and  $K^*(X_t) = X_t$ , and the maximum is given by (9).

(2) If  $X_t > \bar{X}$ , then  $L^*(X_t) \in (0, X_t/q)$  and the maximum is given by (12).

(3) Functions (9) and (12) are continuous and differentiable in  $[0, \bar{X})$  and  $(\bar{X}, \infty)$  respectively. Moreover, they are also continuous at  $X = \bar{X}$ :

$$\begin{aligned} & \lim_{X_t \rightarrow \bar{X}^+} \mathcal{F}(X_t) \\ &= \lim_{L^* \rightarrow 0^+} \left[ A(1 + \lambda L^*)^\theta (X_t - qL^*)^\alpha + (1 - \delta_K)(X_t - qL^*) + (1 - \delta_L)qL^* \right] \\ &= AX_t^\alpha + (1 - \delta_K)X_t = \lim_{X_t \rightarrow \bar{X}^-} \mathcal{F}(X_t) \end{aligned}$$

We apply the Envelope Theorem to prove the differentiability of  $\mathcal{F}$  at  $X_t = \bar{X}$ . Let us define

$$G(X_t, L_t) \equiv A(1 + \lambda L_t)^\theta (X_t - qL_t)^\alpha + (1 - \delta_K)(X_t - qL_t) + (1 - \delta_L)qL_t$$

and denote the first derivatives by  $G_X$  and  $G_L$ .

We know that  $L^*(X_t)$  is solution to  $G_L(X_t, L_t) = 0$ . We observe that, if  $X_t > \bar{X}$  (that is  $L^* > 0$ ), then by the Envelope Theorem,

$$\mathcal{F}'(X_t) = G_X(X_t, L^*(X_t)) + G_L(X_t, L^*(X_t))L'^*(X_t) = G_X(X_t, L^*(X_t))$$

Hence, if  $X_t > \bar{X}$ ,

$$\mathcal{F}'(X_t) = G_X(X_t, L^*(X_t)) = \alpha A [1 + \lambda L^*(X_t)]^\theta [X_t - qL^*(X_t)]^{\alpha-1} + 1 - \delta_K > 0 \quad (27)$$

which proves that,  $\mathcal{F}$  is strictly increasing.

Additionally, we have that

$$\begin{aligned} \lim_{X_t \rightarrow \bar{X}^-} \mathcal{F}'(X_t) &= \alpha AX_t^{\alpha-1} + 1 - \delta_K \\ \lim_{X_t \rightarrow \bar{X}^+} \mathcal{F}'(X_t) &= \lim_{X_t \rightarrow \bar{X}^+} G_X(X_t, L^*(X_t)) = G_X(X_t, 0) = \alpha AX_t^{\alpha-1} + 1 - \delta_K \end{aligned}$$

which proves that  $\mathcal{F}$  is differentiable at  $X_t = \bar{X}$ .

(4) Taking the derivative of (26) with  $L_t = L^*(X_t)$  with respect to  $X_t$ , we obtain

$$qL^{*'}(X_t) = \frac{\alpha q \lambda \theta + \alpha(1 - \alpha)q^2 \frac{P(X_t)}{K^*(X_t)}}{\theta(1 - \theta)\lambda^2 \frac{K^*(X_t)}{P(X_t)} + 2\alpha q \lambda \theta + \alpha(1 - \alpha)q^2 \frac{P(X_t)}{K^*(X_t)}} > 0 \quad (28)$$

$$K^{*'}(X) = 1 - qL^{*'}(X) = \frac{\theta(1 - \theta)\lambda^2 \frac{K^*(X_t)}{P(X_t)} + \alpha q \lambda \theta}{\theta(1 - \theta)\lambda^2 \frac{K^*(X_t)}{P(X_t)} + 2\alpha q \lambda \theta + \alpha(1 - \alpha)q^2 \frac{P(X_t)}{K^*(X_t)}} \quad (29)$$

where  $P(X_t) \equiv 1 + \lambda L^*(X_t)$ .

(5) Focus on the derivative for  $X_t > \bar{X}$ :

$$\mathcal{F}'(X_t) = \alpha A [1 + \lambda L^*(X_t)]^\theta [X_t - qL^*(X_t)]^{\alpha-1} + 1 - \delta_K > 0 \quad (30)$$

The second derivative when  $X_t > \bar{X}$  is given by

$$\mathcal{F}''(X_t) = \alpha A \frac{\lambda \theta K^*(X_t) L^{*\prime}(X_t) - (1 - \alpha) P(X_t) [1 - qL^{*\prime}(X_t)]}{P(X_t)^{1-\theta} K^*(X_t)^{2-\alpha}}$$

Then,  $\mathcal{F}''(X_t) < 0$  if and only if

$$\lambda \theta K^*(X_t) qL^{*\prime}(X_t) - (1 - \alpha) qP(X_t) [1 - qL^{*\prime}(X_t)] < 0 \quad (31)$$

Replacing expressions (28) and (29) for  $qL^{*\prime}(X_t)$  and  $1 - qL^{*\prime}(X_t) = K^{*\prime}(X_t)$  in (31), we find that  $\mathcal{F}''(X_t) < 0$  if and only if  $\gamma < 1$ .

**Proof of Lemma 3**

(1) First, we show that  $\lim_{X_t \rightarrow \infty} L^*(X_t) = \infty$ .

If it was not the case, there should exist  $\nu$  and a sequence  $(X_n)$  with  $\lim_{n \rightarrow \infty} X_n = \infty$  such that  $L^*(X_n) \leq \nu$  for any  $n$ .

Let  $L_n^* \equiv L^*(X_n)$ . According to (26), for any  $n$ , we have

$$\begin{aligned} 0 &= \lambda \theta A (1 + \lambda L_n^*)^{\theta-1} (X_n - qL_n^*)^\alpha \\ &\quad - \alpha q A (1 + \lambda L_n^*)^\theta (X_n - qL_n^*)^{\alpha-1} + q(\delta_K - \delta_L) \end{aligned} \quad (32)$$

and

$$\begin{aligned} \lambda \theta A (1 + \lambda L_n^*)^{\theta-1} (X_n - qL_n^*)^\alpha &\geq \lambda \theta A (1 + \lambda \nu)^{\theta-1} (X_n - q\nu)^\alpha \\ \alpha q A (1 + \lambda L_n^*)^\theta (X_n - qL_n^*)^{\alpha-1} &\leq \alpha q A (1 + \lambda \nu)^\theta (X_n - q\nu)^{\alpha-1} \end{aligned}$$

We observe that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[ \lambda \theta A (1 + \lambda L_n^*)^{\theta-1} (X_n - qL_n^*)^\alpha \right] &\geq \lim_{n \rightarrow \infty} \left[ \lambda \theta A (1 + \lambda \nu)^{\theta-1} (X_n - q\nu)^\alpha \right] = \infty \\ \lim_{n \rightarrow \infty} \left[ \alpha q A (1 + \lambda L_n^*)^\theta (X_n - qL_n^*)^{\alpha-1} \right] &\leq \lim_{n \rightarrow \infty} \left[ \alpha q A (1 + \lambda \nu)^\theta (X_n - q\nu)^{\alpha-1} \right] = 0 \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \left[ \lambda \theta A (1 + \lambda L_n^*)^{\theta-1} (X_n - qL_n^*)^\alpha - \alpha q A (1 + \lambda L_n^*)^\theta (X_n - qL_n^*)^{\alpha-1} + q(\delta_K - \delta_L) \right] = \infty$$

which is in contradiction with (32), and which proves that  $\lim_{X_t \rightarrow \infty} L^*(X_t) = \infty$ .

Next, let us show in a similar manner that  $\lim_{X_t \rightarrow \infty} K^*(X_t) = \infty$ .

If it was not the case, then there should exist  $\kappa > 0$  and a sequence  $(X_n)$  with  $\lim_{n \rightarrow \infty} X_n = \infty$  such that  $K^*(X_n) \leq \kappa$  for any  $n$ .

Let  $K_n^* \equiv K^*(X_n)$ . According to (26), for any  $n$ , we have

$$\lambda \theta A (1 + \lambda L_n^*)^{\theta-1} K_n^{*\alpha} - \alpha q A (1 + \lambda L_n^*)^\theta K_n^{*\alpha-1} + q(\delta_K - \delta_L) = 0 \quad (33)$$



Since  $\lim_{n \rightarrow \infty} L_n^* = \infty$  and  $K_n^* \in [0, \kappa]$ , a bounded set, we have

$$\lim_{n \rightarrow \infty} \left[ \lambda \theta A (1 + \lambda L_n^*)^{\theta-1} K_n^{*\alpha} \right] = 0$$

Moreover,  $\alpha q A (1 + \lambda L_n^*)^\theta K_n^{*\alpha-1} \geq \alpha q A (1 + \lambda L_n^*)^\theta \kappa^{\alpha-1}$  for any  $n$  and

$$\lim_{n \rightarrow \infty} \left[ \alpha q A (1 + \lambda L_n^*)^\theta K_n^{*\alpha-1} \right] \geq \lim_{n \rightarrow \infty} \left[ \alpha q A (1 + \lambda L_n^*)^\theta \kappa^{\alpha-1} \right] = \infty$$

Thus,

$$\lim_{n \rightarrow \infty} \left[ \lambda \theta A (1 + \lambda L_n^*)^{\theta-1} K_n^{*\alpha} - \alpha q A (1 + \lambda L_n^*)^\theta K_n^{*\alpha-1} + q (\delta_K - \delta_L) \right] = -\infty$$

which is in contradiction with (33).

Observe that these results hold for any  $\gamma > 0$ .

(2) We have

$$\mathcal{F}(X_t) = A [1 + \lambda L^*(X_t)]^\theta [X_t - qL^*(X_t)]^\alpha + (\delta_K - \delta_L) qL^*(X_t) + (1 - \delta_K) X_t$$

Hence,

$$\begin{aligned} \frac{\mathcal{F}(X_t)}{X_t} &= A [1 + \lambda L^*(X_t)]^\theta \frac{[X_t - qL^*(X_t)]^\alpha}{X_t} + 1 - \delta_L + (\delta_K - \delta_L) \left[ \frac{qL^*(X_t)}{X_t} - 1 \right] \\ &\leq A \left[ 1 + \frac{\lambda}{q} qL^*(X_t) \right]^\theta \frac{[X_t - qL^*(X_t)]^\alpha}{X_t} + 1 - \delta_L \\ &< A \left( 1 + \frac{\lambda}{q} X_t \right)^\theta X_t^{\alpha-1} + 1 - \delta_L \end{aligned}$$

since  $X_t > \bar{X}$  entails  $L^*(X_t) > 0$ .

$\gamma < 1$  implies

$$\lim_{X_t \rightarrow \infty} \left( 1 + \frac{\lambda}{q} X_t \right)^\theta X_t^{\alpha-1} = 0$$

Therefore,

$$\lim_{X_t \rightarrow \infty} \frac{\mathcal{F}(X_t)}{X_t} \leq 1 - \delta_L < 1$$

(3) Let  $\gamma = 1$ .

First we prove that there exists  $\mu \in (0, \alpha/\theta]$  such that, for any sequence  $(X_n)$  with  $\lim_{n \rightarrow \infty} X_n = \infty$ , we have

$$\mu \leq \liminf_{n \rightarrow \infty} \zeta(X_n)$$

From equation (26), we have

$$\begin{aligned} &\lambda \theta A [1 + \lambda L^*(X_t)]^{\theta-1} [X_t - qL^*(X_t)]^\alpha - \alpha q A [1 + \lambda L^*(X_t)]^\theta [X_t - qL^*(X_t)]^{\alpha-1} \\ &= q (\delta_L - \delta_K) \end{aligned} \tag{34}$$

Since by definition

$$\zeta(X_t) = \frac{X_t - qL^*(X_t)}{qL^*(X_t)}$$

then we have

$$L^*(X_t) = \frac{1}{q} \frac{X_t}{1 + \zeta(X_t)}$$

and (34) becomes  $\mathcal{L}_1(X_t) + \mathcal{L}_2(X_t) = \mathcal{R}(X_t)$ , where

$$\begin{aligned} \mathcal{L}_1(X_t) &\equiv \lambda A X_t^\alpha \left[ 1 + \frac{\lambda}{q} \frac{X_t}{1 + \zeta(X_t)} \right]^{\theta-1} \left[ \theta - \frac{\alpha + \theta}{1 + \zeta(X_t)} \right] \\ \mathcal{L}_2(X_t) &\equiv -\alpha q A X_t^{\alpha-1} \left[ 1 + \frac{\lambda}{q} \frac{X_t}{1 + \zeta(X_t)} \right]^{\theta-1} \\ \mathcal{R}(X_t) &\equiv q(\delta_L - \delta_K) \left[ \frac{\zeta(X_t)}{1 + \zeta(X_t)} \right]^{1-\alpha} \end{aligned}$$

We want to prove that there exist  $\mu > 0$  and  $\nu > 0$  such that, for any  $X_t \geq \nu$ ,

$$\zeta(X_t) \geq \mu$$

Suppose the contrary. In this case, there would exist a sequence  $(X_n)$  with  $\lim_{n \rightarrow \infty} X_n = \infty$  and such that  $\lim_{n \rightarrow \infty} \zeta(X_n) = 0$ .

