Around the asymptotic properties of a two-dimensional parametrized Euler flow

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Abstract

We study the two-dimensional Euler flow solution to $\partial_t X(\cdot, x) = b(X(\cdot, x))$ for x in the torus $\mathbb{T}^2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$, where b is the vector field defined on $\mathbb{T}^2$ by

$$b(x) = b(x_1, x_2) := (-A\cos x_1 - B\sin x_2\,,\ A\sin x_1 + B\cos x_2) \ \text{ with } A, B \in \mathbb{R}\setminus\{0\}.$$

We derive for any $x \in \mathbb{T}^2$, the asymptotics of $X(t, x)$ as t tends to ∞, depending on whether $|A| = |B|$ or $|A| \neq |B|$. In the first case, the orbits of the flow are all bounded. In the second case, it turns out that one of the coordinates of $X(t, x)$ is bounded with an explicit bound, while the other one is equivalent to $a(x)\,t$. The function a does not vanish in $\mathbb{T}^2$ and satisfies uniform bounds which depend on parameters A, B. When $|A| \neq |B|$, we also prove that for any global first integral u of the flow X with a periodic gradient, ∇u has at least a cluster point of roots in $\mathbb{T}^2$. This shows the complexity as well as the interest of this two-dimensional Euler flow.

Keywords: 2D Euler flow, asymptotic expansion, invariant measure, rotation set, first integral

Mathematics Subject Classification: 34E05, 34E10, 37C10, 37C40

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1 Introduction

In this paper we study the asymptotic behavior of the following two-dimensional Euler flow $X(t,x)$, for x in the torus $\mathbb{T}^2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$, solution to the ODEs system

$$\begin{cases} \partial_t X_1(t,x) &= -A\,\cos\big(X_1(t,x)\big) - B\,\sin\big(X_2(t,x)\big) &=: b_1\big(X(t,x)\big) \\ \partial_t X_2(t,x) &= A\,\sin\big(X_1(t,x)\big) + B\,\cos\big(X_2(t,x)\big) &=: b_2\big(X(t,x)\big), \end{cases} \qquad t\in[0,\infty), \quad (1.1)$$

where A, B are two fixed non-zero real parameters.

The vector field b defined by (1.1) represents the velocity solution to the steady Euler equation (see, *e.g.*, [16])

$$\partial_t\omega + \psi_1\,\omega_2 - \psi_2\,\omega_1 = 0, \quad (1.2)$$

where the function ω denotes the (scalar) fluid vorticity, and ψ solution to $-\Delta\psi = \omega$, denotes the stream function. By definition (1.1) we have

$$\omega(x) = (\operatorname{curl} b)(x) = (\partial_{x_1} b_2 - \partial_{x_2} b_1)(x) = A\cos x_1 + B\cos x_2 \quad \text{and} \quad \psi(x) = \omega(x).$$

Conditions of non-linear stability for plane stationary flows were derived by Arnold [1, 2] in the Sixties. According to [16, 7] the study of the stability of the flow (1.1) is actually relevant for the atmospheric flows. For example, Saturn's E Ring may be regarded as a two-dimensional flat torus, considering as in [16] (see in particular [7, Section 1.4.] and the references therein) the non-stationary Euler equation (1.2).

First of all, note that the existence of a global first integral of system (1.1) having a periodic gradient of period $2\pi\mathbb{Z}^2$, is far to be evident. Indeed, when $|A| \neq |B|$, we prove (see Corollary 3.5) that for any first integral $u \in C^1(\mathbb{R}^2)$ of (1.1), *i.e.* satisfying

$$\forall\, x\in\mathbb{T}^2,\ \forall\, t\in[0,\infty), \quad u\big(X(t,x)\big) = u(x), \quad (1.3)$$

with a periodic gradient, the gradient ∇u has at least a cluster point in $\mathbb{T}^2$. This strong constraint compared to the non vanishing property of the very simple vector field b associated with the flow X (1.1) makes difficult or even impossible the derivation of a global first integral of X, but in the same time shows the interest of the flow X.

In the present context, we focus on the asymptotic properties of the flow X through the limits

$$\lim_{t\to\infty} \frac{X(t,x)}{t} \quad \text{for any } x\in\mathbb{T}^2, \quad (1.4)$$

and on the probability measures on $\mathbb{T}^2$ which are invariant for this flow (see Section 3.1). Actually, these two notions are strongly connected in dimension two in view of the Misiurewicz-Ziemian [11, Theorem 3.4] in the case of general two-dimensional flows, or in view of the Peirone result [12, Theorem 3.1] (see also [13, Theorem 4.1]) in the case of the ODE's flow.

On the one hand, in Section 2 we provide (see Theorem 2.1) the limits (1.4) of the flow $X(\cdot,x)$ for each point $x\in\mathbb{T}^2$. Two cases have to be distinguished according to the non-zero parameters A, B (the case $AB = 0$ has been studied in [5, Section 5.3.1]):

- If we have $|A| = |B|$, then the flow X can be computed explicitly thanks to a change of variables (see formula (2.21)), which gives immediately the asymptotics of $X(\cdot,x)$.
- Otherwise we have $|A| \neq |B|$, then the general Lemma 2.5 combined with the case (2) of the proof of [12, Theorem 3.1] allows us to prove that one of the coordinate $X_i(\cdot,x)$, $i = 1,2$, of the flow is bounded with an explicit bound, while the other one $X_{3-i}(t,x)$ is equivalent to $a(x)\,t$ as $t\to\infty$ for some non-zero real $a(x)$. We also provide (see the part *iii*) of Theorem 3.1) uniform bounds from below and above for a, which depend on the two parameters A, B.

On the other hand, in Section 3 we characterize in the case $|A| \neq |B|$ (see Theorem 3.1) the invariant probability measures on $\mathbb{T}^2$ for the flow X (1.1), *i.e.* the probability measures μ on $\mathbb{T}^2$ such that for any $t \in \mathbb{R}$, $\mu \circ X^{-1}(t,\cdot) = \mu \circ X(-t,\cdot) = \mu$ on $\mathbb{T}^2$. We prove that there does not exist any invariant probability measure with Lebesgue's density $\sigma(x)\,dx$ where σ is a non-negative function in $L^1_\sharp(\mathbb{T}^2)$ satisfying some weak positivity condition (see (3.11)). Moreover, we obtain any limit (1.4) of $X(\cdot,x)$ as the mass of the vector field b by a singular probability measure depending on $x \in \mathbb{T}^2$.

