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# Around the asymptotic properties of a two-dimensional parametrized Euler flow

Marc Briane\*

November 14, 2023

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## Abstract

We study the two-dimensional Euler flow solution to  $\partial_t X(\cdot, x) = b(X(\cdot, x))$  for  $x$  in the torus  $\mathbb{T}^2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$ , where  $b$  is the vector field defined on  $\mathbb{T}^2$  by

$$b(x) = b(x_1, x_2) := (-A \cos x_1 - B \sin x_2, A \sin x_1 + B \cos x_2) \text{ with } A, B \in \mathbb{R} \setminus \{0\}.$$

We derive for any  $x \in \mathbb{T}^2$ , the asymptotics of  $X(t, x)$  as  $t$  tends to  $\infty$ , depending on whether  $|A| = |B|$  or  $|A| \neq |B|$ . In the first case, the orbits of the flow are all bounded. In the second case, it turns out that one of the coordinates of  $X(t, x)$  is bounded with an explicit bound, while the other one is equivalent to  $a(x)t$ . The function  $a$  does not vanish in  $\mathbb{T}^2$  and satisfies uniform bounds which depend on parameters  $A, B$ . When  $|A| \neq |B|$ , we also prove that for any global first integral  $u$  of the flow  $X$  with a periodic gradient,  $\nabla u$  has at least a cluster point of roots in  $\mathbb{T}^2$ . This shows the complexity as well as the interest of this two-dimensional Euler flow.

**Keywords:** 2D Euler flow, asymptotic expansion, invariant measure, rotation set, first integral

**Mathematics Subject Classification:** 34E05, 34E10, 37C10, 37C40

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# 1 Introduction

In this paper we study the asymptotic behavior of the following two-dimensional Euler flow  $X(t, x)$ , for  $x$  in the torus  $\mathbb{T}^2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$ , solution to the ODEs system

$$\begin{cases} \partial_t X_1(t, x) = -A \cos(X_1(t, x)) - B \sin(X_2(t, x)) & =: b_1(X(t, x)) \\ \partial_t X_2(t, x) = A \sin(X_1(t, x)) + B \cos(X_2(t, x)) & =: b_2(X(t, x)), \end{cases} \quad t \in [0, \infty), \quad (1.1)$$

where  $A, B$  are two fixed non-zero real parameters.

The vector field  $b$  defined by (1.1) represents the velocity solution to the steady Euler equation (see, *e.g.*, [16])

$$\partial_t \omega + \psi_1 \omega_2 - \psi_2 \omega_1 = 0, \quad (1.2)$$

where the function  $\omega$  denotes the (scalar) fluid vorticity, and  $\psi$  solution to  $-\Delta\psi = \omega$ , denotes the stream function. By definition (1.1) we have

$$\omega(x) = (\text{curl } b)(x) = (\partial_{x_1} b_2 - \partial_{x_2} b_1)(x) = A \cos x_1 + B \cos x_2 \quad \text{and} \quad \psi(x) = \omega(x).$$

Conditions of non-linear stability for plane stationary flows were derived by Arnold [1, 2] in the Sixties. According to [16, 7] the study of the stability of the flow (1.1) is actually relevant for the atmospheric flows. For example, Saturn's E Ring may be regarded as a two-dimensional flat torus, considering as in [16] (see in particular [7, Section 1.4.] and the references therein) the non-stationary Euler equation (1.2).

First of all, note that the existence of a global first integral of system (1.1) having a periodic gradient of period  $2\pi\mathbb{Z}^2$ , is far to be evident. Indeed, when  $|A| \neq |B|$ , we prove (see Corollary 3.5) that for any first integral  $u \in C^1(\mathbb{R}^2)$  of (1.1), *i.e.* satisfying

$$\forall x \in \mathbb{T}^2, \forall t \in [0, \infty), \quad u(X(t, x)) = u(x), \quad (1.3)$$

with a periodic gradient, the gradient  $\nabla u$  has at least a cluster point in  $\mathbb{T}^2$ . This strong constraint compared to the non vanishing property of the very simple vector field  $b$  associated with the flow  $X$  (1.1) makes difficult or even impossible the derivation of a global first integral of  $X$ , but in the same time shows the interest of the flow  $X$ .

In the present context, we focus on the asymptotic properties of the flow  $X$  through the limits

$$\lim_{t \rightarrow \infty} \frac{X(t, x)}{t} \quad \text{for any } x \in \mathbb{T}^2, \quad (1.4)$$

and on the probability measures on  $\mathbb{T}^2$  which are invariant for this flow (see Section 3.1). Actually, these two notions are strongly connected in dimension two in view of the Misiurewicz-Ziemian [11, Theorem 3.4] in the case of general two-dimensional flows, or in view of the Peirone result [12, Theorem 3.1] (see also [13, Theorem 4.1]) in the case of the ODE's flow.

On the one hand, in Section 2 we provide (see Theorem 2.1) the limits (1.4) of the flow  $X(\cdot, x)$  for each point  $x \in \mathbb{T}^2$ . Two cases have to be distinguished according to the non-zero parameters  $A, B$  (the case  $AB = 0$  has been studied in [5, Section 5.3.1]):

- If we have  $|A| = |B|$ , then the flow  $X$  can be computed explicitly thanks to a change of variables (see formula (2.21)), which gives immediately the asymptotics of  $X(\cdot, x)$ .
- Otherwise we have  $|A| \neq |B|$ , then the general Lemma 2.5 combined with the case (2) of the proof of [12, Theorem 3.1] allows us to prove that one of the coordinate  $X_i(\cdot, x)$ ,  $i = 1, 2$ , of the flow is bounded with an explicit bound, while the other one  $X_{3-i}(t, x)$  is equivalent to  $a(x)t$  as  $t \rightarrow \infty$  for some non-zero real  $a(x)$ . We also provide (see the part *iii*) of Theorem 3.1) uniform bounds from below and above for  $a$ , which depend on the two parameters  $A, B$ .

