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Lagrangian intersections and glancing points: typical transitions of phase in semiclassical approximations

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Abstract

Given a semiclassical distribution f_h microlocalized on a Lagrangian manifold Λ_0 , $H \in C^\infty(T^*\mathbf{R}^n)$, and $H = E$ a regular energy surface, we find asymptotic solutions of the PDE $(H(x, \hat{p}) - E)u_h(x, E) = f_h(x)$ in terms of the Maslov canonical operator, when the Hamilton vector field v_H fails to be transverse to Λ_0 at some points.

1 Introduction

Given $H \in C^\infty(T^*\mathbf{R}^n)$, we discuss the semiclassical approximation to the solution of the equation

$$(\widehat{H} - E)u_h(x, y, E) = f_h(x), \quad \widehat{H} = H(x, \widehat{p}), \quad \widehat{p} = -ih\partial_x$$

with a right hand side f_h microlocalized on a smooth Lagrangian submanifold Λ_0 :

$$f_h(x) = [\mathcal{K}_{\Lambda_0}a](x), \quad a : \Lambda_0 \rightarrow \mathbf{R}$$

where a is an amplitude on Λ_0 , and \mathcal{K}_{Λ_0} denotes the Maslov canonical operator. The solution of this problem is formally given by:

$$u_h(x, E) = \frac{i}{h} \int_0^{+\infty} v_h(x, t) \exp(iEt/h) dt$$

via the solution of the Cauchy problem for the Schrödinger equation

$$ih \partial_t v_h = \widehat{H} v_h, \quad v_h(x, 0) = f_h(x). \quad (1.1)$$

We formulate first a set of standard hypotheses, which are sufficient in the framework of formal asymptotics we are considering here.

1. E is non critical for H , i.e. $dH \neq 0$ on $\Sigma_E = \{H = E\}$.
2. E is non trapping energy level for H , in the sense of scattering theory. This allows to cut-off the integral defining $u_h(x, E)$ near $t = +\infty$ as in (2.1);
3. “non-return condition” [16] which is not necessary if we content ourselves to evaluate microlocally $u_h(x, E)$ outside Λ_0 . Otherwise this is a difficult problem involving Sommerfeld radiation condition, see e. g. [5];
4. $\Lambda_0 \cap \Sigma_E$ is compact (isotropic) submanifold, which is fulfilled if H is of elliptic type.

Let $\mathbf{R}_{x,t}^{n+1} = \{(x, t)\}$ be the space-time and $T^*\mathbf{R}_{x,t}^{n+1}$ be its cotangent bundle with the coordinates (x, t, p, E) such that $p dx - E dt$ is the canonical 1-form. To define the semiclassical approximation for v_h one needs:

- the Lagrangian submanifold of $T^*\mathbf{R}^{n+1}$

$$\Lambda = \{(x, p) = g^t(z), t > 0, E = H(z) : z \in \Lambda_0\}$$

where $g^t = \exp tv_H$ is the phase flow of the Hamiltonian vector field v_H generated by H

$$\dot{x} = \partial_p H(x, p), \quad \dot{p} = -\partial_x H(x, p).$$

- the amplitude $b : \Lambda \rightarrow \mathbf{R}$ being the solution to the transport equation with an initial condition $b|_{\Lambda_0} = a$. Then $v_h(x)$ is given by

$$v_h(x, t) = [\mathcal{K}_\Lambda b](x, t) + \mathcal{O}(h), \quad b : \Lambda \rightarrow \mathbf{R}.$$

So we can expect the following semiclassical approximation

$$u_h(x, E) = \frac{i}{h} \int_0^{+\infty} [\mathcal{K}_\Lambda b](x, t) \exp(iEt/h) dt + \mathcal{O}(h). \quad (1.2)$$

The relevant contributions to this integral come from $t = 0$, $t = +\infty$ and the critical points $t \in (0, +\infty)$. See also [14] for higher order approximation.

The Lagrangian submanifold of $T^*\mathbf{R}^n$

$$\Lambda_+^E = \{z \in T^*\mathbf{R}^n : \exists t \geq 0, z \in \exp tv_H(\Lambda_0 \cap \Sigma_E)\}$$

gives the contribution of the critical points $t \in (0, +\infty)$. The following two cases are possible.

1) The Lagrangian submanifold Λ_0 is transversal to Σ_E — it means that E is not a critical value for the restriction H_{Λ_0} . Then v_H is not tangent to Λ_0 and consequently to the submanifold $\Lambda_0 \cap \Sigma_E$. Therefore Λ_+^E is an immersed Lagrangian submanifold with boundary and we say that (Λ_0, Λ_+^E) is a *(Lagrangian) intersecting pair*.