We observe that  $\lim_{n \rightarrow \infty} \mathcal{L}_1(X_n) = -\infty$  because  $\gamma \geq 1$ ,

$$\lim_{n \rightarrow \infty} \left( \lambda A X_n^\alpha \left[ 1 + \frac{\lambda}{q} \frac{X_n}{1 + \zeta(X_n)} \right]^{\theta-1} \right) = \infty$$

and

$$\lim_{n \rightarrow \infty} \left[ \theta - \frac{\gamma}{1 + \zeta(X_n)} \right] = -\alpha$$

Moreover, since  $\alpha < 1$  and  $\theta < 1$ ,

$$\lim_{n \rightarrow \infty} \mathcal{L}_2(X_n) = -\alpha q A \lim_{n \rightarrow \infty} \left( X_n^{\alpha-1} \left[ 1 + \frac{\lambda}{q} \frac{X_n}{1 + \zeta(X_n)} \right]^{\theta-1} \right) = 0$$

Finally, since  $\alpha < 1$ ,

$$\lim_{n \rightarrow \infty} \mathcal{R}(X_n) = q(\delta_L - \delta_K) \lim_{n \rightarrow \infty} \left[ \frac{\zeta(X_n)}{1 + \zeta(X_n)} \right]^{1-\alpha} = 0$$

Then,  $0 = \lim_{n \rightarrow \infty} \mathcal{R}(X_n) = \lim_{n \rightarrow \infty} \mathcal{L}_1(X_n) + \lim_{n \rightarrow \infty} \mathcal{L}_2(X_n) = -\infty$ , a contradiction.

Now, let us prove that, for any sequence  $(X_n)$  with  $\lim_{n \rightarrow \infty} X_n = \infty$ , we have

$$\limsup_{n \rightarrow \infty} \zeta(X_n) \leq \frac{\alpha}{\theta}$$

Equation (26) writes:

$$\lambda\theta A [1 + \lambda L^*(X_t)]^{\theta-1} K^*(X_t)^\alpha + q(\delta_K - \delta_L) = \alpha q A [1 + \lambda L^*(X_t)]^\theta K^*(X_t)^{\alpha-1}$$

Since  $\delta_K \geq \delta_L$ , we have

$$\lambda\theta A [1 + \lambda L^*(X_t)]^{\theta-1} K^*(X_t)^\alpha \leq \alpha q A [1 + \lambda L^*(X_t)]^\theta K^*(X_t)^{\alpha-1}$$

that is

$$\frac{K^*(X_t)}{qL^*(X_t)} \leq \frac{\alpha}{\lambda\theta} \frac{1 + \lambda L^*(X_t)}{L^*(X_t)} \quad (35)$$

Since  $\lim_{X_t \rightarrow \infty} L^*(X_t) = \infty$ , for any sequence  $(X_n)$  such that  $\lim_{n \rightarrow \infty} X_n = \infty$ , we find

$$\limsup_{n \rightarrow \infty} \frac{K^*(X_n)}{qL^*(X_n)} \leq \frac{\alpha}{\theta}$$

(4) Finally, let  $\gamma > 1$ .

It is easy to show that  $\lim_{X_t \rightarrow \bar{X}^-} \zeta(X_t) = \infty$ . Indeed,  $L^*(X_t)$  converges to 0 continuously and  $K^*(X_t)$  to  $\bar{X}$  also continuously ( $L^*(X_t)$  and  $K^*(X_t)$  are differentiable).

Focus now on the asymptotic behavior of  $\zeta(X_t)$  when  $X_t \rightarrow \infty$ . We want to prove that  $\lim_{X_t \rightarrow \infty} \zeta(X_t) = \alpha/\theta$ .

Let us to show first that, if  $\varepsilon > 0$ , then there exists  $\mu > 0$  such that  $\zeta(X_t) \leq \mu$  for any  $X_t \geq \bar{X} + \varepsilon$ .

According to (35), we have

$$\zeta(X_t) \leq \frac{\alpha}{\lambda\theta} \frac{1 + \lambda L^*(X_t)}{L^*(X_t)}$$

for any  $X_t \geq \bar{X} + \varepsilon$ . Since  $\lim_{X_t \rightarrow \infty} L^*(X_t) = \infty$ ,

$$\lim_{X_t \rightarrow \infty} \left[ \frac{\alpha}{\lambda\theta} \frac{1 + \lambda L^*(X_t)}{L^*(X_t)} \right] = \frac{\alpha}{\theta}$$

Let  $\eta > 0$ . Then, there exists  $Y > \bar{X} + \varepsilon$  such that

$$\frac{\alpha}{\lambda\theta} \frac{1 + \lambda L^*(X_t)}{L^*(X_t)} \leq \frac{\alpha}{\theta} + \eta$$

for any  $X_t > Y$ . Let

$$\mu \equiv \max \left\{ \max_{X_t \in [\bar{X} + \varepsilon, Y]} \left[ \frac{\alpha}{\lambda\theta} \frac{1 + \lambda L^*(X_t)}{L^*(X_t)} \right], \frac{\alpha}{\theta} + \eta \right\}$$

Therefore, we obtain

$$\zeta(X_t) \leq \mu \quad (36)$$

for any  $X_t \geq \bar{X} + \varepsilon$ .

(26) is equivalent to

$$\begin{aligned}\lambda\theta AK^*(X_t) - \alpha qA[1 + \lambda L^*(X_t)] &= q(\delta_L - \delta_K)[1 + \lambda L^*(X_t)]^{1-\theta} K^*(X_t)^{1-\alpha} \\ \frac{K^*(X_t)}{qL^*(X_t)} &= \frac{\alpha}{\lambda\theta} \frac{1 + \lambda L^*(X_t)}{L^*(X_t)} + \frac{\delta_L - \delta_K}{\lambda\theta A} \frac{[1 + \lambda L^*(X_t)]^{1-\theta} K^*(X_t)^{1-\alpha}}{L^*(X_t)}\end{aligned}\quad (37)$$

Note that

$$\frac{[1 + \lambda L^*(X_t)]^{1-\theta} K^*(X_t)^{1-\alpha}}{L^*(X_t)} = \left[ \frac{1 + \lambda L^*(X_t)}{L^*(X_t)} \right]^{1-\theta} \left[ \frac{L^*(X_t)}{K^*(X_t)} \right]^{-\theta} K^*(X_t)^{1-\alpha-\theta} \quad (38)$$

According to (36), we have that, for any  $X_t \geq \bar{X} + \varepsilon$ ,

$$\begin{aligned}\frac{K^*(X_t)}{L^*(X_t)} &\leq \mu q \\ \left[ \frac{L^*(X_t)}{K^*(X_t)} \right]^{-\theta} &\leq \left( \frac{1}{\mu q} \right)^{-\theta}\end{aligned}$$

Then, for any  $X_t \geq \bar{X} + \varepsilon$ ,

$$0 \leq \left[ \frac{1 + \lambda L^*(X_t)}{L^*(X_t)} \right]^{1-\theta} \left[ \frac{L^*(X_t)}{K^*(X_t)} \right]^{-\theta} K^*(X_t)^{1-\alpha-\theta} \leq \left[ \frac{1 + \lambda L^*(X_t)}{L^*(X_t)} \right]^{1-\theta} \left( \frac{1}{\mu q} \right)^{-\theta} K^*(X_t)^{1-\alpha-\theta}$$

If  $\gamma > 1$ , we have

$$\lim_{X_t \rightarrow \infty} \left( \left[ \frac{1 + \lambda L^*(X_t)}{L^*(X_t)} \right]^{1-\theta} \left( \frac{1}{\mu q} \right)^{-\theta} K^*(X_t)^{1-\alpha-\theta} \right) = 0$$

and, thus, the limit of (38) also exists with

$$\lim_{X_t \rightarrow \infty} \frac{[1 + \lambda L^*(X_t)]^{1-\theta} K^*(X_t)^{1-\alpha}}{L^*(X_t)} = 0$$

In addition and according to (37), we obtain

$$\lim_{X_t \rightarrow \infty} \frac{K^*(X_t)}{qL^*(X_t)} = \frac{\alpha}{\lambda\theta} \lim_{X_t \rightarrow \infty} \frac{1 + \lambda L^*(X_t)}{L^*(X_t)} = \frac{\alpha}{\theta} \quad (39)$$

Since

$$\frac{qL^*(X_t)}{X_t} = \frac{1}{1 + \zeta(X_t)}$$

then

$$\frac{qL^*(X_t)}{X_t} < \frac{\theta}{\alpha + \theta}$$

for any  $X$  and

$$\lim_{X_t \rightarrow \infty} \frac{qL^*(X_t)}{X_t} = \frac{\theta}{\alpha + \theta}$$

Finally, consider  $\delta_K > \delta_L$ . According to (26), we have

$$\zeta(X_t) \equiv \frac{K^*(X_t)}{qL^*(X_t)} = \frac{\alpha}{\theta} + \frac{\alpha A + (\delta_L - \delta_K)[1 + \lambda L^*(X_t)]^{1-\theta} K^*(X_t)^{1-\alpha}}{\lambda \theta A L^*(X_t)}$$

Notice that

$$\lim_{X \rightarrow \infty} \left( (\delta_L - \delta_K)[1 + \lambda L^*(X)]^{1-\theta} K^*(X)^{1-\alpha} \right) = -\infty$$

Thus, for  $X_t$  large enough,  $\zeta(X_t) < \alpha/\theta$  and, according to (39),  $\lim_{X \rightarrow \infty} \zeta(X_t) = \alpha/\theta$ . ■

#### Proof of Lemma 4

First, we want to prove that, under Assumption 2,  $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$  for any  $X_0 \geq 0$ .

We observe that  $c_t \leq \mathcal{F}(X_t)$  for any  $t$ . If  $X_0 = 0$ , then  $c_t = 0$  for any  $t \geq 0$  and  $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$ .

Let  $X_0 > 0$ . If the feasible sequence  $(X_t)_{t=0}^{\infty}$  is bounded, the sequences  $(\mathcal{F}(X_t))_{t=0}^{\infty}$  and  $(c_t)_{t=0}^{\infty}$  are bounded as well. Thus,  $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$ .

Now, let the sequence  $(X_t)_{t=0}^{\infty}$  be unbounded.

A sequence  $(X_t)_{t=0}^{\infty}$  is feasible if  $X_{t+1} \leq \mathcal{F}(X_t)$  for any  $t$ . Define the maximal sequence  $(\tilde{X}_t)_{t=0}^{\infty}$  by  $\tilde{X}_{t+1} = \mathcal{F}(\tilde{X}_t)$  for any  $t$  with  $\tilde{X}_0 = X_0$ . According to point (3) in Lemma 2,  $\mathcal{F}$  is strictly increasing and  $X_t \leq \tilde{X}_t$  for any  $t$ . Then, if  $(c_t)_{t=0}^{\infty}$  is a sequence of feasible consumptions,  $c_t \leq \mathcal{F}(X_t) \leq \mathcal{F}(\tilde{X}_t) = \tilde{X}_{t+1}$  for any  $t$ , and  $\sum_{t=0}^{\infty} \beta^t \ln c_t \leq \sum_{t=0}^{\infty} \beta^t \ln \mathcal{F}(\tilde{X}_t)$ .

Let us show that there exists  $T$  such that  $\tilde{X}_t > \bar{X}$  for any  $t \geq T$ .

The case  $\tilde{X}_t \leq \bar{X}$  for any  $t$  is impossible. If  $X_0 = \tilde{X}_0 \leq \bar{X}$ , there exists  $T$  such that  $\tilde{X}_T > \bar{X}$ . Indeed, if not,  $\tilde{X}_{t+1} = A\tilde{X}_t^\alpha + (1 - \delta_K)\tilde{X}_t \equiv \xi(\tilde{X}_t)$  for any  $t \geq 0$ . Since  $\xi$  is continuous and strictly concave with  $\xi'(0) = \infty$ ,  $\xi'(\infty) = 1 - \delta_K < 1$ , it crosses the line  $\tilde{X}_{t+1} = \tilde{X}_t$  at  $\tilde{X}_t = 0$  and  $\tilde{X}_t = \hat{X}$ . Thus, according to (16),  $\lim_{t \rightarrow \infty} \tilde{X}_t = \hat{X} > \bar{X}$ , a contradiction.

Let  $\tilde{X}_T > \bar{X}$  for some  $T$ . Since  $\hat{X} = A\hat{X}^\alpha + (1 - \delta_K)\hat{X}$  and, according to (16),  $\bar{X} < \hat{X}$ , we have  $\bar{X} < A\bar{X}^\alpha + (1 - \delta_K)\bar{X}$ . Moreover, by definition of  $\bar{X}$ ,  $\mathcal{F}(\bar{X}) = A\bar{X}^\alpha + (1 - \delta_K)\bar{X}$ . Since  $\mathcal{F}$  is strictly increasing, and if  $\tilde{X}_T > \bar{X}$ , then  $\tilde{X}_{T+1} = \mathcal{F}(\tilde{X}_T) > \mathcal{F}(\bar{X}) = A\bar{X}^\alpha + (1 - \delta_K)\bar{X} > \bar{X}$ . By induction, we obtain  $\tilde{X}_t > \bar{X}$  for any  $t \geq T$ . Clearly, if  $T = 0$ , we have  $\tilde{X}_t > \bar{X}$  for any  $t \geq 0$ .