Notation

- (e_1, e_2) denotes the canonical basis of $\mathbb{R}^2$, and $0_{\mathbb{R}^2}$ denotes the null vector of $\mathbb{R}^2$.
- I_2 is the unit matrix of $\mathbb{R}^{2\times 2}$.
- $R_\perp$ denotes the (2×2) rotation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For any $\xi \in \mathbb{R}^2$, $\xi^\perp$ denotes the perpendicular vector $R_\perp \xi$.
- " $\cdot$ " denotes the scalar product and $|\cdot|$ the euclidean norm in $\mathbb{R}^2$.
- $B(x,R)$ denotes the euclidean open ball of $\mathbb{R}^2$ centered on $x \in \mathbb{R}^2$ and of radius $R > 0$.
- dx denotes the Lebesgue measure on $\mathbb{R}^2$.
- $\mathbb{T}^k$ denotes the torus $\mathbb{R}^k/(2\pi\mathbb{Z}^k)$ for $k = 1, 2$.
- $\mathscr{M}_\sharp(\mathbb{T}^2)$ denotes the set of Radon measures on the torus $\mathbb{T}^2$, and $\mathscr{M}_{\rm p}(\mathbb{T}^2)$ denotes the set of probability measures on $\mathbb{T}^2$.
- $\mathscr{D}'(\Omega)$ denotes the set of the distributions on some open set Ω of $\mathbb{R}^2$.
- $C^k_c(\mathbb{R}^2)$, $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions in $C^k(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$.
- $C^k_\sharp(\mathbb{T}^2)$, resp. $W^{1,k}_\sharp(\mathbb{T}^2)$, $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions $f \in C^k(\mathbb{R}^2)$, resp. $f \in W^{1,k}_\sharp(\mathbb{R}^2)$ which are $2\pi\mathbb{Z}^2$-periodic, *i.e.*

$$\forall\, k \in \mathbb{Z}^2,\ \forall\, x \in \mathbb{R}^2, \quad f(x + 2\pi k) = f(x). \tag{1.5}$$

- For any function $f \in L^1_\sharp(\mathbb{T}^2)$, we denote

$$\left\{ \begin{array}{l} \bar{f}^{\,x_1}(x_2) := \displaystyle\int_{\mathbb{T}^1} f(x_1,x_2)\,dx_1 \\ \bar{f}^{\,x_2}(x_1) := \displaystyle\int_{\mathbb{T}^1} f(x_1,x_2)\,dx_2, \end{array} \right. \quad \text{and} \quad \bar{f} := \int_{\mathbb{T}^2} f(x)\,dx. \tag{1.6}$$

  and for any Borel measure μ on $\mathbb{T}^2$ and any function $f \in L^1(\mathbb{T}^2,\mu)$, we denote

$$\mu(f) := \int_{\mathbb{T}^2} f(x)\,\mu(dx), \tag{1.7}$$

  which is also extended to vector-valued functions in $L^1(\mathbb{T}^2,\mu)^2$.
- C denotes a positive constant which may vary from line to line.

2 Asymptotics of the flow

2.1 The main result

We have the following result.

Theorem 2.1. *Let $A, B \in \mathbb{R} \setminus \{0\}$. Then, the flow X defined by (1.1) satisfies the following asymptotic alternative:*

- *Either we have $(-1)^i \big(|A| - |B|\big) < 0$ for some $i = 1, 2$. Then, we obtain that*

$$\forall\, x \in \mathbb{T}^2, \quad \begin{cases} \Big|X_i(\cdot, x) - x\Big| < 2\pi & \text{in } [0, \infty) \\ \displaystyle\lim_{t \to \infty} \frac{X_{3-i}(t, x)}{t} = a(x) & \text{where } a(x) \neq 0. \end{cases} \tag{2.1}$$

- *Or we have $B = \varepsilon\, A$ with $\varepsilon = \pm 1$. Then, for any $x \in \mathbb{T}^2$, the orbit $X(\mathbb{R}, x)$ is bounded in $\mathbb{R}^2$. More precisely, defining the (2×2) matrix*

$$J := \frac{1}{2} \begin{pmatrix} 1 & \varepsilon \\ -\,\varepsilon & 1 \end{pmatrix} \quad \text{with} \quad J^{-1} := \begin{pmatrix} 1 & -\,\varepsilon \\ \varepsilon & 1 \end{pmatrix} \tag{2.2}$$

and the closed grid of $\mathbb{R}^2$

$$\mathscr{G} := \left(\bigcup_{m \in \mathbb{Z}} \Big[\frac{\pi}{4} + m\pi\Big] \times \mathbb{R} \right) \bigcup \left(\bigcup_{n \in \mathbb{Z}} \mathbb{R} \times \Big[\frac{\pi}{2} + n\pi\Big] \right), \tag{2.3}$$

we get that

$$\forall\, i = 1, 2,\ \forall\, x \in J^{-1}(\mathbb{R}^2 \setminus \mathscr{G}), \quad \Big|X_i(\cdot, x) - x_i\Big| < 2\pi. \tag{2.4}$$

Remark 2.2. The asymptotic behavior is quite different between the cases $|A| \neq |B|$ and $|A| = |B|$. The reason of this difference is that the vector field b of (1.1) does not vanish in the case $|A| \neq |B|$ with no punctual orbit – see the preliminary step of the proof of [12, Theorem 3.1] – contrary to the case $|A| = |B|$ which includes punctual orbits.

Remark 2.3. When $A > 0$ and $B = 0$, in [5, Section 5.3.1] we have got the following asymptotic expansion

$$X(t, x) = x - A\, t\, e_2 + O_K(1), \quad \forall\, t \in [0, \infty),\ \forall\, x \in K \times \mathbb{R}, \tag{2.5}$$

where $O_K(1)$ is bounded uniformly with respect to $t \in [0, \infty)$, to $x_2 \in \mathbb{R}$, and to x_1 in any fixed compact set K of $(-\frac{\pi}{2}, \frac{\pi}{2})$ modulo π. Expansion (2.5) can be thus regarded as a limit expansion as $B \to 0$ of (2.1). However, expansion (2.1) is finer than (2.5) for the coordinate $X_1(t, x)$, while it is the converse for the coordinate $X_2(t, x)$.

Remark 2.4. Assume that $(-1)^i \big(|A| - |B|\big) < 0$ for some $i = 1, 2$. Since the smooth vector field b associated with the flow X (1.1) satisfies

$$\forall\, x = (x_1, x_2) \in \mathbb{R}^2, \quad |b(x)|^2 = A^2 + B^2 + 2\, AB\, \sin(x_1 + x_2) \geq \big(|A| - |B|\big)^2 > 0, \tag{2.6}$$

it does not vanish in $\mathbb{R}^2$. Then, applying the non vanishing final argument of the proof of [3, Theorem 3.1] ([1]) we deduce that the limit function a does not vanish either in $\mathbb{T}^2$. Peirone's [12, Theorem 3.1] allows us to characterize more precisely the function a in (2.1). Indeed, following

[1] This argument is based on the two-dimensional Franks [8, Theorem 3.5].

the case (1) of the proof of [12, Theorem 3.1], it turns out that for any $x \in \mathbb{T}^2$, there exist $T_x > 0$ and an integer $k_x \in \mathbb{Z} \setminus \{0\}$ such that the function a satisfies

$$a(x) = \frac{2\pi\, k_x}{T_x}.$$

Actually, there exists a periodic solution of period T_x in the torus $\mathbb{T}^2$ (rather than in the space $\mathbb{R}^2$) $X(\cdot, z_x)$ to (1.1) with $z_x \in \mathbb{T}^2$, satisfying

$$X\big(T_x, z_x\big) = z_x + 2\pi\, k_x\, e_{3-i} \quad \text{and} \quad \lim_{t\to\infty} \frac{X(t,x)}{t} = a(x)\, e_{3-i}. \tag{2.7}$$

However, it is not obvious to derive explicitly such a periodic solution Z, even for a simple system like (1.1).

