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Möbius Involutions

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Abstract

We study involutions in complex projective geometry. In particular, we construct a geometry on pairs of points in involution.

1 Introduction

We show that Möbius involutions are fundamental for many questions with respect to complex numbers and geometric transformations. In sections 2 and 3 we give an introduction to those questions proving already some results without using Möbius involutions. In sections 4 and 5 we introduce direct and indirect Möbius involutions making all previous results clearer. In section 6 we construct a geometry on pairs of points exchanged by a Möbius involution.

2 Direct and Indirect Similarities

It is classical in two dimensional real Euclidean geometry that an isometry (i.e. a transformation preserving all distances) with three unaligned fixed points A, B, C is already the identity denoted by id . If there are only two fixed points A, B , then it could also be the reflection through the line AB , because any unaligned point C is either fixed or has to be reflected through AB . If there is just one fixed point A , then any other point B is sent to some point B' such that $|AB| = |AB'|$. This could also be achieved with the help of a reflection through the perpendicular bisector of BB' creating an isometry with two fixed points. Since this is either the identity or a line reflection, any isometry with one fixed point is the composition of at most two line reflections. If there is no fixed point at all, we can create one with the help of a single line reflection. Therefore, any isometry is the composition of at most three line reflections.

Any composition of an odd number of line reflections changes the orientation (of every triangle) and is called an indirect or opposite isometry, whereas any composition of an even number of line reflections conserves the orientation and is called a direct isometry. As we have shown already, any direct isometry is the composition of two reflections through lines. Therefore it is either a translation (if those two lines are parallel) or a rotation (about the intersection point of those two lines). In particular, two reflections through lines commute if and only if the lines are identical or perpendicular. In the second case, we get a rotation of 180° about the intersection point also called point reflection (line reflections are indirect involutions whereas point reflections are direct involutions). Finally, the composition of the reflections through three lines is a line reflection if and only if those three lines are parallel or concurrent. Otherwise, it

is a glide reflection, i.e. the composition of a reflection through a line with a translation in the direction of this line (the composition of the reflections through three lines forming a triangle is a glide reflection through one of the sides of its orthic triangle, the perimeter of which is the distance of the translation).

A geometric transformation multiplying all distances by the same dilation factor $k > 0$ is called a similarity. For $k = 1$, it is an isometry, and for $k \neq 1$, it has a unique fixed point F . The most classical example is a homothety with center F : it sends any point P to P' such that $\overrightarrow{FP'} = k \cdot \overrightarrow{FP}$. Therefore any similarity (which is not an isometry) is the composition of a homothety with center F and an isometry with fixed point F . As we have shown already, this isometry is either a reflection through a line containing F providing an indirect similarity, or a rotation about F providing a direct similarity called spiral similarity. In particular, all direct similarities are either spiral similarities with fixed point F or translations. Spiral similarities with angle 180° can also be considered as homotheties with negative dilation factor k .

Let us identify our two dimensional real Euclidean space with the complex numbers \mathbb{C} (i.e. a one dimensional affine space over \mathbb{C}) by distinguishing two special points 0 (necessary and sufficient for defining the addition) and 1 (necessary and sufficient for defining the multiplication), $\mathbb{C} \setminus \{0\} = \mathbb{C}^*$. Then for any fixed $a \in \mathbb{C}^*$, the linear transformation $x \mapsto y = ax$ is a spiral similarity (the identity if $a = 1$) with fixed point 0 , dilation factor $|a|$ and angle $\arg(a)$. For any fixed $b \in \mathbb{C}$, $x \mapsto y = x + b$ is a translation (the identity if $b = 0$), and for any fixed $a \in \mathbb{C} \setminus \{0, 1\}$, the general affine transformation $x \mapsto y = ax + b$ is a spiral similarity with fixed point $F \in \mathbb{C}$:

$$F = aF + b \quad \Leftrightarrow \quad F = \frac{b}{1 - a}, \quad (2.1)$$

dilation factor $|a|$ and angle $\arg(a)$, which we denote by

$$S_{F;a} = S_{F;|a|, \arg(a)}. \quad (2.2)$$

Indeed,

$$y = ax + b \quad \Leftrightarrow \quad y - F = a(x - F). \quad (2.3)$$

In particular, if we choose F to be the special point 0 , then our spiral similarity becomes just a linear transformation.

The reflection through the real axis is just complex conjugation and denoted by $x \mapsto y = \bar{x}$. Therefore indirect similarities are given by $x \mapsto y = a\bar{x} + b$.

For three points or complex numbers A, B, C we define

$$r_{A,B,C} := \frac{A - C}{B - C}, \quad |r_{A,B,C}| = \frac{|AC|}{|BC|}, \quad \arg(r_{A,B,C}) = \angle BCA. \quad (2.4)$$

Then triangles ABC and $A'B'C'$ are directly similar if and only if $r_{A,B,C} = r_{A',B',C'}$, and indirectly similar if and only if $r_{A,B,C} = \overline{r_{A',B',C'}}$. In particular, ABC is degenerated, i.e. A, B, C are collinear, if and only if $r_{A,B,C}$ is a real number, positive if C is outside the segment AB and negative if it is inside. Degenerated triangles can also be similar (or not similar), but it is insufficient to know their angles in order to decide it. The following proposition is evident but useful.

Proposition 1. For any three points A, B, C we have

$$r_{A,B,C} \cdot r_{B,A,C} = 1, \quad r_{A,B,C} + r_{A,C,B} = 1. \quad (2.5)$$

In particular, if two of the three points A, B, C are fixed, then the third one is determined uniquely by $r_{A,B,C}$. \square

More precisely, for fixed points $A \neq B$, we can use the following proposition in order to determine the point C .

Proposition 2. We have

$$r_{A,B,C} = k \Leftrightarrow C = \frac{kB - A}{k - 1}, \quad (2.6)$$

and C can be considered as a one dimensional barycenter of A and B with complex homogeneous coordinates $(-1 : k) = (1 : -k)$.

Proof. We have

$$\frac{A - C}{B - C} = k \Leftrightarrow A - C = k(B - C) \Leftrightarrow C(k - 1) = kB - A. \quad (2.7)$$

\square

If A and B are different from F , then there is a unique spiral similarity with fixed point F sending A to B . We denote it by

$$S_{F;A \rightarrow B} = S_{F;r_{B,A,F}}. \quad (2.8)$$

More generally, for two pairs of points (A, B) and (A', B') there is a unique direct similarity sending A to A' and B to B' . Its fixed point F has to satisfy one of the following equivalent equations :

$$\frac{A' - F}{A - F} = \frac{B' - F}{B - F} \Leftrightarrow \frac{B - F}{A - F} = \frac{B' - F}{A' - F} \Leftrightarrow \quad (2.9)$$

$$(A' - F)(B - F) = (A - F)(B' - F) \Leftrightarrow F \cdot [(B' - B) - (A' - A)] = AB' - A'B \quad (2.10)$$

having a unique solution for F if and only if $ABB'A'$ is not a parallelogram. If $ABB'A'$ is a parallelogram, then there is a unique translation sending A to A' and B to B' , and also another unique translation sending A to B and A' to B' . If $ABB'A'$ is not a parallelogram, however, then there is a unique spiral similarity with fixed point

$$F = \frac{AB' - A'B}{(B' - B) - (A' - A)} \quad (2.11)$$

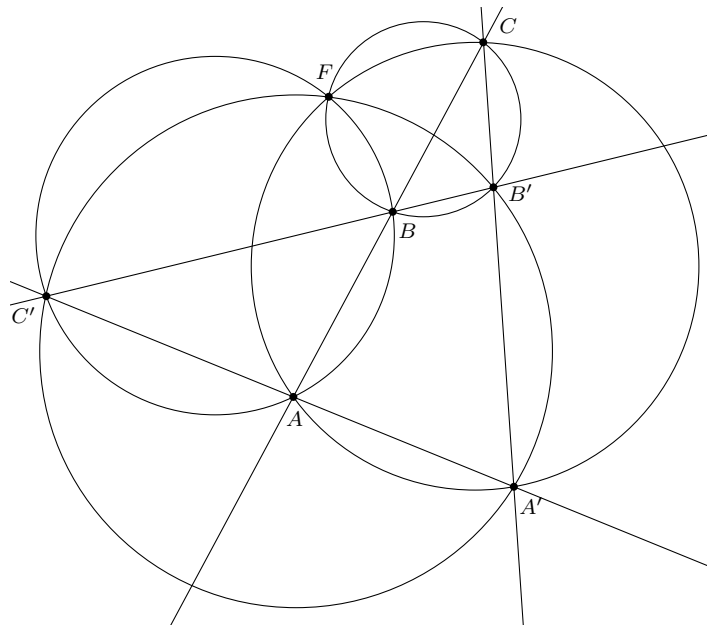
sending A to A' and B to B' , namely

$$S_{F;r_{A',A,F}} = S_{F;r_{B',B,F}}, \quad (2.12)$$

and also another unique spiral similarity with fixed point F sending A to B and A' to B' , namely

$$S_{F;r_{B,A,F}} = S_{F;r_{B',A',F}}. \quad (2.13)$$

Let us suppose that the lines AB and $A'B'$ intersect in C , whereas AA' and BB' intersect in C' . Then the angle of our first spiral similarity, namely $\angle AFA' = \angle BFB'$, is equal to the angle between the lines AB and $A'B'$. Therefore F lies on the circumcircles of the triangles $AA'C$ and $BB'C$. Similarly, the angle of our second spiral similarity, namely $\angle AFB = \angle A'FB'$, is equal to the angle between the lines AA' and BB' . Therefore F lies on the circumcircles of the triangles ABC' and $A'B'C'$. In other words, F is the Miquel point of the quadrilateral defined by the four lines AB , $A'B'$, AA' , BB' .



F coincides with C if and only if the spiral similarity sending A to B and A' to B' is a homothety, i.e if and only if the lines AA' and BB' are parallel. Similarly, F coincides with C' if and only if the spiral similarity sending A to A' and B to B' is a homothety, i.e if and only if the lines AB and $A'B'$ are parallel. F can also be found on four Apollonius circles using the facts that

$$\frac{FA'}{FA} = \frac{FB'}{FB} = \frac{A'B'}{AB} \quad \text{and} \quad \frac{FB}{FA} = \frac{FB'}{FA'} = \frac{BB'}{AA'}. \quad (2.14)$$

For two triangles ABC and $A'B'C'$, there is a direct similarity sending the first to the second if and only if they are directly similar, and there is an indirect similarity sending the first to the second if and only if they are indirectly similar.

3 The Composition of Spiral Similarities

Of course, if we compose several spiral similarities (even with different fixed points), then their dilation factors multiply and their angles add. In particular, if we first apply the spiral similarity $S_{F_1;a_1}$ and after that $S_{F_2;a_2}$, then we get

$$S_{F_2;a_2} \circ S_{F_1;a_1} = S_{F_3;a_1 a_2}, \quad (3.1)$$

where the fixed point F_3 is uniquely determined by F_1 , F_2 , a_1 and a_2 . When reading the beautiful books [5, 6, 7] (also available in English [8, 9, 10, 11, 12]) some 35 years ago, we found, however,

that the dependencies between F_1, F_2, F_3, a_1, a_2 and $a_3 = a_1 a_2$ were not sufficiently studied, and discovered the following results.

