HAL open science

Lagrangian intersections and glancing points: typical transitions of phase in semiclassical approximations

Ilya Bogaevskii, Michel Rouleux

To cite this version:

Ilya Bogaevskii, Michel Rouleux. Lagrangian intersections and glancing points: typical transitions of phase in semiclassical approximations. Days of Diffraction 2023, Jun 2023, St Petersbourg, Russia. hal-04287098

HAL Id: hal-04287098

https://hal.science/hal-04287098v1

Submitted on 15 Nov 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Lagrangian intersections and glancing points: typical transitions of phase in semiclassical approximations

Ilya BOGAEVSKII[1] & Michel ROULEUX[2]

[1]Mech&Math Faculty, Lomonosov Moscow State University, Russia; Scientific Research Institute for System Analysis of RAS, Russia; The University of Liverpool, UK
ibogaevsk@gmail.com

[2]Aix Marseille Univ, Université de Toulon, CPT, CNRS, France
michel.rouleux@univ-tln.fr

Abstract

Given a semiclassical distribution f_h microlocalized on a Lagrangian manifold Λ_0, $H \in C^\infty(T^*\mathbf{R}^n)$, and $H = E$ a regular energy surface, we find asymptotic solutions of the PDE $(H(x,\widehat{p}) - E)\, u_h(x,E) = f_h(x)$ in terms of the Maslov canonical operator, when the Hamilton vector field v_H fails to be transverse to Λ_0 at some points.

1 Introduction

Given $H \in C^\infty(T^*\mathbf{R}^n)$, we discuss the semiclassical approximation to the solution of the equation

$$\big(\widehat{H} - E\big)u_h(x,y,E) = f_h(x), \quad \widehat{H} = H(x,\widehat{p}), \quad \widehat{p} = -ih\partial_x$$

with a right hand side f_h microlocalized on a smooth Lagrangian submanifold Λ_0:

$$f_h(x) = [\mathcal{K}_{\Lambda_0} a](x), \quad a : \Lambda_0 \to \mathbf{R}$$

where a is an amplitude on Λ_0, and $\mathcal{K}_{\Lambda_0}$ denotes the Maslov canonical operator. The solution of this problem is formally given by:

$$u_h(x,E) = \frac{i}{h}\int_0^{+\infty} v_h(x,t)\exp(iEt/h)\,dt$$

via the solution of the Cauchy problem for the Schrödinger equation

$$ih\,\partial_t v_h = \widehat{H} v_h, \quad v_h(x,0) = f_h(x). \tag{1.1}$$

We formulate first a set of standard hypotheses, which are sufficient in the framework of formal asymptotics we are considering here.

1. E is non critical for H, i.e. $dH \neq 0$ on $\Sigma_E = \{H = E\}$.
2. E is non trapping energy level for H, in the sense of scattering theory. This allows to cut-off the integral defining $u_h(x,E)$ near $t = +\infty$ as in (2.1);
3. "non-return condition" [16] which is not necessary if we content ourselves to evaluate microlocally $u_h(x,E)$ outside Λ_0. Otherwise this is a difficult problem involving Sommerfeld radiation condition, see e. g. [5];
4. $\Lambda_0 \cap \Sigma_E$ is compact (isotropic) submanifold, which is fulfilled if H is of elliptic type.

Let $\mathbf{R}^{n+1}_{x,t} = \{(x,t)\}$ be the space-time and $T^*\mathbf{R}^{n+1}_{x,t}$ be its cotangent bundle with the coordinates (x,t,p,E) such that $p\,dx - E\,dt$ is the canonical 1-form. To define the semiclassical approximation for v_h one needs:

- the Lagrangian submanifold of $T^*\mathbf{R}^{n+1}$

  $$\Lambda = \big\{(x,p) = g^t(z), t > 0, E = H(z) : z \in \Lambda_0\big\}$$

  where $g^t = \exp tv_H$ is the phase flow of the Hamiltonian vector field v_H generated by H

  $$\dot{x} = \partial_p H(x,p), \quad \dot{p} = -\partial_x H(x,p).$$

- the amplitude $b : \Lambda \to \mathbf{R}$ being the solution to the transport equation with an initial condition $b|_{\Lambda_0} = a$. Then $v_h(x)$ is given by

  $$v_h(x,t) = [\mathcal{K}_\Lambda b](x,t) + \mathcal{O}(h), \quad b : \Lambda \to \mathbf{R}.$$

So we can expect the following semiclassical approximation

$$u_h(x,E) = \frac{i}{h}\int_0^{+\infty} [\mathcal{K}_\Lambda b](x,t)\exp(iEt/h)\,dt\, + \mathcal{O}(h). \tag{1.2}$$

The relevant contributions to this integral come from $t = 0$, $t = +\infty$ and the critical points $t \in (0, +\infty)$. See also [14] for higher order approximation.

The Lagrangian submanifold of $T^*\mathbf{R}^n$

$$\Lambda_+^E = \{z \in T^*\mathbf{R}^n : \exists\, t \geq 0,\ z \in \exp t v_H(\Lambda_0 \cap \Sigma_E)\}$$

gives the contribution of the critical points $t \in (0, +\infty)$. The following two cases are possible.