On the other hand, in Section 3 we characterize in the case  $|A| \neq |B|$  (see Theorem 3.1) the invariant probability measures on  $\mathbb{T}^2$  for the flow  $X$  (1.1), *i.e.* the probability measures  $\mu$  on  $\mathbb{T}^2$  such that for any  $t \in \mathbb{R}$ ,  $\mu \circ X^{-1}(t, \cdot) = \mu \circ X(-t, \cdot) = \mu$  on  $\mathbb{T}^2$ . We prove that there does not exist any invariant probability measure with Lebesgue's density  $\sigma(x) dx$  where  $\sigma$  is a non-negative function in  $L^1_{\sharp}(\mathbb{T}^2)$  satisfying some weak positivity condition (see (3.11)). Moreover, we obtain any limit (1.4) of  $X(\cdot, x)$  as the mass of the vector field  $b$  by a singular probability measure depending on  $x \in \mathbb{T}^2$ .

## Notation

- $(e_1, e_2)$  denotes the canonical basis of  $\mathbb{R}^2$ , and  $0_{\mathbb{R}^2}$  denotes the null vector of  $\mathbb{R}^2$ .
- $I_2$  is the unit matrix of  $\mathbb{R}^{2 \times 2}$ .
- $R_{\perp}$  denotes the  $(2 \times 2)$  rotation matrix  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . For any  $\xi \in \mathbb{R}^2$ ,  $\xi^{\perp}$  denotes the perpendicular vector  $R_{\perp}\xi$ .
- “ $\cdot$ ” denotes the scalar product and  $|\cdot|$  the euclidean norm in  $\mathbb{R}^2$ .
- $B(x, R)$  denotes the euclidean open ball of  $\mathbb{R}^2$  centered on  $x \in \mathbb{R}^2$  and of radius  $R > 0$ .
- $dx$  denotes the Lebesgue measure on  $\mathbb{R}^2$ .
- $\mathbb{T}^k$  denotes the torus  $\mathbb{R}^k / (2\pi\mathbb{Z}^k)$  for  $k = 1, 2$ .
- $\mathcal{M}_{\sharp}(\mathbb{T}^2)$  denotes the set of Radon measures on the torus  $\mathbb{T}^2$ , and  $\mathcal{M}_p(\mathbb{T}^2)$  denotes the set of probability measures on  $\mathbb{T}^2$ .
- $\mathcal{D}'(\Omega)$  denotes the set of the distributions on some open set  $\Omega$  of  $\mathbb{R}^2$ .
- $C_c^k(\mathbb{R}^2)$ ,  $k \in \mathbb{N} \cup \{\infty\}$ , denotes the space of the real-valued functions in  $C^k(\mathbb{R}^2)$  with compact support in  $\mathbb{R}^2$ .
- $C_{\sharp}^k(\mathbb{T}^2)$ , resp.  $W_{\sharp}^{1,k}(\mathbb{T}^2)$ ,  $k \in \mathbb{N} \cup \{\infty\}$ , denotes the space of the real-valued functions  $f \in C^k(\mathbb{R}^2)$ , resp.  $f \in W_{\sharp}^{1,k}(\mathbb{R}^2)$  which are  $2\pi\mathbb{Z}^2$ -periodic, *i.e.*

$$\forall k \in \mathbb{Z}^2, \forall x \in \mathbb{R}^2, \quad f(x + 2\pi k) = f(x). \quad (1.5)$$

- For any function  $f \in L^1_{\sharp}(\mathbb{T}^2)$ , we denote

$$\begin{cases} \bar{f}^{x_1}(x_2) := \int_{\mathbb{T}^1} f(x_1, x_2) dx_1 \\ \bar{f}^{x_2}(x_1) := \int_{\mathbb{T}^1} f(x_1, x_2) dx_2, \end{cases} \quad \text{and} \quad \bar{f} := \int_{\mathbb{T}^2} f(x) dx. \quad (1.6)$$

and for any Borel measure  $\mu$  on  $\mathbb{T}^2$  and any function  $f \in L^1(\mathbb{T}^2, \mu)$ , we denote

$$\mu(f) := \int_{\mathbb{T}^2} f(x) \mu(dx), \quad (1.7)$$

which is also extended to vector-valued functions in  $L^1(\mathbb{T}^2, \mu)^2$ .

- $C$  denotes a positive constant which may vary from line to line.

## 2 Asymptotics of the flow

### 2.1 The main result

We have the following result.

**Theorem 2.1.** *Let  $A, B \in \mathbb{R} \setminus \{0\}$ . Then, the flow  $X$  defined by (1.1) satisfies the following asymptotic alternative:*

- *Either we have  $(-1)^i (|A| - |B|) < 0$  for some  $i = 1, 2$ . Then, we obtain that*

$$\forall x \in \mathbb{T}^2, \quad \begin{cases} |X_i(\cdot, x) - x| < 2\pi & \text{in } [0, \infty) \\ \lim_{t \rightarrow \infty} \frac{X_{3-i}(t, x)}{t} = a(x) & \text{where } a(x) \neq 0. \end{cases} \quad (2.1)$$

- *Or we have  $B = \varepsilon A$  with  $\varepsilon = \pm 1$ . Then, for any  $x \in \mathbb{T}^2$ , the orbit  $X(\mathbb{R}, x)$  is bounded in  $\mathbb{R}^2$ . More precisely, defining the  $(2 \times 2)$  matrix*

$$J := \frac{1}{2} \begin{pmatrix} 1 & \varepsilon \\ -\varepsilon & 1 \end{pmatrix} \quad \text{with} \quad J^{-1} := \begin{pmatrix} 1 & -\varepsilon \\ \varepsilon & 1 \end{pmatrix} \quad (2.2)$$

and the closed grid of  $\mathbb{R}^2$

$$\mathcal{G} := \left( \bigcup_{m \in \mathbb{Z}} \left[ \frac{\pi}{4} + m\pi \right] \times \mathbb{R} \right) \cup \left( \bigcup_{n \in \mathbb{Z}} \mathbb{R} \times \left[ \frac{\pi}{2} + n\pi \right] \right), \quad (2.3)$$

we get that

$$\forall i = 1, 2, \quad \forall x \in J^{-1}(\mathbb{R}^2 \setminus \mathcal{G}), \quad |X_i(\cdot, x) - x_i| < 2\pi. \quad (2.4)$$

**Remark 2.2.** The asymptotic behavior is quite different between the cases  $|A| \neq |B|$  and  $|A| = |B|$ . The reason of this difference is that the vector field  $b$  of (1.1) does not vanish in the case  $|A| \neq |B|$  with no punctual orbit – see the preliminary step of the proof of [12, Theorem 3.1] – contrary to the case  $|A| = |B|$  which includes punctual orbits.