2) The Lagrangian submanifold Λ_0 is tangent to Σ_E at a point z called *glancing* — it means that z is a critical point of the restriction H_{Λ_0} with the critical value E . Then v_H is tangent to Λ_0 and Λ_+^E may have singularities, see e. g. [6] for applications to the semiclassical context.

Remark: A model problem with the point $t = +\infty$ giving a relevant contribution into the integral (1.2) is investigated in [4].

2 Basic example of intersecting pair

A simple example of a Lagrangian intersecting pair is given by $\tilde{\Lambda}_0 = \{x = 0\}$ (vertical fiber at 0) and

$$\tilde{\Lambda}_+^0 = \{(x, p) \in T^*\mathbf{R}^n : x' = 0, p_n = 0, x_n \geq 0\}, \quad x' = (x_1, \dots, x_{n-1}), \quad p' = (p_1, \dots, p_{n-1}),$$

the flow-out of $\tilde{\Lambda}_0$ by the Hamilton vector field with Hamiltonian p_n . In this case $\hat{H} = -ih\partial_{x_n}$,

$$f_h(x) = [\mathcal{K}_{\tilde{\Lambda}_0} a](x) = \int_{\tilde{\Lambda}_0} e^{ixp/h} a(p) dp = f_1\left(\frac{x}{h}\right)$$

where f_1 decreases rapidly at infinity,

$$v_h(x, t) = f_h(x', x_n - t)$$

and

$$\begin{aligned} u_h(x, 0) &= \frac{i}{h} \int_0^{+\infty} f_h(x', x_n - t) dt = \\ &= \frac{i}{h} \int_0^{+\infty} \Theta_{t_0}(t) f_1\left(\frac{x'}{h}, \frac{x_n - t}{h}\right) dt + \mathcal{O}(h^\infty) \end{aligned} \tag{2.1}$$

for all $x_n \leq t_0/2$, here Θ_{t_0} is a cut-off function equal to 1 on $[0, t_0]$ and 0 near $+\infty$.

This pair of Lagrangian manifolds is actually the paradigm of intersecting pairs, i. e. in some local canonical charts $\Lambda_0 = \tilde{\Lambda}_0$ and $H(x, p) = E + p_n$. In particular (Λ_0, Λ_+^E) are mapped onto $(\tilde{\Lambda}_0, \tilde{\Lambda}_+^0)$ by a canonical transformation sending $\partial\Lambda_+^E = \Lambda_0 \cap \Lambda_+^E$ onto $\partial\tilde{\Lambda}_+^0 = \tilde{\Lambda}_0 \cap \tilde{\Lambda}_+^0$.

3 Main motivation: Helmholtz equation and Bessel cylinder

Most intrinsic formulae are available in the case of the physically relevant case of a Hamiltonian positively homogeneous with respect to p , e. g. the Helmholtz operator. This provides global eikonal coordinates and avoids the microlocal reduction to $H = E + \hat{p}_n$.

Taking into account glancing intersection amounts to “correct” locally the formulae giving the phase function and the half-density in new local coordinates, which are obtained using a normal form.

Let us consider the *Bessel cylinder* $\Lambda_0 \subset T^*\mathbf{R}^n$

$$\Lambda_0 = \{x = \varphi\omega(\psi), p = \omega(\psi) : \varphi \in \mathbf{R}\} \quad (3.1)$$

where $\omega(\psi)$ is a vector on the unit sphere in \mathbf{R}^n .

Our main motivation is the study of Bessel beams, i. e. a wave whose amplitude is described by a Bessel function of the first kind. Assume for instance we are given a non-linear Helmholtz equation on \mathbf{R}^2 of the type

$$-h^2\Delta u - u = F(\varepsilon u), \quad F(0) = 0, \quad F'(0) = 1$$

where F is a smooth function and ε is a small parameter. We expand $u = u_0 + \varepsilon u_1 + \dots$, and find at zero order in ε the equation $(-h^2\Delta - 1)u_0 = 0$. Its radially symmetric solution is given by $u_0 = f_h(x)$ where

$$f_h(x) = (2\pi/h)^{1/2}J_0(|x|/h),$$

and is microlocalized on the Bessel cylinder $\Lambda_0 \subset T^*\mathbf{R}^2$ defined by (3.1) for $n = 2$.