1) The Lagrangian submanifold Λ_0 is transversal to Σ_E — it means that E is not a critical value for the restriction H_{Λ_0}. Then v_H is not tangent to Λ_0 and consequently to the submanifold $\Lambda_0 \cap \Sigma_E$. Therefore Λ_+^E is an immersed Lagrangian submanifold with boundary and we say that (Λ_0, Λ_+^E) is a *(Lagrangian) intersecting pair*.

2) The Lagrangian submanifold Λ_0 is tangent to Σ_E at a point z called *glancing* — it means that z is a critical point of the restriction H_{Λ_0} with the critical value E. Then v_H is tangent to Λ_0 and Λ_+^E may have singularities, see e. g. [6] for applications to the semiclassical context.

Remark: A model problem with the point $t = +\infty$ giving a relevant contribution into the integral (1.2) is investigated in [4].

2 Basic example of intersecting pair

A simple example of a Lagrangian intersecting pair is given by $\widetilde{\Lambda}_0 = \{x = 0\}$ (vertical fiber at 0) and

$$\widetilde{\Lambda}_+^0 = \{(x, p) \in T^*\mathbf{R}^n : x' = 0, p_n = 0, x_n \geq 0\},\ x' = (x_1, \dots, x_{n-1}),\ p' = (p_1, \dots, p_{n-1}),$$

the flow-out of $\widetilde{\Lambda}_0$ by the Hamilton vector field with Hamiltonian p_n. In this case $\widehat{H} = -ih\partial_{x_n}$,

$$f_h(x) = [\mathcal{K}_{\widetilde{\Lambda}_0} a](x) = \int_{\widetilde{\Lambda}_0} e^{ixp/h} a(p)\, dp = f_1\left(\frac{x}{h}\right)$$

where f_1 decreases rapidly at infinity,

$$v_h(x, t) = f_h(x', x_n - t)$$

and

$$\begin{aligned} u_h(x, 0) &= \frac{i}{h} \int_0^{+\infty} f_h(x', x_n - t)\, dt = \\ &\frac{i}{h} \int_0^{+\infty} \Theta_{t_0}(t)\, f_1\left(\frac{x'}{h}, \frac{x_n - t}{h}\right) dt + \mathcal{O}(h^\infty) \end{aligned} \tag{2.1}$$

for all $x_n \leq t_0/2$, here Θ_{t_0} is a cut-off function equal to 1 on $[0, t_0]$ and 0 near $+\infty$.

This pair of Lagrangian manifolds is actually the paradigm of intersecting pairs, i. e. in some local canonical charts $\Lambda_0 = \widetilde{\Lambda}_0$ and $H(x,p) = E + p_n$. In particular (Λ_0, Λ_+^E) are mapped onto $(\widetilde{\Lambda}_0, \widetilde{\Lambda}_+^0)$ by a canonical transformation sending $\partial\Lambda_+^E = \Lambda_0 \cap \Lambda_+^E$ onto $\partial\widetilde{\Lambda}_+^0 = \widetilde{\Lambda}_0 \cap \widetilde{\Lambda}_+^0$.

3 Main motivation: Helmholtz equation and Bessel cylinder

Most intrinsic formulae are available in the case of the physically relevant case of a Hamiltonian positively homogeneous with respect to p, e. g. the Helmholtz operator. This provides global eikonal coordinates and avoids the microlocal reduction to $H = E + \widehat{p}_n$.

Taking into account glancing intersection amounts to "correct" locally the formulae giving the phase function and the half-density in new local coordinates, which are obtained using a normal form.

Let us consider the *Bessel cylinder* $\Lambda_0 \subset T^*\mathbf{R}^n$

$$\Lambda_0 = \{x = \varphi\omega(\psi),\, p = \omega(\psi) : \varphi \in \mathbf{R}\} \tag{3.1}$$

where $\omega(\psi)$ is a vector on the unit sphere in $\mathbf{R}^n$.

Our main motivation is the study of Bessel beams, i. e. a wave whose amplitude is described by a Bessel function of the first kind. Assume for instance we are given a non-linear Helmholtz equation on $\mathbf{R}^2$ of the type

$$-h^2\Delta u - u = F(\varepsilon\, u), \quad F(0) = 0, \quad F'(0) = 1$$

where F is a smooth function and ε is a small parameter. We expand $u = u_0 + \varepsilon\, u_1 + \cdots$, and find at zero order in ε the equation $(-h^2\Delta - 1)u_0 = 0$. Its radially symmetric solution is given by $u_0 = f_h(x)$ where

$$f_h(x) = (2\pi/h)^{1/2} J_0\big(|x|/h\big),$$

and is microlocalized on the Bessel cylinder $\Lambda_0 \subset T^*\mathbf{R}^2$ defined by (3.1) for $n = 2$.

At first order in ε we get the Helmholtz equation

$$(-h^2\Delta - 1)u_1 = f_h, \quad H = p^2$$

where $u_1 = u(x, 1)$ in our previous notation, and

$$u_h(x, 1) = -J_1(|x|/h)(|x|/2h)$$

is its radially symmetric solution. We observe that all points of Λ_0 turn to be glancing because $\Lambda_0 \subset \Sigma_1$ and $H_{\Lambda_0} \equiv 1$.