**Remark 2.3.** When  $A > 0$  and  $B = 0$ , in [5, Section 5.3.1] we have got the following asymptotic expansion

$$X(t, x) = x - A t e_2 + O_K(1), \quad \forall t \in [0, \infty), \quad \forall x \in K \times \mathbb{R}, \quad (2.5)$$

where  $O_K(1)$  is bounded uniformly with respect to  $t \in [0, \infty)$ , to  $x_2 \in \mathbb{R}$ , and to  $x_1$  in any fixed compact set  $K$  of  $(-\frac{\pi}{2}, \frac{\pi}{2})$  modulo  $\pi$ . Expansion (2.5) can be thus regarded as a limit expansion as  $B \rightarrow 0$  of (2.1). However, expansion (2.1) is finer than (2.5) for the coordinate  $X_1(t, x)$ , while it is the converse for the coordinate  $X_2(t, x)$ .

**Remark 2.4.** Assume that  $(-1)^i (|A| - |B|) < 0$  for some  $i = 1, 2$ . Since the smooth vector field  $b$  associated with the flow  $X$  (1.1) satisfies

$$\forall x = (x_1, x_2) \in \mathbb{R}^2, \quad |b(x)|^2 = A^2 + B^2 + 2AB \sin(x_1 + x_2) \geq (|A| - |B|)^2 > 0, \quad (2.6)$$

it does not vanish in  $\mathbb{R}^2$ . Then, applying the non vanishing final argument of the proof of [3, Theorem 3.1] <sup>(1)</sup> we deduce that the limit function  $a$  does not vanish either in  $\mathbb{T}^2$ . Peirone's [12, Theorem 3.1] allows us to characterize more precisely the function  $a$  in (2.1). Indeed, following

<sup>1</sup>This argument is based on the two-dimensional Franks [8, Theorem 3.5].

the case (1) of the proof of [12, Theorem 3.1], it turns out that for any  $x \in \mathbb{T}^2$ , there exist  $T_x > 0$  and an integer  $k_x \in \mathbb{Z} \setminus \{0\}$  such that the function  $a$  satisfies

$$a(x) = \frac{2\pi k_x}{T_x}.$$

Actually, there exists a periodic solution of period  $T_x$  in the torus  $\mathbb{T}^2$  (rather than in the space  $\mathbb{R}^2$ )  $X(\cdot, z_x)$  to (1.1) with  $z_x \in \mathbb{T}^2$ , satisfying

$$X(T_x, z_x) = z_x + 2\pi k_x e_{3-i} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = a(x) e_{3-i}. \quad (2.7)$$

However, it is not obvious to derive explicitly such a periodic solution  $Z$ , even for a simple system like (1.1).

There is an alternative to derive the limit (2.7). By virtue of [11, Theorem 2.4, 3.4] and [11, Remark 2.5] (see also [4, Appendix A]) involving the properties of the Herman rotation set [10] associated with a flow, for any  $x \in \mathbb{T}^2$ , there exists a probability measure  $\mu_x$  on  $\mathbb{T}^2$  invariant for the flow (1.1) such that

$$\lim_{t \rightarrow \infty} \frac{X(t, x)}{t} = \mu_x(b) = \int_{\mathbb{T}^2} b(y) \mu_x(dy). \quad (2.8)$$

Using Liouville's theorem (see, *e.g.*, [14, Lecture 11]) the invariance of the probability measure  $\mu_x$  is characterized by

$$\operatorname{div}(b \mu_x) = 0 \quad \text{in } \mathbb{T}^2. \quad (2.9)$$

But once again, it is difficult to get generally an explicit invariant probability measure for a given flow. This is the aim of Theorem 3.1 below for the two-dimensional Euler flow (1.1).

## 2.2 Proof of Theorem 2.1

### 2.2.1 Proof of the case $|A| \neq |B|$

The proof is based on the following general result.

**Lemma 2.5.** *Let  $f$  be a function in  $W_{\#}^{1,\infty}(\mathbb{R}/T\mathbb{Z})$ ,  $T > 0$  (i.e.  $f$  is Lipschitz and periodic of period  $T$  in  $\mathbb{R}$ ), let  $x_0 \in \mathbb{R}$ , and let  $x$  be a function in  $C^1([0, \infty))$  such that  $f(x(0)) \neq f(x_0)$ . Then, we have the two following implications:*

$$\begin{cases} x' + f(x) < f(x_0) & \text{in } [0, \infty) \Rightarrow x - x(0) < T & \text{in } [0, \infty), \\ x' + f(x) > f(x_0) & \text{in } [0, \infty) \Rightarrow x - x(0) > -T & \text{in } [0, \infty), \end{cases} \quad (2.10)$$

where  $f(x) := f \circ x$ .

Assume for instance that  $|A| > |B|$ , and set

$$\alpha := \frac{A}{|A|} = \pm 1 \quad \text{and} \quad \beta := \frac{B}{|A|} \in (-1, 1). \quad (2.11)$$

Then, consider the new coordinates  $y := x/|A|$  which leads us to the flow  $Y(t, y) := X(t, x)/|A|$  solution to the ODEs system

$$\begin{cases} \partial_t Y_1(t, y) &= -\alpha \cos(|A| Y_1(t, y)) - \beta \sin(|A| Y_2(t, y)) \\ \partial_t Y_2(t, y) &= \alpha \sin(|A| Y_1(t, y)) + \beta \cos(|A| Y_2(t, y)). \end{cases} \quad (2.12)$$

Fix  $y = (y_1, y_2) \in \mathbb{R}^2$ , and let  $\theta \in [0, \pi/|A|]$  be such that

$$|\beta| < \alpha \cos(|A|\theta) \neq \pm \alpha \cos(|A|y_1).$$

Then, the first equation of (2.12) implies that

$$\begin{cases} \left| \partial_t Y_1(\cdot, y) + \alpha \cos(|A|Y_1(\cdot, y)) \right| < \alpha \cos(|A|\theta) & \text{in } [0, \infty) \\ \cos(|A|\theta) \neq \pm \cos(|A|y_1). \end{cases}$$

Therefore, applying the first inequality of (2.10) to the function  $x(\cdot) := Y_1(\cdot, y_1)$  with  $x_0 := \theta$  and  $f(\cdot) := \alpha \cos(|A|\cdot)$  (which is  $|A|$ -Lipschitz and periodic of period  $T := 2\pi/|A|$  in  $\mathbb{R}$ ), then applying the second inequality of (2.10) to the function  $x(\cdot) := Y_1(\cdot, y_1)$  with  $x_0 := \pi/|A| - \theta$  and  $f$ , we get that

$$|Y_1(\cdot, y) - y_1| < T \quad \text{in } [0, \infty),$$

which implies the first estimate of (2.1) for  $i = 1$ . The proof is quite similar for  $i = 3$ , replacing  $Y_1$  by  $Y_2$  and permuting the roles of  $A$  and  $B$ .