Lemma 1. *For any six points A, B, C, A', B', C' we have*

$$S_{B';C \rightarrow A} \circ S_{A';B \rightarrow C} \circ S_{C';A \rightarrow B} = id \Leftrightarrow r_{A,C,B'} \cdot r_{C,B,A'} \cdot r_{B,A,C'} = 1. \quad (3.2)$$

Proof. First of all,

$$S_{C';A \rightarrow B} = S_{C';r_{B,A,C'}}, \quad S_{A';B \rightarrow C} = S_{A';r_{C,B,A'}}, \quad S_{B';C \rightarrow A} = S_{B';r_{A,C,B'}}. \quad (3.3)$$

Since the composition of those spiral similarities has A as fixed point, we know that

$$S_{B';C \rightarrow A} \circ S_{A';B \rightarrow C} \circ S_{C';A \rightarrow B} = S_{A;r_{A,C,B'} \cdot r_{C,B,A'} \cdot r_{B,A,C'}}, \quad (3.4)$$

and this is the identity if and only if

$$r_{A,C,B'} \cdot r_{C,B,A'} \cdot r_{B,A,C'} = 1. \quad (3.5)$$

□

Corollary 1. *For any six points A, B, C, A', B', C' we have*

$$S_{B';C \rightarrow A} \circ S_{A';B \rightarrow C} \circ S_{C';A \rightarrow B} = id \Leftrightarrow \quad (3.6)$$

$$S_{C';A \rightarrow B} \circ S_{B';C \rightarrow A} \circ S_{A';B \rightarrow C} = id \Leftrightarrow \quad (3.7)$$

$$S_{A';B \rightarrow C} \circ S_{C';A \rightarrow B} \circ S_{B';C \rightarrow A} = id. \quad \square \quad (3.8)$$

Corollary 2. *For any six points A, B, C, A', B', C' we have*

$$S_{B';C \rightarrow A} \circ S_{A';B \rightarrow C} \circ S_{C';A \rightarrow B} = id \Leftrightarrow \quad (3.9)$$

$$S_{B;C' \rightarrow A'} \circ S_{A;B' \rightarrow C'} \circ S_{C;A' \rightarrow B'} = id \quad (3.10)$$

Proof. According to our preceding lemma, it is necessary and sufficient to prove that

$$\frac{A - B'}{C - B'} \cdot \frac{C - A'}{B - A'} \cdot \frac{B - C'}{A - C'} = 1 \Leftrightarrow \quad (3.11)$$

$$\frac{A' - B}{C' - B} \cdot \frac{C' - A}{B' - A} \cdot \frac{B' - C}{A' - C} = 1, \quad (3.12)$$

but this equivalence is evident. □

If the triples (A, B, C) and (A', B', C') are related as in the preceding lemma and corollaries, we used to call them "in friendship" (in our next section, we will see a better characterization of friendship). Our preceding corollary shows that friendship is invariant under the simultaneous exchanges $A \leftrightarrow A', B \leftrightarrow B'$ and $C \leftrightarrow C'$. In the following theorem we will see that it is even invariant if we perform just some of those exchanges, for example the exchange $A \leftrightarrow A'$ or the additional exchange $B \leftrightarrow B'$.

Theorem 1. For any six points A, B, C, A', B', C' we have

$$S_{B';C \mapsto A} \circ S_{A';B \mapsto C} \circ S_{C';A \mapsto B} = id \Leftrightarrow \quad (3.13)$$

$$S_{B';C \mapsto A'} \circ S_{A;B \mapsto C} \circ S_{C';A' \mapsto B} = id \Leftrightarrow \quad (3.14)$$

$$S_{B;C \mapsto A'} \circ S_{A;B' \mapsto C} \circ S_{C';A' \mapsto B'} = id. \quad (3.15)$$

More generally, those identities are invariant under any of the exchanges $A \leftrightarrow A', B \leftrightarrow B', C \leftrightarrow C'$.

Proof. According to our preceding lemma, it is necessary and sufficient to show that the identity

$$r_{A,C,B'} \cdot r_{C,B,A'} \cdot r_{B,A,C'} = 1 \quad (3.16)$$

is invariant under any of the exchanges $A \leftrightarrow A', B \leftrightarrow B', C \leftrightarrow C'$. But it is equivalent to

$$\frac{A - B'}{C - B'} \cdot \frac{C - A'}{B - A'} \cdot \frac{B - C'}{A - C'} = 1 \Leftrightarrow \quad (3.17)$$

$$(A - B')(C - A')(B - C') = (C - B')(B - A')(A - C') \Leftrightarrow \quad (3.18)$$

$$AA'(C + C' - B - B') + BB'(A + A' - C - C') + CC'(B + B' - A - A') = 0. \quad (3.19)$$

□

Corollary 3. Let A, B, C, A', B', C' be six points such that

$$\frac{AB'}{CB'} \cdot \frac{CA'}{BA'} \cdot \frac{BC'}{AC'} = 1 \quad (3.20)$$

and

$$\angle CB'A + \angle BA'C + \angle AC'B = 0^\circ \pmod{360^\circ}, \quad (3.21)$$

then

$$\angle B'C'A' = \angle B'AC + \angle CBA' \pmod{360^\circ}, \quad (3.22)$$

$$\angle C'A'B' = \angle C'BA + \angle ACB' \pmod{360^\circ}, \quad (3.23)$$

$$\angle A'B'C' = \angle A'CB + \angle BAC' \pmod{360^\circ}. \quad (3.24)$$

Proof. According to our preceding lemma, the first two relations of our corollary are equivalent to

$$S_{B';C \mapsto A} \circ S_{A';B \mapsto C} \circ S_{C';A \mapsto B} = id. \quad (3.25)$$

According to our preceding theorem, this identity is equivalent to

$$S_{B;C \mapsto A'} \circ S_{A;B' \mapsto C} \circ S_{C';A' \mapsto B'} = id \Leftrightarrow \quad (3.26)$$

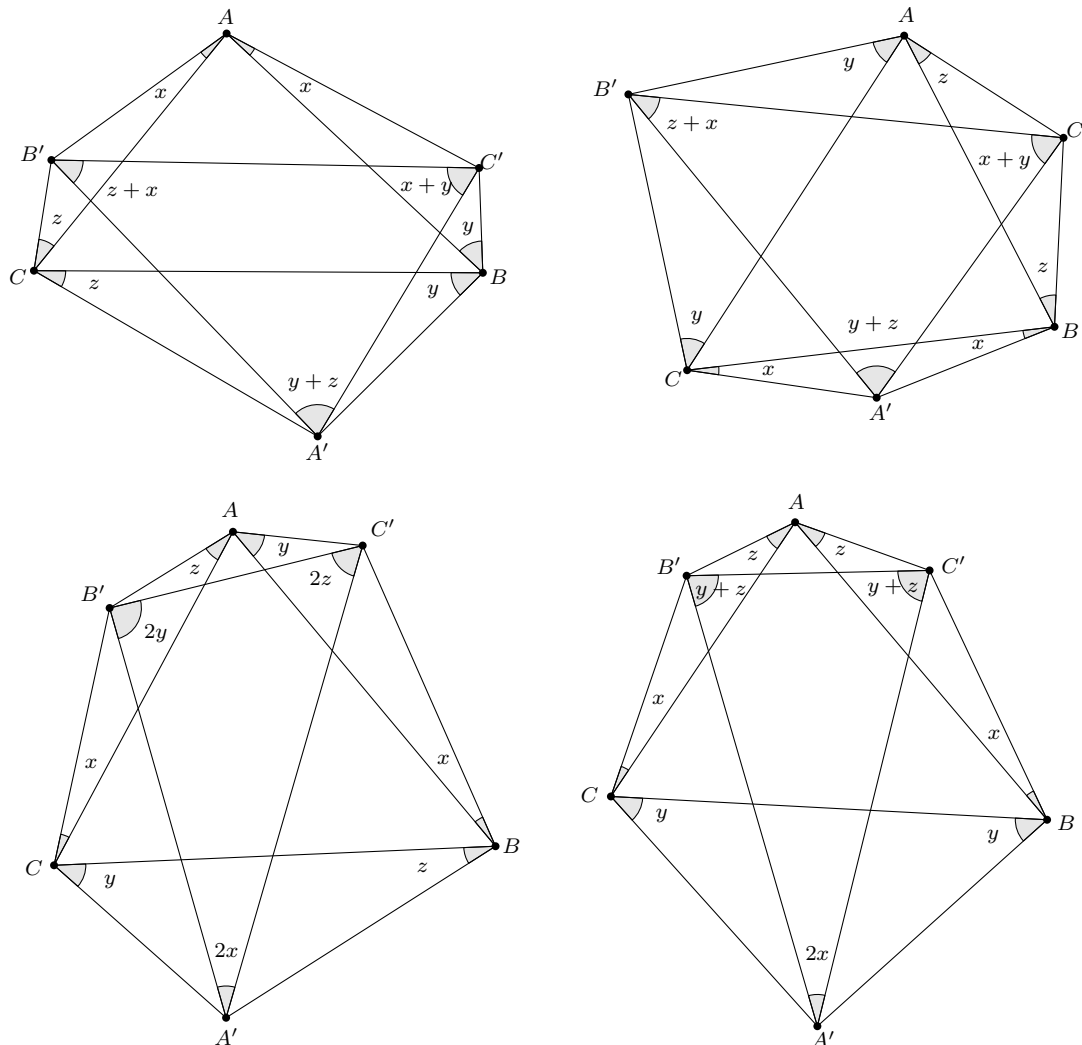
$$r_{A',C,B} \cdot r_{C,B',A} \cdot r_{B',A',C'} = 1. \quad (3.27)$$

If we look at the arguments in the last equation, then we get

$$\angle CBA' + \angle B'AC + \angle A'C'B' = 0^\circ \pmod{360^\circ}, \quad (3.28)$$

proving the first identity of our corollary. The other two identities are obtained in the same way. □

Example. Let x, y, z be three (oriented) angles such that $x + y + z = 90^\circ$. If they appear in the triangles $C'AB$, $A'BC$, $B'CA$ at the vertices A, B, C as in the following figures, then our preceding corollary allows us to calculate the angles of $A'B'C'$ with the help of x, y, z only, without knowing the angles of ABC .



Indeed, the law of sines allows us to calculate each of the quotients $\frac{AB'}{CB'}$, $\frac{CA'}{BA'}$, $\frac{BC'}{AC'}$ showing that their product is equal to 1. Moreover, $\angle CB'A + \angle BA'C + \angle AC'B = 3 \cdot 180^\circ - 2(x + y + z) = 360^\circ$.

4 Direct Projective Involutions

We have already seen that among the similarities, line reflections are the only indirect involutions whereas point reflections are the only direct involutions. Of course, $z \mapsto f(z) = \frac{1}{z}$ is also an involution (since $f(f(z)) = z$), but not a similarity since

$$r_{\frac{1}{A}, \frac{1}{B}, \frac{1}{C}} = \frac{\frac{1}{A} - \frac{1}{C}}{\frac{1}{B} - \frac{1}{C}} = \frac{BC}{AC} \cdot \frac{C - A}{C - B} = \frac{B}{A} \cdot r_{A, B, C} \quad (4.1)$$

is neither conserved nor conjugated, because the factor $\frac{B}{A}$ is disturbing. This factor, however, disappears as soon as we consider cyclic products such as

$$r_{A,C,B'} \cdot r_{C,B,A'} \cdot r_{B,A,C'} \quad (4.2)$$

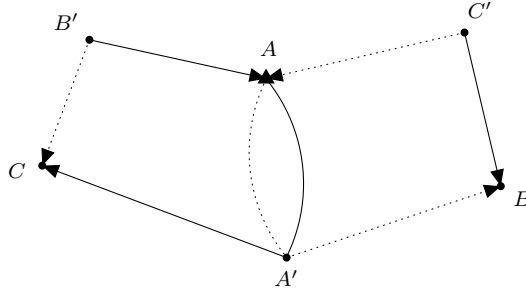
widely used in our preceding section, or the more classical cross-ratio, which we can define as a cyclic product of two factors instead of three ones :

$$b_{A,B;C,D} := r_{ABC} \cdot r_{BAD}. \quad (4.3)$$

Therefore those cyclic products are invariant under the involution $z \mapsto f(z) = \frac{1}{z}$, which means that this involution is a direct projective transformation. Actually, we have the identity

$$r_{A,C,B'} \cdot r_{C,B,A'} \cdot r_{B,A,C'} = b_{A,C;B',A'} \cdot b_{A,B;A',C'} \quad (4.4)$$

represented by the following figure showing its multiplications and divisions.