For the Helmholtz equation with variable coefficients, we have the following:

Proposition 3.1 *Let Λ_0 be the n-dimensional Bessel cylinder (3.1) and $H \in C^\infty(T^*\mathbf{R}^n)$ be homogeneous of degree m with respect to p. Then $z = (x,p) \in \Lambda_0$ is a glancing point at energy E iff*

$$\begin{aligned} &\partial_p H(z) + \varphi \partial_x H(z) = mH\omega(\psi), \\ &\langle -\partial_x H(z), \omega(\psi)\rangle = 0, \quad H(z) = E \end{aligned} \tag{3.2}$$

Proof: We complete $\omega(\psi)$ in $\mathbf{S}^{n-1}$ into a (direct) orthonormal basis $\omega^\perp(\psi) = \big(\omega_1(\psi), \cdots, \omega_{n-1}(\psi)\big)$ of $\mathbf{R}^n$, and denote by $\omega^\perp(\psi)\delta\psi = \omega_1(\psi)\delta\psi_1 + \cdots + \omega_{n-1}(\psi)\delta\psi_{n-1}$ a section of $T\mathbf{S}^{n-1}$, $\delta\psi_j \in \mathbf{R}$. The tangent space $T_z\Lambda_0$ has the parametric equations

$$\delta X = \omega(\psi)\delta\varphi + \varphi\omega^\perp(\psi)\delta\psi, \ \delta P = \omega^\perp(\psi)\delta\psi, \ (\delta\varphi, \delta\psi) \in \mathbf{R}^n$$

so $v_H \in T_z\Lambda_0$ iff there exist $(\delta\varphi, \delta\psi)$ such that

$$\partial_p H = \omega(\psi)\delta\varphi + \varphi\omega^\perp(\psi)\delta\psi - \partial_x H = \omega^\perp(\psi)\delta\psi$$

Taking scalar products with $\omega(\psi), \omega^\perp(\psi)$, and using that $(\omega(\psi), \omega^\perp(\psi))$ form a basis of $\mathbf{R}^n$, readily shows that relations

$$\partial_p H + \varphi\partial_x H = \langle \partial_p H, P(\psi)\rangle\omega(\psi), \ \langle -\partial_x H, \omega(\psi)\rangle = 0 \tag{3.3}$$

are necessary and sufficient for $v_H \in T_z\Lambda_0$.

We set $\mathcal{H}(\varphi,\psi) = H|_{\Lambda_0}$. Then

$$\nabla\mathcal{H}(\varphi,\psi) = \big(\langle \partial_x H, \omega^\perp(\psi)\rangle, \varphi\langle \partial_x H, \omega^\perp(\psi)\rangle + \langle \partial_p H, \omega^\perp(\psi)\rangle\big)$$

so (3.3) readily gives $\nabla\mathcal{H}(\varphi,\psi) = 0$. Therefore $z = (x,p) \in \Lambda_0$ is a glancing point.

Now, if H is positively homogeneous of degree m with respect to p, using Euler identity, we get $\delta\varphi = \langle \partial_p H, P(\psi)\rangle = mH$, and (3.2) holds iff for $v_H \in T_z\Lambda_+^E \cap T_z\Lambda_0$ when $H = E$. ♣

In particular when $H(z) = \frac{|p|^m}{\rho(x)}$ is a conformal metric, with ρ a smooth positive function on $\mathbf{R}^n$, z is a glancing point iff

$$\begin{aligned} &\text{either}: \ \varphi \neq 0 \text{ and } \nabla\rho = 0, \\ &\text{or}: \ \varphi = 0 \text{ and } \langle \nabla\rho(0), \omega(\psi)\rangle = 0. \end{aligned}$$

Example: Let $n = 2$, $m = 1$, $H(z) = \frac{|p|}{\rho(x)}$, with $\rho(x) = \frac{1}{2}(1 + (x - x_0)^2)$. If $x_0 = \varphi\omega(\psi) \neq 0$, we have $\rho^4(x_0) \det \nabla^2(H|_{\Lambda_0}) = \varphi^2$, $\rho^2(x_0) \operatorname{Tr} \nabla^2(H|_{\Lambda_0}) = -(1 + \varphi^2)$. Critical energy is given by $E = H(z) = \frac{|p|}{\rho(x)}$, i. e. $E_0 = 1/\rho(x_0)$.