In the following, without loss of generality, we focus on the case  $\tilde{X}_t > \bar{X}$  for any  $t \geq 0$ .

We consider three cases: (1)  $\gamma < 1$ , (2)  $\gamma = 1$ , (3)  $1 < \gamma < 1/\beta$ .

(1)  $\gamma < 1$ .

According to point (3) of Lemma 2,  $\mathcal{F}$  is  $C^1$ , and, since  $\gamma < 1$ , according to point (5),  $\mathcal{F}$  is strictly concave. Then,  $\mathcal{F}(X_t)/X_t$  is a continuous and strictly decreasing function. Moreover, according to point (2) of Lemma 3,

$$\lim_{X_t \rightarrow 0^+} \frac{\mathcal{F}(X_t)}{X_t} = \infty \text{ and } \lim_{X_t \rightarrow \infty} \frac{\mathcal{F}(X_t)}{X_t} \leq 1 - \delta_L < 1$$

Thus,  $\mathcal{F}(X_t)/X_t$  crosses 1 once and, therefore, there exists a unique fixed point  $Z > \bar{X}$  such that  $Z = \mathcal{F}(Z)$ .

If  $\tilde{X}_T \leq Z$  for some  $T$ , then  $\tilde{X}_{T+1} = \mathcal{F}(\tilde{X}_T) < \mathcal{F}(Z) = Z$ . By induction, we can conclude that  $\tilde{X}_t \leq Z$  for any  $t \geq T$ .

If  $\tilde{X}_T > Z$  for some  $T$ , then  $\tilde{X}_{T+1} = \mathcal{F}(\tilde{X}_T) < \tilde{X}_T$  and, again by induction, we conclude that  $\tilde{X}_t \leq \tilde{X}_T$  for any  $t \geq T$ .

Therefore,  $\tilde{X}_t \leq \max\{Z, \tilde{X}_T\}$  for any  $t$ . This implies  $\sum_{t=0}^{\infty} \beta^t \ln c_t \leq \sum_{t=0}^{\infty} \beta^t \ln \tilde{X}_{t+1} < \infty$ .

(2)  $\gamma = 1$ .

As above,  $\tilde{X}_t > \bar{X}$  for any  $t \geq 0$ . Then, according to point (5) of Lemma 2,  $\mathcal{F}''(X_t) = 0$  for any  $X_t > \bar{X}$  and, thus,  $\mathcal{F}(X_t) = aX_t + b$  for any  $X_t > \bar{X}$ . Therefore,  $\tilde{X}_{t+1} = \mathcal{F}(\tilde{X}_t) = a\tilde{X}_t + b$  for any  $t \geq 0$ .

We can distinguish two subcases depending on  $a$ : (2.1)  $0 < a < 1$ , (2.2)  $a \geq 1$ .

(2.1) Let  $0 < a < 1$  and  $Z$  satisfy  $Z = aZ + b$ .

If  $\tilde{X}_t \leq Z$  then  $\tilde{X}_{t+1} \leq Z$  and, by induction,  $\tilde{X}_t \leq Z$  for all  $t$ . The sequence  $(\tilde{X}_t)_{t=0}^{\infty}$  is bounded and we are fine:

$$\begin{aligned} \sum_{t=T}^{\infty} \beta^t \ln c_t &= \sum_{t=T}^{\infty} \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}] \leq \sum_{t=T}^{\infty} \beta^t \ln \mathcal{F}(\tilde{X}_t) \\ &\leq \sum_{t=T}^{\infty} \beta^t \ln \mathcal{F}(Z) = \frac{\beta^T}{1-\beta} \ln \mathcal{F}(Z) < \infty \end{aligned}$$

If  $\tilde{X}_T > Z$  for some  $T$ , then  $\tilde{X}_{T+1} = a\tilde{X}_T + b < \tilde{X}_T$  and, by induction,  $\tilde{X}_t \leq \tilde{X}_T$  for any  $t \geq T$ . Therefore,  $\tilde{X}_t \leq \max\{Z, \tilde{X}_T\}$  for any  $t$ . This implies  $\sum_{t=0}^{\infty} \beta^t \ln c_t \leq \sum_{t=0}^{\infty} \beta^t \ln \tilde{X}_{t+1} < \infty$ .

(2.2) Let  $a \geq 1$ . Then,  $\tilde{X}_{t+1} = a\tilde{X}_t + b > \tilde{X}_t$  and the sequence  $(\tilde{X}_t)_{t=0}^{\infty}$  converges to infinity when  $t$  goes to infinity.

Let  $\varepsilon > 0$ . There exists  $T$  such that

$$\tilde{X}_t > \frac{1}{\varepsilon} \frac{b}{a}$$

for any  $t \geq T$ . Clearly,

$$\tilde{X}_{t+1} = \left(1 + \frac{1}{\tilde{X}_t} \frac{b}{a}\right) a\tilde{X}_t < (1 + \varepsilon) a\tilde{X}_t$$

for any  $t \geq T$ .

Since

$$\tilde{X}_{T+1} > \tilde{X}_T > \frac{1}{\varepsilon} \frac{b}{a}$$

we get also  $\tilde{X}_{T+2} < (1 + \varepsilon) a \tilde{X}_{T+1}$ . By induction,  $\tilde{X}_{T+t} < [(1 + \varepsilon) a]^t \tilde{X}_T$  and, thus,

$$\begin{aligned} \sum_{t=T+1}^{\infty} \beta^t \ln c_t &\leq \sum_{t=T+1}^{\infty} \beta^t \ln X_{t+1} = \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \ln X_{T+t+2} \\ &\leq \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \ln \tilde{X}_{T+t+2} < \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \ln \left( [(1 + \varepsilon) a]^{t+2} \tilde{X}_T \right) \\ &= \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \left( (t+2) \ln [(1 + \varepsilon) a] + \ln \tilde{X}_T \right) < \infty \end{aligned}$$

because  $\sum_{t=0}^{\infty} t \beta^t = \beta / (1 - \beta)^2 < \infty$ . Then,  $\sum_{t=0}^{\infty} \beta^t \ln c_t = \sum_{t=0}^T \beta^t \ln c_t + \sum_{t=T+1}^{\infty} \beta^t \ln c_t < \infty$ .

(3)  $1 < \gamma < 1/\beta$ .

We know that  $\tilde{X}_t > \bar{X}$  for any  $t \geq 0$  and that

$$\begin{aligned} \tilde{X}_{t+1} &\equiv \mathcal{F}(\tilde{X}_t) \\ &= A \left[ 1 + \lambda L^*(\tilde{X}_t) \right]^{\theta} \left[ \tilde{X}_t - qL^*(\tilde{X}_t) \right]^{\alpha} \\ &\quad + (1 - \delta_K) \left[ \tilde{X}_t - qL^*(\tilde{X}_t) \right] + (1 - \delta_L) qL^*(\tilde{X}_t) \end{aligned}$$

We observe that  $\tilde{X}_t \equiv \tilde{K}^*(\tilde{X}_t) + qL^*(\tilde{X}_t)$  and

$$\begin{aligned} \tilde{X}_{t+1} &= \mathcal{F}(\tilde{X}_t) \\ &= A \left[ 1 + \frac{\lambda}{q} qL^*(\tilde{X}_t) \right]^{\theta} \left[ \tilde{X}_t - qL^*(\tilde{X}_t) \right]^{\alpha} \\ &\quad + (1 - \delta_K) \left[ \tilde{X}_t - qL^*(\tilde{X}_t) \right] + (1 - \delta_L) qL^*(\tilde{X}_t) \\ &\leq A \left( 1 + \frac{\lambda}{q} \tilde{X}_t \right)^{\theta} \tilde{X}_t^{\alpha} + (1 - \delta_K) \tilde{X}_t + (1 - \delta_L) \tilde{X}_t \\ &= A \left( 1 + \frac{\lambda}{q} \tilde{X}_t \right)^{\theta} \tilde{X}_t^{\alpha} + (2 - \delta_K - \delta_L) \tilde{X}_t \\ &= A \left( \frac{\lambda}{q} \right)^{\theta} \left( \tilde{X}_t + \frac{q}{\lambda} \right)^{\theta} \tilde{X}_t^{\alpha} + (2 - \delta_K - \delta_L) \tilde{X}_t \\ &= a \left( \tilde{X}_t + b \right)^{\theta} \tilde{X}_t^{\alpha} + c \tilde{X}_t \end{aligned}$$

where  $a \equiv A (\lambda/q)^{\theta} > 0$ ,  $b \equiv q/\lambda > 0$  and  $c \equiv 2 - \delta_K - \delta_L$ .

Define

$$\tilde{Y}_{t+1} \equiv a \left( \tilde{Y}_t + b \right)^{\gamma} + c \tilde{Y}_t$$

with  $\tilde{Y}_0 = \tilde{X}_0$ . Clearly,

$$\tilde{Y}_{t+1} = a \left( \tilde{Y}_t + b \right)^\alpha \left( \tilde{Y}_t + b \right)^\theta + c\tilde{Y}_t \geq a \left( \tilde{Y}_t + b \right)^\theta \tilde{Y}_t^\alpha + c\tilde{Y}_t$$

and

$$\tilde{Y}_1 \geq a \left( \tilde{Y}_0 + b \right)^\theta \tilde{Y}_0^\alpha + c\tilde{Y}_0 = a \left( \tilde{X}_0 + b \right)^\theta \tilde{X}_0^\alpha + c\tilde{X}_0 \geq \mathcal{F} \left( \tilde{X}_0 \right) = \tilde{X}_1$$

$$\tilde{Y}_2 \geq a \left( \tilde{Y}_1 + b \right)^\theta \tilde{Y}_1^\alpha + c\tilde{Y}_1 \geq a \left( \tilde{X}_1 + b \right)^\theta \tilde{X}_1^\alpha + c\tilde{X}_1 \geq \mathcal{F} \left( \tilde{X}_1 \right) = \tilde{X}_2$$

Thus, by induction,  $\tilde{Y}_t \geq \tilde{X}_t$  for any  $t \geq 0$ .

Since  $\gamma > 1$  we can find  $\varepsilon > 0$  such that  $a(b/\varepsilon)^\gamma > b/\varepsilon$ . Clearly,  $\tilde{Y}_t > b/\varepsilon$  implies

$$\tilde{Y}_{t+1} \equiv a \left( \tilde{Y}_t + b \right)^\gamma + c\tilde{Y}_t > a\tilde{Y}_t^\gamma > a \left( \frac{b}{\varepsilon} \right)^\gamma > \frac{b}{\varepsilon}$$

and

$$\begin{aligned} \tilde{Y}_{t+1} &\equiv a \left( \tilde{Y}_t + b \right)^\gamma + c\tilde{Y}_t \\ &= \left[ a \left( 1 + \frac{b}{\tilde{Y}_t} \right)^\gamma + \frac{c}{\tilde{Y}_t^{\gamma-1}} \right] \tilde{Y}_t^\gamma < a(1+\varepsilon)^\gamma \tilde{Y}_t^\gamma \end{aligned}$$

If  $\tilde{Y}_t \leq b/\varepsilon$  for any  $t$ , since  $c_t \leq F(\tilde{X}_t) \leq F(\tilde{Y}_t) \leq F(b/\varepsilon)$ , the sum  $\sum_{t=0}^{\infty} \beta^t \ln c_t$  is finite-valued.

Suppose instead that  $\tilde{Y}_T > b/\varepsilon$  for some  $T$ . Then,  $\tilde{Y}_t > b/\varepsilon$  for any  $t \geq T$  and  $\tilde{Y}_{t+1} < a(1+\varepsilon)^\gamma \tilde{Y}_t^\gamma$  for any  $t \geq T$ . Hence,  $\tilde{X}_{T+1} \leq \tilde{Y}_{T+1} < a(1+\varepsilon)^\gamma \tilde{Y}_T^\gamma$  and

$$\tilde{X}_{T+t} \leq \tilde{Y}_{T+t} < [a(1+\varepsilon)^\gamma]^{\frac{t-1}{\gamma-1}} \tilde{Y}_T^{\gamma^t} \quad (40)$$

for any  $t \geq 1$ . In particular, we get

$$\tilde{X}_{T+t+2} \leq \tilde{Y}_{T+t+2} < [a(1+\varepsilon)^\gamma]^{\frac{t+2-1}{\gamma-1}} \tilde{Y}_T^{\gamma^{t+2}}$$



and, thus,

$$\begin{aligned}
\sum_{t=T+1}^{\infty} \beta^t \ln c_t &\leq \sum_{t=T+1}^{\infty} \beta^t \ln X_{t+1} = \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \ln X_{T+t+2} \\
&\leq \beta^{T+1} \sum_{t=0}^{\infty} \beta^s \ln \tilde{X}_{T+t+2} \leq \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \ln \tilde{Y}_{T+t+2} \\
&< \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \ln \left( [a(1+\varepsilon)\gamma]^{\frac{\gamma^{t+2}-1}{\gamma-1}} \tilde{Y}_T^{\gamma^{t+2}} \right) \\
&\leq \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \left( \gamma^{t+2} \ln \tilde{Y}_T + \frac{\gamma^{t+2}-1}{\gamma-1} \ln [a(1+\varepsilon)\gamma] \right) \\
&\leq \beta^{T+1} \sum_{t=0}^{\infty} \beta^t \left[ \gamma^{t+2} (\ln \tilde{Y}_T)^+ + \frac{\gamma^{t+2}-1}{\gamma-1} (\ln [a(1+\varepsilon)\gamma])^+ \right] \\
&< \infty
\end{aligned}$$

because  $\beta\gamma < 1$ . Then,  $\sum_{t=0}^{\infty} \beta^t \ln c_t = \sum_{t=0}^T \beta^t \ln c_t + \sum_{t=T+1}^{\infty} \beta^t \ln c_t < \infty$ .