There is an alternative to derive the limit (2.7). By virtue of [11, Theorem 2.4, 3.4] and [11, Remark 2.5] (see also [4, Appendix A]) involving the properties of the Herman rotation set [10] associated with a flow, for any $x \in \mathbb{T}^2$, there exists a probability measure μ_x on $\mathbb{T}^2$ invariant for the flow (1.1) such that

$$\lim_{t\to\infty} \frac{X(t,x)}{t} = \mu_x(b) = \int_{\mathbb{T}^2} b(y)\, \mu_x(dy). \tag{2.8}$$

Using Liouville's theorem (see, *e.g.*, [14, Lecture 11]) the invariance of the probability measure μ_x is characterized by

$$\operatorname{div}(b\,\mu_x) = 0 \quad \text{in } \mathbb{T}^2. \tag{2.9}$$

But once again, it is difficult to get generally an explicit invariant probability measure for a given flow. This is the aim of Theorem 3.1 below for the two-dimensional Euler flow (1.1).

2.2 Proof of Theorem 2.1

2.2.1 Proof of the case $|A| \neq |B|$

The proof is based on the following general result.

Lemma 2.5. *Let f be a function in $W^{1,\infty}_\sharp(\mathbb{R}/T\mathbb{Z})$, $T > 0$ (i.e. f is Lispschitz and periodic of period T in $\mathbb{R}$), let $x_0 \in \mathbb{R}$, and let x be a function in $C^1([0,\infty))$ such that $f(x(0)) \neq f(x_0)$. Then, we have the two following implications:*

$$\begin{cases} x' + f(x) < f(x_0) \quad \text{in } [0,\infty) \;\Rightarrow\; x - x(0) < T \quad \text{in } [0,\infty), \\ x' + f(x) > f(x_0) \quad \text{in } [0,\infty) \;\Rightarrow\; x - x(0) > -T \quad \text{in } [0,\infty), \end{cases} \tag{2.10}$$

where $f(x) := f \circ x$.

Assume for instance that $|A| > |B|$, and set

$$\alpha := \frac{A}{|A|} = \pm 1 \quad \text{and} \quad \beta := \frac{B}{|A|} \in (-1,1). \tag{2.11}$$

Then, consider the new coordinates $y := x/|A|$ which leads us to the flow $Y(t,y) := X(t,x)/|A|$ solution to the ODEs system

$$\begin{cases} \partial_t Y_1(t,y) &= -\,\alpha \cos\big(|A|\, Y_1(t,y)\big) - \beta\, \sin\big(|A|\, Y_2(t,y)\big) \\ \partial_t Y_2(t,y) &= \alpha\, \sin\big(|A|\, Y_1(t,y)\big) + \beta\, \cos\big(|A|\, Y_2(t,y)\big). \end{cases} \tag{2.12}$$

Fix $y = (y_1, y_2) \in \mathbb{R}^2$, and let $\theta \in [0, \pi/|A|]$ be such that

$$|\beta| < \alpha \cos(|A|\,\theta) \neq \pm\,\alpha \cos(|A|\,y_1).$$

Then, the first equation of (2.12) implies that

$$\begin{cases} \Big|\,\partial_t Y_1(\cdot, y) + \alpha \cos\big(|A|\,Y_1(\cdot, y)\big)\,\Big| < \alpha \cos(|A|\,\theta) & \text{in } [0, \infty) \\ \cos(|A|\,\theta) \neq \pm \cos(|A|\,y_1). \end{cases}$$

Therefore, applying the first inequality of (2.10) to the function $x(\cdot) := Y_1(\cdot, y_1)$ with $x_0 := \theta$ and $f(\cdot) := \alpha \cos(|A|\,\cdot)$ (which is $|A|$-Lipschitz and periodic of period $T := 2\pi/|A|$ in $\mathbb{R}$), then applying the second inequality of (2.10) to the function $x(\cdot) := Y_1(\cdot, y_1)$ with $x_0 := \pi/|A| - \theta$ and f, we get that

$$\Big|Y_1(\cdot, y) - y_1\Big| < T \quad \text{in } [0, \infty),$$

which implies the first estimate of (2.1) for $i = 1$. The proof is quite similar for $i = 3$, replacing Y_1 by Y_2 and permuting the roles of A and B.

On the other hand, let $i := 1, 2$. Then, recalling Remark 2.4 combined with the first estimate of (2.1) (we have just proved before) we get that for any $x \in \mathbb{T}^2$,

$$\exists \lim_{t\to\infty} \frac{X(t,x)}{t} \neq 0_{\mathbb{R}^2} \quad \text{and} \quad \lim_{t\to\infty} \frac{X_i(t,x)}{t} = 0,$$

which yields immediately the limit of (2.1).

Therefore, the proof of the case $|A| \neq |B|$ is done. □

Proof of Lemma 2.5. First of all, there exists a unique $n \in \mathbb{Z}$ such that

$$x_0 + n\,T \leq x(0) < x_0 + (n+1)\,T.$$

Moreover due to the periodicity of f and to $f(x(0)) \neq f(x_0)$, we have $x(0) \neq x_0 + n\,T$. Hence, we deduce that

$$x_0 + n\,T < x(0) < x_0 + (n+1)\,T. \tag{2.13}$$

Now, assume that the left hand-side of the first implication of (2.10) holds, and consider the solution y to the ODE

$$y' + f(y) = f(x_0) \quad \text{in } [0, \infty), \quad y(0) = x(0). \tag{2.14}$$

Assume by contradiction that there exists $t \in [0, \infty)$ such that $y(t) = x_0 + n\,T$. Then, y and the constant function $y_s := x_0 + n\,T$ are both solutions to the ODE: $u' + f(u) = f(x_0)$ in $[0, \infty)$, and they agree at point t. Then, by an uniqueness argument we get that $y = y_s$, which due to (2.13) and (2.14) is not satisfied at the point 0. This leads us to a contraction. Similarly, the trajectory $y([0, \infty))$ cannot meet the point $x_0 + (n+1)\,T$. Therefore, by a connectedness argument we obtain that

$$x_0 + n\,T < y < x_0 + (n+1)\,T \quad \text{in } [0, \infty). \tag{2.15}$$

This combined with (2.13) implies that

$$|y - x(0)| < T \quad \text{in } [0, \infty). \tag{2.16}$$

Next, define the function $z := x - y$. Subtracting the equality (2.14) to the inequality satisfied by x, and using that f is k-Lipschitz for some $k > 0$, we get that

$$z' < f(x_0) - f(x) + f(y) - f(x_0) \leq k\,|z| \quad \text{in } [0,\infty). \tag{2.17}$$