Remark: Let $n = 2$, $H(z)$ positively homogeneous of degree $m = 1$. Away from glancing points, Λ_+^E is parametrized near $t = 0$ by (ψ, θ^E), where $\theta^E = \theta^E(t, \varphi)$ and $\theta^E : \mathbf{R}^2 \to \mathbf{R}$ is a submersion, such that

$$H\big(X(t, \varphi, \psi), P(t, \varphi, \psi)\big) = E$$

When $\rho(x) = \rho(|x|)$, we have $\varphi = \varphi(E)$ on Λ_+^E, so that we can take $\theta^E(t, \varphi) = Et$ (or equivalently $\theta^E(t, \varphi) = \varphi(E) + Et$, which is the eikonal on Λ_+^E), and Λ_+^E is parametrized by (t, ψ). Conversely, if Λ_+^E is parametrized by (t, ψ), the fact that Λ_+^E is Lagrangian implies the symmetry relation $\langle \dot{X}, P_\psi \rangle = \langle \dot{P}, X_\psi \rangle$. For $H = \frac{|p|}{\rho(x)}$, taking the limit $t \to 0$ in Hamilton equations gives $\varphi \langle \nabla\rho, \omega^\perp(\psi) \rangle = 0$, which shows that ρ is radially symmetric.

4 Eikonal coordinates and generating families in the case $m = 1$

Let $\iota : L \to T^*\mathbf{R}_x^d$ be a smooth immersed Lagrangian manifold, the 1-form $p\,dx$ is closed on L, and so locally $p\,dx = dS$. If $p\,dx \neq 0$ then S can be chosen as a (local) coordinate on L. Following [7], [10] we say that S is the *eikonal* or the *action* on L which can be completed to a system of *eikonal coordinates* on L.

Let $L = \Lambda_0 \subset T^*\mathbf{R}_x^n$ the Bessel cylinder (3.1). Since $p\,dx = d\varphi$ on Λ_0 we get that (φ, ψ) are eikonal coordinates.

Let $L = \Lambda \subset T^*\mathbf{R}_{x,t}^{n+1}$ be the integral manifold in the extended phase space. If $m = 1$ then according to Euler identity

$$\dot{S} = p\dot{x} - H = p\,\partial_p H - H = 0$$

along the trajectories and (φ, ψ, t) are eikonal coordinates on Λ. Let

$$\Phi : \mathbf{R}^N \times \mathbf{R}_x^d \to \mathbf{R}, \quad \theta \in \mathbf{R}^N$$

be a smooth function such that the $N \times (d + N)$ matrix $\big(\partial_{\theta\theta}^2 \Phi, \partial_{\theta x}^2 \Phi\big)$ has rank N on the critical set

$$C_\Phi = \big\{(\theta, x) \in \mathbf{R}^d \times \mathbf{R}^N : \partial_\theta \Phi(\theta, x) = 0\big\}. \tag{4.1}$$

In other words, C_Φ defined by the equations (4.1) is a smooth submanifold in the sense of the implicit function theorem. Then

$$\iota_\Phi : C_\Phi \to T^*\mathbf{R}^d_x, \quad (\theta, x) \mapsto \big(x, \partial_x \Phi(\theta, x)\big) \tag{4.2}$$

is a Lagrangian (i. e. $\iota_\Phi^* dp \wedge dx = 0$) immersion, $L = \iota_\Phi(C_\Phi)$ is an immersed Lagrangian submanifold, and Φ is called a *generating family* or *phase function* of L — see e. g. [3], [13].

It is standard to show that the number N of θ-variables can be reduced to $N \le d$, and the minimal possible N is equal to the co-rank of the projection $L \to \mathbf{R}^d_x$.

If the system (4.1) is degenerate in the sense of the implicit function theorem then $L = \iota_\Phi(C_\Phi)$ is a singular (isotropic) submanifold. In particular its dimension can be less than d. See e. g. [6] for a general discussion.

Let $d = n$ and $L = \Lambda_0 \subset T^*\mathbf{R}^n_x$ be Bessel cylinder (3.1). Following [8], it is convenient to use among the θ-parameters a Lagrange multiplier λ and get for Λ_0 the generating family

$$\begin{aligned} &\Phi_0(x, \theta) = \varphi + \lambda\langle \omega(\psi), x - \varphi\omega(\psi)\rangle = (1-\lambda)\varphi + \lambda\langle \omega(\psi), x\rangle, \\ &\theta = (\lambda, \varphi, \psi) \in \mathbf{R}^{n+1}. \end{aligned}$$

Let now

$$p = P(\varphi, \psi, t), \quad x = X(\varphi, \psi, t)$$

be the Hamiltonian trajectory with an initial condition on the Bessel cylinder (3.1):

$$P(\varphi, \psi, 0) = \omega(\psi), \quad X(\varphi, \psi, 0) = \varphi\omega(\psi)$$

We apply then (4.2) to $d = n+1$ and $L = \Lambda$ being the flow out of Λ_0 by v_H in $T^*\mathbf{R}^{n+1}$.

Proposition 4.1 *Let $H(x, p)$ be positively homogeneous of degree 1 with respect to p. Then*

$$\Phi(\theta, x, t) = \varphi + \lambda\langle P(\varphi, \psi, t), x - X(\varphi, \psi, t)\rangle, \ \theta = (\lambda, \varphi, \psi) \in \mathbf{R}^{n+1}$$

is a generating family for $\Lambda \subset T^*\mathbf{R}^{n+1}_{x,t}$ at the points satisfying the inequality $\det(P, P_\psi) \ne 0$, which holds at least for small t (because $\det(P, \partial_\psi P) = 1$ for $t = 0$). In particular, Φ satisfies the initial condition $\Phi(\theta, 0, x) = \Phi_0(\theta, x)$.