Our projective involution $z \mapsto f(z) = \frac{1}{z}$ has 1 and -1 as fixed points. Any other z is distinct from $\frac{1}{z}$ and exchanged with it. This is even true for 0, exchanged with a unique point at infinity, added to the complex numbers \mathbb{C} and denoted by ∞ . For fixed points $A \neq B$, we have

$$r_{A,B;C} = 1 \Leftrightarrow C = \infty. \quad (4.5)$$

We have $\arg(b_{A,B;C,D}) = \angle BCA + \angle ADB \pmod{360^\circ}$. Therefore $b_{A,B;C,D}$ is a positive real number if and only if $\angle ADB = \angle ACB \pmod{360^\circ}$, i.e. if and only if either A, B, C, D are concyclic and the segments AB and CD do not intersect, or A, B, C, D are collinear and C, D are either both inside or both outside the segment AB . Similarly, $b_{A,B;C,D}$ is a negative real number if and only if $\angle BCA + \angle ADB = 180^\circ \pmod{360^\circ}$, i.e. if and only if either A, B, C, D are concyclic and the segments AB and CD do intersect, or A, B, C, D are collinear and one of C, D is inside and the other outside the segment AB .

Proposition 3. *For any four points A, B, C, D we have*

$$b_{A,B;C,D} = b_{B,A;D,C} = b_{C,D;A,B} = b_{D,C;B,A}, \quad (4.6)$$

$$b_{A,B;C,D} \cdot b_{B,A;C,D} = b_{A,B;C,D} \cdot b_{A,B;D,C} = 1, \quad b_{A,B;C,D} + b_{A,C;B,D} = 1. \quad (4.7)$$

In particular, if three of the four points A, B, C, D are fixed, then the fourth one is determined uniquely by $b_{A,B;C,D}$.

Proof. Only the very last equation is not directly evident from the definition $b_{A,B;C,D} = r_{ABC} \cdot r_{BAD}$. It can be proven by a direct calculation, but it is easier to use a translation in order to get $D = 0$ and then our direct projective involution $z \mapsto f(z) = \frac{1}{z}$ in order to get $D = \infty$ and $b_{A,B;C,D} = r_{ABC}$. Since this leaves $b_{A,B;C,D}$ invariant, the very last equation follows from (2.5). \square

Remark. Our preceding proposition shows that for distinct $A, B, C, D, b_{A,B;C,D} \in \{0, 1, \infty\}$ is not possible. The identity $b_{A,B;C,D} + b_{A,C;B,D} = 1$ has two important corollaries. On the one hand, $b_{A,B;C,D}$ is real if and only if $b_{A,C;B,D}$ is real, i.e. $\angle ADB = \angle ACB \pmod{180^\circ}$ if and only if $\angle ADC = \angle ABC \pmod{180^\circ}$, which is classical for cyclic quadrilaterals. On the other hand, the triangle inequality tells us

$$|b_{A,B;C,D}| + |b_{A,C;B,D}| \geq 1 \quad \Leftrightarrow \quad AC \cdot BD + AB \cdot CD \geq AD \cdot BC \quad (4.8)$$

with equality if and only if $b_{A,B;C,D}$ and $b_{A,C;B,D}$ are real and positive, i.e. if and only if $ABCD$ is a cyclic quadrilateral and the segments AD and BC intersect, or A, B, C, D are collinear and exactly one of the points B, C lies on the segment AD . This is Ptolemy's inequality.

It is natural to consider lines as circles containing the point ∞ . Then $b_{A,B;C,D}$ is real if and only if A, B, C, D are concyclic, and this property is preserved by our involution $z \mapsto f(z) = \frac{1}{z}$. More precisely, we can say that lines passing through 0 remain lines through 0, lines avoiding 0 are exchanged with circles passing through 0, and all other circles remain circles. Since $f'(z) = -z^{-2}$ exists, f acts locally as a spiral similarity. In particular, angles between circles or lines are preserved.

These properties immediately carry over to general direct projective transformations (also called Möbius transformations) defined for fixed $a, b, c, d \in \mathbb{C}$ such that $ad \neq bc$ by

$$z \mapsto f(z) = \frac{az + b}{cz + d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz + d} \quad (4.9)$$

because those transformations can be written as the composition of a direct similarity $z \mapsto cz + d$ with our favorite involution and another direct similarity, namely the multiplication by $b - \frac{ad}{c}$ and the addition of $\frac{a}{c}$ (if $c = 0$, then f is already a direct similarity).

For any three points A, B, C and any other three points A', B', C' , there is a general direct projective transformation sending A to A' , B to B' and C to C' . For any fourth point D , however, its image D' is then uniquely determined by $b_{A,B;C,D} = b_{A',B';C',D'}$. Of course, every direct projective transformation has two fixed points : the solutions of the quadratic equation $f(z) = z$.

For arbitrary points A, B and C we can choose 0 and 1 in such a way that $A = -1, B = 1, C = z$ (for some $z \in \mathbb{C}$), and $M = 0$ is the midpoint of the segment AB . Then there is a unique point $D = 1/z$, for which we have $b_{A,B;C,D} = b_{A,B;D,C}$ because our favorite involution conserves the cross-ratio and exchanges C with D while A and B remain fixed. But our preceding proposition tells us that we always have $b_{A,B;D,C} = 1/b_{A,B;C,D}$. Therefore $b_{A,B;C,D}^2 = 1$. Since $b_{A,B;C,D} = 1$ is impossible for distinct points, it follows that $b_{A,B;C,D} = -1$. By definition, this means that $A, B; C, D$ are four *harmonic* points. In other words, for any three points $A, B; C$ there is a unique point D such that $A, B; C, D$ are harmonic, and we can suppose then that

$A = -1, B = 1, C = z$ and $D = 1/z$ (we could also suppose that $C = -1, D = 1, A = z$ and $B = 1/z$).

Since -1 is a (negative) real number, four harmonic points $A, B; C, D$ are concyclic or collinear, and $|b_{A,B;C,D}| = 1$ implies $AC/BC = AD/BD$. Moreover, we have

$$\frac{C-M}{B-M} = \frac{B-M}{D-M} = \frac{C-A}{D-A} = z, \quad \frac{C-M}{A-M} = \frac{A-M}{D-M} = \frac{C-B}{D-B} = -z. \quad (4.10)$$

Therefore the three triangles CMB, BMD, CAD are similar, and the three triangles CMA, AMD, CBD are also similar. In particular, it follows that

$$\frac{A-C}{D-C} = \frac{M-C}{B-C}, \quad \frac{A-C}{M-C} = \frac{D-C}{B-C}, \quad \frac{A-D}{C-D} = \frac{M-D}{B-D}, \quad \frac{A-D}{M-D} = \frac{C-D}{B-D}. \quad (4.11)$$

We have proven the following proposition.

Proposition 4. *Let $A, B; C, D$ be four harmonic points, i.e. $b_{A,B;C,D} = -1$, and let M be the midpoint of the segment AB . Then we have the following identities of Newton*

$$(C-M)(D-M) = (B-M)^2 = (A-M)^2, \quad (4.12)$$

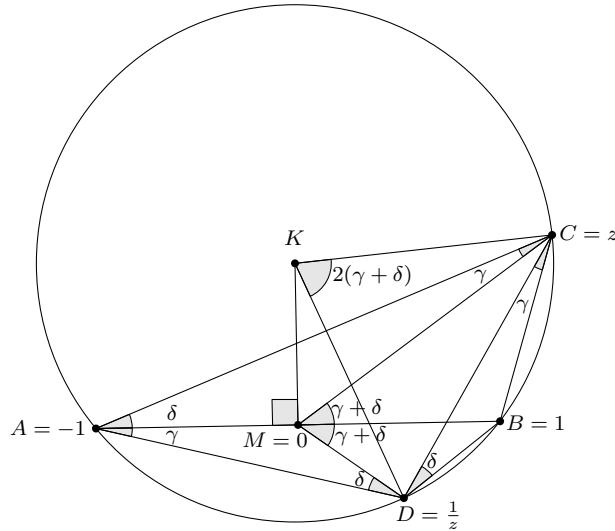
and MacLaurin

$$(A-C)(B-C) = (M-C)(D-C), \quad (A-D)(B-D) = (M-D)(C-D). \quad (4.13)$$

Similar identities hold for the midpoint of the segment CD . If K is the center of the circumscribed circle of $ABCD$, then

$$\angle DKC = 2 \cdot \angle DAC = \angle DMB + \angle BMC = \angle DMC, \quad (4.14)$$

such that $CDMK$ is a cyclic quadrilateral. Moreover, AB and MK are perpendicular bisectors of $\angle DMC$. \square

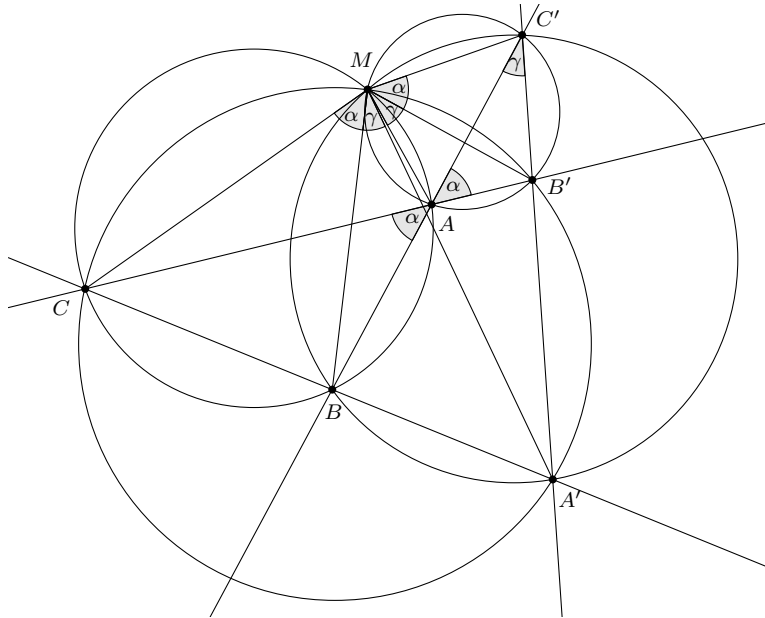


If M , C and D are arbitrary points, then there always exists a direct projective involution with center M (i.e. exchanging M and ∞) which exchanges C and D . Indeed, it is natural to put first $M = 0$, and then $C \cdot D = A^2 = B^2$ in order to find its fixed points B and $A = -B$ as unique solutions of this equation. After that we can put $A = -1$ and $B = 1$, and our favorite involution does what we want it to do. Of course, the involution exchanging M and ∞ as well as C and D is unique, because any direct projective transformation defined on three points is already unique. For given M , C and D , we can first construct the perpendicular bisectors of $\angle DMC$, then according to our preceding proposition K using the fact that $CDMK$ is a cyclic quadrilateral (or that K lies on the perpendicular bisector of CD) and finally the fixed points A and B .

Similarly, if A and B are arbitrary points, then there exists a unique direct projective involution having them as fixed points. Indeed, if C and D are exchanged in it, then $b_{A,B;C,D} = b_{A,B;D,C}$ meaning that $b_{A,B;C,D} = -1$ and $A, B; C, D$ are harmonic : for every C there is a unique D exchanged with it.

Remark. The equation $C \cdot D = A^2 = B^2$ cannot be solved in real numbers if $C \cdot D$ is negative. Therefore direct complex projective involutions (exchanging ∞ with some point $M \neq \infty$) are always of the form $z \mapsto f(z) = \frac{1}{z}$, but direct real projective involutions can also be of the form $z \mapsto f(z) = -\frac{1}{z}$ having no fixed points. If ∞ is a fixed point, however, we have a direct affine involution, i.e. a point reflection which can be written in the form $z \mapsto f(z) = -z$. In any case, a direct complex projective (or affine) involution always has two distinct fixed points.