At first order in ε we get the Helmholtz equation

$$(-h^2\Delta - 1)u_1 = f_h, \quad H = p^2$$

where $u_1 = u(x, 1)$ in our previous notation, and

$$u_h(x, 1) = -J_1(|x|/h)(|x|/2h)$$

is its radially symmetric solution. We observe that all points of Λ_0 turn to be glancing because $\Lambda_0 \subset \Sigma_1$ and $H_{\Lambda_0} \equiv 1$.

For the Helmholtz equation with variable coefficients, we have the following:

Proposition 3.1 *Let Λ_0 be the n -dimensional Bessel cylinder (3.1) and $H \in C^\infty(T^*\mathbf{R}^n)$ be homogeneous of degree m with respect to p . Then $z = (x, p) \in \Lambda_0$ is a glancing point at energy E iff*

$$\begin{aligned} \partial_p H(z) + \varphi \partial_x H(z) &= mH\omega(\psi), \\ \langle -\partial_x H(z), \omega(\psi) \rangle &= 0, \quad H(z) = E \end{aligned} \quad (3.2)$$

Proof: We complete $\omega(\psi)$ in \mathbf{S}^{n-1} into a (direct) orthonormal basis $\omega^\perp(\psi) = (\omega_1(\psi), \dots, \omega_{n-1}(\psi))$ of \mathbf{R}^n , and denote by $\omega^\perp(\psi)\delta\psi = \omega_1(\psi)\delta\psi_1 + \dots + \omega_{n-1}(\psi)\delta\psi_{n-1}$ a section of $T\mathbf{S}^{n-1}$, $\delta\psi_j \in \mathbf{R}$. The tangent space $T_z\Lambda_0$ has the parametric equations

$$\delta X = \omega(\psi)\delta\varphi + \varphi\omega^\perp(\psi)\delta\psi, \quad \delta P = \omega^\perp(\psi)\delta\psi, \quad (\delta\varphi, \delta\psi) \in \mathbf{R}^n$$

so $v_H \in T_z\Lambda_0$ iff there exist $(\delta\varphi, \delta\psi)$ such that

$$\partial_p H = \omega(\psi)\delta\varphi + \varphi\omega^\perp(\psi)\delta\psi - \partial_x H = \omega^\perp(\psi)\delta\psi$$

Taking scalar products with $\omega(\psi), \omega^\perp(\psi)$, and using that $(\omega(\psi), \omega^\perp(\psi))$ form a basis of \mathbf{R}^n , readily shows that relations

$$\partial_p H + \varphi \partial_x H = \langle \partial_p H, P(\psi) \rangle \omega(\psi), \quad \langle -\partial_x H, \omega(\psi) \rangle = 0 \quad (3.3)$$

are necessary and sufficient for $v_H \in T_z\Lambda_0$.

We set $\mathcal{H}(\varphi, \psi) = H|_{\Lambda_0}$. Then

$$\nabla \mathcal{H}(\varphi, \psi) = (\langle \partial_x H, \omega^\perp(\psi) \rangle, \varphi \langle \partial_x H, \omega^\perp(\psi) \rangle + \langle \partial_p H, \omega^\perp(\psi) \rangle)$$

so (3.3) readily gives $\nabla \mathcal{H}(\varphi, \psi) = 0$. Therefore $z = (x, p) \in \Lambda_0$ is a glancing point.

Now, if H is positively homogeneous of degree m with respect to p , using Euler identity, we get $\delta\varphi = \langle \partial_p H, P(\psi) \rangle = mH$, and (3.2) holds iff for $v_H \in T_z\Lambda_+^E \cap T_z\Lambda_0$ when $H = E$. ♣

In particular when $H(z) = \frac{|p|^m}{\rho(x)}$ is a conformal metric, with ρ a smooth positive function on \mathbf{R}^n , z is a glancing point iff

$$\begin{aligned} \text{either : } & \varphi \neq 0 \text{ and } \nabla\rho = 0, \\ \text{or : } & \varphi = 0 \text{ and } \langle \nabla\rho(0), \omega(\psi) \rangle = 0. \end{aligned}$$

Example: Let $n = 2$, $m = 1$, $H(z) = \frac{|p|}{\rho(x)}$, with $\rho(x) = \frac{1}{2}(1 + (x - x_0)^2)$. If $x_0 = \varphi\omega(\psi) \neq 0$, we have $\rho^4(x_0) \det \nabla^2(H|_{\Lambda_0}) = \varphi^2$, $\rho^2(x_0) \text{Tr} \nabla^2(H|_{\Lambda_0}) = -(1 + \varphi^2)$. Critical energy is given by $E = H(z) = \frac{|p|}{\rho(x)}$, i. e. $E_0 = 1/\rho(x_0)$.