Since by (2.14) $z(0) = 0$, inequality (2.17) shows that $z'(0) < 0$, which implies that $z < 0$ in some interval $[0, t_0]$ with $t_0 > 0$. We can then consider

$$S := \sup\big\{t \in [t_0,\infty) : z < 0 \text{ in } [t_0, t]\big\} \in [t_0,\infty].$$

Assume by contradiction that $S < \infty$. Then, we have

$$z < 0 \quad \text{in } [0, S) \quad \text{and} \quad z(S) = 0,$$

which combined with (2.17) implies that

$$\forall\, t \in [0, S), \quad \frac{d}{dt}\big(e^{k\,t} z(t)\big) = e^{k\,t}\,\big(z'(t) + k\,z(t)\big) < 0.$$

Hence, it follows that $0 = e^{k\,S} z(S) < e^0\, z(0) = 0$, which leads us to a contradiction. Therefore, we have $S = \infty$, and by the same argument as above, we get that $z = x - y < 0$ in $[0,\infty)$. Finally, recalling (2.16) we obtain that $x < y < x(0) + T$.

The proof of the second implication of (2.10) is quite similar leading us to the inequality in $[0,\infty)$, which concludes the proof of Lemma 2.5. □

2.2.2 Proof of the case $|A| = |B|$

Assume that $B = \varepsilon\, A$ with $\varepsilon = \pm 1$. Then, consider the new coordinates

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_1 + \varepsilon\, x_2 \\ -\,\varepsilon\, x_1 + x_2 \end{pmatrix} = J\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = J\,x. \tag{2.18}$$

where the marix J is defined by (2.2).
By (1.1) we are led to the new flow $Y(t,y) := J\,X(t,x)$ solution to the ODEs system

$$\begin{cases} \partial_t Y_1(t,y) &= \dfrac{A}{2}\Big(-\cos\big(X_1(t,x)\big) - \varepsilon\,\sin\big(X_2(t,x)\big) + \varepsilon\,\sin\big(X_1(t,x)\big) + \cos\big(X_2(t,x)\big)\Big) \\ \partial_t Y_2(t,y) &= \dfrac{A}{2}\Big(\varepsilon\,\cos\big(X_1(t,x)\big) + \sin\big(X_2(t,x)\big) + \sin\big(X_1(t,x)\big) + \varepsilon\,\cos\big(X_2(t,x)\big)\Big), \end{cases}$$

or equivalently, using trigonometrical formulas we have

$$\begin{cases} \partial_t Y_1(t,y) &= A\,\sin\big(\frac{X_1(t,x)+X_2(t,x)}{2}\big)\,\sin\big(\frac{X_1(t,x)-X_2(t,x)}{2}\big) \\ &\quad + \varepsilon\,A\,\cos\big(\frac{X_1(t,x)+X_2(t,x)}{2}\big)\,\sin\big(\frac{X_1(t,x)-X_2(t,x)}{2}\big) \\ \partial_t Y_2(t,y) &= A\,\sin\big(\frac{X_1(t,x)+X_2(t,x)}{2}\big)\,\cos\big(\frac{X_1(t,x)-X_2(t,x)}{2}\big) \\ &\quad + \varepsilon\,A\,\cos\big(\frac{X_1(t,x)+X_2(t,x)}{2}\big)\,\cos\big(\frac{X_1(t,x)-X_2(t,x)}{2}\big). \end{cases}$$

This combined with (2.18) thus implies the new system depending on $\varepsilon = \pm 1$,

$$\begin{cases} \partial_t Y_1(t,y) &= -\,\varepsilon\,A\Big(\sin\big(Y_{\frac{3-\varepsilon}{2}}(t,y)\big) + \varepsilon\,\cos\big(Y_{\frac{3-\varepsilon}{2}}(t,y)\big)\Big)\,\sin\big(Y_{\frac{3+\varepsilon}{2}}(t,y)\big) \\ \partial_t Y_2(t,y) &= A\Big(\sin\big(Y_{\frac{3-\varepsilon}{2}}(t,y)\big) + \varepsilon\,\cos\big(Y_{\frac{3-\varepsilon}{2}}(t,y)\big)\Big)\,\cos\big(Y_{\frac{3+\varepsilon}{2}}(t,y)\big). \end{cases} \tag{2.19}$$

Now, assume for example that $\varepsilon = 1$. Then, system (2.19) reads as

$$\begin{cases} \partial_t Y_1(t,y) &= -A\sqrt{2}\,\sin\big(Y_1(t,y)+\frac{\pi}{4}\big)\,\sin\big(Y_2(t,y)\big) \\ \partial_t Y_2(t,y) &= A\sqrt{2}\,\sin\big(Y_1(t,y)+\frac{\pi}{4}\big)\,\cos\big(Y_2(t,y)\big). \end{cases} \tag{2.20}$$

It is clear that for any $m \in \mathbb{Z}$ and $y_2 \in \mathbb{R}$, $Y(\cdot,y) = y := (-\frac{\pi}{4} + m\pi, y_2)$ is a stationary solution to system (2.20). Moreover, we can check that for any $y_1 \in -\frac{\pi}{4} + 2n\pi + (-\pi,\pi)$,

$$\begin{cases} Y_1(t,y) = -\dfrac{\pi}{4} + 2n\pi + 2\arctan\Big[\, e^{(-1)^{n+1}A\sqrt{2}\,t}\tan\Big(\dfrac{y_1}{2}+\dfrac{\pi}{8}\Big)\Big] \\ Y_2(t,y) = \dfrac{\pi}{2} + n\pi, \end{cases} \qquad t \in \mathbb{R}, \tag{2.21}$$

is also solution to system (2.20). Then, collecting all these solutions we obtain that the closed set $\mathscr{G}$ defined by (2.3) is composed of trajectories of the flow Y: either stationary points along lines parallel to direction y_2, or closed line segments parallel to direction y_1. Hence, since the trajectories of the flow do not intersect, for any $y \in \mathbb{R}^2 \setminus \mathscr{G}$, the trajectory $Y(\mathbb{R},y)$ lies in one of the connected components of $\mathbb{R}^2 \setminus \mathscr{G}$ which are the open squares of side π defined by

$$Q_{m,n} := \Big(-\frac{\pi}{4} + m\pi, -\frac{\pi}{4} + (m+1)\,\pi\Big) \times \Big(\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\,\pi\Big) \quad \text{for } (m,n) \in \mathbb{Z}^2. \tag{2.22}$$