Finally, we want to prove that the limit of the utility function along any feasible path converges to a finite real number or diverges to  $-\infty$ .

From the initial part of this lemma, we know that, under Assumption 2,  $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$  for any  $X_0 \geq 0$ .

Let  $\ln^- \equiv \min\{0, \ln\}$  and  $\ln^+ \equiv \max\{0, \ln\}$ .

The sequence  $\sum_{t=0}^T \beta^t (\ln [\mathcal{F}(X_t) - X_{t+1}])^-$  is decreasing in  $T$ , so that it converges to a finite non-positive number or diverges to  $-\infty$ .

We know that, for any  $\mu > 0$ , there is  $T$  such that  $\sum_{t=T}^{\infty} \beta^t (\ln [\mathcal{F}(X_t) - X_{t+1}])^+ < \mu$ . This implies  $\sum_{t=0}^{\infty} \beta^t (\ln [\mathcal{F}(X_t) - X_{t+1}])^+ < \infty$ . Indeed, otherwise,  $\sum_{t=T}^{\infty} \beta^t (\ln [\mathcal{F}(X_t) - X_{t+1}])^+ = \infty$  for any  $T$ . Since  $\sum_{t=0}^{\infty} \beta^t (\ln [\mathcal{F}(X_t) - X_{t+1}])^+$  is increasing, it converges to a finite non-negative number.

Notice that

$$\begin{aligned}
&\sum_{t=0}^T \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}] \\
&= \sum_{t=0}^T \beta^t (\ln [\mathcal{F}(X_t) - X_{t+1}])^- + \sum_{t=0}^T \beta^t (\ln [\mathcal{F}(X_t) - X_{t+1}])^+
\end{aligned}$$

Then,  $\sum_{t=0}^T \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}]$  converges to a finite number or diverges to  $-\infty$ . ■

### Proof of Proposition 5

The set  $\Pi(X_0)$  is compact in the product topology.

We want to show that  $U(\mathbf{X}) \equiv \sum_{t=0}^{\infty} \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}]$  is upper semi-continuous in  $\Pi(X_0)$ .

Let  $X_0$  be given. Consider the maximal sequence  $(\tilde{X}_t)_{t=0}^{\infty}$  with  $\tilde{X}_{t+1} = \mathcal{F}(\tilde{X}_t)$  for any  $t$ , and  $\tilde{X}_0 = X_0$ .

Consider a sufficiently small  $\varepsilon \in (0, b/\bar{X})$ .

There are two cases: either (1)  $\tilde{X}_t \leq b/\varepsilon$  for any  $t \geq 0$ , or (2) there exists  $T$  such that  $\tilde{X}_T > b/\varepsilon$ .

(1) If  $\tilde{X}_t \leq b/\varepsilon$  for any  $t \geq 0$ , then,

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}] &\leq \sum_{t=0}^{\infty} \beta^t \ln \mathcal{F}(\tilde{X}_t) \leq \sum_{t=0}^{\infty} \beta^t \ln \mathcal{F}\left(\frac{b}{\varepsilon}\right) \\ &= \frac{1}{1-\beta} \ln \mathcal{F}\left(\frac{b}{\varepsilon}\right) < \infty \end{aligned}$$

Given  $\mu > 0$ , there exists  $T_0$  such that, for any  $T \geq T_0$  and any  $\mathbf{X} \in \Pi(X_0)$ , we have

$$\sum_{t=T}^{\infty} \beta^t (\ln [\mathcal{F}(X_t) - X_{t+1}])^+ \leq \mu$$

where  $\ln^+ \equiv \max\{0, \ln\}$ .

Let  $\mathbf{X}^n \in \Pi(X_0)$  such that  $\mathbf{X}^n \rightarrow \mathbf{X} \in \Pi(X_0)$  in the product topology.

Let  $\mu$  be given. Then, for any  $T \geq T_0$  and any  $n$ ,

$$\begin{aligned} U(\mathbf{X}^n) &\leq \sum_{t=0}^T \beta^t \ln [\mathcal{F}(X_t^n) - X_{t+1}^n] + \sum_{t=T+1}^{\infty} \beta^t (\ln [\mathcal{F}(X_t^n) - X_{t+1}^n])^+ \\ &\leq \sum_{t=0}^T \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}] + \mu \end{aligned}$$

This implies that

$$\begin{aligned} \limsup_n U(\mathbf{X}^n) &\leq \lim_{n \rightarrow \infty} \sum_{t=0}^T \beta^t \ln [\mathcal{F}(X_t^n) - X_{t+1}^n] + \mu \\ &= \sum_{t=0}^T \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}] + \mu \end{aligned}$$

Let  $T \rightarrow \infty$ . We have

$$\limsup_n U(\mathbf{X}^n) \leq \sum_{t=0}^{\infty} \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}] + \mu = U(\mathbf{X}) + \mu$$

Since  $\mu$  is an arbitrary positive number, we get  $\limsup_n U(\mathbf{X}^n) \leq U(\mathbf{X})$ , that is  $U$  is upper semi-continuous.

(2) Now, assume that there exists  $T$  such that  $\tilde{X}_T > b/\varepsilon$ .

As in Lemma 4, we consider the following subcases: (2.1)  $\gamma < 1$ , (2.2)  $\gamma = 1$ ,

(2.3)  $1 < \gamma < 1/\beta$ .

(2.1) If  $\gamma < 1$ , according to point (1) in the proof of Lemma 4, we have  $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$ .

(2.2) Let  $\gamma = 1$ . As above,  $\tilde{X}_t > \bar{X}$  for any  $t \geq 0$ . Then,  $\tilde{X}_{t+1} = \mathcal{F}(\tilde{X}_t) = a\tilde{X}_t + b$  for any  $t \geq 0$ . If  $a < 1$ , we follow the point (2.1) in the proof of Lemma 4 to get  $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$ . Similarly, if  $a \geq 1$ , we follow the point (2.2) in the proof of Lemma 4 to obtain  $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$ .

(2.3) Let  $1 < \gamma < 1/\beta$ . Since  $\varepsilon \in (0, b/\bar{X})$ , we have  $b/\varepsilon > \bar{X}$ . Thus,  $\tilde{X}_T > b/\varepsilon$  implies  $\tilde{X}_T > \bar{X}$ . We know that, if  $\tilde{X}_T > \bar{X}$  for some  $T$ , then, under Assumption 2,  $\tilde{X}_t > \bar{X}$  and

$$\begin{aligned} \tilde{X}_{t+1} &= A \left[ 1 + \frac{\lambda}{q} qL^*(\tilde{X}_t) \right]^\theta [\tilde{X}_t - qL^*(\tilde{X}_t)]^\alpha \\ &\quad + (1 - \delta_K) [\tilde{X}_t - qL^*(\tilde{X}_t)] + (1 - \delta_L) qL^*(\tilde{X}_t) \end{aligned}$$

for any  $t \geq T$ . Since  $\gamma > 1$ , we can find  $\varepsilon > 0$  such that  $a(b/\varepsilon)^\gamma > b/\varepsilon$ .

Let  $\tilde{Y}_{t+1} \equiv a(\tilde{Y}_t + b)^\gamma + c\tilde{Y}_t$  for any  $t \geq T$  with  $\tilde{Y}_0 = \tilde{X}_0$ . It is easy to see that  $\tilde{Y}_t > b/\varepsilon$  implies  $\tilde{Y}_{t+1} > b/\varepsilon$  and  $\tilde{Y}_{t+1} < a(1+\varepsilon)^\gamma \tilde{Y}_t^\gamma$ .

If  $\tilde{Y}_t \leq b/\varepsilon$  for any  $t$ , since  $c_t \leq \mathcal{F}(\tilde{X}_t) \leq \mathcal{F}(\tilde{Y}_t) \leq \mathcal{F}(b/\varepsilon)$ , we have that the sum  $\sum_{t=0}^{\infty} \beta^t \ln c_t$  is finite-valued.

If  $\tilde{Y}_T > b/\varepsilon$  for some  $T$ , then,  $\tilde{Y}_t > b/\varepsilon$  for any  $t \geq T$ . Hence,  $\tilde{X}_{T+1} \leq \tilde{Y}_{T+1} < a(1+\varepsilon)^\gamma \tilde{Y}_T^\gamma$  and (40) holds. Applying the same argument of the point (3) in the proof of Lemma 4, we get  $\sum_{t=0}^{\infty} \beta^t \ln c_t < \infty$ .

Therefore, in all the three subcases: (2.1), (2.2), (2.3), given  $\mu > 0$ , there exists  $T_0$  such that, for any  $T \geq T_0$  and any  $\mathbf{X} \in \Pi(X_0)$ , we have  $\sum_{t=T}^{\infty} \beta^t (\ln [\mathcal{F}(X_t) - X_{t+1}])^+ \leq \mu$ .

Let  $\mathbf{X}^n \in \Pi(X_0) \rightarrow \mathbf{X} \in \Pi(X_0)$ . Then, for any  $T \geq T_0$  and any  $n$ ,

$$\begin{aligned} U(\mathbf{X}^n) &\leq \sum_{t=0}^T \beta^t \ln [\mathcal{F}(X_t^n) - X_{t+1}^n] + \sum_{t=T+1}^{\infty} \beta^t (\ln [\mathcal{F}(X_t^n) - X_{t+1}^n])^+ \\ &\leq \sum_{t=0}^T \beta^t \ln [\mathcal{F}(X_t^n) - X_{t+1}^n] + \mu \end{aligned}$$

This implies that

$$\begin{aligned} \limsup_n U(\mathbf{X}^n) &\leq \lim_{n \rightarrow \infty} \sum_{t=0}^T \beta^t \ln [\mathcal{F}(X_t^n) - X_{t+1}^n] + \mu \\ &= \sum_{t=0}^T \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}] + \mu \end{aligned}$$

Let  $T \rightarrow \infty$ . We have

$$\limsup_n U(\mathbf{X}^n) \leq \sum_{t=0}^{\infty} \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}] + \mu = U(\mathbf{X}) + \mu$$

Since  $\mu$  is arbitrary, we get  $\limsup_n U(\mathbf{X}^n) \leq U(\mathbf{X})$ , that is  $U$  is upper semi-continuous in  $\Pi(X_0)$ .

Therefore, there exists  $\mathbf{X}$  that maximizes  $U(\mathbf{X})$  in  $\Pi(X_0)$ .

In order to have  $U(\mathbf{X}) > -\infty$ , we observe the following.

Consider  $X_0 \in (0, \bar{X}]$ . According to Assumption 2,

$$X_0 < \hat{X} = \left( \frac{A}{\delta_K} \right)^{\frac{1}{1-\alpha}}$$

In this case, since  $X_0 < AX_0^\alpha + (1 - \delta_K)X_0$  and  $\mathcal{F}(X_0) - X_0 = AX_0^\alpha + (1 - \delta_K)X_0 - X_0 > 0$ ,  $\mathbf{X} = (X_0, X_0, \dots)$  is feasible from  $X_0$ . Then,

$$\sum_{t=0}^{\infty} \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}] = \sum_{t=0}^{\infty} \beta^t \ln [\mathcal{F}(X_0) - X_0] = \frac{\ln [\mathcal{F}(X_0) - X_0]}{1 - \beta} > -\infty$$

Clearly,  $\max U(\mathbf{X}) \geq \sum_{t=0}^{\infty} \beta^t \ln [AX_0^\alpha + (1 - \delta_K)X_0 - X_0] > -\infty$ .

If  $X_0 > \bar{X}$ , consider the sequence  $(X_0, \bar{X}, \bar{X}, \dots)$ . We have  $\mathcal{F}(X_0) > \mathcal{F}(\bar{X}) > \bar{X}$  because, if  $\tilde{X}_T > \bar{X}$  for some  $T$ , then, under Assumption 2,  $\tilde{X}_t > \bar{X}$  for any  $t \geq T$ . Thus, the sequence  $(X_0, \bar{X}, \bar{X}, \dots)$  is feasible and

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \ln [\mathcal{F}(X_t) - X_{t+1}] &= \ln [\mathcal{F}(X_0) - \bar{X}] + \sum_{t=1}^{\infty} \beta^t \ln [\mathcal{F}(\bar{X}) - \bar{X}] \\ &= \ln [\mathcal{F}(X_0) - \bar{X}] + \frac{\beta}{1 - \beta} \ln [\mathcal{F}(\bar{X}) - \bar{X}] > -\infty \end{aligned}$$

Clearly,  $\max U(\mathbf{X}) \geq \ln [\mathcal{F}(X_0) - \bar{X}] + \sum_{t=1}^{\infty} \beta^t \ln [A\bar{X}^\alpha + (1 - \delta_K)\bar{X} - \bar{X}] > -\infty$ . ■

### Proof of Lemma 6

Since  $\mathcal{F}$  is increasing, the optimal sequence  $(X_t^*)_{t=0}^{\infty}$  is monotonic (see Amir (1996) and Le Van and Dana (2002)).