On the other hand, let  $i := 1, 2$ . Then, recalling Remark 2.4 combined with the first estimate of (2.1) (we have just proved before) we get that for any  $x \in \mathbb{T}^2$ ,

$$\exists \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} \neq 0_{\mathbb{R}^2} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{X_i(t, x)}{t} = 0,$$

which yields immediately the limit of (2.1).

Therefore, the proof of the case  $|A| \neq |B|$  is done.  $\square$

*Proof of Lemma 2.5.* First of all, there exists a unique  $n \in \mathbb{Z}$  such that

$$x_0 + nT \leq x(0) < x_0 + (n+1)T.$$

Moreover due to the periodicity of  $f$  and to  $f(x(0)) \neq f(x_0)$ , we have  $x(0) \neq x_0 + nT$ . Hence, we deduce that

$$x_0 + nT < x(0) < x_0 + (n+1)T. \quad (2.13)$$

Now, assume that the left hand-side of the first implication of (2.10) holds, and consider the solution  $y$  to the ODE

$$y' + f(y) = f(x_0) \quad \text{in } [0, \infty), \quad y(0) = x(0). \quad (2.14)$$

Assume by contradiction that there exists  $t \in [0, \infty)$  such that  $y(t) = x_0 + nT$ . Then,  $y$  and the constant function  $y_s := x_0 + nT$  are both solutions to the ODE:  $u' + f(u) = f(x_0)$  in  $[0, \infty)$ , and they agree at point  $t$ . Then, by an uniqueness argument we get that  $y = y_s$ , which due to (2.13) and (2.14) is not satisfied at the point 0. This leads us to a contradiction. Similarly, the trajectory  $y([0, \infty))$  cannot meet the point  $x_0 + (n+1)T$ . Therefore, by a connectedness argument we obtain that

$$x_0 + nT < y < x_0 + (n+1)T \quad \text{in } [0, \infty). \quad (2.15)$$

This combined with (2.13) implies that

$$|y - x(0)| < T \quad \text{in } [0, \infty). \quad (2.16)$$

Next, define the function  $z := x - y$ . Subtracting the equality (2.14) to the inequality satisfied by  $x$ , and using that  $f$  is  $k$ -Lipschitz for some  $k > 0$ , we get that

$$z' < f(x_0) - f(x) + f(y) - f(x_0) \leq k|z| \quad \text{in } [0, \infty). \quad (2.17)$$

Since by (2.14)  $z(0) = 0$ , inequality (2.17) shows that  $z'(0) < 0$ , which implies that  $z < 0$  in some interval  $[0, t_0]$  with  $t_0 > 0$ . We can then consider

$$S := \sup \{t \in [t_0, \infty) : z < 0 \text{ in } [t_0, t]\} \in [t_0, \infty).$$

Assume by contradiction that  $S < \infty$ . Then, we have

$$z < 0 \quad \text{in } [0, S) \quad \text{and} \quad z(S) = 0,$$

which combined with (2.17) implies that

$$\forall t \in [0, S), \quad \frac{d}{dt}(e^{kt}z(t)) = e^{kt}(z'(t) + kz(t)) < 0.$$

Hence, it follows that  $0 = e^{kS}z(S) < e^0z(0) = 0$ , which leads us to a contradiction. Therefore, we have  $S = \infty$ , and by the same argument as above, we get that  $z = x - y < 0$  in  $[0, \infty)$ . Finally, recalling (2.16) we obtain that  $x < y < x(0) + T$ .

The proof of the second implication of (2.10) is quite similar leading us to the inequality in  $[0, \infty)$ , which concludes the proof of Lemma 2.5.  $\square$

## 2.2.2 Proof of the case $|A| = |B|$

Assume that  $B = \varepsilon A$  with  $\varepsilon = \pm 1$ . Then, consider the new coordinates

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_1 + \varepsilon x_2 \\ -\varepsilon x_1 + x_2 \end{pmatrix} = J \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Jx. \quad (2.18)$$

where the matrix  $J$  is defined by (2.2).

By (1.1) we are led to the new flow  $Y(t, y) := JX(t, x)$  solution to the ODEs system

$$\begin{cases} \partial_t Y_1(t, y) &= \frac{A}{2} \left( -\cos(X_1(t, x)) - \varepsilon \sin(X_2(t, x)) + \varepsilon \sin(X_1(t, x)) + \cos(X_2(t, x)) \right) \\ \partial_t Y_2(t, y) &= \frac{A}{2} \left( \varepsilon \cos(X_1(t, x)) + \sin(X_2(t, x)) + \sin(X_1(t, x)) + \varepsilon \cos(X_2(t, x)) \right), \end{cases}$$

or equivalently, using trigonometrical formulas we have

$$\begin{cases} \partial_t Y_1(t, y) &= A \sin\left(\frac{X_1(t, x) + X_2(t, x)}{2}\right) \sin\left(\frac{X_1(t, x) - X_2(t, x)}{2}\right) \\ &\quad + \varepsilon A \cos\left(\frac{X_1(t, x) + X_2(t, x)}{2}\right) \sin\left(\frac{X_1(t, x) - X_2(t, x)}{2}\right) \\ \partial_t Y_2(t, y) &= A \sin\left(\frac{X_1(t, x) + X_2(t, x)}{2}\right) \cos\left(\frac{X_1(t, x) - X_2(t, x)}{2}\right) \\ &\quad + \varepsilon A \cos\left(\frac{X_1(t, x) + X_2(t, x)}{2}\right) \cos\left(\frac{X_1(t, x) - X_2(t, x)}{2}\right). \end{cases}$$