Therefore, we get that

$$\forall\, i = 1,2,\ \forall\, y \in \mathbb{R}^2 \setminus \mathscr{G}, \quad \Big|Y_i(\cdot,y) - y_i\Big| < \pi.$$

Finally, this combined with the change of coordinates (2.18) yields the desired estimates (2.4). The case $\varepsilon = -1$ is quite similar, which concludes the proof of Theorem 2.1. □

3 Invariant measures and rotation set

3.1 Some recalls of ergodic theory

Let b a vector field in $C^1_\sharp(\mathbb{T}^2)$. A probability measure μ in $\mathscr{M}_\sharp(\mathbb{T}^2)$ is said to *invariant for some flow X* associated with the vector field b by

$$\begin{cases} \partial_t X(t,x) = b\big(X(t,x)\big), \quad t \in \mathbb{R} \\ X(0,x) = x \in \mathbb{T}^2, \end{cases} \tag{3.1}$$

if one has

$$\forall\, t \in \mathbb{R},\ \forall\, \psi \in C^0_\sharp(\mathbb{T}^2), \quad \int_{\mathbb{T}^2} \psi\big(X(t,y)\big)\,d\mu(y) = \int_{\mathbb{T}^2} \psi(y)\,d\mu(y). \tag{3.2}$$

By virtue of Liouville's theorem (see, *e.g.*, [9, Theorem 1, Section 2.2]) condition (3.2) means that

$$\operatorname{div}(b\,\mu) = 0 \quad \text{in } \mathbb{T}^2, \tag{3.3}$$

or equivalently, with the variational formulation (see, *e.g.*, [4, Proposition])

$$\forall\, \varphi \in C^1_\sharp(\mathbb{T}^2), \quad \int_{\mathbb{T}^2} b(x)\cdot\nabla\varphi(x)\,\mu(dx) = 0. \tag{3.4}$$

Alternatively, define the set

$$\mathscr{I}_b := \big\{\mu \in \mathscr{M}_{\mathrm{p}}(\mathbb{T}^2) : \mu \text{ invariant for the flow } X\big\}, \tag{3.5}$$

where $\mathscr{M}_{\mathrm{p}}(\mathbb{T}^2)$ is the set of probability measures on $\mathbb{T}^2$. Also define the two following non-empty subsets of $\mathbb{R}^2$:

- The set of all the limit points of the sequences $\big(X(n,x)/n\big)_{n\geq 1}$ for $x \in \mathbb{T}^2$ (denoted by $\rho_{\mathrm{p}}(b)$ in [11])

$$\mathsf{A}_b := \bigcup_{x\in\mathbb{T}^2} \left[\bigcap_{n\geq 1} \overline{\left\{ \frac{X(k,x)}{k} : k \geq n \right\}} \right]. \tag{3.6}$$

  When b does not vanish in $\mathbb{T}^2$, by virtue of Peirone's [12, Theorem 3.1] the former definition is reduced to

$$\mathsf{A}_b = \left\{ \lim_{t\to\infty} \frac{X(t,x)}{t} : x \in \mathbb{T}^2 \right\}. \tag{3.7}$$

- The so-called Herman [10] rotation set (recall notation (1.7))

$$\mathsf{C}_b := \big\{ \mu(b) : \mu \in \mathscr{I}_b \big\} \tag{3.8}$$

  which is a compact convex subset of $\mathbb{R}^2$.

An implicit consequence of the Misiurewicz-Ziemian results [11, Theorem 2.4, Remark 2.5] and [11, Corollary 2.6] shows that

$$\mathsf{A}_b \subset \mathsf{C}_b = \mathrm{conv}\,(\mathsf{A}_b).$$

However, in dimension two [11, Theorem 3.4 (c)] shows that actually the set A_b (3.6) is convex which thus implies that

$$\mathsf{C}_b = \mathsf{A}_b. \tag{3.9}$$

When in addition b does not vanish in $\mathbb{T}^2$, we thus deduce from (3.7) that

$$\mathsf{C}_b = \big\{ \mu(b) : \mu \in \mathscr{I}_b \big\} = \left\{ \lim_{t\to\infty} \frac{X(t,x)}{t} : x \in \mathbb{T}^2 \right\} = \mathsf{A}_b. \tag{3.10}$$

This links closely the invariant probability measures for the flow X (3.1) and the limit points of the flow.

3.2 The case of the two-dimensional Euler flow

The invariant probability measures (3.2) and the Herman rotation set (3.8) for the flow X defined by (1.1) are characterized as follows.

Theorem 3.1. *Assume that $|A| \neq |B|$ for the vector field b defined in* (1.1)*. Then, we have the following results:*

i) There does not exist any non-negative function $\sigma \in L^1_\sharp(\mathbb{T}^2)$ whose integrals over its two sections are a.e. positive, i.e.

$$\begin{cases} \overline{\sigma}^{x_2}(x_1) = \displaystyle\int_{\mathbb{T}^1} \sigma(x_1,x_2)\,dx_2 > 0 \quad a.e.\ x_1 \in \mathbb{T}^1 \\ \overline{\sigma}^{x_1}(x_2) = \displaystyle\int_{\mathbb{T}^1} \sigma(x_1,x_2)\,dx_1 > 0 \quad a.e.\ x_2 \in \mathbb{T}^1, \end{cases} \tag{3.11}$$

such that the $\sigma(x)\,dx$ is an invariant probability measure for the flow X.

ii) Moreover, for any $\zeta \in \mathsf{C}_b$, there exist $T \in (0,\infty)$ and $z \in \mathbb{T}^2$ such that the orbit $X(\cdot,z)$ is periodic of period T in the torus $\mathbb{T}^2$, i.e.