Note that Φ is the 1-jet on Λ of the solution of the Hamilton-Jacobi equation

$$\partial_t \Psi + H(x, \partial_x \Psi) = 0, \quad \Psi|_{t=0} = \langle x, \omega(\psi)\rangle$$

defining v_h in (1.1), namely $\partial_t\Phi + H(x, \partial_x\Phi) = 0$ after the substitution $\lambda = 1$, $x = X(\varphi, \psi, t)$.

Let $z_0 \in \Lambda_0$ be a non-glancing point. It means that z_0 is not a critical point of the restriction $H|_{\Lambda_0}$.

Proposition 4.2 *If $H(x, p)$ is positively homogeneous of degree 1 with respect to p and $E = H(z_0)$ then*

$$\Phi_+^E(\theta_+, x) = \Phi(\lambda, \varphi, \psi, x, t) + Et,\ \ \theta_+ = (\lambda, \varphi, \psi, t) \in \mathbf{R}^{n+2}$$

is a generating family for $\Lambda_+^E \subset T^*\mathbf{R}_x^n$ at the points that are close to the Hamiltonian trajectory starting at z_0 and satisfy the inequality $\det(P, P_\psi) \neq 0$*, which holds at least for small t.*

5 Invariant density using the eikonal coordinates

Let $y = (y_1, \cdots, y_d)$ be some local coordinates on C_Φ extended locally to smooth functions on $\mathbf{R}^d \times \mathbf{R}^N$. Then the non vanishing real function

$$F[\Phi, dy] = \frac{dy \wedge d(\partial_\theta\Phi)}{dx \wedge d\theta} = \frac{dy_1 \wedge \cdots \wedge dy_d \wedge d(\partial_{\theta_1}\Phi) \wedge \cdots \wedge d(\partial_{\theta_N}\Phi)}{dx_1 \wedge \cdots \wedge dx_d \wedge d\theta_1 \wedge \cdots \wedge d\theta_N} \tag{5.1}$$

is well-defined near C_Φ as the quotient of two volume forms.

The restriction of this function to C_Φ is important for computations of the Maslov canonical operator on a Lagrangian submanifold $L = \iota(C_\Phi)$ via its generating family Φ — see e. g. [7], [10].

Proposition 5.1 *Let $H(x, p)$ be positively homogeneous of degree 1 with respect to p, Φ be the generating family for $\Lambda \subset T^*\mathbf{R}_{x,t}^{n+1}$ from Proposition 4.1, $\theta = (\lambda, \varphi, \psi)$, and $y = (\varphi, \psi, t)$. Then*

$$F[\Phi, dy]|_{C_\Phi} = \pm \det(P, \partial_\psi P).$$

6 Specific results near a glancing point

If Λ_+^E is an immersed Lagrangian submanifold we can compute the contribution of the critical points $t \in (0, +\infty)$ in the integral (1.2) using the Maslov

canonical operator on Λ_+^E (see [2]). At glancing points Λ_+^E can become singular and this procedure does not work.

Here we consider glancing points which are non-degenerate critical points of $H|_{\Lambda_0}$ in the case $n = 2$. We use (x, y, p_x, p_y) instead of (x_1, x_2, p_1, p_2), let

$$\pi_t : (p_x, p_y, E, x, y, t) \mapsto (x, y, t),$$
$$\pi_E : (p_x, p_y, E, x, y, t) \mapsto (x, y, E)$$

be the natural projections.

Theorem 6.1 *Let $z_0 \in \Lambda_0$ be a glancing point being a non-degenerate minimum or maximum of $H|_{\Lambda_0}$, $z \in \Lambda$ be a point of the trajectory starting at $z_0 \in \Lambda_0$ and the point $\pi_t(z)$ does not belong to the caustic of Λ. Then in a neighbourhood of the point z there exists a canonical map*

$$\big(p_x, p_y, E, x, y, t\big) \mapsto \big(p_\xi, p_\eta, \varepsilon, \xi, \eta, \tau\big),$$

i. e.

$$dp_\xi \wedge d\xi + dp_\eta \wedge d\eta - d\,\varepsilon \wedge d\tau = dp_x \wedge dx + dp_y \wedge dy - dE \wedge dt,$$

such that
- $\pi_E(p_\xi, p_\eta, \varepsilon, \xi, \eta, \tau) = (\xi, \eta, \varepsilon)$;
- ε *depends only on* E *and does not depend on the other coordinates* (p_x, p_y, x, y, t);
- *the Lagrangian submanifold* Λ *is defined by the following formulas*

$$S(\xi, \eta, \tau) = -(\tau^3/3 + \xi^2\tau), \quad p_\xi = \partial_\xi S = -2\xi\tau,$$
$$p_\eta = \partial_\eta S = 0, \quad \varepsilon = -\partial_\tau S = \tau^2 + \xi^2.$$

A proof of this theorem with the help of standard methods of Singularity Theory is going to be published later.