Remark: Let $n = 2$, $H(z)$ positively homogeneous of degree $m = 1$. Away from glancing points, Λ_+^E is parametrized near $t = 0$ by (ψ, θ^E) , where $\theta^E = \theta^E(t, \varphi)$ and $\theta^E : \mathbf{R}^2 \rightarrow \mathbf{R}$ is a submersion, such that

$$H(X(t, \varphi, \psi), P(t, \varphi, \psi)) = E$$

When $\rho(x) = \rho(|x|)$, we have $\varphi = \varphi(E)$ on Λ_+^E , so that we can take $\theta^E(t, \varphi) = Et$ (or equivalently $\theta^E(t, \varphi) = \varphi(E) + Et$, which is the eikonal on Λ_+^E), and Λ_+^E is parametrized by (t, ψ) . Conversely, if Λ_+^E is parametrized by (t, ψ) , the fact that Λ_+^E is Lagrangian implies the symmetry relation $\langle \dot{X}, P_\psi \rangle = \langle \dot{P}, X_\psi \rangle$. For $H = \frac{|p|}{\rho(x)}$, taking the limit $t \rightarrow 0$ in Hamilton equations gives $\varphi \langle \nabla \rho, \omega^\perp(\psi) \rangle = 0$, which shows that ρ is radially symmetric.

4 Eikonal coordinates and generating families in the case $m = 1$

Let $\iota : L \rightarrow T^*\mathbf{R}_x^d$ be a smooth immersed Lagrangian manifold, the 1-form $p dx$ is closed on L , and so locally $p dx = dS$. If $p dx \neq 0$ then S can be chosen as a (local) coordinate on L . Following [7], [10] we say that S is the *eikonal* or the *action* on L which can be completed to a system of *eikonal coordinates* on L .

Let $L = \Lambda_0 \subset T^*\mathbf{R}_x^n$ the Bessel cylinder (3.1). Since $p dx = d\varphi$ on Λ_0 we get that (φ, ψ) are eikonal coordinates.

Let $L = \Lambda \subset T^*\mathbf{R}_{x,t}^{n+1}$ be the integral manifold in the extended phase space. If $m = 1$ then according to Euler identity

$$\dot{S} = p\dot{x} - H = p \partial_p H - H = 0$$

along the trajectories and (φ, ψ, t) are eikonal coordinates on Λ . Let

$$\Phi : \mathbf{R}^N \times \mathbf{R}_x^d \rightarrow \mathbf{R}, \quad \theta \in \mathbf{R}^N$$

be a smooth function such that the $N \times (d + N)$ matrix $(\partial_{\theta\theta}^2 \Phi, \partial_{\theta x}^2 \Phi)$ has rank N on the critical set

$$C_\Phi = \{(\theta, x) \in \mathbf{R}^d \times \mathbf{R}^N : \partial_\theta \Phi(\theta, x) = 0\}. \quad (4.1)$$

In other words, C_Φ defined by the equations (4.1) is a smooth submanifold in the sense of the implicit function theorem. Then

$$\iota_\Phi : C_\Phi \rightarrow T^*\mathbf{R}_x^d, \quad (\theta, x) \mapsto (x, \partial_x \Phi(\theta, x)) \quad (4.2)$$

is a Lagrangian (i. e. $\iota_\Phi^* dp \wedge dx = 0$) immersion, $L = \iota_\Phi(C_\Phi)$ is an immersed Lagrangian submanifold, and Φ is called a *generating family* or *phase function* of L — see e. g. [3], [13].

It is standard to show that the number N of θ -variables can be reduced to $N \leq d$, and the minimal possible N is equal to the co-rank of the projection $L \rightarrow \mathbf{R}_x^d$.

If the system (4.1) is degenerate in the sense of the implicit function theorem then $L = \iota_\Phi(C_\Phi)$ is a singular (isotropic) submanifold. In particular its dimension can be less than d . See e. g. [6] for a general discussion.