$$\exists\, k \in \mathbb{Z}^2, \quad X(T,z) = z + 2\pi\, k, \tag{3.12}$$

and

$$\zeta = \mu_{T,z}(b) \quad \textit{where} \quad \mu_{T,z}(\varphi) := \frac{1}{T}\int_0^T \varphi\big(X(t,z)\big)\,dt \ \ \textit{for } \varphi \in C^0_\sharp(\mathbb{T}^2), \tag{3.13}$$

is an invariant probability measure which is singular with respect to Lebesgue's measure. In other words, Herman's rotation set is characterized by

$$\mathsf{C}_b = \big\{\mu_{T,z}(b)\,,\, T>0,\, z\in\mathbb{T}^2 : X(T,z)-z \in 2\pi\mathbb{Z}^2\big\}. \tag{3.14}$$

iii) Finally, assume that $|A|>|B|$. *Denoting*

$$\left\{\begin{array}{l} \xi_\mu := \mu(\cos x_1)\,e_1 + \mu(\sin x_1)\,e_2 \\ \eta_\mu := \mu(\sin x_2)\,e_1 + \mu(\cos x_2)\,e_2, \end{array}\right. \quad \textit{for } \mu\in\mathscr{I}_b, \tag{3.15}$$

the asymptotics flow function a *defined by* (2.1) *satisfies the bounds from below and above (with the convention:* $\inf_{\emptyset} = \infty$*)*

$$\left\{\begin{array}{c} \displaystyle\inf_{x\in\mathbb{T}^2}|a(x)| \geq \min\left(m_{AB}\left(|A|-|B|\right),\, \inf_{\{\mu\in\mathscr{I}_b\,:\,|\xi_\mu|\leq|\eta_\mu|\}}\left|A\,\xi_\mu + B\,\eta_\mu\right|\right) > 0, \\ \displaystyle m_{AB} := \frac{1}{\sqrt{2}}\inf_{\mu\in\mathscr{I}_b}\sqrt{|\xi_\mu|^2+|\eta_\mu|^2} \in (0,1], \end{array}\right. \tag{3.16}$$

and

$$\left\{\begin{array}{l} \displaystyle\sup_{x\in\mathbb{T}^2}|a(x)| \leq \sqrt{\big(|A|+|B|\big)^2 - M_{AB}^2}\,, \\ \displaystyle M_{AB} := \inf_{\mu\in\mathscr{I}_b}\int_{\mathbb{T}^2}|b-\mu(b)|\,\mu(dx) > 0. \end{array}\right. \tag{3.17}$$

Remark 3.2. Due to (2.6) the vector field b in (1.1) does not vanish when $|A|\neq|B|$. Hence, thanks to the first step of the proof of Peirone's [12, Theorem 3.1], for any periodic orbit in the torus the integer vector k in (3.12) is not null.

Remark 3.3. The part *i*) of Theorem 3.1 means that for any non-negative function σ in $L^1_\sharp(\mathbb{T}^2)$ satisfying the weak positivity condition (3.11), the vector field $\sigma\,b$ is not realizable as a divergence free field, or in conductivity term, as a current field. The dual problem (which is equivalent in dimension two) *i.e.* the realizability of a gradient field as an electric field has been addressed in [6] with a very simple counter-example [6, Example 2.7], but involving both regular (at least continuous) and positive functions σ. Here, the two-dimensional Euler flow induced by the vector field b (1.1) provides a more definite counter-example of realizability, since here the functions σ are only assumed to be in $L^1_\sharp(\mathbb{T}^2)$ rather than regular, and are only non-negative with condition (3.11) rather than positive a.e. in $\mathbb{T}^2$.

Remark 3.4. On the one hand, as $|A|-|B|$ tends to 0, the bound from below (3.16) clearly converges to 0. This asymptotic result is consistent with the picture of bounded orbits in the case $|A|=|B|$ (see above the second part of Theorem 2.1 and Section 2.2.2), which due to the set equality (3.9) implies that the Herman rotation set C_b (3.8) is reduced to the unit set $\{0_{\mathbb{R}_2}\}$.

On the other hand, the bound from above (3.17) has actually a general scope. Indeed, the proof of the part *iii*) of Theorem 3.1 below shows that the vector field $b\in C^1(\mathbb{T}^2)^2$ needs to satisfy the two following conditions

$$\big\{x\in\mathbb{T}^2 : b(x)=0_{\mathbb{R}^2}\big\} = \emptyset \quad \text{and} \quad \forall\,\xi\in\mathbb{R}^2,\ \ \#\big\{x\in\mathbb{T}^2 : b(x)=\xi\big\} < \infty. \tag{3.18}$$

Then, we still deduce from (3.10) the extension of (3.17)

$$\begin{cases} \displaystyle\sup_{x\in\mathbb{T}^2}\left|\lim_{t\to\infty}\frac{X(t,x)}{t}\right| \le \sqrt{\|b\|^2_{L^\infty(\mathbb{T}^2)^2}-M_b^2}\,, \\ \qquad M_b := \displaystyle\inf_{\mu\in\mathscr{I}_b}\int_{\mathbb{T}^2}|b-\mu(b)|\,\mu(dx) > 0. \end{cases} \tag{3.19}$$

Corollary 3.5. *Assume there exists a first integral u in $C^1(\mathbb{R}^2)$ of the flow (1.1) having a periodic gradient of period $2\pi\mathbb{Z}^2$. Then, ∇u has at least a cluster point of roots in the torus $\mathbb{T}^2$.*

Proof of Theorem 3.1.

Proof of part i). Assume for instance that $|A| > |B|$, and that there exists an invariant probability measure $\sigma(x)\,dx$ for the flow X (1.1) with Lebesgue's density $\sigma \in L^1_\sharp(\mathbb{T}^2)$ satisfying the weak positivity (3.11). By Liouville's theorem (3.3) combined with a classical duality result there exists a function ψ with $\nabla\psi \in L^1_\sharp(\mathbb{T}^2)$ such that (recall the third notation)

$$\sigma\, b = \nabla^\perp\psi \ \text{ in } \mathbb{R}^2, \tag{3.20}$$

or equivalently,

$$\begin{cases} \sigma(x)\big(A\,\cos x_1 + B\,\sin x_2\big) = \partial_{x_2}\psi(x) \\ \sigma(x)\big(A\,\sin x_1 + B\,\cos x_2\big) = \partial_{x_1}\psi(x) \end{cases} \quad \text{a.e. } x=(x_1,x_2)\in\mathbb{T}^2. \tag{3.21}$$

Recalling (2.6) equalities (3.10) hold true. This combined with (2.1) and (3.20) implies that there exists $x_\sigma \in \mathbb{T}^2$ such that

$$\overline{\nabla\psi} = -\big(\overline{\sigma\, b}\big)^\perp = -\left(\lim_{t\to\infty}\frac{X(t,x_\sigma)}{t}\right)^\perp = -\,a(x_\sigma)\,e_2^\perp = a(x_\sigma)\,e_1, \quad \text{with } a(x_\sigma)\neq 0.$$

Then, since the function $\psi_\sharp : y \mapsto \big(\psi(y) - \overline{\nabla\psi}\cdot y\big)$ is in $W^{1,1}_\sharp(\mathbb{T}^2)$, we get that

$$\overline{\partial_{x_2}\psi}^{\,x_2}(x_1) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\partial_{x_2}\psi(x_1,x_2)\,dx_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\partial_{x_2}\psi_\sharp(x_1,x_2)\,dx_2 = 0 \quad \text{a.e. } x_1\in\mathbb{T}^1. \tag{3.22}$$

Therefore, integrating the second equality of (3.21) with respect to variable x_2 and taking into account (3.11) and (3.22), it follows that

$$|\cos x_1| = \left|-\frac{B}{A}\,\frac{\overline{\sigma\,\sin x_2}^{\,x_2}(x_1)}{\overline{\sigma}^{\,x_2}(x_1)}\right| \le \frac{|B|}{|A|} \quad \text{a.e. } x_1\in\mathbb{T}^1, \tag{3.23}$$

which, making x_1 tend to 0, contradicts our assumption $|A| > |B|$.