This combined with (2.18) thus implies the new system depending on  $\varepsilon = \pm 1$ ,

$$\begin{cases} \partial_t Y_1(t, y) &= -\varepsilon A \left( \sin\left(Y_{\frac{3-\varepsilon}{2}}(t, y)\right) + \varepsilon \cos\left(Y_{\frac{3-\varepsilon}{2}}(t, y)\right) \right) \sin\left(Y_{\frac{3+\varepsilon}{2}}(t, y)\right) \\ \partial_t Y_2(t, y) &= A \left( \sin\left(Y_{\frac{3-\varepsilon}{2}}(t, y)\right) + \varepsilon \cos\left(Y_{\frac{3-\varepsilon}{2}}(t, y)\right) \right) \cos\left(Y_{\frac{3+\varepsilon}{2}}(t, y)\right). \end{cases} \quad (2.19)$$



Now, assume for example that  $\varepsilon = 1$ . Then, system (2.19) reads as

$$\begin{cases} \partial_t Y_1(t, y) = -A\sqrt{2} \sin\left(Y_1(t, y) + \frac{\pi}{4}\right) \sin\left(Y_2(t, y)\right) \\ \partial_t Y_2(t, y) = A\sqrt{2} \sin\left(Y_1(t, y) + \frac{\pi}{4}\right) \cos\left(Y_2(t, y)\right). \end{cases} \quad (2.20)$$

It is clear that for any  $m \in \mathbb{Z}$  and  $y_2 \in \mathbb{R}$ ,  $Y(\cdot, y) = y := \left(-\frac{\pi}{4} + m\pi, y_2\right)$  is a stationary solution to system (2.20). Moreover, we can check that for any  $y_1 \in -\frac{\pi}{4} + 2n\pi + (-\pi, \pi)$ ,

$$\begin{cases} Y_1(t, y) = -\frac{\pi}{4} + 2n\pi + 2 \arctan \left[ e^{(-1)^{n+1}A\sqrt{2}t} \tan\left(\frac{y_1}{2} + \frac{\pi}{8}\right) \right] \\ Y_2(t, y) = \frac{\pi}{2} + n\pi, \end{cases} \quad t \in \mathbb{R}, \quad (2.21)$$

is also solution to system (2.20). Then, collecting all these solutions we obtain that the closed set  $\mathcal{G}$  defined by (2.3) is composed of trajectories of the flow  $Y$ : either stationary points along lines parallel to direction  $y_2$ , or closed line segments parallel to direction  $y_1$ . Hence, since the trajectories of the flow do not intersect, for any  $y \in \mathbb{R}^2 \setminus \mathcal{G}$ , the trajectory  $Y(\mathbb{R}, y)$  lies in one of the connected components of  $\mathbb{R}^2 \setminus \mathcal{G}$  which are the open squares of side  $\pi$  defined by

$$Q_{m,n} := \left(-\frac{\pi}{4} + m\pi, -\frac{\pi}{4} + (m+1)\pi\right) \times \left(\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\pi\right) \quad \text{for } (m, n) \in \mathbb{Z}^2. \quad (2.22)$$

Therefore, we get that

$$\forall i = 1, 2, \forall y \in \mathbb{R}^2 \setminus \mathcal{G}, \quad |Y_i(\cdot, y) - y_i| < \pi.$$

Finally, this combined with the change of coordinates (2.18) yields the desired estimates (2.4). The case  $\varepsilon = -1$  is quite similar, which concludes the proof of Theorem 2.1.  $\square$

### 3 Invariant measures and rotation set

#### 3.1 Some recalls of ergodic theory

Let  $b$  a vector field in  $C_{\#}^1(\mathbb{T}^2)$ . A probability measure  $\mu$  in  $\mathcal{M}_{\#}(\mathbb{T}^2)$  is said to *invariant for some flow  $X$*  associated with the vector field  $b$  by

$$\begin{cases} \partial_t X(t, x) = b(X(t, x)), \quad t \in \mathbb{R} \\ X(0, x) = x \in \mathbb{T}^2, \end{cases} \quad (3.1)$$

if one has

$$\forall t \in \mathbb{R}, \forall \psi \in C_{\#}^0(\mathbb{T}^2), \quad \int_{\mathbb{T}^2} \psi(X(t, y)) d\mu(y) = \int_{\mathbb{T}^2} \psi(y) d\mu(y). \quad (3.2)$$

By virtue of Liouville's theorem (see, *e.g.*, [9, Theorem 1, Section 2.2]) condition (3.2) means that

$$\operatorname{div}(b\mu) = 0 \quad \text{in } \mathbb{T}^2, \quad (3.3)$$

or equivalently, with the variational formulation (see, *e.g.*, [4, Proposition])

$$\forall \varphi \in C_{\#}^1(\mathbb{T}^2), \quad \int_{\mathbb{T}^2} b(x) \cdot \nabla \varphi(x) \mu(dx) = 0. \quad (3.4)$$

Alternatively, define the set

$$\mathcal{I}_b := \left\{ \mu \in \mathcal{M}_p(\mathbb{T}^2) : \mu \text{ invariant for the flow } X \right\}, \quad (3.5)$$

where  $\mathcal{M}_p(\mathbb{T}^2)$  is the set of probability measures on  $\mathbb{T}^2$ . Also define the two following non-empty subsets of  $\mathbb{R}^2$ :

- The set of all the limit points of the sequences  $(X(n, x)/n)_{n \geq 1}$  for  $x \in \mathbb{T}^2$  (denoted by  $\rho_p(b)$  in [11])

$$A_b := \bigcup_{x \in \mathbb{T}^2} \left[ \bigcap_{n \geq 1} \overline{\left\{ \frac{X(k, x)}{k} : k \geq n \right\}} \right]. \quad (3.6)$$

When  $b$  does not vanish in  $\mathbb{T}^2$ , by virtue of Peirone's [12, Theorem 3.1] the former definition is reduced to

$$A_b = \left\{ \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} : x \in \mathbb{T}^2 \right\}. \quad (3.7)$$

- The so-called Herman [10] rotation set (recall notation (1.7))

$$C_b := \{ \mu(b) : \mu \in \mathcal{I}_b \} \quad (3.8)$$

which is a compact convex subset of  $\mathbb{R}^2$ .

An implicit consequence of the Misiurewicz-Ziemian results [11, Theorem 2.4, Remark 2.5] and [11, Corollary 2.6] shows that

$$A_b \subset C_b = \text{conv}(A_b).$$

However, in dimension two [11, Theorem 3.4 (c)] shows that actually the set  $A_b$  (3.6) is convex which thus implies that

$$C_b = A_b. \quad (3.9)$$

When in addition  $b$  does not vanish in  $\mathbb{T}^2$ , we thus deduce from (3.7) that

$$C_b = \{ \mu(b) : \mu \in \mathcal{I}_b \} = \left\{ \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} : x \in \mathbb{T}^2 \right\} = A_b. \quad (3.10)$$

This links closely the invariant probability measures for the flow  $X$  (3.1) and the limit points of the flow.