Let $d = n$ and $L = \Lambda_0 \subset T^*\mathbf{R}_x^n$ be Bessel cylinder (3.1). Following [8], it is convenient to use among the θ -parameters a Lagrange multiplier λ and get for Λ_0 the generating family

$$\begin{aligned} \Phi_0(x, \theta) &= \varphi + \lambda \langle \omega(\psi), x - \varphi \omega(\psi) \rangle = (1 - \lambda)\varphi + \lambda \langle \omega(\psi), x \rangle, \\ \theta &= (\lambda, \varphi, \psi) \in \mathbf{R}^{n+1}. \end{aligned}$$

Let now

$$p = P(\varphi, \psi, t), \quad x = X(\varphi, \psi, t)$$

be the Hamiltonian trajectory with an initial condition on the Bessel cylinder (3.1):

$$P(\varphi, \psi, 0) = \omega(\psi), \quad X(\varphi, \psi, 0) = \varphi \omega(\psi)$$

We apply then (4.2) to $d = n + 1$ and $L = \Lambda$ being the flow out of Λ_0 by v_H in $T^*\mathbf{R}^{n+1}$.

Proposition 4.1 *Let $H(x, p)$ be positively homogeneous of degree 1 with respect to p . Then*

$$\Phi(\theta, x, t) = \varphi + \lambda \langle P(\varphi, \psi, t), x - X(\varphi, \psi, t) \rangle, \quad \theta = (\lambda, \varphi, \psi) \in \mathbf{R}^{n+1}$$

is a generating family for $\Lambda \subset T^\mathbf{R}_{x,t}^{n+1}$ at the points satisfying the inequality $\det(P, P_\psi) \neq 0$, which holds at least for small t (because $\det(P, \partial_\psi P) = 1$ for $t = 0$). In particular, Φ satisfies the initial condition $\Phi(\theta, 0, x) = \Phi_0(\theta, x)$.*

Note that Φ is the 1-jet on Λ of the solution of the Hamilton-Jacobi equation

$$\partial_t \Psi + H(x, \partial_x \Psi) = 0, \quad \Psi|_{t=0} = \langle x, \omega(\psi) \rangle$$

defining v_h in (1.1), namely $\partial_t \Phi + H(x, \partial_x \Phi) = 0$ after the substitution $\lambda = 1$, $x = X(\varphi, \psi, t)$.

Let $z_0 \in \Lambda_0$ be a non-glancing point. It means that z_0 is not a critical point of the restriction $H|_{\Lambda_0}$.

Proposition 4.2 *If $H(x, p)$ is positively homogeneous of degree 1 with respect to p and $E = H(z_0)$ then*

$$\Phi_+^E(\theta_+, x) = \Phi(\lambda, \varphi, \psi, x, t) + Et, \quad \theta_+ = (\lambda, \varphi, \psi, t) \in \mathbf{R}^{n+2}$$

is a generating family for $\Lambda_+^E \subset T^*\mathbf{R}_x^n$ at the points that are close to the Hamiltonian trajectory starting at z_0 and satisfy the inequality $\det(P, P_\psi) \neq 0$, which holds at least for small t .

5 Invariant density using the eikonal coordinates

Let $y = (y_1, \dots, y_d)$ be some local coordinates on C_Φ extended locally to smooth functions on $\mathbf{R}^d \times \mathbf{R}^N$. Then the non vanishing real function

$$F[\Phi, dy] = \frac{dy \wedge d(\partial_\theta \Phi)}{dx \wedge d\theta} = \frac{dy_1 \wedge \dots \wedge dy_d \wedge d(\partial_{\theta_1} \Phi) \wedge \dots \wedge d(\partial_{\theta_N} \Phi)}{dx_1 \wedge \dots \wedge dx_d \wedge d\theta_1 \wedge \dots \wedge d\theta_N} \quad (5.1)$$

is well-defined near C_Φ as the quotient of two volume forms.

The restriction of this function to C_Φ is important for computations of the Maslov canonical operator on a Lagrangian submanifold $L = \iota(C_\Phi)$ via its generating family Φ — see e. g. [7], [10].

Proposition 5.1 *Let $H(x, p)$ be positively homogeneous of degree 1 with respect to p , Φ be the generating family for $\Lambda \subset T^*\mathbf{R}_{x,t}^{n+1}$ from Proposition 4.1, $\theta = (\lambda, \varphi, \psi)$, and $y = (\varphi, \psi, t)$. Then*

$$F[\Phi, dy]|_{C_\Phi} = \pm \det(P, \partial_\psi P).$$

6 Specific results near a glancing point

If Λ_+^E is an immersed Lagrangian submanifold we can compute the contribution of the critical points $t \in (0, +\infty)$ in the integral (1.2) using the Maslov

canonical operator on Λ_+^E (see [2]). At glancing points Λ_+^E can become singular and this procedure does not work.