Let us consider the following simplest example. If

$$H = p_y, \quad S_0(x, y) = x^2 y + y^3/3, \quad z_0 = 0,$$
$$\Lambda_0 = \left\{p_x = \partial_x S_0 = 2xy,\ p_y = \partial_y S_0 = x^2 + y^2\right\}.$$

Then

$$S(x, y, t) = x^2(y - t) + (y - t)^3/3,$$
$$\Lambda = \{p_x = \partial_x S,\ p_y = \partial_y S,\ E = -\partial_t S\}.$$

The trajectory starting at z_0 is $p_x = p_y = x = 0$, $y = t$, let z be its point for $t = T$. Then the canonical transformation

$$p_\xi = p_x, \quad p_\eta = p_y - E, \quad \varepsilon = E,$$
$$\xi = x, \quad \eta = y - T, \quad \tau = t - y$$

gives $S = -x^2\tau - \tau^3/3$ according to Theorem 6.1.

Theorem 6.1 implies the following: for $z \in \Lambda$ as above, if $U_z \subset \Lambda$ is a sufficiently small neighbourhood of z, then for any smooth function $b : \Lambda \to \mathbf{R}$ vanishing outside of U_z, there exists smooth function $S, B : \mathbf{R}^3_{x,y,t} \to \mathbf{R}$ vanishing outside of $\pi_t(U_z)$ such that

$$\Lambda \cap U_z = \{p_x = \partial_x S,\ p_y = \partial_y S,\ E = -\partial_t S\} \cap U_z,$$
$$[\mathcal{K}_\Lambda b](x, y, t) = B(x, y, t) \exp\left\{\frac{iS(x, y, t)}{h}\right\}.$$

Therefore the integral (1.2) is rewritten in terms of the Maslov canonical operator on the same Λ but in the space $T^*\mathbf{R}^3_{x,y,E}$:

$$\int_0^{+\infty} [\mathcal{K}_\Lambda b](x, y, t) \exp\left\{\frac{iEt}{h}\right\} dt = \int_{-\infty}^{+\infty} B(x, y, t) \exp\left\{\frac{i(S(x, y, t) + Et)}{h}\right\} dt.$$

and we can express this value in the coordinates of Theorem 6.1 like in [9],[15].

7 The transition of Λ^E_+ and its phase

Now we describe what happens with the singular Lagrangian submanifold Λ^E_+ as $\sigma = E - E_0$ changes its sign. More precisely, let us consider a submanifold

$$\operatorname{loc} \Lambda^E_+ = \{z \in T^*\mathbf{R}^n : \exists\, t \geq 0, z \in \exp tv_H(\Lambda_0 \cap U_{z_0} \cap \Sigma_E)\} \subset \Lambda^E_+$$

where U_{z_0} is a sufficiently small neighbourhood of the initial point $z_0 \in \Lambda_0$ from Theorem 6.1.

Corollary 7.1 *In a neighbourhood of the point $\pi_E(z)$ there exist new local coordinates*

$$\big(\xi(x, y, E), \eta(x, y, E), \varepsilon(E)\big), \quad \varepsilon(E_0) = 0$$

such that the singular Lagrangian submanifold $\operatorname{loc} \Lambda^E_+$ is defined by the following formulas

$$S(\xi, \eta, \tau) = -(\tau^3/3 + \xi^2\tau), \quad p_\xi = \partial_\xi S = -2\xi\tau,$$
$$p_\eta = \partial_\eta S = 0, \quad \varepsilon = -\partial_\tau S = \tau^2 + \xi^2.$$

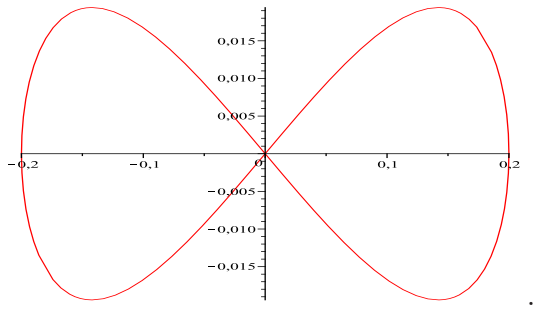

Figure 1: The Lagrangian curve "∞"

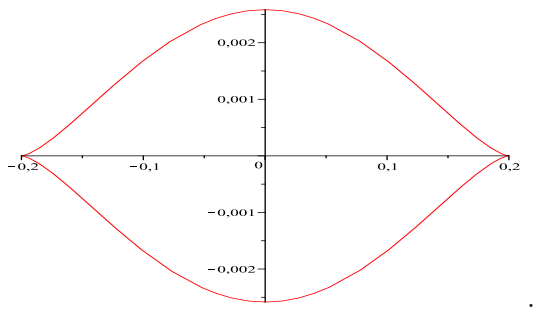

Figure 2: A section of the phase graph

The Lagrangian submanifold $\operatorname{loc}\Lambda_+^E$ is empty if $\varepsilon(E) < 0$. If $\varepsilon(E) > 0$ then $\operatorname{loc}\Lambda_+^E$ becomes the cylinder over the Lagrangian curve "∞" shown in Fig. 1. The graph of the phase in this case is the cylinder over the curve with two cusps shown in Fig. 2. If $\varepsilon(E) = 0$ then $\operatorname{loc}\Lambda_+^E$ degenerates into an interval of the Hamiltonian phase trajectory starting at z_0.