It cannot converge to zero, otherwise, for any  $t$  large enough,  $\mathcal{F}(X_t^*) = AX_t^{*\alpha} + (1 - \delta_K)X_t^*$ . Since  $\mathcal{F}'(0) = \infty$ , the Euler equation would lead to a contradiction.

There is a period  $T$  such that  $X_T^* > \bar{X}$ . Otherwise, the optimal path  $(X_t^*)$  would satisfy  $\mathcal{F}(X_t^*) = AX_t^{*\alpha} + (1 - \delta_K)X_t^*$  for any  $t$ . Besides, it must converge to the steady state  $\tilde{X}$  which is larger than  $\bar{X}$ . By Assumption 2, we get a contradiction.

The optimal path either converges to a steady state larger than the threshold  $\bar{X}$  or diverges to  $+\infty$ . ■

### Proof of Proposition 8

Let  $\gamma < 1$ .

From the Euler equation,  $X^*$  is a steady state if and only if  $\mathcal{F}'(X^*) = 1/\beta$ . From points (3) and (5) of Lemma 2,  $\mathcal{F}$  is strictly concave and differentiable on  $(0, +\infty)$ .

We recall that  $1 - \delta_K + \alpha A \tilde{X}^{\alpha-1} = 1/\beta$  and that  $\bar{X} < \tilde{X}$  implies  $\mathcal{F}'(\bar{X}) = 1 - \delta_K + \alpha A \bar{X}^{\alpha-1} > 1/\beta$ . We know that the stationary state  $X^*$  satisfies  $1/\beta = \mathcal{F}'(X^*)$ .

$\mathcal{F}'(X^*) = 1/\beta < \mathcal{F}'(\bar{X})$  implies that  $X^* > \bar{X}$ . Indeed,  $\mathcal{F}$  is strictly concave. We want to show that  $X^* > \tilde{X}$ .

We have

$$\mathcal{F}'(X_t) = \alpha A [1 + \lambda L^*(X_t)]^\theta [X_t - qL^*(X_t)]^{\alpha-1} + 1 - \delta_K$$

We know that  $X_t > \bar{X}$  implies  $L^*(X_t) > 0$ , that is  $X_t - qL^*(X_t) < X_t$  and  $[X_t - qL^*(X_t)]^{\alpha-1} > X_t^{\alpha-1}$ . In addition,  $[1 + \lambda L^*(X_t)]^\theta > 1$ .

Then,  $\mathcal{F}'(X_t) > \alpha A X_t^{\alpha-1} + 1 - \delta_K$  for any  $X_t > \bar{X}$ . Since  $\tilde{X} > \bar{X}$ , we get  $\mathcal{F}'(\tilde{X}) > \alpha A \tilde{X}^{\alpha-1} + 1 - \delta_K = 1/\beta = \mathcal{F}'(X^*)$ .

We know also from Lemma 2 that  $\mathcal{F}$  is strictly concave. We obtain  $\tilde{X} < X^*$ . If  $\delta_K = \delta_L$ , (26) becomes

$$\lambda \theta (X_t - qL_t) = \alpha q (1 + \lambda L_t) \quad (41)$$

Since

$$X_t = (1 + \zeta_t) qL_t \quad (42)$$

we obtain

$$L_t = \frac{\alpha}{\lambda(\theta\zeta_t - \alpha)}$$

(41) entails  $\lambda\theta K_t = \alpha q + \alpha q \lambda L_t$  and we find

$$\zeta_t \equiv \frac{K_t}{qL_t} = \frac{\alpha}{\lambda\theta} \frac{1}{L_t} + \frac{\alpha}{\theta} \quad (43)$$

Thus,

$$\lim_{t \rightarrow \infty} \zeta_t = \frac{\alpha}{\theta} \left( 1 + \frac{1}{\lambda \lim_{t \rightarrow \infty} L_t} \right) = \frac{\alpha}{\theta} \left( 1 + \frac{1}{\lambda L^*} \right) > \frac{\alpha}{\theta}$$

■

#### Proof of Proposition 10

(2) Let  $\gamma = 1$  and  $\delta_K \geq \delta_L$ . Consider only  $X_0 \geq \bar{X}$  and  $X \geq X_0$ . In this case,

$$\mathcal{F}(\bar{X}) = A\bar{X}^\alpha + (1 - \delta_K)\bar{X} = \mathcal{F}'(\bar{X})\bar{X} + Z$$

where

$$\mathcal{F}'(\bar{X}) = \alpha A \bar{X}^{\alpha-1} + 1 - \delta_K > \frac{1}{\beta} \quad (44)$$

and  $Z \equiv (1 - \alpha) A \bar{X}^\alpha$ . From the first-order condition (30), we know that, for any  $X > \bar{X}$ ,

$$\lambda \theta A \left( \frac{1 + \lambda L}{X - qL} \right)^{\theta-1} (X - qL)^{\gamma-1} - \alpha q A \left( \frac{1 + \lambda L}{X - qL} \right)^\theta (X - qL)^{\gamma-1+q} (\delta_K - \delta_L) = 0$$

and, since  $\gamma = 1$ ,

$$\lambda\theta A \left( \frac{1 + \lambda L}{X - qL} \right)^{\theta-1} - \alpha q A \left( \frac{1 + \lambda L}{X - qL} \right)^{\theta} + q(\delta_K - \delta_L) = 0 \quad (45)$$

Then,

$$\frac{1 + \lambda L^*(X)}{X - qL^*(X)}$$

is constant for any  $X \geq \bar{X}$  and equal to  $1/\bar{X}$  (since  $L^*(\bar{X}) = 0$ ). Hence,

$$L_t = L^*(X_t) = \frac{X_t - \bar{X}}{q + \lambda \bar{X}} \quad (46)$$

Moreover,

$$K_t = X_t - qL_t = \bar{X} \frac{q + \lambda X_t}{q + \lambda \bar{X}} \quad (47)$$

and

$$\frac{K_t}{qL_t} = \bar{X} \frac{\frac{\lambda}{q} X_t + 1}{X_t - \bar{X}} \quad (48)$$

For any  $X \geq \bar{X}$ , we obtain

$$\begin{aligned} \mathcal{F}'(X) &= \alpha A [1 + \lambda L^*(X)]^{\theta} [X - qL^*(X)]^{\alpha-1} + 1 - \delta_K \\ &= \alpha A \left[ \frac{1 + \lambda L^*(X)}{X - qL^*(X)} \right]^{\theta} [X - qL^*(X)]^{\alpha+\theta-1} + 1 - \delta_K \\ &= \alpha A \left( \frac{1}{\bar{X}} \right)^{1-\alpha} + 1 - \delta_K = \mathcal{F}'(\bar{X}) \end{aligned}$$

For any  $X_{t+1} \geq \bar{X}$ , the Euler equation implies

$$\frac{c_{t+1}}{c_t} = \beta \mathcal{F}'(X_{t+1}) = \beta \mathcal{F}'(\bar{X}) > 1$$

Optimal consumption grows at a constant rate  $> 1$ .

$$c_t = c_0 [\beta \mathcal{F}'(\bar{X})]^t \quad (49)$$

Let us compute  $\mathcal{F}(X_t)$  for  $X_t > \bar{X}$ :

$$\begin{aligned} \mathcal{F}(X_t) &= A(1 + \lambda L_t)^{\theta} (X_t - qL_t)^{\alpha} + (1 - \delta_K)(X_t - qL_t) + (1 - \delta_L)qL_t \\ &= A(1 + \lambda L_t)^{1-\alpha} (X_t - qL_t)^{\alpha} + (1 - \delta_K)(X_t - qL_t) + (1 - \delta_L)qL_t \\ &= A \left( \frac{1 + \lambda L_t}{X_t - qL_t} \right)^{1-\alpha} (X_t - qL_t) + (1 - \delta_K)(X_t - qL_t) + (1 - \delta_L)qL_t \\ &= (A\bar{X}^{\alpha-1} + \delta_L - \delta_K)(X_t - qL_t) + (1 - \delta_L)X_t \end{aligned}$$

Using (46) to replace  $L_t$ , we get

$$\begin{aligned}\mathcal{F}(X_t) &= (A\bar{X}^{\alpha-1} + \delta_L - \delta_K) \left( X_t - q \frac{X_t - \bar{X}}{q + \lambda\bar{X}} \right) + (1 - \delta_L) X_t \\ &= \left[ 1 - \delta_L + (A\bar{X}^{\alpha-1} + \delta_L - \delta_K) \frac{\lambda\bar{X}}{q + \lambda\bar{X}} \right] X_t + (A\bar{X}^{\alpha-1} + \delta_L - \delta_K) \frac{q\bar{X}}{q + \lambda\bar{X}} \\ &= aX_t + b\end{aligned}$$

which is linear in  $X_t$ , where

$$a \equiv 1 - \delta_L + (A\bar{X}^{\alpha-1} + \delta_L - \delta_K) \frac{\lambda\bar{X}}{q + \lambda\bar{X}} \quad (50)$$

$$b \equiv (A\bar{X}^{\alpha-1} + \delta_L - \delta_K) \frac{q\bar{X}}{q + \lambda\bar{X}} \quad (51)$$

Notice that  $\bar{X}$  is solution to (45)

$$A\bar{X}^{\alpha-1} = \frac{q(\delta_K - \delta_L)}{\alpha q - \lambda\theta\bar{X}}$$

Then, the optimal path  $(X_t)$  satisfies for any  $t$ :

$$\begin{aligned}X_{t+1} &= \mathcal{F}(X_t) - c_t = \mathcal{F}(X_t) - c_0 [\beta\mathcal{F}'(\bar{X})]^t \\ &= aX_t + b - c_0 [\beta\mathcal{F}'(\bar{X})]^t \\ &= aX_t + b - c_0 (a\beta)^t\end{aligned} \quad (52)$$

since  $\mathcal{F}'(\bar{X}) = \mathcal{F}'(X_t) = a$ .

Now, we want to prove that the solution to the difference equation:

$$x_{t+1} = ax_t + b - c(a\beta)^t \quad (53)$$

with  $x_0 \neq 0$ ,  $0 < \beta < 1$  and

$$a > 1/\beta > 1, b > 0, c > 0 \quad (54)$$

is given by

$$x_{t+1} = a^{t+1} \left( x_0 - \frac{b}{1-a} - \frac{c}{a} \frac{1 - \beta^{t+1}}{1 - \beta} \right) + \frac{b}{1-a} \quad (55)$$

that is by

$$x_{t+1} = a^{t+1} \left( x_0 - \frac{b}{1-a} - \frac{c}{a} \frac{1}{1-\beta} \right) + \frac{c}{a} \frac{(a\beta)^{t+1}}{1-\beta} + \frac{b}{1-a} \quad (56)$$

Let us prove (56) by induction.

For  $t = 0$ , (56) gives

$$x_1 = a \left( x_0 - \frac{b}{1-a} - \frac{c}{a} \right) + \frac{b}{1-a} = ax_0 + b - c$$

that is (53).

Now, let (56) be true at period  $t$ . Check it for period  $t + 1$ .

(53) implies

$$x_{t+2} = ax_{t+1} + b - c(\beta a)^{t+1}$$

Replacing (56), we have

$$x_{t+2} = a^{t+2} \left( x_0 - \frac{b}{1-a} - \frac{c}{a} \frac{1 - \beta^{t+1}}{1 - \beta} \right) + \frac{ab}{1-a} + b - c(\beta a)^{t+1} \quad (57)$$

Since

$$\frac{ab}{1-a} + b = \frac{b}{1-a}$$

and

$$\begin{aligned} & -a^{t+2} \frac{c}{a} \frac{1 - \beta^{t+1}}{1 - \beta} - c(\beta a)^{t+1} \\ &= -a^{t+2} \frac{c}{a} \frac{1}{1 - \beta} [1 - \beta^{t+1} + \beta^{t+1} (1 - \beta)] \\ &= -a^{t+2} \frac{c}{a} \frac{1}{1 - \beta} (1 - \beta^{t+2}) \end{aligned}$$

(57) becomes

$$x_{t+2} = a^{t+2} \left( x_0 - \frac{b}{1-a} - \frac{c}{a} \frac{1 - \beta^{t+2}}{1 - \beta} \right) + \frac{b}{1-a}$$

Thus, (56) is verified also for  $t + 1$ . Then, it holds for any  $t \geq 0$  proving that (55) is the solution to (53).