### 3.2 The case of the two-dimensional Euler flow

The invariant probability measures (3.2) and the Herman rotation set (3.8) for the flow  $X$  defined by (1.1) are characterized as follows.

**Theorem 3.1.** *Assume that  $|A| \neq |B|$  for the vector field  $b$  defined in (1.1). Then, we have the following results:*

- i) *There does not exist any non-negative function  $\sigma \in L^1_{\sharp}(\mathbb{T}^2)$  whose integrals over its two sections are a.e. positive, i.e.*

$$\begin{cases} \bar{\sigma}^{x_2}(x_1) = \int_{\mathbb{T}^1} \sigma(x_1, x_2) dx_2 > 0 & \text{a.e. } x_1 \in \mathbb{T}^1 \\ \bar{\sigma}^{x_1}(x_2) = \int_{\mathbb{T}^1} \sigma(x_1, x_2) dx_1 > 0 & \text{a.e. } x_2 \in \mathbb{T}^1, \end{cases} \quad (3.11)$$

*such that the  $\sigma(x) dx$  is an invariant probability measure for the flow  $X$ .*

- ii) *Moreover, for any  $\zeta \in C_b$ , there exist  $T \in (0, \infty)$  and  $z \in \mathbb{T}^2$  such that the orbit  $X(\cdot, z)$  is periodic of period  $T$  in the torus  $\mathbb{T}^2$ , i.e.*

$$\exists k \in \mathbb{Z}^2, \quad X(T, z) = z + 2\pi k, \quad (3.12)$$

and

$$\zeta = \mu_{T,z}(b) \quad \text{where} \quad \mu_{T,z}(\varphi) := \frac{1}{T} \int_0^T \varphi(X(t, z)) dt \quad \text{for } \varphi \in C_{\sharp}^0(\mathbb{T}^2), \quad (3.13)$$

is an invariant probability measure which is singular with respect to Lebesgue's measure. In other words, Herman's rotation set is characterized by

$$\mathbf{C}_b = \{ \mu_{T,z}(b), T > 0, z \in \mathbb{T}^2 : X(T, z) - z \in 2\pi\mathbb{Z}^2 \}. \quad (3.14)$$

iii) Finally, assume that  $|A| > |B|$ . Denoting

$$\begin{cases} \xi_\mu := \mu(\cos x_1) e_1 + \mu(\sin x_1) e_2 \\ \eta_\mu := \mu(\sin x_2) e_1 + \mu(\cos x_2) e_2, \end{cases} \quad \text{for } \mu \in \mathcal{I}_b, \quad (3.15)$$

the asymptotic flow function  $a$  defined by (2.1) satisfies the bounds from below and above (with the convention:  $\inf \emptyset = \infty$ )

$$\begin{cases} \inf_{x \in \mathbb{T}^2} |a(x)| \geq \min \left( m_{AB} (|A| - |B|), \inf_{\{\mu \in \mathcal{I}_b : |\xi_\mu| \leq |\eta_\mu|\}} |A \xi_\mu + B \eta_\mu| \right) > 0, \\ m_{AB} := \frac{1}{\sqrt{2}} \inf_{\mu \in \mathcal{I}_b} \sqrt{|\xi_\mu|^2 + |\eta_\mu|^2} \in (0, 1], \end{cases} \quad (3.16)$$

and

$$\begin{cases} \sup_{x \in \mathbb{T}^2} |a(x)| \leq \sqrt{(|A| + |B|)^2 - M_{AB}^2}, \\ M_{AB} := \inf_{\mu \in \mathcal{I}_b} \int_{\mathbb{T}^2} |b - \mu(b)| \mu(dx) > 0. \end{cases} \quad (3.17)$$

**Remark 3.2.** Due to (2.6) the vector field  $b$  in (1.1) does not vanish when  $|A| \neq |B|$ . Hence, thanks to the first step of the proof of Peirone's [12, Theorem 3.1], for any periodic orbit in the torus the integer vector  $k$  in (3.12) is not null.

**Remark 3.3.** The part *i*) of Theorem 3.1 means that for any non-negative function  $\sigma$  in  $L_{\sharp}^1(\mathbb{T}^2)$  satisfying the weak positivity condition (3.11), the vector field  $\sigma b$  is not realizable as a divergence free field, or in conductivity term, as a current field. The dual problem (which is equivalent in dimension two) *i.e.* the realizability of a gradient field as an electric field has been addressed in [6] with a very simple counter-example [6, Example 2.7], but involving both regular (at least continuous) and positive functions  $\sigma$ . Here, the two-dimensional Euler flow induced by the vector field  $b$  (1.1) provides a more definite counter-example of realizability, since here the functions  $\sigma$  are only assumed to be in  $L_{\sharp}^1(\mathbb{T}^2)$  rather than regular, and are only non-negative with condition (3.11) rather than positive a.e. in  $\mathbb{T}^2$ .

**Remark 3.4.** On the one hand, as  $|A| - |B|$  tends to 0, the bound from below (3.16) clearly converges to 0. This asymptotic result is consistent with the picture of bounded orbits in the case  $|A| = |B|$  (see above the second part of Theorem 2.1 and Section 2.2.2), which due to the set equality (3.9) implies that the Herman rotation set  $\mathbf{C}_b$  (3.8) is reduced to the unit set  $\{0_{\mathbb{R}^2}\}$ .