Proof of part ii). Let $\zeta \in \mathsf{C}_b$. By (3.10) there exists $x\in\mathbb{T}^2$ such that

$$\zeta = \lim_{t\to\infty}\frac{X(t,x)}{t}. \tag{3.24}$$

Then, consider the T_x-periodic (in the torus $\mathbb{T}^2$) orbit $X(\cdot,z_x)$ satisfying (2.7) for some $i=1,2$, and consider the probability measure μ_{T_x,z_x} defined by (3.13). Due to the semi-group property satisfied by the flow, we have for any $s\in\mathbb{R}$ and any $\varphi\in C^0_\sharp(\mathbb{T}^2)$,

$$\begin{aligned}\mu_{T_x,z_x}\big(\varphi(X(s,\cdot))\big) &= \frac{1}{T_x}\int_0^{T_x}\varphi\big(X(s,X(t,z_x))\big)\,dt = \frac{1}{T_x}\int_0^{T_x}\varphi\big(X(s+t,z_x)\big)\,dt \\ &= \frac{1}{T_x}\int_s^{s+T_x}\varphi\big(X(t,z_x)\big)\,dt = \frac{1}{T_x}\int_0^{T_x}\varphi\big(X(t,z_x)\big)\,dt,\end{aligned}$$

since the function $\varphi\big(X(\cdot,z_x)\big)$ is T_x-periodic in $\mathbb{R}$ taking into account the first equality of (2.7). Hence, μ_{T_x,z_x} is an invariant probability measure for the flow X.
Moreover, again using (2.7) we get that

$$\mu_{T_x,z_x}(b)=\frac{1}{T_x}\int_0^{T_x} b\big(X(t,z_x)\big)\,dt=\frac{1}{T_x}\int_0^{T_x}\partial_t X(t,z_x)\,dt=\frac{X(T_x,z_x)-z_x}{T_x}=a(x)\,e_{3-i},$$

which combined with (3.24) implies the desired equalities $\zeta=\mu_{T_x,z_x}(b)$ and (3.14).

Proof of bound (3.16). By the definition of the vector field b in (1.1) and the definitions (3.15) of the vectors ξ_μ,η_μ, we have

$$\forall\,\mu\in\mathscr{I}_b,\quad \mu(b)=A\,\xi_\mu+B\,\eta_\mu.$$

Since the vector field b does not vanish in $\mathbb{T}^2$ by (2.6), from the set equality (3.10) and the non vanishing property of the function a in (2.1), we deduce that

$$\inf_{x\in\mathbb{T}^2}|a(x)|=\inf_{\mu\in\mathscr{I}_b}|\mu(b)|\quad\text{and}\quad\forall\,\mu\in\mathscr{I}_b,\ \ |\mu(b)|=\big|A\,\xi_\mu+B\,\eta_\mu\big|>0. \tag{3.25}$$

This combined with the continuity of the mapping

$$\mu\in\mathscr{I}_b\mapsto\big|A\,\xi_\mu+B\,\eta_\mu\big|$$

defined on the compact set of invariant probability measures $\mathscr{I}_b$ for the weak-$*$ measures topology, implies that

$$\inf_{\mu\in\mathscr{I}_b}\big|A\,\xi_\mu+B\,\eta_\mu\big|=\min\left(\inf_{\{\mu\in\mathscr{I}_b\,:\,|\xi_\mu|\geq|\eta_\mu|\}}\big|A\,\xi_\mu+B\,\eta_\mu\big|,\inf_{\{\mu\in\mathscr{I}_b\,:\,|\xi_\mu|\leq|\eta_\mu|\}}\big|A\,\xi_\mu+B\,\eta_\mu\big|\right)>0. \tag{3.26}$$

Then, by the Cauchy-Schwarz inequality we have

$$|\xi_\mu|^2+|\eta_\mu|^2\leq\sum_{i=1}^{2}\big(\mu(\cos^2 x_i)+\mu(\sin^2 x_i)\big)=2.$$

Again applying the Cauchy-Schwarz inequality the constant m_{AB} of (3.16) thus satisfies the estimate

$$\begin{aligned}1\geq m_{AB}=\frac{1}{\sqrt{2}}\inf_{\mu\in\mathscr{I}_b}\sqrt{|\xi_\mu|^2+|\eta_\mu|^2}\ &\geq\frac{1}{\sqrt{2\,(A^2+B^2)}}\inf_{\mu\in\mathscr{I}_b}\big(A\,|\xi_\mu|+B\,|\eta_\mu|\big)\\&\geq\frac{1}{\sqrt{2\,(A^2+B^2)}}\inf_{\mu\in\mathscr{I}_b}\big|A\,\xi_\mu+B\,\eta_\mu\big|>0.\end{aligned}$$

Now, focus on the second infimum of (3.26). Using the triangle inequality we have for any $\mu\in\mathscr{I}_b$ satisfying $|\xi_\mu|\geq|\eta_\mu|$ which also implies that $|\xi_\mu|\geq m_{AB}$,

$$\big|A\,\xi_\mu+B\,\eta_\mu\big|=|\xi_\mu|\left|A\,\frac{\xi_\mu}{|\xi_\mu|}+B\,\frac{\eta_\mu}{|\xi_\mu|}\right|\geq|\xi_\mu|\left(|A|-|B|\,\frac{|\eta_\mu|}{|\xi_\mu|}\right)\geq m_{AB}\big(|A|-|B|\big).$$

Therefore, recalling (3.25) and (3.26) we obtain immediately the bound (3.16) with $m_{AB}>0$.

Proof of bound (3.17). Assume by contradiction that there exists an invariant probability measure $\mu\in\mathscr{I}_b$ for the flow X, satisfying $b=\mu(b)$ μ-a.e. in $\mathbb{T}^2$. Then, the set J of the points

$z \in \mathbb{T}^2$ solutions to the equation $b(z) = \mu(b)$ is clearly finite (we can check that $1 \leq \#J \leq 4$). Hence, there exists a non-empty subset I of J such that the probability measure μ reads as

$$\mu = \sum_{z \in I} \mu(\{z\})\, \delta_z \ \text{ on } \mathbb{T}^2, \quad \text{with} \quad \forall\, z \in I,\ \mu(\{z\}) > 0 \ \text{ and } \ \sum_{z \in I} \mu(\{z\}) = 1.$$

Hence, recalling (3.4) we have

$$\forall\, \varphi \in C^1_\sharp(\mathbb{T}^2), \quad \int_{\mathbb{T}^2} b(x) \cdot \nabla \varphi(x)\, \mu(dx) = \sum_{z \in I} \mu(\{z\})\, b(z) \cdot \nabla \varphi(z) = 0,$$

which due to the arbitrariness of the function φ implies that for any $z \in I$, $b(z) = 0_{\mathbb{R}^2}$, and leads us to contradiction. Therefore, we get that

$$\forall\, \mu \in \mathscr{I}_b, \quad \int_{\mathbb{T}^2} |b - \mu(b)|\, \mu(dx) > 0.$$

This combined with the compactness of the set $\mathscr{I}_b$ of the invariant probability measures implies that the infimum M_{AB} of (3.17) is positive.