Equation (52) is exactly equation (55) with  $c = c_0$ . Notice that all restrictions in (54) are satisfied. Indeed, according to (44),

$$a = \mathcal{F}'(\bar{X}) > \frac{1}{\beta}$$

and, according to (50) and (51),

$$b = [a - (1 - \delta_L)] \frac{q}{\lambda} > \left[ \frac{1}{\beta} - (1 - \delta_L) \right] \frac{q}{\lambda} > 0 \quad (58)$$

Therefore, using to (56), the solution becomes

$$\begin{aligned} X_{t+1} &= [\mathcal{F}'(\bar{X})]^{t+1} \left( X_0 + \frac{q}{\lambda} \left[ 1 + \frac{\delta_L}{\mathcal{F}'(\bar{X}) - 1} \right] - \frac{1}{1 - \beta} \frac{c_0}{\mathcal{F}'(\bar{X})} \right) \\ &+ \frac{\beta}{1 - \beta} c_0 [\beta \mathcal{F}'(\bar{X})]^t - \frac{q}{\lambda} \left[ 1 + \frac{\delta_L}{\mathcal{F}'(\bar{X}) - 1} \right] \end{aligned} \quad (59)$$



According to (58) and using that by (54)  $a = \mathcal{F}'(\bar{X}) > 1$ ,

$$\frac{b}{1-a} = \frac{q}{\lambda} \left( \frac{\delta_L}{1-a} - 1 \right) = -\frac{q}{\lambda} \left[ 1 + \frac{\delta_L}{\mathcal{F}'(\bar{X}) - 1} \right] < 0$$

We need  $X_t \geq 0$  for any  $t$ . If the term multiplying  $\mathcal{F}'(\bar{X})$  in (59) was negative, i.e. if

$$X_0 + \frac{q}{\lambda} \left[ 1 + \frac{\delta_L}{\mathcal{F}'(\bar{X}) - 1} \right] - \frac{1}{1-\beta} \frac{c_0}{\mathcal{F}'(\bar{X})} < 0$$

and since by (54)  $\mathcal{F}'(\bar{X}) > 1/\beta > 1$ , then  $X_t$  would become negative sooner or later.

Hence, this term must be positive, which implies that

$$\frac{1}{1-\beta} \frac{c_0}{\mathcal{F}'(\bar{X})} \leq X_0 + \frac{q}{\lambda} \left[ 1 + \frac{\delta_L}{\mathcal{F}'(\bar{X}) - 1} \right] \quad (60)$$

establishing an upper bound on  $c_0$ .

The value of the program is  $V(c_0) = \sum_{t=0}^{\infty} \beta^t \ln(c_0 [\beta \mathcal{F}'(\bar{X})]^t)$  and  $V$  is increasing in  $c_0$ . From (60), the maximum is attained at

$$c_0 = \bar{c}_0 \equiv (1-\beta) \mathcal{F}'(\bar{X}) \left( X_0 + \frac{q}{\lambda} \left[ 1 + \frac{\delta_L}{\mathcal{F}'(\bar{X}) - 1} \right] \right)$$

Note that this implies that the first term in (59) becomes zero.

Therefore, by (49) the optimal paths are given by

$$\begin{aligned} c_t &= \bar{c}_0 [\beta \mathcal{F}'(\bar{X})]^t \\ X_{t+1} &= \bar{c}_0 [\beta \mathcal{F}'(\bar{X})]^t \frac{\beta}{1-\beta} - z \end{aligned}$$

for any  $t$ , where  $z > 0$  is defined by (17). Clearly, when  $t \rightarrow \infty$ ,

$$X_{t+1} \sim \bar{c}_0 [\beta \mathcal{F}'(\bar{X})]^t \frac{\beta}{1-\beta}$$

From (46), (47) and (48),  $L_{t+1}$ ,  $K_{t+1}$  and the ratio  $\frac{K_{t+1}}{qL_{t+1}}$  obtain

$$\begin{aligned} L_{t+1} &= \frac{X_{t+1} - \bar{X}}{q + \lambda \bar{X}} = \frac{\bar{c}_0 [\beta \mathcal{F}'(\bar{X})]^t \frac{\beta}{1-\beta} - z - \bar{X}}{q + \lambda \bar{X}} \\ K_{t+1} &= \bar{X} \frac{\frac{q}{\lambda} + X_{t+1}}{\frac{q}{\lambda} + \bar{X}} = \bar{X} \frac{\frac{q}{\lambda} + \bar{c}_0 [\beta \mathcal{F}'(\bar{X})]^t \frac{\beta}{1-\beta} - z}{\frac{q}{\lambda} + \bar{X}} \\ \frac{K_{t+1}}{qL_{t+1}} &= \bar{X} \frac{\frac{\lambda}{q} X_{t+1} + 1}{X_{t+1} - \bar{X}} \end{aligned}$$

Thus,

$$\lim_{t \rightarrow \infty} K_t = \lim_{t \rightarrow \infty} L_t = \infty$$

Taking the limit of the ratio we have

$$\lim_{t \rightarrow \infty} \frac{K_t}{qL_t} = \lim_{t \rightarrow \infty} \left( \bar{X} \frac{\frac{\lambda}{q} X_t + 1}{X_t - \bar{X}} \right) = \bar{X} \frac{\lambda}{q} \leq \frac{\alpha}{\theta}$$

where the last inequality follows from (8). Note that this result on the limit is in line with point (3) of Lemma 3.

Finally, we observe that, asymptotically, physical, living and total capital, and consumption grow at the same rate:

$$\frac{L_{t+1}}{L_t} \sim \frac{K_{t+1}}{K_t} \sim \frac{X_{t+1}}{X_t} \sim \frac{c_{t+1}}{c_t} = \beta \mathcal{F}'(\bar{X}) > 1$$

In this respect, we can say that the unbounded growth is asymptotically balanced. ■

**Proof of Proposition 11**

Let  $1 < \gamma < 1/\beta$ . In this case,  $\mathcal{F}$  is strictly convex for  $X_t > \bar{X}$ . Hence  $\mathcal{F}'$  is increasing and  $\mathcal{F}'(X_t) > \mathcal{F}'(\bar{X})$  for any  $X_t > \bar{X}$ . Since  $\mathcal{F}'(\bar{X}) > 1/\beta$ , there exists no steady state. The optimal path  $(X_t)_{t=0}^{\infty}$  diverges to infinity:  $\lim_{t \rightarrow \infty} X_t = \infty$ . Then, according to points (1) and (4) of Lemma 3,  $\lim_{t \rightarrow \infty} L_t = \lim_{t \rightarrow \infty} K_t = \infty$  and

$$\lim_{t \rightarrow \infty} \frac{K_t}{qL_t} = \frac{\alpha}{\theta}$$

with  $K_t/(qL_t) < \alpha/\theta$  for  $t$  large enough.

In the following, we consider the two possible cases according to Assumption 1:  $\delta_K = \delta_L$  and  $\delta_K > \delta_L$ .

(1) Let  $\delta_K = \delta_L$ . Following the same argument as at the end of the proof of Proposition 8, we obtain again equation (43):

$$\zeta_t = \frac{\alpha}{\lambda\theta} \frac{1}{L_t} + \frac{\alpha}{\theta}$$

However, under the assumptions of Proposition 11,  $L_t$  diverges to  $\infty$ . Thus,  $\zeta_t > \alpha/\theta$  for any  $t$  and, since  $\lim_{t \rightarrow \infty} L_t = \infty$ , we have  $\lim_{t \rightarrow \infty} \zeta_t = \alpha/\theta$ .

(2) Let now  $\delta_K > \delta_L$ .

The first order condition in (26) can be written as:

$$\lambda\theta AK^*(X) - \alpha qA [1 + \lambda L^*(X)] = q(\delta_L - \delta_K) [1 + \lambda L^*(X)]^{1-\theta} K^*(X)^{1-\alpha}$$

Since  $\zeta(X_t) \equiv K^*(X_t)/[qL^*(X_t)]$ , we can divide the above expression to obtain

$$\lambda\theta A\zeta(X_t) = \alpha A \left[ \lambda + \frac{1}{L^*(X_t)} \right] + (\delta_L - \delta_K) q^\theta \zeta(X_t)^\theta K^*(X_t)^{1-\gamma} \left[ \lambda + \frac{1}{L^*(X_t)} \right]^{1-\theta}$$

and, taking derivatives on both sides with respect to  $X_t$ :

$$\begin{aligned}
& \zeta'(X_t) \left( \lambda \theta A + \theta (\delta_K - \delta_L) q^\theta \zeta(X_t)^{\theta-1} K^*(X_t)^{1-\gamma} \left[ \lambda + \frac{1}{L^*(X_t)} \right]^{1-\theta} \right) \\
= & \frac{L^{*'}(X_t)}{L^*(X_t)^2} \left( -\alpha A + (1-\theta) (\delta_K - \delta_L) q^\theta \zeta(X_t)^\theta K^*(X_t)^{1-\gamma} \left[ \lambda + \frac{1}{L^*(X_t)} \right]^{-\theta} \right) \\
& + K^{*'}(X_t) (\gamma - 1) q^\theta (\delta_K - \delta_L) \zeta(X_t)^\theta K^*(X_t)^{-\gamma} \left[ \lambda + \frac{1}{L^*(X_t)} \right]^{1-\theta} \quad (61)
\end{aligned}$$

We have

$$K^*(X_t)^{-\gamma} K^{*'}(X_t) = q^{1-\gamma} L^*(X_t)^{2-\gamma} \frac{K^{*'}(X_t)}{qL^{*'}(X_t)} \frac{L^{*'}(X_t)}{L^*(X_t)^2} \left[ \frac{K^*(X_t)}{qL^*(X_t)} \right]^{-\gamma} \quad (62)$$

(28) and (29) imply

$$\frac{K^{*'}(X_t)}{qL^{*'}(X_t)} = \frac{\alpha q \lambda \theta + \theta (1-\theta) \lambda^2 \frac{K^*(X_t)}{1+\lambda L^*(X_t)}}{\alpha q \lambda \theta + \alpha (1-\alpha) q^2 \frac{1+\lambda L^*(X_t)}{K^*(X_t)}}$$

since  $P(X_t) \equiv 1 + \lambda L^*(X_t)$ .

This means that, for any  $X_t$  sufficiently large, the term  $K^{*'}(X_t) / [qL^{*'}(X_t)]$  is uniformly bounded above and below away from zero. We also have that  $\zeta(X_t) \equiv K^*(X_t) / [qL^*(X_t)]$  is uniformly bounded away from zero and from  $+\infty$ .

(61) can be rewritten as

$$\zeta'(X_t) R(X_t) = S(X_t)$$

where

$$R(X_t) \equiv \lambda \theta A + \theta (\delta_K - \delta_L) q^\theta \zeta(X_t)^{\theta-1} K^*(X_t)^{1-\gamma} \left[ \lambda + \frac{1}{L^*(X_t)} \right]^{1-\theta} > 0 \quad (63)$$

and

$$S(X_t) \equiv \sigma(X_t) \frac{L^{*'}(X_t)}{L^*(X_t)^2}$$

with

$$\begin{aligned}
\sigma(X_t) \equiv & -\alpha A + (1-\theta) (\delta_K - \delta_L) q^\theta \zeta(X_t)^\theta K^*(X_t)^{1-\gamma} \left[ \lambda + \frac{1}{L^*(X_t)} \right]^{-\theta} \\
& + (\gamma - 1) (\delta_K - \delta_L) q^{-\alpha} \zeta(X_t)^{-\alpha} L^*(X_t)^{2-\gamma} \frac{K^{*'}(X_t)}{L^{*'}(X_t)} \left[ \lambda + \frac{1}{L^*(X_t)} \right]^{1-\theta} \quad (64)
\end{aligned}$$

because of (62).

Taking the limit of the second term in (64)

$$\begin{aligned} & \lim_{X_t \rightarrow \infty} \left( (1 - \theta) (\delta_K - \delta_L) q^\theta \zeta(X_t)^\theta K^*(X_t)^{1-\gamma} \left[ \lambda + \frac{1}{L^*(X_t)} \right]^{-\theta} \right) \\ &= (1 - \theta) (\delta_K - \delta_L) q^\theta \lambda^{-\theta} \lim_{X_t \rightarrow \infty} \left[ \zeta(X_t)^\theta K^*(X_t)^{1-\gamma} \right] = 0 \end{aligned}$$

Moreover, the limit of the third term of (64)

$$\begin{aligned} & \lim_{X_t \rightarrow \infty} \left[ (\gamma - 1) (\delta_K - \delta_L) q^{-\alpha} \zeta(X_t)^{-\alpha} L^*(X_t)^{2-\gamma} \frac{K^{*'}(X_t)}{L^{*'}(X_t)} \left[ \lambda + \frac{1}{L^*(X_t)} \right]^{1-\theta} \right] \\ &= (\gamma - 1) (\delta_K - \delta_L) q^{-\alpha} \lambda^{1-\theta} \lim_{X_t \rightarrow \infty} \left[ \zeta(X_t)^{-\alpha} L^*(X_t)^{2-\gamma} \frac{K^{*'}(X_t)}{L^{*'}(X_t)} \right] = \infty \end{aligned}$$

Hence, for any  $X_t$  large enough,  $S(X_t) > 0$  and  $\zeta'(X_t) > 0$ .

Let us prove that  $\zeta'(X_t) < 0$  when  $X_t$  is sufficiently close to  $\bar{X}$  from the right.

We have

$$\zeta'(X_t) = \frac{\sigma(X_t) L^{*'}(X_t)}{R(X_t) L^*(X_t)^2} \quad (65)$$

where  $R(X_t) > 0$  is given by (63) and  $\sigma(X_t)$  by (64).