Here we consider glancing points which are non-degenerate critical points of $H|_{\Lambda_0}$ in the case $n = 2$. We use (x, y, p_x, p_y) instead of (x_1, x_2, p_1, p_2) , let

$$\begin{aligned}\pi_t &: (p_x, p_y, E, x, y, t) \mapsto (x, y, t), \\ \pi_E &: (p_x, p_y, E, x, y, t) \mapsto (x, y, E)\end{aligned}$$

be the natural projections.

Theorem 6.1 *Let $z_0 \in \Lambda_0$ be a glancing point being a non-degenerate minimum or maximum of $H|_{\Lambda_0}$, $z \in \Lambda$ be a point of the trajectory starting at $z_0 \in \Lambda_0$ and the point $\pi_t(z)$ does not belong to the caustic of Λ . Then in a neighbourhood of the point z there exists a canonical map*

$$(p_x, p_y, E, x, y, t) \mapsto (p_\xi, p_\eta, \varepsilon, \xi, \eta, \tau),$$

i. e.

$$dp_\xi \wedge d\xi + dp_\eta \wedge d\eta - d\varepsilon \wedge d\tau = dp_x \wedge dx + dp_y \wedge dy - dE \wedge dt,$$

such that

- $\pi_E(p_\xi, p_\eta, \varepsilon, \xi, \eta, \tau) = (\xi, \eta, \varepsilon)$;
- ε depends only on E and does not depend on the other coordinates (p_x, p_y, x, y, t) ;
- the Lagrangian submanifold Λ is defined by the following formulas

$$\begin{aligned}S(\xi, \eta, \tau) &= -(\tau^3/3 + \xi^2\tau), \quad p_\xi = \partial_\xi S = -2\xi\tau, \\ p_\eta &= \partial_\eta S = 0, \quad \varepsilon = -\partial_\tau S = \tau^2 + \xi^2.\end{aligned}$$

A proof of this theorem with the help of standard methods of Singularity Theory is going to be published later.

Let us consider the following simplest example. If

$$\begin{aligned}H &= p_y, \quad S_0(x, y) = x^2y + y^3/3, \quad z_0 = 0, \\ \Lambda_0 &= \{p_x = \partial_x S_0 = 2xy, \quad p_y = \partial_y S_0 = x^2 + y^2\}.\end{aligned}$$

Then

$$\begin{aligned}S(x, y, t) &= x^2(y - t) + (y - t)^3/3, \\ \Lambda &= \{p_x = \partial_x S, \quad p_y = \partial_y S, \quad E = -\partial_t S\}.\end{aligned}$$

The trajectory starting at z_0 is $p_x = p_y = x = 0$, $y = t$, let z be its point for $t = T$. Then the canonical transformation

$$\begin{aligned} p_\xi &= p_x, & p_\eta &= p_y - E, & \varepsilon &= E, \\ \xi &= x, & \eta &= y - T, & \tau &= t - y \end{aligned}$$

gives $S = -x^2\tau - \tau^3/3$ according to Theorem 6.1.

Theorem 6.1 implies the following: for $z \in \Lambda$ as above, if $U_z \subset \Lambda$ is a sufficiently small neighbourhood of z , then for any smooth function $b : \Lambda \rightarrow \mathbf{R}$ vanishing outside of U_z , there exists smooth function $S, B : \mathbf{R}_{x,y,t}^3 \rightarrow \mathbf{R}$ vanishing outside of $\pi_t(U_z)$ such that

$$\begin{aligned} \Lambda \cap U_z &= \{p_x = \partial_x S, p_y = \partial_y S, E = -\partial_t S\} \cap U_z, \\ [\mathcal{K}_\Lambda b](x, y, t) &= B(x, y, t) \exp \left\{ \frac{iS(x, y, t)}{h} \right\}. \end{aligned}$$

Therefore the integral (1.2) is rewritten in terms of the Maslov canonical operator on the same Λ but in the space $T^*\mathbf{R}_{x,y,E}^3$:

$$\int_0^{+\infty} [\mathcal{K}_\Lambda b](x, y, t) \exp \left\{ \frac{iEt}{h} \right\} dt = \int_{-\infty}^{+\infty} B(x, y, t) \exp \left\{ \frac{i(S(x, y, t) + Et)}{h} \right\} dt.$$

and we can express this value in the coordinates of Theorem 6.1 like in [9],[15].