On the other hand, the bound from above (3.17) has actually a general scope. Indeed, the proof of the part *iii*) of Theorem 3.1 below shows that the vector field  $b \in C^1(\mathbb{T}^2)^2$  needs to satisfy the two following conditions

$$\{x \in \mathbb{T}^2 : b(x) = 0_{\mathbb{R}^2}\} = \emptyset \quad \text{and} \quad \forall \xi \in \mathbb{R}^2, \# \{x \in \mathbb{T}^2 : b(x) = \xi\} < \infty. \quad (3.18)$$

Then, we still deduce from (3.10) the extension of (3.17)

$$\left\{ \begin{array}{l} \sup_{x \in \mathbb{T}^2} \left| \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} \right| \leq \sqrt{\|b\|_{L^\infty(\mathbb{T}^2)^2}^2 - M_b^2}, \\ M_b := \inf_{\mu \in \mathcal{I}_b} \int_{\mathbb{T}^2} |b - \mu(b)| \mu(dx) > 0. \end{array} \right. \quad (3.19)$$

**Corollary 3.5.** *Assume there exists a first integral  $u$  in  $C^1(\mathbb{R}^2)$  of the flow (1.1) having a periodic gradient of period  $2\pi\mathbb{Z}^2$ . Then,  $\nabla u$  has at least a cluster point of roots in the torus  $\mathbb{T}^2$ .*

*Proof of Theorem 3.1.*

*Proof of part i).* Assume for instance that  $|A| > |B|$ , and that there exists an invariant probability measure  $\sigma(x) dx$  for the flow  $X$  (1.1) with Lebesgue's density  $\sigma \in L^1_{\#}(\mathbb{T}^2)$  satisfying the weak positivity (3.11). By Liouville's theorem (3.3) combined with a classical duality result there exists a function  $\psi$  with  $\nabla\psi \in L^1_{\#}(\mathbb{T}^2)$  such that (recall the third notation)

$$\sigma b = \nabla^\perp \psi \quad \text{in } \mathbb{R}^2, \quad (3.20)$$

or equivalently,

$$\begin{cases} \sigma(x) (A \cos x_1 + B \sin x_2) = \partial_{x_2} \psi(x) \\ \sigma(x) (A \sin x_1 + B \cos x_2) = \partial_{x_1} \psi(x) \end{cases} \quad \text{a.e. } x = (x_1, x_2) \in \mathbb{T}^2. \quad (3.21)$$

Recalling (2.6) equalities (3.10) hold true. This combined with (2.1) and (3.20) implies that there exists  $x_\sigma \in \mathbb{T}^2$  such that

$$\overline{\nabla\psi} = -(\overline{\sigma b})^\perp = -\left( \lim_{t \rightarrow \infty} \frac{X(t, x_\sigma)}{t} \right)^\perp = -a(x_\sigma) e_2^\perp = a(x_\sigma) e_1, \quad \text{with } a(x_\sigma) \neq 0.$$

Then, since the function  $\psi_{\#} : y \mapsto (\psi(y) - \overline{\nabla\psi} \cdot y)$  is in  $W_{\#}^{1,1}(\mathbb{T}^2)$ , we get that

$$\overline{\partial_{x_2} \psi}^{x_2}(x_1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_{x_2} \psi(x_1, x_2) dx_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_{x_2} \psi_{\#}(x_1, x_2) dx_2 = 0 \quad \text{a.e. } x_1 \in \mathbb{T}^1. \quad (3.22)$$

Therefore, integrating the second equality of (3.21) with respect to variable  $x_2$  and taking into account (3.11) and (3.22), it follows that

$$|\cos x_1| = \left| -\frac{B}{A} \frac{\overline{\sigma \sin x_2}^{x_2}(x_1)}{\overline{\sigma}_{x_2}(x_1)} \right| \leq \frac{|B|}{|A|} \quad \text{a.e. } x_1 \in \mathbb{T}^1, \quad (3.23)$$

which, making  $x_1$  tend to 0, contradicts our assumption  $|A| > |B|$ .

*Proof of part ii).* Let  $\zeta \in \mathbf{C}_b$ . By (3.10) there exists  $x \in \mathbb{T}^2$  such that

$$\zeta = \lim_{t \rightarrow \infty} \frac{X(t, x)}{t}. \quad (3.24)$$

Then, consider the  $T_x$ -periodic (in the torus  $\mathbb{T}^2$ ) orbit  $X(\cdot, z_x)$  satisfying (2.7) for some  $i = 1, 2$ , and consider the probability measure  $\mu_{T_x, z_x}$  defined by (3.13). Due to the semi-group property satisfied by the flow, we have for any  $s \in \mathbb{R}$  and any  $\varphi \in C_{\#}^0(\mathbb{T}^2)$ ,

$$\begin{aligned} \mu_{T_x, z_x}(\varphi(X(s, \cdot))) &= \frac{1}{T_x} \int_0^{T_x} \varphi(X(s, X(t, z_x))) dt = \frac{1}{T_x} \int_0^{T_x} \varphi(X(s+t, z_x)) dt \\ &= \frac{1}{T_x} \int_s^{s+T_x} \varphi(X(t, z_x)) dt = \frac{1}{T_x} \int_0^{T_x} \varphi(X(t, z_x)) dt, \end{aligned}$$

since the function  $\varphi(X(\cdot, z_x))$  is  $T_x$ -periodic in  $\mathbb{R}$  taking into account the first equality of (2.7). Hence,  $\mu_{T_x, z_x}$  is an invariant probability measure for the flow  $X$ . Moreover, again using (2.7) we get that

$$\mu_{T_x, z_x}(b) = \frac{1}{T_x} \int_0^{T_x} b(X(t, z_x)) dt = \frac{1}{T_x} \int_0^{T_x} \partial_t X(t, z_x) dt = \frac{X(T_x, z_x) - z_x}{T_x} = a(x) e_{3-i},$$

which combined with (3.24) implies the desired equalities  $\zeta = \mu_{T_x, z_x}(b)$  and (3.14).

*Proof of bound (3.16).* By the definition of the vector field  $b$  in (1.1) and the definitions (3.15) of the vectors  $\xi_\mu, \eta_\mu$ , we have

$$\forall \mu \in \mathcal{S}_b, \quad \mu(b) = A \xi_\mu + B \eta_\mu.$$

Since the vector field  $b$  does not vanish in  $\mathbb{T}^2$  by (2.6), from the set equality (3.10) and the non vanishing property of the function  $a$  in (2.1), we deduce that

$$\inf_{x \in \mathbb{T}^2} |a(x)| = \inf_{\mu \in \mathcal{S}_b} |\mu(b)| \quad \text{and} \quad \forall \mu \in \mathcal{S}_b, \quad |\mu(b)| = |A \xi_\mu + B \eta_\mu| > 0. \quad (3.25)$$