Observe that

$$\lim_{X_t \rightarrow \bar{X}^+} \sigma(X_t) = -\alpha A + (1 - \theta) (\delta_K - \delta_L) \bar{X}^{1-\alpha} \quad (66)$$

because

$$\begin{aligned} & q^{-\alpha} \zeta(X_t)^{-\alpha} L^*(X_t)^{2-\gamma} \frac{K^{*'}(X_t)}{L^{*'}(X_t)} \left[ \lambda + \frac{1}{L^*(X_t)} \right]^{1-\theta} \\ & \sim q^{-\alpha} \zeta(X_t)^{-\alpha} L^*(X_t)^{2-\gamma} L^*(X_t)^{\theta-1} \frac{K^{*'}(X_t)}{L^{*'}(X_t)} \\ & = q^{-\alpha} q^{1-\theta} \zeta(X_t)^{-\alpha} K^*(X_t)^{\theta-1} L^*(X_t)^{2-\gamma} \frac{K^{*'}(X_t)}{L^{*'}(X_t)} \left[ \frac{qL^*(X_t)}{K^*(X_t)} \right]^{\theta-1} \\ & = q^{1-\gamma} \zeta(X_t)^{1-\gamma} K^*(X_t)^{\theta-1} L^*(X_t)^{2-\gamma} \frac{K^{*'}(X_t)}{L^{*'}(X_t)} \rightarrow 0 \end{aligned}$$

since  $1 - \gamma < 0 < 2 - \gamma$ ,  $\lim_{X_t \rightarrow \bar{X}^+} L^*(X_t) = 0$ ,  $\lim_{X_t \rightarrow \bar{X}^+} K^*(X_t) = \bar{X}$ ,  $\lim_{X_t \rightarrow \bar{X}^+} \zeta(X_t)^{1-\gamma} = 0$  and

$$\lim_{X_t \rightarrow \bar{X}^+} \frac{K^{*'}(X_t)}{L^{*'}(X_t)} = q \frac{\alpha q \lambda \theta + \theta (1 - \theta) \lambda^2 \bar{X}}{\alpha q \lambda \theta + \alpha (1 - \alpha) q^2 \frac{1}{\bar{X}}}$$

$\bar{X}$  satisfies (7), that is

$$(\delta_K - \delta_L) \bar{X}^{1-\alpha} = \alpha A - \lambda \theta A \frac{\bar{X}}{q}$$

Replacing in (66), we find

$$\lim_{X_t \rightarrow \bar{X}^+} \sigma(X_t) = -\alpha A + (1 - \theta) \left( \alpha A - \lambda \theta A \frac{\bar{X}}{q} \right) = -\theta A \left[ \alpha + \lambda (1 - \theta) \frac{\bar{X}}{q} \right] < 0$$

Hence, when  $X_t$  is sufficiently close to  $\bar{X}$  from the right, we get  $\sigma(X_t)/R(X_t) < 0$  and, according to (65),  $\zeta'(X) < 0$ . ■

**Proof of Proposition 12**

Under the proposition's assumptions  $\gamma > 1$  and  $\delta_K = \delta_L$ . In this case, it is possible to write the dynamics in the variable  $\zeta_t$ , which, we know, converges to  $\alpha/\theta$  (see Proposition 11).

The existence of a limit for this reduced dynamics will allow us to simulate the trajectory  $(\zeta_t)$  and then, the trajectories  $(X_t)$  and  $(c_t)$ .

We know that

$$\mathcal{F}(X_{t+1}) = A(1 + \lambda L_{t+1})^\theta (X_{t+1} - qL_{t+1})^\alpha + (1 - \delta_K) X_{t+1}$$

and that according to (30),

$$\mathcal{F}'(X_{t+1}) = \alpha A(1 + \lambda L_{t+1})^\theta (X_{t+1} - qL_{t+1})^{\alpha-1} + 1 - \delta_K$$

Moreover,  $c_t = \mathcal{F}(X_t) - X_{t+1}$ ,  $L_t = \alpha/[\lambda(\theta\zeta_t - \alpha)]$  and

$$X_t = (1 + \zeta_t) qL_t = \frac{q}{\lambda} \frac{\alpha\zeta_t + \alpha}{\theta\zeta_t - \alpha} \equiv X(\zeta_t) \quad (67)$$

The Euler equation becomes

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{c_{t+1}}{c_t} = \beta \mathcal{F}'(X_{t+1})$$

that is (19), where  $\psi$  is defined by (20).

Besides, (19) obtains solving the "new" optimization problem:  $\max \sum_{t=0}^{\infty} \beta^t \ln(\psi(\zeta_t, \zeta_{t+1}))$  with  $\zeta_t \geq \alpha/\theta$  for any  $t$ .

The solution will be interior because of  $\ln$ . Taking the derivative of the intertemporal utility with respect to  $\zeta_{t+1}$ , we find

$$\frac{\psi_2(\zeta_t, \zeta_{t+1})}{\psi(\zeta_t, \zeta_{t+1})} + \beta \frac{\psi_1(\zeta_{t+1}, \zeta_{t+2})}{\psi(\zeta_{t+1}, \zeta_{t+2})} = 0$$

Since  $\psi(\zeta_t, \zeta_{t+1}) \equiv \mathcal{F}(X(\zeta_t)) - X(\zeta_{t+1})$ , we have

$$\begin{aligned} \psi_1(\zeta_{t+1}, \zeta_{t+2}) &= \mathcal{F}'(X(\zeta_{t+1})) X'(\zeta_{t+1}) \\ \psi_2(\zeta_t, \zeta_{t+1}) &\equiv -X'(\zeta_{t+1}) \end{aligned}$$

and we recover (19). ■

**Proof of Proposition 13**

We study now the case  $\gamma > 1$  with  $\delta_K > \delta_L$ . From (26):

$$\lambda\theta A \left( \frac{1 + \lambda L_t}{X_t - qL_t} \right)^{\theta-1} = \alpha q A \left( \frac{1 + \lambda L_t}{X_t - qL_t} \right)^\theta + q(\delta_L - \delta_K) K_t^{1-\gamma} \quad (68)$$

for any  $t$ , where  $K_t = X_t - qL_t$ . We know that (26) implies that  $L_t = L^*(X_t)$  is a strictly increasing function of  $X_t$ .

Define  $\varphi(X_t)$  according to (22). (68) reduces to

$$\lambda\theta A \varphi(X_t)^{\theta-1} = \alpha q A \varphi(X_t)^\theta + q(\delta_L - \delta_K) K^*(X_t)^{1-\gamma} \quad (69)$$

Taking derivatives in (69) with respect to  $X_t$ , we find

$$\theta A \varphi'(X_t) \varphi(X_t)^{\theta-2} [\lambda(\theta-1) - \alpha q \varphi(X_t)] = q(\gamma-1)(\delta_K - \delta_L) K^*(X_t)^{-\gamma} K^{*\prime}(X_t) \quad (70)$$

$\theta < 1$ ,  $\gamma > 1$  and  $K^{*\prime}(X_t) > 0$  imply  $\varphi'(X_t) < 0$ . Since  $\varphi$  is strictly monotonic, we can define the inverse function  $X_t = \varphi^{-1}(\varphi_t)$  for any  $\varphi_t > 0$ .

According to (69), we find

$$\lambda\theta A \left[ \lim_{X_t \rightarrow \infty} \varphi(X_t) \right]^{\theta-1} = \alpha q A \left[ \lim_{X_t \rightarrow \infty} \varphi(X_t) \right]^\theta \quad (71)$$

since  $\gamma > 1$  and  $\lim_{X_t \rightarrow \infty} K^*(X_t)^{1-\gamma} = 0$ . Thus,  $\lim_{X_t \rightarrow \infty} \varphi(X_t) = l > 0$ , otherwise, according to (71),  $\lim_{X_t \rightarrow \infty} \varphi(X_t) = 0$  would imply that  $\infty = 0$ , which is a contradiction. Then, (71) becomes  $\lambda\theta A l^{\theta-1} = \alpha q A l^\theta$ , that is  $l = \lambda\theta/(\alpha q) > 0$ , which is not surprising:

$$\begin{aligned} \lim_{X_t \rightarrow \infty} \varphi(X_t) &= \lim_{X_t \rightarrow \infty} \frac{1 + \lambda L^*(X_t)}{K^*(X_t)} = \lim_{X_t \rightarrow \infty} \frac{1}{K^*(X_t)} + \frac{\lambda}{q} \lim_{X_t \rightarrow \infty} \frac{qL^*(X_t)}{K^*(X_t)} \\ &= 0 + \frac{\lambda}{q} \lim_{X_t \rightarrow \infty} \frac{1}{\zeta(X_t)} = \frac{\lambda}{q} \frac{\theta}{\alpha} \end{aligned}$$

Now, consider the Euler equation:

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \beta \mathcal{F}'(X_{t+1}) = \beta \left( \alpha A [1 + \lambda L^*(X_{t+1})]^\theta [K^*(X_{t+1})]^{\alpha-1} + 1 - \delta_K \right) \\ &= \beta \left( \alpha A \varphi(X_{t+1})^\theta [K^*(X_{t+1})]^{\alpha+\theta-1} + 1 - \delta_K \right) \end{aligned}$$

Clearly,  $\lim_{t \rightarrow \infty} (c_{t+1}/c_t) = \infty$  since  $\lim_{X_t \rightarrow \infty} \varphi(X_t) = l > 0$  and  $\lim_{X_t \rightarrow \infty} K^*(X_t)^{\alpha+\theta-1} = \infty$ .

For the numerical exercises, we will consider the Euler equation  $c_{t+1}/c_t = \beta \mathcal{F}'(X_{t+1})$  with  $c_t = \mathcal{F}(X_t) - X_{t+1}$ :

$$\frac{\mathcal{F}(X_{t+1}) - X_{t+2}}{\mathcal{F}(X_t) - X_{t+1}} = \beta \mathcal{F}'(X_{t+1})$$

Using  $X_t = \varphi^{-1}(\varphi_t)$ , we obtain

$$\frac{\mathcal{F}(\varphi^{-1}(\varphi_{t+1})) - \varphi^{-1}(\varphi_{t+2})}{\mathcal{F}(\varphi^{-1}(\varphi_t)) - \varphi^{-1}(\varphi_{t+1})} = \beta \mathcal{F}'(\varphi^{-1}(\varphi_{t+1}))$$

that is (21), a recursive transition function.

The trajectory is determined by (21) jointly with the transversality condition:

$$\lim_{t \rightarrow \infty} \varphi_t = l = \frac{\lambda \theta}{\alpha q}$$

■

**Proof of Theorem 15.**

First, we prove that  $(K_t^*)$  maximizes the firm's profit. Define

$$\Delta^T \equiv \sum_{t=0}^T p_t^* [F(K_t^*, L_t^*) - K_{t+1}^* - qL_{t+1}^*] - \sum_{t=0}^T p_t^* [F(K_t, L_t) - K_{t+1} - qL_{t+1}^*]$$

where  $K_0^* = K_0$  and  $0 \leq K_{t+1} \leq F(K_t, L_t) - qL_{t+1}^* = L(K_t, L_t^*, L_{t+1}^*)$ . We have:

$$\begin{aligned} \Delta^T &= \sum_{t=0}^{T-1} (-p_t^* (K_{t+1}^* - K_{t+1}) + p_{t+1}^* [F(K_{t+1}^*, L_{t+1}^*) - F(K_{t+1}, L_{t+1}^*)]) \\ &\quad - p_T^* (K_{T+1}^* - K_{T+1}) \end{aligned}$$

Since  $F(K_t^*, L_t^*) - F(K_t, L_t) \geq F_K(K_t^*, L_t^*) (K_t^* - K_t)$ , we obtain

$$\Delta^T \geq \sum_{t=0}^{T-1} [-p_t^* + p_{t+1}^* F_K(K_{t+1}^*, L_{t+1}^*)] (K_{t+1}^* - K_{t+1}) - p_T^* (K_{T+1}^* - K_{T+1})$$

The Euler equation implies  $-p_t^* + p_{t+1}^* F_K(K_{t+1}^*, L_{t+1}^*) = 0$ , that is  $\Delta^T \geq -p_T^* (K_{T+1}^* - K_{T+1}) \geq -p_T^* K_{T+1}^*$ .

To finish this proof, we have to show that  $\lim_{T \rightarrow \infty} p_T^* K_{T+1}^* = 0$ . As above, we focus on the three cases: (1)  $\gamma < 1$ , (2)  $\gamma = 1$ , and (3)  $1 < \gamma < 1/\beta$ .

(1)  $\gamma < 1$ . Since  $c_T^* = F(K_T^*, L_T^*) - K_{T+1}^* - qL_{T+1}^*$ , we get

$$p_T^* K_{T+1}^* = \beta^T u'(c_T^*) K_{T+1}^* = \beta^T \frac{K_{T+1}^*}{c_T^*} = \beta^T \frac{\frac{K_{T+1}^*}{L_{T+1}^*}}{\frac{F(K_T^*, L_T^*)}{L_{T+1}^*} - \frac{K_{T+1}^*}{L_{T+1}^*} - q} \equiv \beta^T \rho_T^* \quad (72)$$

where the ratio  $\rho_T^*$  is uniformly bounded because  $(K_t^*, L_t^*)$  converges to the steady state  $(K^*, L^*)$ . According to (72)  $\lim_{T \rightarrow \infty} \beta^T = 0$  implies that  $\lim_{T \rightarrow \infty} p_T^* K_{T+1}^* = \lim_{T \rightarrow \infty} \beta^T \rho_T^* = 0$ .