7 The transition of Λ_+^E and its phase

Now we describe what happens with the singular Lagrangian submanifold Λ_+^E as $\sigma = E - E_0$ changes its sign. More precisely, let us consider a submanifold

$$\text{loc } \Lambda_+^E = \{z \in T^*\mathbf{R}^n : \exists t \geq 0, z \in \exp tv_H(\Lambda_0 \cap U_{z_0} \cap \Sigma_E)\} \subset \Lambda_+^E$$

where U_{z_0} is a sufficiently small neighbourhood of the initial point $z_0 \in \Lambda_0$ from Theorem 6.1.

Corollary 7.1 *In a neighbourhood of the point $\pi_E(z)$ there exist new local coordinates*

$$(\xi(x, y, E), \eta(x, y, E), \varepsilon(E)), \quad \varepsilon(E_0) = 0$$

such that the singular Lagrangian submanifold $\text{loc } \Lambda_+^E$ is defined by the following formulas

$$\begin{aligned} S(\xi, \eta, \tau) &= -(\tau^3/3 + \xi^2\tau), & p_\xi &= \partial_\xi S = -2\xi\tau, \\ p_\eta &= \partial_\eta S = 0, & \varepsilon &= -\partial_\tau S = \tau^2 + \xi^2. \end{aligned}$$

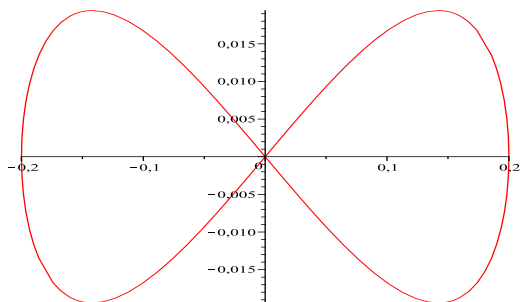


Figure 1: The Lagrangian curve “ ∞ ”

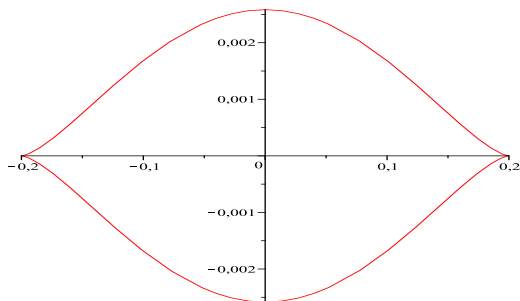


Figure 2: A section of the phase graph

The Lagrangian submanifold $\text{loc } \Lambda_+^E$ is empty if $\varepsilon(E) < 0$. If $\varepsilon(E) > 0$ then $\text{loc } \Lambda_+^E$ becomes the cylinder over the Lagrangian curve “ ∞ ” shown in Fig. 1. The graph of the phase in this case is the cylinder over the curve with two cusps shown in Fig. 2. If $\varepsilon(E) = 0$ then $\text{loc } \Lambda_+^E$ degenerates into an interval of the Hamiltonian phase trajectory starting at z_0 .

The above transition is realized, for example, if $H = (p_x^2 + p_y^2)/\rho(x, y)$, ρ has a non-degenerate maximum or minimum at the origin, and $f_h = J_0(\sqrt{(x-a)^2 + y^2}/h)$, $a \neq 0$. In this case Λ_0 is the cylinder $p_x = \cos \phi$, $p_y = \sin \phi$, $x = a + \tau \cos \phi$, $y = \tau \sin \phi$. As the “source” $(a, 0)$ goes to $-\infty$ f_h becomes the simple wave $\exp \frac{ix}{h}$ and Λ_0 is the plane $p_x = 1$, $p_y = 0$. Here is the simplest example $f_h = \exp \frac{ix}{h}$, $\varrho = 1 + x^2 + y^2$ where one can get all explicit formulas. Our first figure is the dependence between y (the horizontal axe) and p_y (the vertical axe) if $x = 0$. The second figure is the dependence between y (the horizontal axe) and the phase (the vertical axe). The first curve is the derivative of the second one.

8 Glancing hypersurfaces and Lagrangian intersections

We discuss here a possible application of non transverse Lagrangian intersections to problems of diffraction by an obstacle. Let M be a smooth manifold, since we are working locally, we will assume $M = \mathbf{R}^n$. Let F, G be two smooth hypersurfaces of T^*M , intersecting transversally at z . Recall [12, Definition 21.4.6] that F and G are said to be *glancing* at z iff the Hamilton foliation of $F = \{f = 0\}$ and $G = \{g = 0\}$ (locally near z) are simply tangent at z .

Stated otherwise, we have $f(z) = g(z) = \{f, g\}(z) = 0$ (Poisson bracket), but the second Poisson brackets $\{f, \{f, g\}\}(z), \{g, \{g, f\}\}(z)$ are non zero.