This combined with the continuity of the mapping

$$\mu \in \mathcal{S}_b \mapsto |A \xi_\mu + B \eta_\mu|$$

defined on the compact set of invariant probability measures  $\mathcal{S}_b$  for the weak-\* measures topology, implies that

$$\inf_{\mu \in \mathcal{S}_b} |A \xi_\mu + B \eta_\mu| = \min \left( \inf_{\{\mu \in \mathcal{S}_b : |\xi_\mu| \geq |\eta_\mu|\}} |A \xi_\mu + B \eta_\mu|, \inf_{\{\mu \in \mathcal{S}_b : |\xi_\mu| \leq |\eta_\mu|\}} |A \xi_\mu + B \eta_\mu| \right) > 0. \quad (3.26)$$

Then, by the Cauchy-Schwarz inequality we have

$$|\xi_\mu|^2 + |\eta_\mu|^2 \leq \sum_{i=1}^2 (\mu(\cos^2 x_i) + \mu(\sin^2 x_i)) = 2.$$

Again applying the Cauchy-Schwarz inequality the constant  $m_{AB}$  of (3.16) thus satisfies the estimate

$$\begin{aligned} 1 \geq m_{AB} &= \frac{1}{\sqrt{2}} \inf_{\mu \in \mathcal{S}_b} \sqrt{|\xi_\mu|^2 + |\eta_\mu|^2} \geq \frac{1}{\sqrt{2(A^2 + B^2)}} \inf_{\mu \in \mathcal{S}_b} (A |\xi_\mu| + B |\eta_\mu|) \\ &\geq \frac{1}{\sqrt{2(A^2 + B^2)}} \inf_{\mu \in \mathcal{S}_b} |A \xi_\mu + B \eta_\mu| > 0. \end{aligned}$$

Now, focus on the second infimum of (3.26). Using the triangle inequality we have for any  $\mu \in \mathcal{S}_b$  satisfying  $|\xi_\mu| \geq |\eta_\mu|$  which also implies that  $|\xi_\mu| \geq m_{AB}$ ,

$$|A \xi_\mu + B \eta_\mu| = |\xi_\mu| \left| A \frac{\xi_\mu}{|\xi_\mu|} + B \frac{\eta_\mu}{|\xi_\mu|} \right| \geq |\xi_\mu| \left( |A| - |B| \frac{|\eta_\mu|}{|\xi_\mu|} \right) \geq m_{AB} (|A| - |B|).$$

Therefore, recalling (3.25) and (3.26) we obtain immediately the bound (3.16) with  $m_{AB} > 0$ .

*Proof of bound (3.17).* Assume by contradiction that there exists an invariant probability measure  $\mu \in \mathcal{S}_b$  for the flow  $X$ , satisfying  $b = \mu(b)$   $\mu$ -a.e. in  $\mathbb{T}^2$ . Then, the set  $J$  of the points

$z \in \mathbb{T}^2$  solutions to the equation  $b(z) = \mu(b)$  is clearly finite (we can check that  $1 \leq \#J \leq 4$ ). Hence, there exists a non-empty subset  $I$  of  $J$  such that the probability measure  $\mu$  reads as

$$\mu = \sum_{z \in I} \mu(\{z\}) \delta_z \text{ on } \mathbb{T}^2, \quad \text{with } \forall z \in I, \mu(\{z\}) > 0 \text{ and } \sum_{z \in I} \mu(\{z\}) = 1.$$

Hence, recalling (3.4) we have

$$\forall \varphi \in C_{\#}^1(\mathbb{T}^2), \quad \int_{\mathbb{T}^2} b(x) \cdot \nabla \varphi(x) \mu(dx) = \sum_{z \in I} \mu(\{z\}) b(z) \cdot \nabla \varphi(z) = 0,$$

which due to the arbitrariness of the function  $\varphi$  implies that for any  $z \in I$ ,  $b(z) = 0_{\mathbb{R}^2}$ , and leads us to contradiction. Therefore, we get that

$$\forall \mu \in \mathcal{I}_b, \quad \int_{\mathbb{T}^2} |b - \mu(b)| \mu(dx) > 0.$$

This combined with the compactness of the set  $\mathcal{I}_b$  of the invariant probability measures implies that the infimum  $M_{AB}$  of (3.17) is positive.

Next, by the Cauchy-Schwarz inequality we have for any  $\mu \in \mathcal{I}_b$ ,

$$\begin{aligned} M_{AB}^2 &\leq \int_{\mathbb{T}^2} |b - \mu(b)|^2 \mu(dx) = \mu(|b|^2) - |\mu(b)|^2 \\ &= A^2 + B^2 + 2AB \left( \int_{\mathbb{T}^2} \sin(x_1 + x_2) \mu(dx) \right) - |\mu(b)|^2 \\ &\leq (|A| + |B|)^2 - |\mu(b)|^2, \end{aligned}$$

which implies that

$$\forall \mu \in \mathcal{I}_b, \quad |\mu(b)|^2 \leq (|A| + |B|)^2 - M_{AB}^2.$$

This combined with (2.1) and (3.10) thus yields the desired bound (3.17).

The proof of Theorem 3.1 is now complete.  $\square$

*Proof of Corollary 3.5.* Assume by contradiction that for  $|A| \neq |B|$ , there exists a first integral  $u$  in  $C^1(\mathbb{R}^2)$  of system (1.1), whose periodic gradient  $\nabla u \in C_{\#}^0(\mathbb{T}^2)^2$  has a finite number of roots in the torus  $\mathbb{T}^2$ . Hence, taking the derivative of (1.3) with respect to  $t$ , we have

$$\forall t \in [0, \infty), \quad \nabla u(X(t, x)) \cdot b(X(t, x)) = 0,$$

and for  $t = 0$ , we get that  $\nabla u \cdot b = 0$  in  $\mathbb{T}^2$ . Since the vector field does not vanish in  $\mathbb{T}^2$  (recall (2.6)), there exists a function  $\sigma \in C_{\#}^0(\mathbb{T}^2)$  such that  $\nabla^{\perp} u = \sigma b$  in  $\mathbb{T}^2$ . The continuous function  $\sigma$  does not vanish in  $\mathbb{R}^2 \setminus \{\nabla u = 0\}$  which by assumption is a connected open set of  $\mathbb{R}^2$ . It follows that  $\sigma$  has a constant sign, say positive, in  $\mathbb{R}^2 \setminus \{\nabla u = 0\}$ . This combined with the continuity of  $\sigma$  implies that the weak positivity condition (3.11) holds true, and thus contradicts the part *i*) of Theorem 3.1.  $\square$

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