More generally, for any four points A, A', B, B' there is a (unique) direct projective (or affine) involution exchanging A with A' and B with B' . On the one hand, this follows from the fact that we always have $b_{A,A';B,B'} = b_{A',A;B',B}$ by (4.6). On the other hand, we can find the center M of this involution (i.e. the point exchanged with ∞) by solving the equation $(A' - M)(A - M) = (B' - M)(B - M)$. But we have already solved this equation (2.10) and found the Miquel point of the quadrilateral defined by the four lines $AB, AB', A'B, A'B'$ as solution if $ABA'B'$ is not a parallelogram



(if $ABA'B'$ is a parallelogram, the reflection through the common midpoint of AA' and BB' is an

affine involution exchanging A with A' and B with B'). If $AB \cap A'B' = C'$ and $AB' \cap A'B = C$, then we find the same Miquel point M as center of the involution exchanging B with B' and C with C' . Therefore there exists a direct projective involution with center M exchanging simultaneously A with A' , B with B' and C with C' .

In general, if there exists a direct projective involution exchanging simultaneously A with A' , B with B' and C with C' , then the invariance of the cross-ratio implies for example

$$b_{A,C;B',A'} = b_{A',C';B,A}. \quad (4.15)$$

This relation is actually equivalent to the existence of a direct projective involution exchanging simultaneously A with A' , B with B' and C with C' . Indeed, there is a unique direct projective involution exchanging simultaneously A with A' and B with B' , but then, for every C , this involution determines uniquely C' , for which our relation (4.15) is satisfied. Our relation (4.15), however, also determines uniquely C' , because C' appears only once in $b_{A',C';B,A}$ and not at all in $b_{A,C;B',A'}$.

If we use the relations (4.6) and (4.7), then we see that our condition (4.15) is equivalent to the identity

$$b_{A,C;B',A'} \cdot b_{A,B;A',C'} = 1. \quad (4.16)$$

Using (4.4), it follows that a direct projective involution exchanging simultaneously A with A' , B with B' and C with C' exists if and only if

$$r_{A,C;B'} \cdot r_{C,B;A'} \cdot r_{B,A;C'} = 1, \quad (4.17)$$

see [2]. (For the points of the preceding figure, this identity is Menelaus's theorem.) Therefore (3.2) and (3.15) prove the following theorem.

Theorem 2. *For any six points A, A', B, B', C, C' , a direct projective involution exchanging simultaneously A with A' , B with B' and C with C' exists if and only if*

$$S_{B';C \rightarrow A} \circ S_{A';B \rightarrow C} \circ S_{C';A \rightarrow B} = id. \quad (4.18)$$

□

A direct projective involution defines a perfect matching on all points (including ∞) except two distinct fixed points. Therefore two direct projective involutions define a partition of all points into even cycles and paths (possibly with 0 edges or an infinite number of edges) starting and ending at the fixed points. They commute if and only if all cycles are of length 2 or 4, and all path contain 0 or 1 edge, where a path with 1 edge joins the two fixed points from the same involution, and a path without edges joins two fixed points from different involutions. If there is a path without edges (i.e. a common fixed point of the two involutions), then there cannot be a path with one edge. Therefore we have two paths without edges in that case. This means that our involutions have the same fixed points and coincide for this reason. We have proven the first part of the following theorem.

Theorem 3. *If we compose two different commuting direct projective involutions, we get a third one commuting with both of them. They partition all points (including ∞) into blocks of size four (on which they represent all three possible perfect matchings) and three blocks of size two (on*

which two of them represent the matching and the third one its two fixed points). In particular, each of our three pairwise commuting direct projective involutions exchanges the fixed points of the other two.

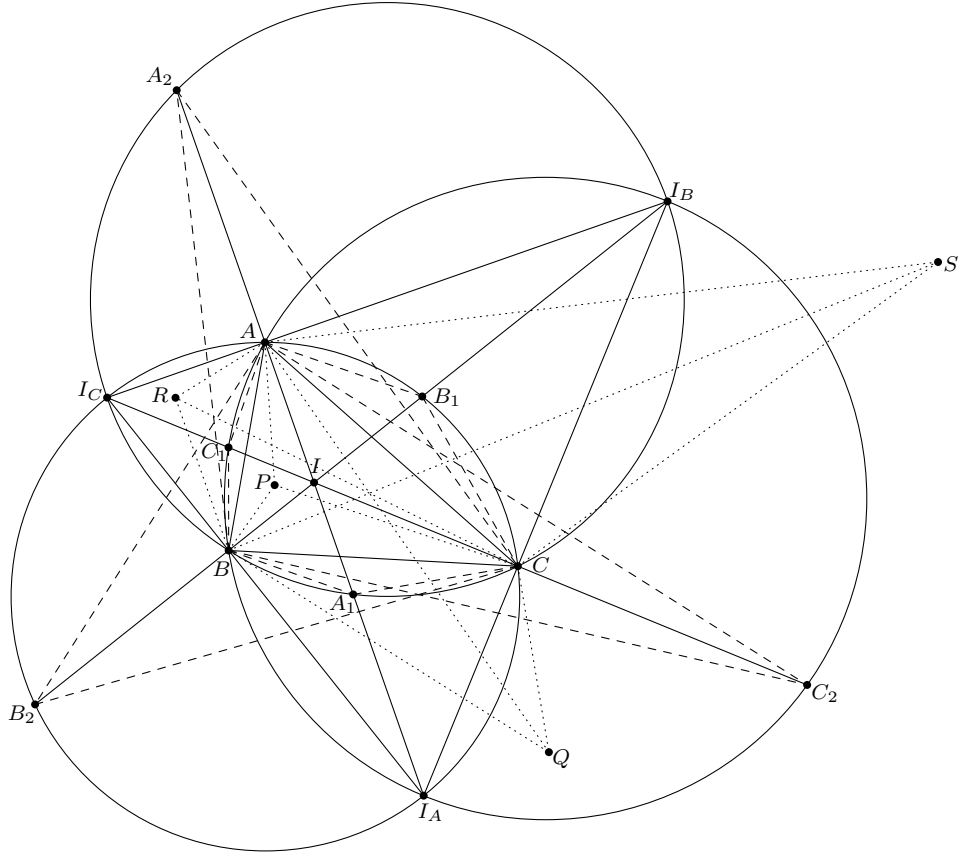
On the other hand, two commuting direct projective involutions can be uniquely defined by two perfect matchings on a block of size four. Moreover, a direct projective involution commutes with another one if and only if it exchanges its fixed points.

Proof. Any direct projective involution can be defined uniquely by a perfect matching on a block of size four, and if the commutation relation of two projective transformations is valid on three points, then it is valid everywhere by the invariance of the cross-ratio.

If a direct projective involution with fixed points A and B exchanges the fixed points C and D of another one, then $A, B; C, D$ are four harmonic points. Therefore our direct projective involution with fixed points C and D exchanges A and B . But since our two direct projective involutions commute on the four element set $\{A, B, C, D\}$, they commute everywhere. \square

If we have three pairwise commuting direct projective involutions for which the point ∞ belongs to a block of size two, then we can consider that two of our involutions exchange it with a point which we call 0 , whereas the third one has 0 and ∞ as its fixed points : it is the involution $z \mapsto -z$. If we call 1 and -1 the fixed points of one of the other two involutions, then it becomes $z \mapsto \frac{1}{z}$, so that the third involution is $z \mapsto -\frac{1}{z}$, with fixed points i and $-i$.

If we have three pairwise commuting direct projective involutions for which the point ∞ belongs to a block of size four, however, then our three involutions exchange it with three points A, B, C . The involution with center A (exchanging A and ∞ as well as B and C) has two fixed points A_1 and A_2 on the bisector of $\angle BAC$ such that $AB \cdot AC = AA_1^2 = AA_2^2$. Analogous relations hold for the fixed points B_1, B_2, C_1, C_2 . All other points are on blocks of size four $PQRS$, and a reflection through the line A_1A_2 exchanges AP, AQ, AR, AS in pairs. Analogous results hold for the reflections through the lines B_1B_2 and C_1C_2 :



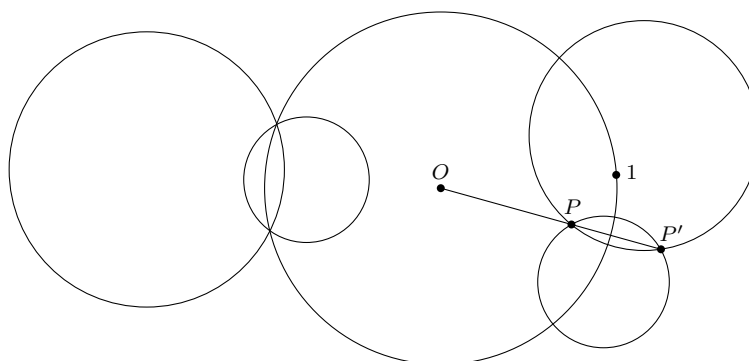
When constructing the angle bisectors of ABC we find its incenter and excenters $\{I, I_A, I_B, I_C\}$ forming an orthocentric system (each of them is the orthocenter of the other three). The circle having any two of them as diameter always contains two of the points $\{A, B, C\}$, and there are many similar triangles showing in particular that our involution $A \leftrightarrow \infty, B \leftrightarrow C$ also exchanges $I \leftrightarrow I_A$ and $I_B \leftrightarrow I_C$: one of our many blocks of size four is $\{I, I_A, I_B, I_C\}$. The midpoint of $I_B I_C$ is the center of the circumcircle of $I_B I_C B C$, and it lies on the perpendicular bisector of BC and on the circumcircle of ABC , the Euler circle of any triangle formed by three points among $\{I, I_A, I_B, I_C\}$. This confirms our constructions of the fixed points (harmonic points) of the involution $A \leftrightarrow \infty, B \leftrightarrow C$, which are A_1, A_2 lying on the circle of diameter $I_B I_C$. Finally, we get $AA_1^2 = AA_2^2 = AI_B \cdot AI_C = AI \cdot AI_A = AB \cdot AC = AB_1 \cdot AB_2 = AC_1 \cdot AC_2 = AP \cdot AQ = AR \cdot AS$, and $A_1 A_2$ is the common bisector for all those pairs of points (forming harmonic points with them). In particular, the lines AQ, BS and CR pass through the isogonal conjugate of P .

Remark. A Miquel point is often associated to four points P, Q, R, S , but in reality, it must be associated to four lines and lies on the circumcircles of the four triangles defined by any three of them. On four points P, Q, R, S we can define three commuting direct projective involutions, namely $P \leftrightarrow Q, R \leftrightarrow S$ (we call its center A), $P \leftrightarrow S, Q \leftrightarrow R$ (we call its center B) and $P \leftrightarrow R, Q \leftrightarrow S$ (we call its center C). P, Q, R, S define six lines, and A is the Miquel point of the four ones different from PQ, RS , whereas B and C are the Miquel points of the four ones different from PS, QR or PR, QS , respectively. As we have seen earlier, if we define $A' := PQ \cap RS$, $B' := PS \cap QR$, $C' := PR \cap QS$, then our commuting involutions with center A, B, C exchange $B' \leftrightarrow C', C' \leftrightarrow A', A' \leftrightarrow B'$, respectively. In particular, A', B', C' belong to the same block of

size four.

5 Indirect Projective Involutions

There is an involution much more famous than our direct projective involution $z \mapsto f(z) = \frac{1}{z}$, namely the classical inversion with respect to the unit circle $z \mapsto f(z) = \frac{1}{\bar{z}}$. It does not preserve the cross-ratio, but it just conjugates it. It can be considered as our direct projective involution composed with a line reflection through the real axis, and therefore it preserves the set of circles and lines in the same way. The orientation of angles, however, is inverted. In particular, a circle remains fixed under the classical inversion if and only if it is the unit circle or a circle perpendicular to it :



By calculating the power of O we also see that a circle is fixed if and only if it passes through two points $P \leftrightarrow P'$ exchanged in our inversion, i.e. O, P and P' are collinear and $OP \cdot OP' = 1$. On the other hand, we can construct the partner P' of P by constructing circles perpendicular to the unit circle and passing through P . This shows the following important lemma.