The above transition is realized, for example, if $H = (p_x^2 + p_y^2)/\rho(x, y)$, ρ has a non-degenerate maximum or minimum at the origin, and $f_h = J_0\big(\sqrt{(x-a)^2 + y^2}/h\big)$, $\quad a \neq 0$. In this case Λ_0 is the cylinder $p_x = \cos\phi$, $p_y = \sin\phi$, $x = a + \tau\cos\phi$, $y = \tau\sin\phi$. As the "source" $(a, 0)$ goes to $-\infty$ f_h becomes the simple wave $\exp\frac{ix}{h}$ and Λ_0 is the plane $p_x = 1$, $p_y = 0$. Here is the simplest example $f_h = \exp\frac{ix}{h}$, $\quad \varrho = 1 + x^2 + y^2$ where one can get all explicit formulas. Our first figure is the dependence between y (the horizontal axe) and p_y (the vertical axe) if $x = 0$. The second figure is the dependence between y (the horizontal axe) and the phase (the vertical axe). The first curve is the derivative of the second one.

8 Glancing hypersurfaces and Lagrangian intersections

We discuss here a possible application of non transverse Lagrangian intersections to problems of diffraction by an obstacle. Let M be a smooth manifold, since we are working locally, we will assume $M = \mathbf{R}^n$. Let F, G be two smooth hypersurfaces of T^*M, intersecting transversally at z. Recall [12, Definition 21.4.6] that F and G are said to be *glancing* at z iff the Hamilton foliation of $F = \{f = 0\}$ and $G = \{g = 0\}$ (locally near z) are simply tangent at z.

Stated otherwise, we have $f(z) = g(z) = \{f, g\}(z) = 0$ (Poisson bracket), but the second Poisson brackets $\{f, \{f, g\}\}(z), \{g, \{g, f\}\}(z)$ are non zero.

By the theorem of equivalence of glancing hypersurfaces of Melrose [12, Theorem 21.4.8] there are local symplectic coordinates (x, ξ) vanishing at z such that F, G are defined resp. by $x_1 = 0$ and $\xi_1^2 - x_1 - \xi_2 = 0$. Then $g = \xi_1^2 - x_1 - \xi_2 = 0$ will be the "normal form" of H in these coordinates. (We use the notation g for the normal form of H, or $H - E$).

We apply this theorem to G being the energy surface $H = E$ (i. e. $g = 0$) and F an auxiliary hypersurface intersecting G transversally at a glancing point z. We want to find some germs of Lagrangian manifolds Λ such that Λ is transverse to F at z but (Λ, G) has glancing intersection at z. This means that $T_z\Lambda \cap (T_zF)^\sigma = \{0\}$ and $\mathbf{R}v_H(z) = (T_zG)^\sigma \subset T_z\Lambda$, where superscript σ denotes symplectic orthogonal. Assume $n = 2$ for simplicity.

Define Λ locally near z by $f_1 = f_2 = 0$, with $\{f_i, f_j\} = 0$. Consider the symmetric matrix

$$A_z = A_z(\Lambda, G) = \begin{pmatrix} \{f_2, \{f_2, g\}\} & -\{f_1, \{f_2, g\}\} \\ -\{f_2, \{f_1, g\}\} & \{f_1, \{f_1, g\}\} \end{pmatrix}(z)$$

and the vector

$$B_z = B_z(\Lambda, G) = \begin{pmatrix} \{g, \{g, f_1\}\} \\ \{g, \{g, f_2\}\} \end{pmatrix}(z).$$

Let z be a *glancing point* for the pair (Λ, G) i. e.

$$g(z) = f_1(z) = f_2(z) = 0, \ \{g, f_1\}(z) = \{g, f_2\}(z) = 0.$$

We distinguish the following 10 possibilities for the 2-jets of $(\Lambda, G)_z$ (not covering the entire classification of [18]):

1. $\det A_z > 0, \ B_z \neq 0$;
2. $\det A_z > 0, \ B_z = 0$;
3. $\det A_z < 0, \ {}^tB_zA_zB_z \neq 0$;

4. $\det A_z < 0,\ {}^tB_zA_zB_z = 0,\ B_z \neq 0;$

5. $\det A_z < 0,\ B_z = 0;$

6. $\det A_z = 0,\ {}^tB_zA_zB_z \neq 0;$

7. $\det A_z = {}^tB_zA_zB_z = 0,\ A_z \neq 0,\ B_z \neq 0;$

8. $\det A_z = \ B_z = 0,\ A_z \neq 0;$

9. $A_z = 0,\ B_z \neq 0;$

10. $A_z = B_z = 0.$

We know that a general Lagrangian manifold can be parametrized in the mixed representation, so when $n = 2$ by one of the following cases

$$\begin{aligned}
(I)\ \Lambda &= \{p = \partial_x\phi\}\,,\\
(II)\ \Lambda &= \{x = -\partial_p\phi\}\,,\\
(III)\ \Lambda &= \{x_1 = -\partial_{p_1}\phi, p_2 = \partial_{x_2}\phi\}\,,\\
(IV)\ \Lambda &= \{p_1 = \partial_{x_1}\phi, x_2 = -\partial_{p_2}\phi\}\,.
\end{aligned}$$