By the theorem of equivalence of glancing hypersurfaces of Melrose [12, Theorem 21.4.8] there are local symplectic coordinates (x, ξ) vanishing at z such that F, G are defined resp. by $x_1 = 0$ and $\xi_1^2 - x_1 - \xi_2 = 0$. Then $g = \xi_1^2 - x_1 - \xi_2 = 0$ will be the “normal form” of H in these coordinates. (We use the notation g for the normal form of H , or $H - E$).

We apply this theorem to G being the energy surface $H = E$ (i. e. $g = 0$) and F an auxiliary hypersurface intersecting G transversally at a glancing point z . We want to find some germs of Lagrangian manifolds Λ such that Λ is transverse to F at z but (Λ, G) has glancing intersection at z . This means that $T_z\Lambda \cap (T_zF)^\sigma = \{0\}$ and $\mathbf{R}v_H(z) = (T_zG)^\sigma \subset T_z\Lambda$, where superscript σ denotes symplectic orthogonal. Assume $n = 2$ for simplicity.

Define Λ locally near z by $f_1 = f_2 = 0$, with $\{f_i, f_j\} = 0$. Consider the symmetric matrix

$$A_z = A_z(\Lambda, G) = \begin{pmatrix} \{f_2, \{f_2, g\}\} & -\{f_1, \{f_2, g\}\} \\ -\{f_2, \{f_1, g\}\} & \{f_1, \{f_1, g\}\} \end{pmatrix} (z)$$

and the vector

$$B_z = B_z(\Lambda, G) = \begin{pmatrix} \{g, \{g, f_1\}\} \\ \{g, \{g, f_2\}\} \end{pmatrix} (z).$$

Let z be a *glancing point* for the pair (Λ, G) i. e.

$$g(z) = f_1(z) = f_2(z) = 0, \quad \{g, f_1\}(z) = \{g, f_2\}(z) = 0.$$

We distinguish the following 10 possibilities for the 2-jets of $(\Lambda, G)_z$ (not covering the entire classification of [18]):

1. $\det A_z > 0, B_z \neq 0;$
2. $\det A_z > 0, B_z = 0;$
3. $\det A_z < 0, {}^t B_z A_z B_z \neq 0;$

4. $\det A_z < 0, {}^t B_z A_z B_z = 0, B_z \neq 0;$
5. $\det A_z < 0, B_z = 0;$
6. $\det A_z = 0, {}^t B_z A_z B_z \neq 0;$
7. $\det A_z = {}^t B_z A_z B_z = 0, A_z \neq 0, B_z \neq 0;$
8. $\det A_z = B_z = 0, A_z \neq 0;$
9. $A_z = 0, B_z \neq 0;$
10. $A_z = B_z = 0.$

We know that a general Lagrangian manifold can be parametrized in the mixed representation, so when $n = 2$ by one of the following cases

$$\begin{aligned}
(I) \quad \Lambda &= \{p = \partial_x \phi\}, \\
(II) \quad \Lambda &= \{x = -\partial_p \phi\}, \\
(III) \quad \Lambda &= \{x_1 = -\partial_{p_1} \phi, p_2 = \partial_{x_2} \phi\}, \\
(IV) \quad \Lambda &= \{p_1 = \partial_{x_1} \phi, x_2 = -\partial_{p_2} \phi\}.
\end{aligned}$$

We have

Proposition 8.1 *Assume Λ as above to be parametrized by a quadratic phase $\phi = \phi_0$. In Cases (I), (II), (III) ϕ_0 are one-parameter families taking values:*

$$\begin{aligned}
(I) \quad \phi_0(x) &= \frac{1}{2}(ax_1^2 - 2x_1x_2); \\
(II) \quad \phi_0(\xi) &= \frac{1}{2}(2\xi_1\xi_2 + c\xi_2^2); \\
(III) \quad \phi_0(x_2, \xi_1) &= \frac{1}{2}b(\xi_1 + x_2)^2
\end{aligned}$$

respectively, where $a, b, c \neq 0$. The manifold Λ is transverse to F at $z = 0$, and the corresponding matrices $A_z(\Lambda, G), B_z(\Lambda, G)$ above are then given by:

$$\begin{aligned}
(I) \quad A_z &= 2 \begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix}, \quad B_z = 2 \begin{pmatrix} -a \\ 1 \end{pmatrix}; \\
(II) \quad A_z &= 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_z = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \\
(III) \quad A_z &= 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_z = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{aligned}$$

At last, Case (IV) does not occur.

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