Lemma 2. *The property of points being exchanged by an inversion remains invariant under inversions of the whole figure.* \square

Let us recall some classical facts about pencils of circles in two dimensional real (or complex projective geometry). For two fixed points, which we can consider to be $(-1; 0)$ and $(1; 0)$, the set of circles passing through them is called an elliptic pencil of circles. The centers of those circles lie on the axis $x = 0$ and are given by $(x; y) = (0; c)$. Their equations are $x^2 + (y - c)^2 = c^2 + 1$. The radical axis of any two of those circles is $y = 0$: it is considered to be part of the elliptic pencil of circles. A point on this axis is given by $(x; y) = (d; 0)$, and its common power with respect to all circles of our elliptic pencil is given by $(d - 1)(d + 1) = d^2 - 1$. Therefore a circle with center $(d; 0)$ and equation $(x - d)^2 + y^2 = d^2 - 1$ is perpendicular to all circles of our elliptic pencil, and it passes through the two complex fixed points $(0; i)$ and $(0; -i)$. The set of all those circles is called a hyperbolic pencil of circles. The radical axis of any two of them is $x = 0$ (the common power of $(0; c)$ is $c^2 + 1$) : it is considered to be part of the hyperbolic pencil.

Remark. There exists a two dimensional complex projective involution exchanging the points $(0; i)$ and $(0; -i)$ with the so called cyclic points (circular points at infinity in the complex projective plane that are contained in the complexification of every real circle). It fixes the real plane and defines on it a two dimensional projective involution conserving all lines and all circles

belonging to our hyperbolic pencil, while exchanging their common radical axis with the line at infinity.

Finally, all circles passing through $(0;0)$ and tangent to the line $y = 0$ form a parabolic pencil; and all circles passing through $(0;0)$ and tangent to the line $x = 0$ form the perpendicular parabolic pencil.

An inversion with center $(-1;0)$ transforms our elliptic pencil of circles into a pencil of concurrent lines, whereas its perpendicular hyperbolic pencil becomes a pencil of concentric circles. Two perpendicular parabolic pencils of circles can be transformed into two perpendicular pencils of parallel lines. An inversion with respect to a circle defines an involution on all circles of its pencil whereas all circles of its perpendicular pencil remain fixed.

Any point of the plane (different from the 0, 1 or 2 special ones) belongs to a unique circle of a pencil of circles, and to a unique circle of its perpendicular pencil. Any two circles belong to a unique pencil of circles : elliptic if they are intersecting, hyperbolic if they are disjoint and parabolic if they are tangent. In particular, an inversion allows us to transform two intersecting circles into two intersecting lines, two disjoint circles into two concentric ones, and two tangent circles into two parallel lines. If a circle is perpendicular to two circles of a pencil, then it belongs to the perpendicular pencil.

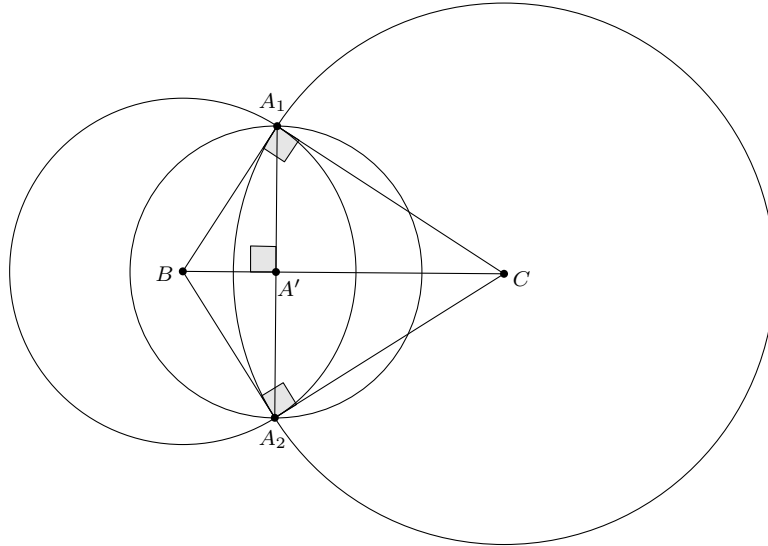
Remark. Usually, one proves that three circles belong to the same pencil by showing that there are two points having the same power with respect to all of them. For example, if the lines p, q, r, s form a quadrilateral with vertices $A := s \cap p$, $A' := q \cap r$, $B := r \cap s$, $B' := p \cap q$, $C := p \cap r$, $C' := q \cap s$, then the circles of diameter AA' , BB' , CC' belong to the same pencil because the orthocenters of $\{p, q, r\}$, $\{p, q, s\}$, $\{p, r, s\}$, $\{q, r, s\}$ all have the same power with respect to them.

Lemma 3. *The inversions with respect to two different circles or lines K and K' commute with each other if and only if K and K' are perpendicular.*

Proof. By our preceding lemma, we can suppose that K and K' are either concentric or two lines. But in those cases, the proof is evident. \square

If two perpendicular circles K and K' have centers B and C and intersect in A_1 and A_2 , then the composition of the inversions with respect to K and K' is equal to the composition of the reflections with respect to the line A_1A_2 and the perpendicular circle of diameter A_1A_2 . Indeed, it is sufficient to proof this after an inversion sending A_2 to ∞ , but then it becomes evident : the composition of the reflections through two perpendicular lines always is a point reflection.

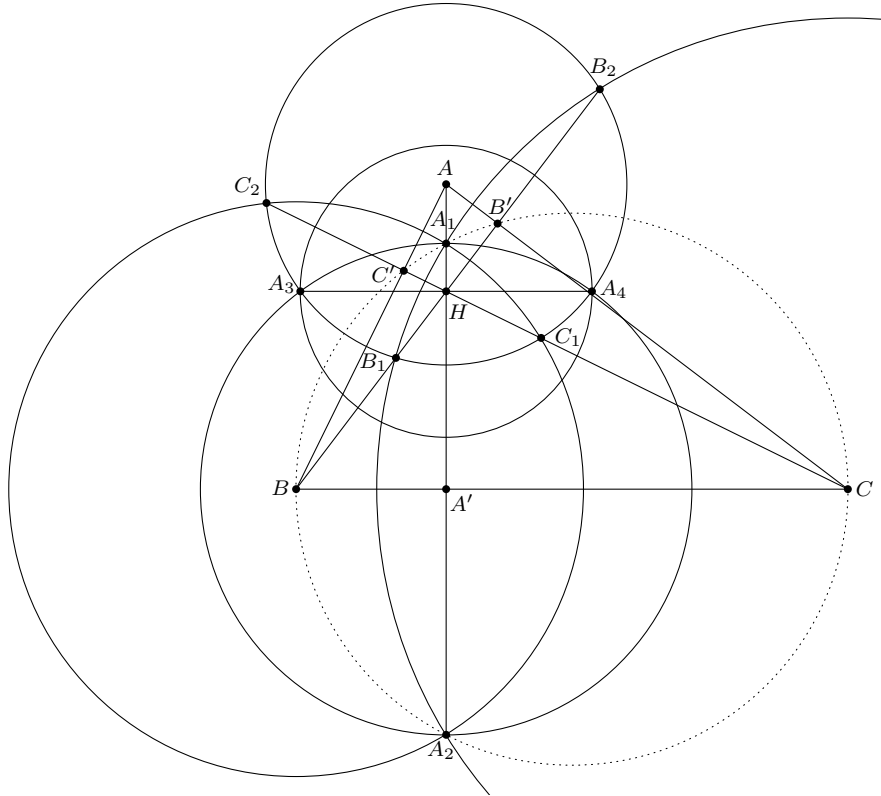
The inversion with respect to K fixes A_1 and A_2 while exchanging $B \leftrightarrow \infty$ and $C \leftrightarrow A'$, where $A' := A_1A_2 \cap BC$. The inversion with respect to K' fixes A_1 and A_2 while exchanging $C \leftrightarrow \infty$ and $B \leftrightarrow A'$. Their composition is a direct projective involution with fixed points A_1 and A_2 exchanging $B \leftrightarrow C$ and $A' \leftrightarrow \infty$. This confirms the classical identities following from similar triangles : $BA' \cdot BC = BA_1^2$, $CA' \cdot CB = CA_1^2$ and $A'B \cdot A'C = A'A_1^2$.



Let ABC be an arbitrary acute triangle with vertices ABC , orthocenter H and orthic triangle $A'B'C'$. Let us suppose that the circle of diameter BC intersects AA' in A_1 and A_2 . The points B_1, B_2 and C_1, C_2 are defined similarly. Then we have the classical identities $BB' \cdot BH = BC' \cdot BA = BA' \cdot BC = BA_1^2 = BA_2^2 = BC_1^2 = BC_2^2$ and $A'B' \cdot A'C' = A'A \cdot A'H = A'B \cdot A'C = A'A_1^2 = A'A_2^2$, etc. Therefore the inversion with center B and radius $BA_1 = BA_2 = BC_1 = BC_2$ exchanges $B \leftrightarrow \infty, B' \leftrightarrow H, C' \leftrightarrow A, A' \leftrightarrow C$ and commutes with the analogous inversions with center A or C . The composition of our inversions with center B and C gives a direct projective involution with fixed points A_1 and A_2 exchanging $A' \leftrightarrow \infty, B' \leftrightarrow C', A \leftrightarrow H, B \leftrightarrow C$. It can be represented as the composition of an inversion with center A' and radius $A'A_1 = A'A_2$ with a reflection through AA' .

Since our circle with center A (and radius $AB_1 = AB_2 = AC_1 = AC_2$) is perpendicular to our circles with center B and C , it must belong to the perpendicular pencil and is also perpendicular to the circle with diameter A_1A_2 intersecting it in two points A_3, A_4 . Since H has the same power with respect to our circles with center A, B, C and also with diameter A_1A_2 , it lies on A_3A_4 . Therefore the composition of an inversion with center A' and radius $A'A_1 = A'A_2$ with a reflection through AA' and our inversion with center A can be simplified as the composition of an inversion with center H and radius $HA_3 = HA_4$ and line reflections through A_1A_2 and A_3A_4 . Finally, the composition of our three pairwise commuting inversions with center A, B, C is an inversion with center H (their radical center) and radius $HA_3 = HA_4$ composed with a point reflection with center H . This is also called an inversion with negative radius and exchanges $H \leftrightarrow \infty, A' \leftrightarrow A, B' \leftrightarrow B, C' \leftrightarrow C$ (the identity $HA' \cdot HA = HB' \cdot HB = HC' \cdot HC$ is classical). Those pairwise commuting inversions are the last theorem of [1].

Finally, we have three inversions with center A, B, C , three direct projective involutions with center A', B', C' and axis AA', BB', CC' , respectively, and one inversion with center H and negative radius. Any two of them commute, and their composition gives the one with a collinear center, if we consider that A', B', C' are also collinear. Actually, we have seen at the end of the previous section that three pairwise commuting direct projective involutions are naturally embedded in this picture.

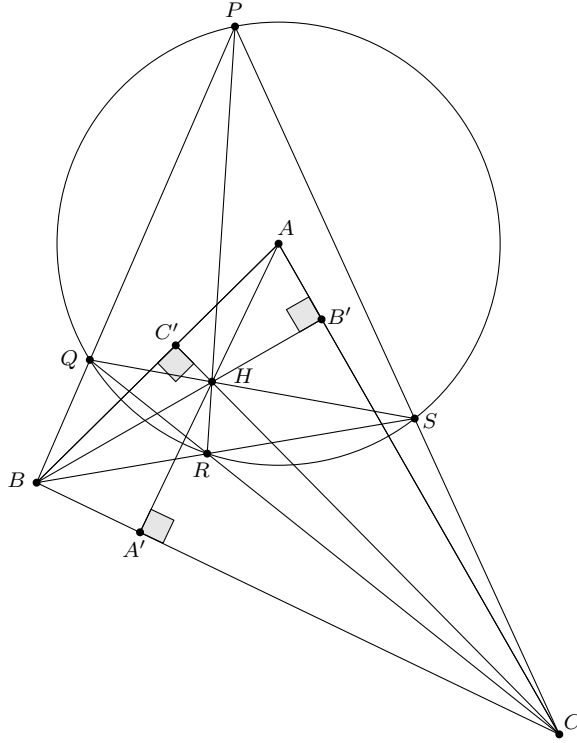


As mentioned at the beginning of this section, the fixed circles of the inversion $z \mapsto f(z) = \frac{1}{\bar{z}}$ are the unit circle and all circles perpendicular to it.