(2)  $\gamma = 1$ . According to (4), we have

$$\frac{F(K_T^*, L_T^*)}{L_{T+1}^*} = \left[ A \left( \frac{1 + \lambda L_T^*}{K_T^*} \right)^\theta \frac{K_T^*}{L_T^*} + (1 - \delta_K) \frac{K_T^*}{L_T^*} + q(1 - \delta_L) \right] \frac{L_T^*}{L_{T+1}^*}$$

since  $\gamma \equiv \alpha + \theta = 1$ . The growth path is asymptotically balanced and, by (18),

$\lim_{T \rightarrow \infty} (L_{T+1}^*/L_T^*) = \beta \mathcal{F}'(\bar{X})$ . Then,

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{F(K_T^*, L_T^*)}{L_{T+1}^*} &= \left[ A \lambda^\theta \lim_{T \rightarrow \infty} \left( \frac{K_T^*}{L_T^*} \right)^{1-\theta} + (1 - \delta_K) \lim_{T \rightarrow \infty} \frac{K_T^*}{L_T^*} + q(1 - \delta_L) \right] \lim_{T \rightarrow \infty} \frac{L_T^*}{L_{T+1}^*} \\ &= \frac{\lambda \bar{X} (A \bar{X}^{-\theta} + 1 - \delta_K) + q(1 - \delta_L)}{\beta \mathcal{F}'(\bar{X})} \end{aligned} \quad (73)$$

since  $\lim_{T \rightarrow \infty} K_T^* = \infty$  and  $\lim_{T \rightarrow \infty} (K_T^*/L_T^*) = \lambda \bar{X}$ .

$\rho_T^*$  in (72) is uniformly bounded because of (73) and  $\lim_{T \rightarrow \infty} (K_T^*/L_T^*) = \lambda \bar{X}$ . Then, according to (72),  $\lim_{T \rightarrow \infty} \beta^T = 0$  implies  $\lim_{T \rightarrow \infty} \rho_T^* K_{T+1}^* = 0$ .

(3)  $1 < \gamma < 1/\beta$ . We know that

$$\begin{aligned} c_{T+1}^* &= \beta c_T^* F_K(K_{T+1}^*, L_{T+1}^*) = \beta c_T^* \left[ \alpha A (1 + \lambda L_{T+1}^*)^\theta K_{T+1}^{*\alpha-1} + 1 - \delta_K \right] \\ &= \beta \frac{c_T^*}{K_{T+1}^*} \alpha A (1 + \lambda L_{T+1}^*)^\theta K_{T+1}^{*\alpha} + \beta c_T^* (1 - \delta_K) \end{aligned}$$

Hence,

$$\frac{c_{T+1}^*}{K_{T+2}^*} = \beta \frac{c_T^*}{K_{T+1}^*} \left[ \alpha A \left( \frac{1 + \lambda L_{T+1}^*}{K_{T+1}^*} \right)^\theta K_{T+1}^{*\alpha+\theta-1} + 1 - \delta_K \right] \frac{K_{T+1}^*}{K_{T+2}^*} \quad (74)$$

From the budget constraint, at equilibrium, we have:

$$\begin{aligned} K_{T+2}^* + qL_{T+2}^* &\leq A (1 + \lambda L_{T+1}^*)^\theta K_{T+1}^{*\alpha} + (1 - \delta_K) K_{T+1}^* + q(1 - \delta_L) L_{T+1}^* - c_{T+1}^* \\ &\leq A (1 + \lambda L_{T+1}^*)^\theta K_{T+1}^{*\alpha} + (1 - \delta_K) K_{T+1}^* + q(1 - \delta_L) L_{T+1}^* \end{aligned}$$

that is, dividing by  $K_{T+2}^*$ , and rearranging

$$\frac{1 + \frac{qL_{T+2}^*}{K_{T+2}^*}}{A \left( \frac{1 + \lambda L_{T+1}^*}{K_{T+1}^*} \right)^\theta K_{T+1}^{*\alpha+\theta-1} + 1 - \delta_K + (1 - \delta_L) \frac{qL_{T+1}^*}{K_{T+1}^*}} \leq \frac{K_{T+1}^*}{K_{T+2}^*} \quad (75)$$

Replacing (75) in (74), we get

$$\frac{c_{T+1}^*}{K_{T+2}^*} \geq \beta \frac{c_T^*}{K_{T+1}^*} \frac{\left( 1 + \frac{qL_{T+2}^*}{K_{T+2}^*} \right) \left[ \alpha A \left( \frac{1 + \lambda L_{T+1}^*}{K_{T+1}^*} \right)^\theta K_{T+1}^{*\alpha+\theta-1} + 1 - \delta_K \right]}{A \left( \frac{1 + \lambda L_{T+1}^*}{K_{T+1}^*} \right)^\theta K_{T+1}^{*\alpha+\theta-1} + 1 - \delta_K + (1 - \delta_L) \frac{qL_{T+1}^*}{K_{T+1}^*}}$$

Since  $\gamma \equiv \alpha + \theta > 1$ ,  $\lim_{T \rightarrow \infty} (qL_T^*/K_T^*) = \theta/\alpha$  and  $\lim_{T \rightarrow \infty} K_T^* = \infty$ , we find

$$\begin{aligned} &\lim_{T \rightarrow \infty} \frac{\left( 1 + \frac{qL_{T+2}^*}{K_{T+2}^*} \right) \left[ \alpha A \left( \frac{1 + \lambda L_{T+1}^*}{K_{T+1}^*} \right)^\theta K_{T+1}^{*\alpha+\theta-1} + 1 - \delta_K \right]}{A \left( \frac{1 + \lambda L_{T+1}^*}{K_{T+1}^*} \right)^\theta K_{T+1}^{*\alpha+\theta-1} + 1 - \delta_K + (1 - \delta_L) \frac{qL_{T+1}^*}{K_{T+1}^*}} \\ &= \alpha \left( 1 + \lim_{T \rightarrow \infty} \frac{qL_{T+2}^*}{K_{T+2}^*} \right) = \alpha \left( 1 + \frac{\theta}{\alpha} \right) = \gamma \end{aligned}$$



Fix  $\varepsilon \in (0, \gamma - 1)$ . Then, there exists  $T_0$  such that, for any  $T \geq T_0$ , we have  $c_{T+1}^*/K_{T+2}^* \geq \beta(\gamma - \varepsilon)c_T^*/K_{T+1}^*$  and, by induction,  $c_{T_0+\tau+1}^*/K_{T_0+\tau+2}^* \geq [\beta(\gamma - \varepsilon)]^{\tau+1} c_{T_0}^*/K_{T_0+1}^*$  or, equivalently,

$$\frac{K_{T_0+\tau+2}^*}{c_{T_0+\tau+1}^*} \leq \frac{1}{[\beta(\gamma - \varepsilon)]^{\tau+1}} \frac{K_{T_0+1}^*}{c_{T_0}^*} \quad (76)$$

Therefore,

$$\lim_{T \rightarrow \infty} p_T^* K_{T+1}^* = \lim_{T \rightarrow \infty} \frac{\beta^T K_{T+1}^*}{c_T^*} \leq \frac{\beta^{T_0} K_{T_0+1}^*}{c_{T_0}^*} \lim_{T \rightarrow \infty} (\gamma - \varepsilon)^{T_0 - T} = 0$$

since  $\gamma - \varepsilon > 1$ .

Let us prove that the consumer does maximize her overall utility when  $0 < \gamma < 1/\beta$ , that is, if  $(c_t)$  satisfies  $\sum_{t=0}^{\infty} \beta^t u(c_t) > \sum_{t=0}^{\infty} \beta^t u(c_t^*)$ , then  $\sum_{t=0}^{\infty} p_t^* c_t > \pi^* + r^* K_0$ .

(1) From (P0), we have  $c_t^* + K_{t+1}^* + qL_{t+1}^* = F(K_t^*, L_t^*)$  for any  $t$ . Thus,  $p_t^* c_t^* = p_t^* F(K_t^*, L_t^*) - p_t^* K_{t+1}^* - qp_t^* L_{t+1}^*$ . Let us show that the series  $\sum_{t=0}^{\infty} p_t^* c_t^*$ ,  $\sum_{t=0}^{\infty} p_t^* K_{t+1}^*$ ,  $\sum_{t=0}^{\infty} p_t^* L_{t+1}^*$  and  $\sum_{t=0}^{\infty} p_t^* F(K_t^*, L_t^*)$  are bounded away from  $\infty$  and, thus,  $\sum_{t=0}^{\infty} p_t^* c_t^* = \pi^* + r^* K_0$  with  $\sum_{t=0}^{\infty} p_t^* c_t^* < \infty$  and  $\pi^* + r^* K_0 = \sum_{t=0}^{\infty} [p_t^* F(K_t^*, L_t^*) - p_t^* K_{t+1}^* - qp_t^* L_{t+1}^*] < \infty$ .

Note that  $p_t^* \equiv \beta^t u'(c_t^*) = \beta^t / c_t^*$ . Then,  $\sum_{t=0}^{\infty} p_t^* c_t^* = \sum_{t=0}^{\infty} \beta^t = 1/(1 - \beta) < \infty$ . Moreover, since  $\sum_{t=0}^{\infty} p_t^* c_t^* = \sum_{t=0}^{\infty} p_t^* [F(K_t^*, L_t^*) - K_{t+1}^* - qL_{t+1}^*]$  and  $\sum_{t=0}^{\infty} p_t^* c_t^* < \infty$ , we get  $\sum_{t=0}^{\infty} p_t^* F(K_t^*, L_t^*) = \sum_{t=0}^{\infty} p_t^* c_t^* + \sum_{t=0}^{\infty} p_t^* K_{t+1}^* + q \sum_{t=0}^{\infty} p_t^* L_{t+1}^* < \infty$ , provided that  $\sum_{t=0}^{\infty} p_t^* K_{t+1}^* < \infty$  and  $\sum_{t=0}^{\infty} p_t^* L_{t+1}^* < \infty$ .

First, since the sequence  $(L_t^*/K_t^*)$  is bounded away from 0 and from  $\infty$ , if  $\sum_{t=0}^{\infty} p_t^* K_{t+1}^* < \infty$ , then we would have that  $\sum_{t=0}^{\infty} p_t^* L_{t+1}^* = \sum_{t=0}^{\infty} p_t^* K_{t+1}^* (L_{t+1}^*/K_{t+1}^*) < \infty$ .

In this regard, let us show that  $\sum_{t=0}^{\infty} p_t^* K_{t+1}^* < \infty$ . We need to distinguish two cases:  $\gamma \leq 1$  and  $1 < \gamma < 1/\beta$ . If  $\gamma \leq 1$ , we have that  $\rho_T^*$  in (72) is uniformly bounded and  $\sum_{t=0}^{\infty} p_t^* K_{t+1}^* = \sum_{t=0}^{\infty} \beta^t \rho_T^* < \infty$ . In the second case,  $1 < \gamma < 1/\beta$ , and according to (76), we have

$$\begin{aligned} \sum_{t=0}^{\infty} p_t^* K_{t+1}^* &= \sum_{t=0}^{\infty} \frac{\beta^t K_{t+1}^*}{c_t^*} = \sum_{t=0}^{T_0} \frac{\beta^t K_{t+1}^*}{c_t^*} + \sum_{\tau=0}^{\infty} \frac{\beta^{T_0+\tau+1} K_{T_0+\tau+2}^*}{c_{T_0+\tau+1}^*} \\ &\leq \sum_{t=0}^{T_0} \frac{\beta^t K_{t+1}^*}{c_t^*} + \sum_{\tau=0}^{\infty} \frac{\beta^{T_0+\tau+1}}{[\beta(\gamma - \varepsilon)]^{\tau+1}} \frac{K_{T_0+1}^*}{c_{T_0}^*} \\ &= \sum_{t=0}^{T_0} \frac{\beta^t K_{t+1}^*}{c_t^*} + \frac{\beta^{T_0} K_{T_0+1}^*}{(\gamma - \varepsilon) c_{T_0}^*} \sum_{\tau=0}^{\infty} \left( \frac{1}{\gamma - \varepsilon} \right)^{\tau} \\ &= \sum_{t=0}^{T_0} \frac{\beta^t K_{t+1}^*}{c_t^*} + \frac{\beta^{T_0} K_{T_0+1}^*}{c_{T_0}^*} \frac{1}{\gamma - \varepsilon - 1} < \infty \end{aligned}$$

Therefore,  $\sum_{t=0}^{\infty} p_t^* F(K_t^*, L_t^*) < \infty$ .

Finally, let us show that, if  $(c_t)$  satisfies  $\sum_{t=0}^{\infty} \beta^t u(c_t) > \sum_{t=0}^{\infty} \beta^t u(c_t^*)$ , then  $\sum_{t=0}^{\infty} p_t^* c_t > \pi^* + r^* K_0$ . Indeed,  $0 > \sum_{t=0}^{\infty} \beta^t u(c_t^*) - \sum_{t=0}^{\infty} \beta^t u(c_t) \geq \sum_{t=0}^{\infty} \beta^t u'(c_t^*) (c_t^* - c_t)$ . Thus,  $0 > \sum_{t=0}^{\infty} p_t^* c_t^* - \sum_{t=0}^{\infty} p_t^* c_t$  and  $\sum_{t=0}^{\infty} p_t^* c_t > \sum_{t=0}^{\infty} p_t^* c_t^* = \pi^* + r^* K_0$ . To finish the proof, take  $r^* = p_0^* F_K(K_0, L_0)$ . ■

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