Next, by the Cauchy-Schwarz inequality we have for any $\mu \in \mathscr{I}_b$,

$$\begin{aligned} M_{AB}^2 \leq \int_{\mathbb{T}^2} |b - \mu(b)|^2\, \mu(dx) \ &= \mu(|b|^2) - |\mu(b)|^2 \\ &= A^2 + B^2 + 2AB \left(\int_{\mathbb{T}^2} \sin(x_1 + x_2)\, \mu(dx) \right) - |\mu(b)|^2 \\ &\leq \big(|A| + |B|\big)^2 - |\mu(b)|^2, \end{aligned}$$

which implies that

$$\forall\, \mu \in \mathscr{I}_b, \quad |\mu(b)|^2 \leq \big(|A| + |B|\big)^2 - M_{AB}^2.$$

This combined with (2.1) and (3.10) thus yields the desired bound (3.17).

The proof of Theorem 3.1 is now complete. □

Proof of Corollary 3.5. Assume by contradiction that for $|A| \neq |B|$, there exists a first integral u in $C^1(\mathbb{R}^2)$ of system (1.1), whose periodic gradient $\nabla u \in C^0_\sharp(\mathbb{T}^2)^2$ has a finite number of roots in the torus $\mathbb{T}^2$. Hence, taking the derivative of (1.3) with respect to t, we have

$$\forall\, t \in [0, \infty), \quad \nabla u\big(X(t,x)\big) \cdot b\big(X(t,x)\big) = 0,$$

and for $t = 0$, we get that $\nabla u \cdot b = 0$ in $\mathbb{T}^2$. Since the vector field does not vanish in $\mathbb{T}^2$ (recall (2.6)), there exists a function $\sigma \in C^0_\sharp(\mathbb{T}^2)$ such that $\nabla^\perp u = \sigma\, b$ in $\mathbb{T}^2$. The continuous function σ does not vanish in $\mathbb{R}^2 \setminus \{\nabla u = 0\}$ which by assumption is a connected open set of $\mathbb{R}^2$. It follows that σ has a constant sign, say positive, in $\mathbb{R}^2 \setminus \{\nabla u = 0\}$. This combined with the continuity of σ implies that the weak positivity condition (3.11) holds true, and thus contradicts the part $i)$ of Theorem 3.1. □

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References

[1] V.I. Arnold: "Sur la topologie des écoulements stationnaires des fluides parfait" (French), *C. R. Acad. Sci. Paris*, **261** (1965), 17-20. *Vladimir I. Arnold – Collected Works Vol. II. Hydrodynamics, Bifurcation Theory, and Algebraic Geometry* 1965-1972, 14-18. Ed. by A.B. Givental, B.A. Khesin, A.N. Varchenko, V.A. Vassiliev, O.Ya. Viro, Springer-Verlag, Berlin, 2014, 465 pp.

[2] V.I. Arnold: "On conditions for non-linear stability of plane stationary curvilinear flows of an ideal fluid." (Russian) *Dokl. Akad. Nauk SSSR*, **162** (1965), 975-978. *Vladimir I. Arnold – Collected Works Vol. II. Hydrodynamics, Bifurcation Theory, and Algebraic Geometry* 1965-1972, 19-23. Ed. by A.B. Givental, B.A. Khesin, A.N. Varchenko, V.A. Vassiliev, O.Ya. Viro, Springer-Verlag, Berlin, 2014, 465 pp.

[3] M. Briane & L. Hervé: "Asymptotics of ODE's flow on the torus through a singleton condition and a perturbation result. Applications", *Dis. Con. Dyn. Sys.*, **42** (7) (2022), 3431-3463.

[4] M. Briane & L. Hervé: "Asymptotics of ODE's flow on the torus through a singleton condition and a perturbation result. Applications", *Dis. Con. Dyn. Sys.*, **42** (7) (2022), 3431-3463.

[5] M. Briane & L. Hervé: "Fine asymptotic expansion of the ODE's flow", *J. Differential Equations*, **373** (2023), 327-358.

[6] M. Briane, G.W. Milton & A. Treibergs : "Which electric fields are realizable in conducting materials?", *ESAIM: Math. Model. Numer. Anal.*, **48** (2) (2014), 307-323.

[7] A. Constantin & P. Germain : "Stratospheric planetary flows from the perspective of the Euler equation on a rotating sphere", *Arch. Ration. Mech. Anal.*, **245** (2022), 587-644.

[8] J. Franks: "Recurrence and fixed points of surface homeomorphisms", *Ergodic Theory Dynam. Systems*, **8** (1988), Charles Conley Memorial Issue, 99-107.

[9] I.P. Cornfeld, S.V. Fomin & Ya.G. Sinaĭ: *Ergodic Theory*, translated from the Russian by A.B. Sosinskii, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **245**, Springer-Verlag, New York, 1982, 486 pp.

[10] M.R. Herman: "Existence et non existence de tores invariants par des difféomorphismes symplectiques" (French), [Existence and nonexistence of tori invariant under symplectic diffeomorphisms], *Séminaire sur les Équations aux Dérivées Partielles 1987-1988*, **XIV**, École Polytech. Palaiseau, 1988, 24 pp.

[11] M. Misiurewicz & K. Ziemian: "Rotation sets for maps of tori", *J. London Math. Soc. (2)*, **40** (3) (1989), 490-506.

[12] R. Peirone: "Convergence of solutions of linear transport equations", *Ergodic Theory Dynam. Systems*, **23** (3) (2003), 919-933.

[13] R. Peirone: "Homogenization of ODE's in $\mathbb{R}^N$", *Ann. Mat. Pura Appl.* (4) **198** (3) (2019), 869-879.

[14] Ya.G. Sinai: *Introduction to Ergodic Theory*, Translated by V. Scheffer, Mathematical Notes **18**, Princeton University Press, Princeton, N.J., 1976, 144 pp.

[15] T. Tassa: "Homogenization of two-dimensional linear flows with integral invariance", *SIAM J. Appl. Math.*, **57** (5) (1997), 1390-1405.

[16] D. Wirosoetisno & T.G. Shepherd : "Nonlinear stability of Euler flows in two-dimensional periodic domains", *Geophys. Astrophys. Fluid Dynam.*, **90** (3-4) (1999), 229-246.