The fixed circles of the inversion with negative radius $z \mapsto f(z) = -\frac{1}{\bar{z}}$ are the unit circle and all circles intersecting it in two opposite vertices with respect to O . Indeed, a fixed circle must contain both interior and exterior points of the unit circle. Therefore it has to intersect it in two points, and two opposite points are then necessary and sufficient.

Finally, the fixed circles of our direct projective involution $z \mapsto f(z) = \frac{1}{z}$ are the unit circle, the pencil of all circles passing through the fixed points -1 and 1 , and its perpendicular pencil.

For any four points P, Q, R, S , there exists a direct projective involution exchanging $P \leftrightarrow Q$, $R \leftrightarrow S$ because $b_{P,Q,R,S} = b_{Q,P,S,R}$. An indirect projective involution, however, exists if and only if P, Q, R, S are collinear or concyclic, because the cross-ratio is conjugated, and invariant if and only if it is real. If P, Q, R, S are concyclic in that order, then we have an inversion with center $B := PQ \cap RS$ exchanging $P \leftrightarrow Q$, $R \leftrightarrow S$, an inversion with center $C := QR \cap SP$ exchanging $Q \leftrightarrow R$, $S \leftrightarrow P$, an inversion with negative radius and center $H := PR \cap QS$ exchanging $P \leftrightarrow R$, $Q \leftrightarrow S$, and an inversion with center A (the center of the circumscribed circle of $PQRS$) fixing P, Q, R, S . Those transformations commute pairwise, because the commutation relation holds on the set $\{P, Q, R, S\}$ and then (cross-ratio) everywhere. Therefore we get exactly our previous picture : H is the orthocenter of ABC and A', B', C' are the three Miquel points of P, Q, R, S , see the end of our previous section.



Actually, the triangle BCH is autopolar. In total, we have seven commuting involutions (with center A, B, C, H, A', B', C') and we know their actions on $A, B, C, H, A', B', C', P, Q, R, S$ perfectly. In particular, every line not passing through the center of an involution becomes a circle. Therefore we can find many additional circles in the figure.

Lemma 4. *A pencil of circles and its perpendicular pencil define two commuting projective involutions on any fixed circle or line K not belonging to any of the two pencils.*

Proof. If a circle of our pencil intersects K in A and A' , then we have to show that the exchange $A \leftrightarrow A'$ is a projective transformation (it is of course an involution), and that this transformation commutes with the one obtained from the perpendicular pencil. But those affirmations become evident if we transform our perpendicular pencils by an inversion into pencils of concurrent lines and concentric circles. \square

For any two lines or slopes a and b , it is classical to define (a, b) as the angle, defined modulo 180° , we have to turn a counterclockwise so that it becomes parallel to b . Of course, for any three lines or slopes a, b, c we have $(a, b) + (b, c) = (a, c) \pmod{180^\circ}$: Chasles's relation well known for vectors. Therefore, it is natural to fix some line o as "origin" and to define $a := (o, a)$, $b := (o, b)$, etc. Then the relation $(o, a) + (a, b) = (o, b) \pmod{180^\circ}$ can be rewritten as

$$(a, b) = b - a \pmod{180^\circ}. \tag{5.1}$$

In this context, expressions like $a + b$ do depend on the choice of o , but barycentric expressions such as $\frac{a+b}{2}$ do not: it is defined modulo 90° and provides the slopes of the two perpendicular bisectors of the slopes a and b .

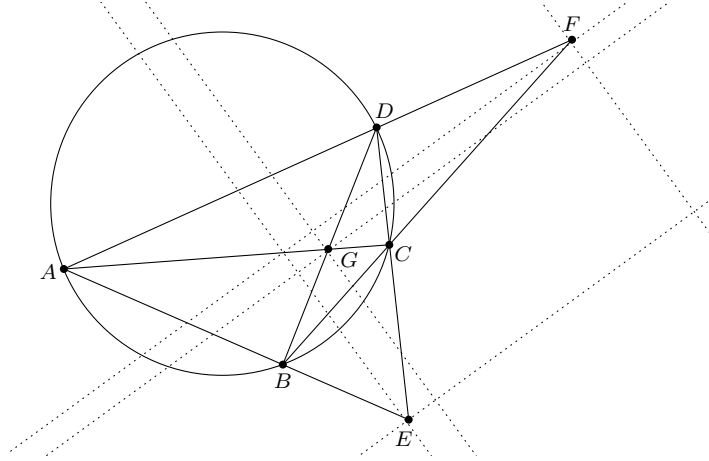
It is classical that $ABCD$ is a cyclic quadrilateral if and only if we have for its slopes modulo 180° one of the following equivalent relations :

$$(AC, BC) = (AD, BD) \Leftrightarrow BC - AC = BD - AD \Leftrightarrow AD + BC = AC + BD. \quad (5.2)$$

Therefore the more symmetrical relation is actually

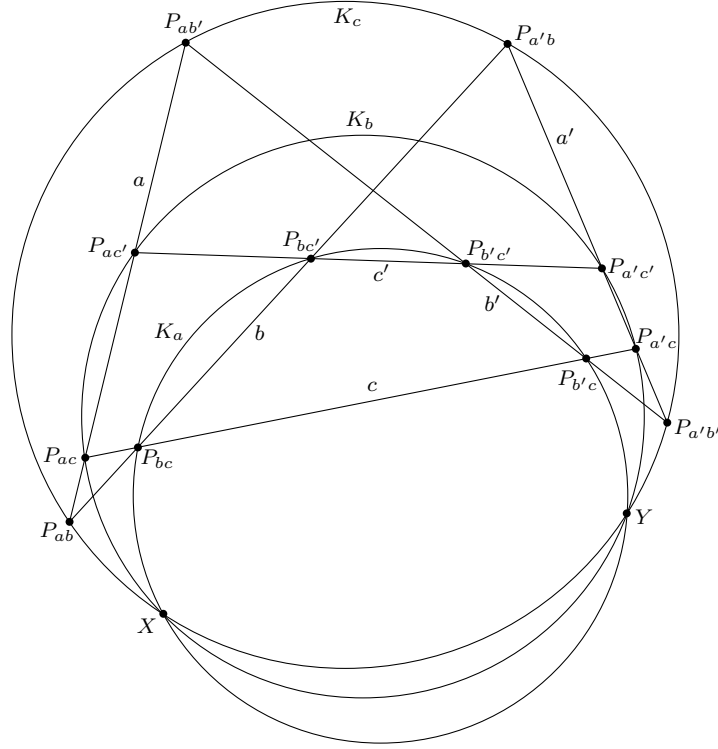
$$\frac{AD + BC}{2} = \frac{AC + BD}{2} = \frac{AB + CD}{2} \pmod{90^\circ} \quad (5.3)$$

showing that the slopes of the bisectors of two opposite sides of a cyclic quadrilateral do not depend on the choice of those opposite sides :



In fact, a reflection through any of the two perpendicular slopes of our bisectors exchanges the slopes of $AD \leftrightarrow BC$, $AC \leftrightarrow BD$, $AB \leftrightarrow CD$. It is Desargues's involution on the line at infinity, see [4].

Two pairs of lines $\{a, a'\}$ and $\{b, b'\}$ define a unique quadrilateral with vertices $P_{ab} := a \cap b$, $P_{ab'} := a \cap b'$, $P_{a'b} := a' \cap b$, $P_{a'b'} := a' \cap b'$. It is cyclic if and only if $\frac{a+a'}{2} = \frac{b+b'}{2} \pmod{90^\circ}$. If $\{c, c'\}$ is a third pair of lines, then $\{a, a'\}$ and $\{c, c'\}$ also define a unique quadrilateral with vertices $P_{ac} := a \cap c$, $P_{ac'} := a \cap c'$, $P_{a'c} := a' \cap c$, $P_{a'c'} := a' \cap c'$. It is cyclic if and only if $\frac{a+a'}{2} = \frac{c+c'}{2} \pmod{90^\circ}$. But in this case we also have $\frac{b+b'}{2} = \frac{c+c'}{2} \pmod{90^\circ}$. Therefore the quadrilateral with vertices $P_{bc} := b \cap c$, $P_{bc'} := b \cap c'$, $P_{b'c} := b' \cap c$, $P_{b'c'} := b' \cap c'$ is also cyclic.



Lemma 5. Let $\{a, a'\}$, $\{b, b'\}$ and $\{c, c'\}$ be three pairs of lines such that

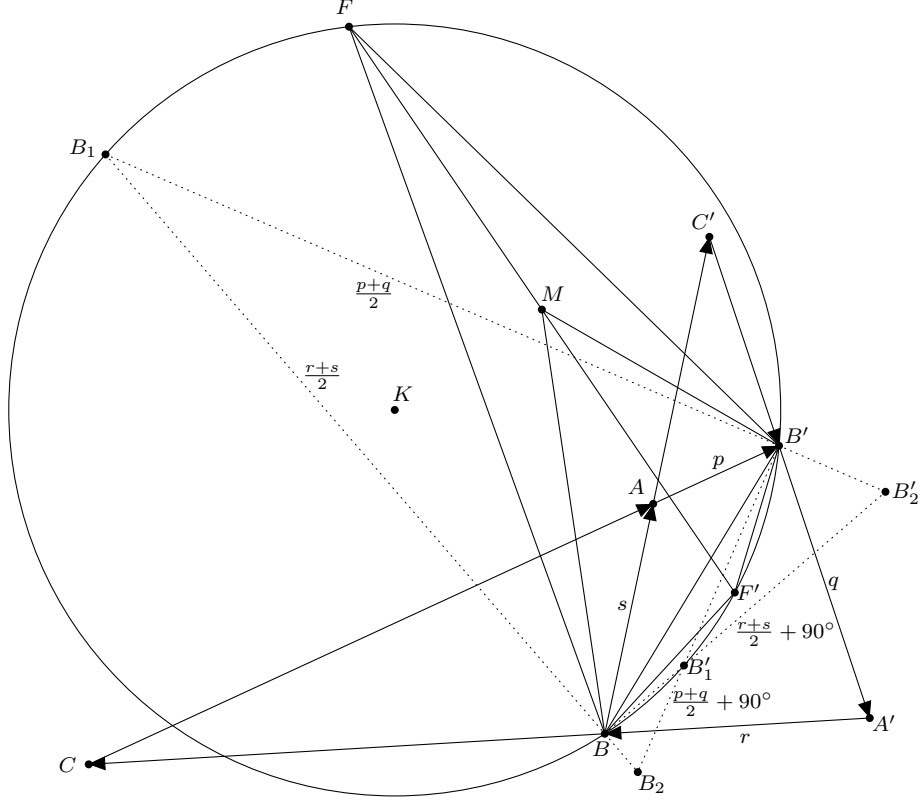
$$\frac{a + a'}{2} = \frac{b + b'}{2} = \frac{c + c'}{2} \pmod{90^\circ}. \quad (5.4)$$

Then the pairs $\{a, a'\}$ and $\{b, b'\}$ define a cyclic quadrilateral with circumscribed circle K_c . Similarly, the pairs $\{a, a'\}$ and $\{c, c'\}$ define K_b , whereas $\{b, b'\}$ and $\{c, c'\}$ define K_a . Then K_a , K_b and K_c belong to the same pencil of circles.

Proof. Let K'_a be the unique circle which passes through P_{bc} and belongs to the pencil defined by K_b and K_c . Then K'_a , K_b and K_c define a projective involution on the line c , and the pencil of conics passing through the four points P_{ab} , $P_{ab'}$, $P_{a'b}$, $P_{a'b'}$ also defines a projective involution on the line c . This second pencil contains K_c but also the two degenerated conics formed by the two lines $\{a, a'\}$ or $\{b, b'\}$. Therefore our two involutions defined on c both exchange the intersection points of c with K_c (which can be complex) as well as $P_{ac} \leftrightarrow P_{a'c}$. But since those two involutions coincide on two pairs of points, they coincide everywhere, and in particular, K'_a must pass through $P_{b'c}$, exchanged with P_{bc} .