We have

Proposition 8.1 *Assume Λ as above to be parametrized by a quadratic phase $\phi = \phi_0$. In Cases (I), (II), (III) ϕ_0 are one-parameter families taking values:*
(I) $\phi_0(x) = \frac{1}{2}(ax_1^2 - 2x_1x_2)$;
(II) $\phi_0(\xi) = \frac{1}{2}(2\xi_1\xi_2 + c\xi_2^2)$;
(III) $\phi_0(x_2, \xi_1) = \frac{1}{2}b(\xi_1 + x_2)^2$
respectively, where $a, b, c \neq 0$. The manifold Λ is transverse to F at $z = 0$, and the corresponding matrices $A_z(\Lambda, G), B_z(\Lambda, G)$ above are then given by:
(I) $A_z = 2\begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix}$, $B_z = 2\begin{pmatrix} -a \\ 1 \end{pmatrix}$;
(II) $A_z = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B_z = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$;
(III) $A_z = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B_z = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
At last, Case (IV) does not occur.

References

[1] A. Anikin, S. Dobrokhotov, V. Nazaikinskii, M. Rouleux. The Maslov canonical operator on a pair of Lagrangian manifolds and asymptotic solutions of stationary equations with localized right-hand sides. *Doklady Math.*, **96**(1), pp. 406–410, 2017.

[2] A. Anikin, S. Dobrokhotov, V. Nazaikinskii, M. Rouleux. Lagrangian manifolds and the construction of asymptotics for (pseudo)differential equations with localized right-hand sides. *Theor. Math. Phys.*, **214**(1), pp. 1–23, 2023.

[3] V. Arnold, S. Gusein-Zade, A. Varchenko. *Singularities of Differentiable Maps I. The classification of critical points, caustics and wave fronts.* Translated by Ian Porteous and Mark Reynolds. Birkhäuser, Boston, MA. 1985.

[4] I. Bogaevskii, S. Dobrokhotov, A. Tolchennikov, Arnold's Lagrangian singularity in the asymptotics of solutions of the model two-dimensional Helmholtz equation with localized right-hand side. *Theor. Math. Phys.*, 2024, in press.

[5] F. Castella. The radiation condition at infinity for the high-frequency Helmholtz equation with source term: A wave-packet approach. *J. Funct. Analysis*, **223**(1), pp. 204–257, 2005.

[6] Y. Colin de Verdière. Singular Lagrangian manifolds and semiclassical analysis. *Duke Math. J.*, **116**(2), pp. 263–298, 2003.

[7] S. Dobrokhotov, G. Makrakis, V. Nazaikinskii, T. Tudorovskii. New formulas for Maslov's canonical operator in a neighborhood of focal points and caustics in 2D semiclassical asymptotics. *Theor. Math. Phys.*, **177**(2), pp. 1679–1605, 2013.

[8] S. Dobrokhotov, V. Nazaikinskii. *Private communication.*

[9] S. Dobrokhotov, V. Nazaikinskii, Lagrangian manifolds and efficient short-wave asymptotics in a neighborhood of a caustic cus. *Math. Notes*, **108**(3), pp. 318–338, 2020.

[10] S. Dobrokhotov, V. Nazaikinskii, A. Shafarevich. New integral representations of Maslov canonical operator in singular charts. *Izv. Math.*, **81**(2), pp. 286–328, 2017.

[11] S. Dobrokhotov, V. Nazaikinskii, A. Tsvetkova. Asymptotics of the localized Bessel beams and Lagrangian manifolds. *J. Communications Technology and Electronics*, **26**(6), pp. 625–638, 2023.

[12] L. Hörmander. *The Analysis of Linear Partial Differential Operators I–IV.* Springer.

[13] V. Ivrii. *Microlocal Analysis and Precise Spectral Asymptotics.* Springer-Verlag, Berlin, 1998.

[14] M. Karasev, V. Maslov. Asymptotic and geometric quantization. *Russ. Math. Surveys*, **39**, pp. 133–205, 1984.

[15] V. Nazaikinskii, F. Tolchennikov, Constructive implementation of semiclassical asymptotic formulas in a neighborhood of a generic caustic cusp. *Russ. J. Math. Phys.*, **29**, pp. 558–567, 2022.

[16] R. B. Melrose, G. A. Uhlmann. Lagrangian intersection and the Cauchy problem. *Comm. Pure Appl. Math.*, **32**(4), pp. 483–519, 1979.

[17] M. Rouleux. Semiclassical Green functions and Lagrangian intersection. Applications to the propagation of Bessel beams in non-homogeneous media. Submitted.

[18] V. Zakalyukin, O. Myashnichenko. Lagrange singularities under symplectic reduction. *Funct. Anal. Appl.*, **32**(1), pp. 1–9, 1998.