Similarly, K'_a , K_b and K_c and the pencil of conics passing through P_{ac} , $P_{ac'}$, $P_{a'c}$, $P_{a'c'}$ define the same involution on b proving that K'_a also passes through $P_{bc'}$. Therefore K'_a and K_a coincide. This concludes our proof. \square

Let p, q, r and s be four lines defining a quadrilateral with vertices $A := s \cap p$, $A' := q \cap r$, $B := r \cap s$, $B' := p \cap q$, $C := p \cap r$, $C' := q \cap s$. We orient p, q, r, s as in the following figure in order to be able to calculate with their slopes modulo 360° .



In particular, the Miquel point M of $\{p, q, r, s\}$ does not only lie on the circumcircles of $\{p, q, r\}$, $\{p, q, s\}$, $\{p, r, s\}$, $\{q, r, s\}$: it is also the center of many spiral similarities allowing us to calculate the following angles at M .

Lemma 6. *Modulo 360° we have*

$$\angle AMA' = q + r - s - p, \quad \angle BMB' = p + q - r - s, \quad \angle CMC' = s + q - p - r. \quad \square \quad (5.5)$$

But M is also the center of the direct projective involution exchanging $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$. Its fixed points $\{F, F'\}$ lie on the common bisector of $\angle AMA'$, $\angle BMB'$, $\angle CMC'$ and form harmonic points with $\{A, A'\}$, $\{B, B'\}$ and $\{C, C'\}$.

Lemma 7. *Modulo 180° we have*

$$(FA, FA') = (F'A, F'A') = \frac{q+r}{2} - \frac{s+p}{2}, \quad (5.6)$$

$$(FB, FB') = (F'B, F'B') = \frac{p+q}{2} - \frac{r+s}{2}, \quad (5.7)$$

$$(FC, FC') = (F'C, F'C') = \frac{s+q}{2} - \frac{p+r}{2}. \quad \square \quad (5.8)$$

The bisector at B' with slope $\frac{p+q}{2}$ and the bisector at B with slope $\frac{r+s}{2}$ intersect in a point which we denote B_1 , whereas the perpendicular bisectors at B' and B intersect in B'_1 . We denote B_2 and B'_2 the other two intersection points of our bisectors at B' and B , see the preceding figure. Since

$$(B_1B, B_1B') = (B'_1B, B'_1B') = \frac{p+q}{2} - \frac{r+s}{2} \pmod{180^\circ}, \quad (5.9)$$

the circumcircle of $BB'B_1B'_1$ belongs to the elliptic pencil of circles passing through F and F' . Therefore it is fixed under the direct projective involution $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$.

Since the circumcircle of $BB'B_2B'_2$ is perpendicular to the circumcircle of $BB'B_1B'_1$ and passes through B, B' (exchanged in our involution), it is also fixed in our involution. Therefore it belongs to the perpendicular pencil of circles.

If the points $A_1, A'_1, A_2, A'_2, C_1, C'_1, C_2, C'_2$ are defined in the same way, then we get the following proposition.

Proposition 5. *The circumcircles of $AA'A_1A'_1$, $BB'B_1B'_1$ and $CC'C_1C'_1$ belong to the elliptic pencil of circles passing through F and F' , whereas the circumcircles of $AA'A_2A'_2$, $BB'B_2B'_2$ and $CC'C_2C'_2$ belong to the perpendicular hyperbolic pencil of circles. All six circles are fixed under the direct projective involution $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$, $M \leftrightarrow \infty$ with fixed points F and F' . \square*

The triangle defined by its sides $\{p, q, r\}$ usually has one incenter and three excenters, but since its sides are oriented, it is natural to consider that it has a unique incenter denoted I_s , see the following figure. Similarly, I_p, I_q, I_r are the unique incenters of the triangles with oriented sides $\{q, r, s\}$, $\{r, s, p\}$, $\{s, p, q\}$, respectively. If we change the orientation of p and r (or of q and s) simultaneously, then those unique incenters become I'_p, I'_q, I'_r, I'_s . We have

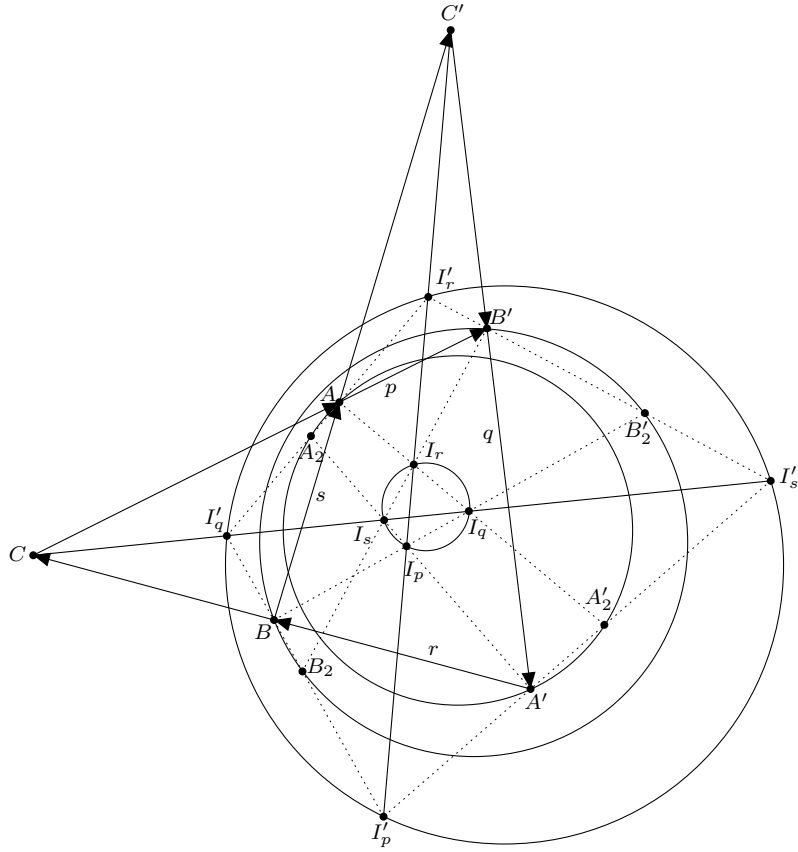
$$\frac{I'_p I'_r + I'_q I'_s}{2} = \frac{I_p I_q + I_r I_s}{2} = \frac{I'_p I'_q + I'_r I'_s}{2} = \frac{p + q + r + s}{4} \pmod{90^\circ} \quad (5.10)$$

showing that $I_p I_q I_r I_s, I'_p I'_q I'_r I'_s$ and $BB'B_2B'_2$ are cyclic quadrilaterals the circumcircles of which belong to the same pencil. Similarly,

$$\frac{I'_p I'_r + I'_q I'_s}{2} = \frac{I_s I_p + I_q I_r}{2} = \frac{I'_s I'_p + I'_q I'_r}{2} = \frac{p + q + r + s}{4} \pmod{90^\circ} \quad (5.11)$$

show that $I_p I_q I_r I_s, I'_p I'_q I'_r I'_s$ and $AA'A_2A'_2$ are cyclic quadrilaterals the circumcircles of which belong to the same pencil. Those two pencils, however, coincide because they have two circles in common defining a pencil uniquely. Therefore the circumcircles of $I_p I_q I_r I_s$ and $I'_p I'_q I'_r I'_s$ belong to the hyperbolic pencil we have already seen in the preceding proposition.

This argument can be applied for all kinds of reorientations of the sides of $\{p, q, r, s\}$ and proves the following theorem.



Theorem 4. Let $\{p, q, r, s\}$ be a quadrilateral with oriented sides as in our figures, and let F, F' be the fixed points of the direct projective involution exchanging its opposite vertices pairwise, and its Miquel point with ∞ . If we reorient an odd number of sides of $\{p, q, r, s\}$, then the circumcircle of $I_p I_q I_r I_s$ belongs to the elliptic pencil of circles passing through F and F' . If we reorient an even number of sides of $\{p, q, r, s\}$, then the circumcircle of $I_p I_q I_r I_s$ belongs to the perpendicular hyperbolic pencil of circles. In both cases, the circumcircles are fixed under our direct projective involution. \square

Remark. The preceding theorem was conjectured by Auguste Ramondou because it provides an easy proof for problem 6 of the IMO 2011, solved by only 6 students during the competition.

6 Geometry on Pairs of Points in Involution

We have already seen that for any four fixed points A, A', B, B' there is a unique direct projective involution exchanging $A \leftrightarrow A'$ and $B \leftrightarrow B'$. Let us denote our pairs of points in involution by $\{A, A'\} = \dot{A}$ and $\{B, B'\} = \dot{B}$. We suppose that our involution is not affine. Therefore we can choose 0 and 1 in such a way that $AA' = BB' = 1$. Motivated by (2.6) we define for any point C

$$r_{\dot{A}, \dot{B}, C} := r_{A, B, C} \cdot r_{A', B', C} = \frac{(A - C)(A' - C)}{(B - C)(B' - C)}. \quad (6.1)$$

Replacing A' by $\frac{1}{A}$ and B' by $\frac{1}{B}$ we get the following equivalence :

$$\begin{aligned}
r_{\hat{A},\hat{B},C} = k &\Leftrightarrow (A - C)\left(\frac{1}{A} - C\right) = k(B - C)\left(\frac{1}{B} - C\right) \\
&\Leftrightarrow \left[k\left(B + \frac{1}{B}\right) - \left(A + \frac{1}{A}\right)\right] \cdot C = (k - 1)(C^2 + 1) \\
&\Leftrightarrow C^2 - \frac{k\left(B + \frac{1}{B}\right) - \left(A + \frac{1}{A}\right)}{k - 1} \cdot C + 1 = 0.
\end{aligned} \tag{6.2}$$

This is a quadratic equation in C having two solutions (which we call C and C') uniquely determined by Vieta's formulas :

$$C \cdot C' = 1, \quad \frac{C + C'}{2} = \frac{k\frac{B + \frac{1}{B}}{2} - \frac{A + \frac{1}{A}}{2}}{k - 1}. \tag{6.3}$$

Therefore our direct projective involution exchanging $A \leftrightarrow A'$ and $B \leftrightarrow B'$ also exchanges $C \leftrightarrow C'$. If we define $\{C, C'\} = \hat{C}$,

$$r_{\hat{A},\hat{B},\hat{C}} := r_{\hat{A},\hat{B},C} = r_{\hat{A},\hat{B},C'}, \tag{6.4}$$

and

$$\hat{A} := \frac{A + A'}{2}, \quad \hat{B} := \frac{B + B'}{2}, \quad \hat{C} := \frac{C + C'}{2}, \tag{6.5}$$

(the midpoints of AA' , BB' , CC'), then we get an equivalence as in (2.6) :

$$r_{\hat{A},\hat{B},\hat{C}} = k \quad \Leftrightarrow \quad \hat{C} = \frac{k\hat{B} - \hat{A}}{k - 1} \quad \Leftrightarrow \quad r_{\hat{A},\hat{B},\hat{C}} = k. \tag{6.6}$$

We have proven the following theorem.

Theorem 5. *If a direct projective involution exchanges $A \leftrightarrow A'$, $B \leftrightarrow B'$ and $C \leftrightarrow C'$, then*

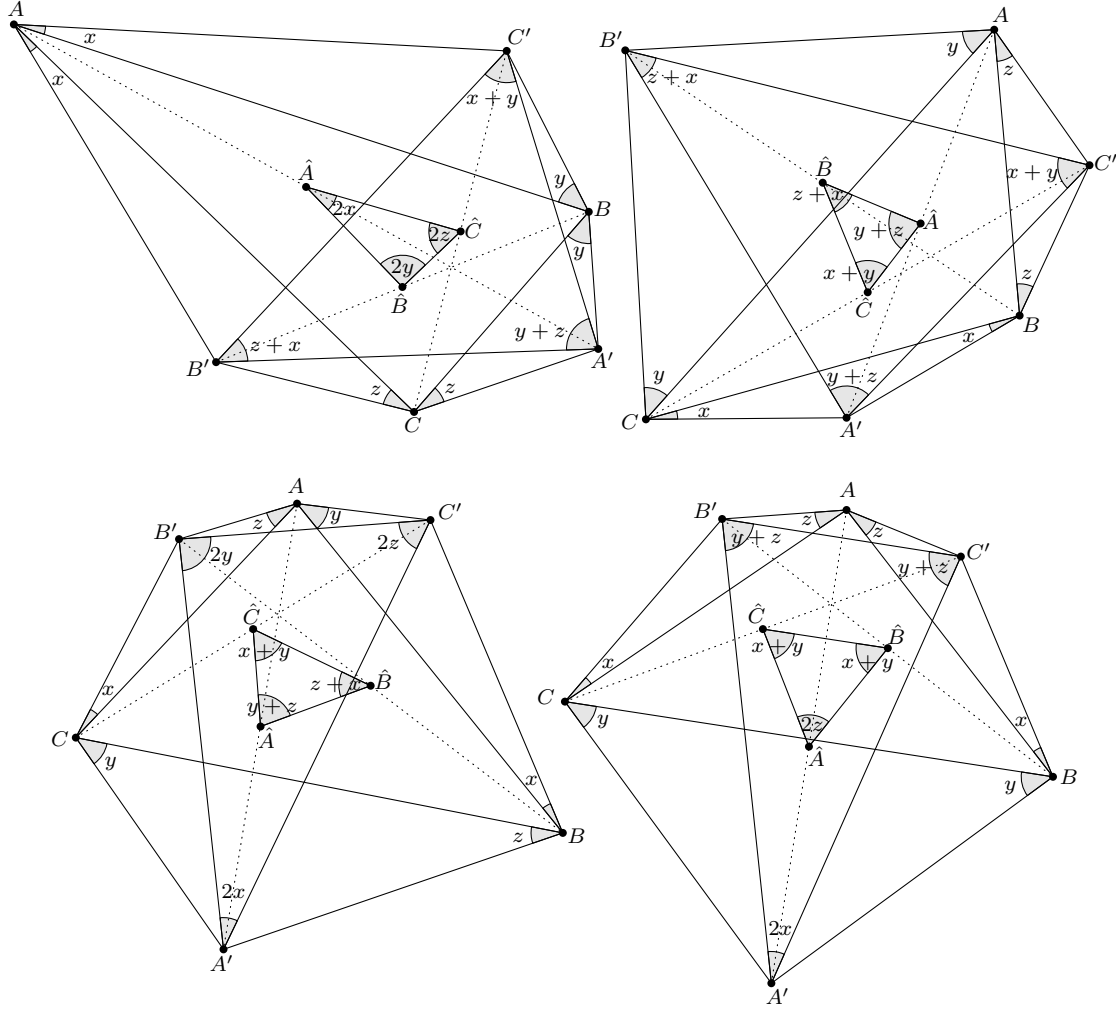
$$r_{\hat{A},\hat{B},\hat{C}} = r_{\hat{A},\hat{B},C} = r_{\hat{A},\hat{B},C'} = r_{\hat{A},\hat{B},\hat{C}}. \tag{6.7}$$

In particular,

$$\begin{aligned}
\angle \hat{B}\hat{C}\hat{A} &= \angle BCA + \angle B'CA' = \angle BCA' + \angle B'CA \\
&= \angle BC'A + \angle B'C'A' = \angle BC'A' + \angle B'C'A \pmod{360^\circ}.
\end{aligned} \tag{6.8}$$

If \hat{A} and \hat{B} are given, then for every complex number k there is a unique pair of points $\hat{C} = \{C, C'\}$ exchanged in the same involution as \hat{A} and \hat{B} and such that $r_{\hat{A},\hat{B},\hat{C}} = k$. It can be found by Vieta's relations or by solving a quadratic equation. \square

Example. Let x, y, z be three (oriented) angles such that $x + y + z = 90^\circ$. If they appear in the triangles $C'AB$, $A'BC$, $B'CA$ at the vertices A, B, C as in the following figures (already seen earlier), then our preceding theorem allows us to calculate the angles of $\hat{A}\hat{B}\hat{C}$ with the help of x, y, z only, without knowing the angles of ABC .



If A^*, B^*, C^* are the midpoints of BC, CA, AB , respectively, then the following triangles are similar : $ABC' \sim B^*A^*\hat{C}$, $BCA' \sim C^*B^*\hat{A}$, $CAB' \sim A^*C^*\hat{B}$, and Serge Bidallier remarked that the identity

$$S_{\hat{B};C^* \mapsto A^*} \circ S_{\hat{A};B^* \mapsto C^*} \circ S_{\hat{C};A^* \mapsto B^*} = id. \quad (6.9)$$

can be used in order to calculate the angles of $\hat{A}\hat{B}\hat{C}$. This identity, however, is equivalent to the existence of a direct projective involution exchanging $\hat{A} \leftrightarrow A^*$, $\hat{B} \leftrightarrow B^*$, $\hat{C} \leftrightarrow C^*$, which follows directly from our involution $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$, invariant under all permutations within the pairs \hat{A}, \hat{B} or \hat{C} . This proves the following theorem.

Theorem 6. *Let $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$ be a direct projective involution, and let us choose $A'' \in \{A, A'\}$, $B'' \in \{B, B'\}$ and $C'' \in \{C, C'\}$. If A^*, B^*, C^* are the midpoints of $B''C'', C''A'', A''B''$, respectively, and $\hat{A}, \hat{B}, \hat{C}$ are the midpoints of AA', BB', CC' , respectively, then there exists a direct projective involution $\hat{A} \leftrightarrow A^*$, $\hat{B} \leftrightarrow B^*$, $\hat{C} \leftrightarrow C^*$. \square*

For our pairs of points in involution $\hat{A} = \{A, A'\}$, $\hat{B} = \{B, B'\}$ and $\hat{C} = \{C, C'\}$ we have already shown that $r_{\hat{A}, \hat{B}, \hat{C}}$ is equal to the more classical $r_{\hat{A}, \hat{B}, \hat{C}}$. This directly implies the following corollary.

Corollary 4. *We have*

$$r_{\dot{A},\dot{B},\dot{C}} \cdot r_{\dot{B},\dot{A},\dot{C}} = 1, \quad r_{\dot{A},\dot{B},\dot{C}} + r_{\dot{A},\dot{C},\dot{B}} = 1. \quad (6.10)$$

In particular, $r_{\dot{A},\dot{B},\dot{C}}$ is real if and only if $r_{\dot{A},\dot{C},\dot{B}}$ is real. \square

The complex number $r_{\dot{A},\dot{B},\dot{C}} = \frac{(A-C)(A'-C)}{(B-C)(B'-C)}$ is real if and only if the slopes of CA , CA' , CB , CB' satisfy

$$CA + CA' - CB - CB' = 0^\circ \pmod{180^\circ} \Leftrightarrow \frac{CA + CA'}{2} = \frac{CB + CB'}{2} \pmod{90^\circ}, \quad (6.11)$$

that is if and only if there is a line reflection exchanging simultaneously the slopes $CA \leftrightarrow CA'$ and $CB \leftrightarrow CB'$.

Corollary 5. *If $\dot{A} = \{A, A'\}$, $\dot{B} = \{B, B'\}$ and $\dot{C} = \{C, C'\}$ are three pairs of points in involution, then their midpoints \hat{A} , \hat{B} and \hat{C} are collinear if and only if $r_{\dot{A},\dot{B},\dot{C}}$ is real, and this happens if and only if there is a line reflection exchanging simultaneously the slopes $CA \leftrightarrow CA'$ and $CB \leftrightarrow CB'$. In that case we also call \dot{A} , \dot{B} and \dot{C} collinear.*

A more classical formulation in this context is the following :

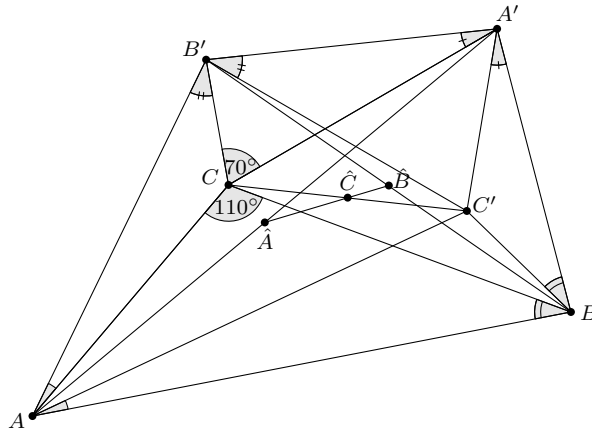
Corollary 6. *Let $ABA'B'$ be a convex quadrilateral and C a point in its interior. Then C has an isogonal conjugate if and only if $\angle ACB + \angle A'CB' = 180^\circ$.*

Proof. First of all, identifying CA, CA', CB, CB' with their slopes, we get

$$\begin{aligned} \angle ACB + \angle A'CB' = 180^\circ &\Leftrightarrow (CB - CA) + (CB' - CA') = 0^\circ \pmod{180^\circ} \\ &\Leftrightarrow \frac{CA + CA'}{2} = \frac{CB + CB'}{2} \pmod{90^\circ} \\ &\Leftrightarrow r_{\dot{A},\dot{B},\dot{C}} \in \mathbb{R}. \end{aligned} \quad (6.12)$$

If this condition is satisfied, then the unique direct projective involution $A \leftrightarrow A'$, $B \leftrightarrow B'$ exchanges C with some point C' . It is the isogonal conjugate of C by the results of our preceding corollary.

On the other hand, if C has an isogonal conjugate C' , then we can consider the unique direct projective involution $B \leftrightarrow B'$, $C \leftrightarrow C'$. If it exchanges A with some point A'' , then C and C' are isogonal conjugates in $B'ABA''$ because $r_{\dot{B},\dot{C},\dot{A}} \in \mathbb{R}$. But this implies $A'' = A'$ and finishes our proof. \square



Remark. I proposed the preceding corollary as an exercise for an Animath test in 2009. The students found two proofs : One based on the fact that the condition $\angle ACB + \angle A'CB' = 180^\circ$ (or the condition for isogonal conjugation) holds if and only if the composition of the line reflections through CA , CB , CA' and CB' is the identity; and another based on the fact that the projections of C to the sides of the convex quadrilateral are concyclic.

In the particular cas $C = C'$ our preceding corollary implies the following one.

Corollary 7. *If $ABA'B'$ is a circumscribed quadrilateral with incenter I , then there exists a direct projective involution with fixed point I exchanging $A \leftrightarrow A'$ and $B \leftrightarrow B'$. In particular, the midpoints of AA' and BB' are collinear with I .*

We have already seen that there is a direct projective involution exchanging the opposite vertices of a complete quadrilateral $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$. Moreover \hat{A} , \hat{B} and \hat{C} are collinear, because $r_{\hat{A},\hat{B},\hat{C}}$ is a real number. Therefore the midpoints of AA' , BB' and CC' are collinear (a well-known classical fact).

Last but not least, if $\hat{A} = \{A, A'\}$, $\hat{B} = \{B, B'\}$, $\hat{C} = \{C, C'\}$ and $\hat{D} = \{D, D'\}$ are four pairs of points in involution, then we can define

$$b_{\hat{A},\hat{B},\hat{C},\hat{D}} := r_{\hat{A},\hat{B},\hat{C}} \cdot r_{\hat{B},\hat{A},\hat{D}}, \quad (6.13)$$

which is invariant under projective transformations. Once again, we get

$$b_{\hat{A},\hat{B},\hat{C},\hat{D}} = b_{\hat{A},\hat{B},\hat{C},\hat{D}}, \quad (6.14)$$

where $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are the respective midpoints of AA', BB', CC', DD' . We can define that $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ are concyclic if and only if $b_{\hat{A},\hat{B},\hat{C},\hat{D}}$ is a real number. But in order to calculate $b_{\hat{A},\hat{B},\hat{C},\hat{D}}$ we only need to know one of the points C, C' and one of the points D, D' . In other words, the whole classical geometry can be developed for pairs of points in involution instead of just points.

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