Space time convergence of implicit discretization strategies for the mixed formulation of linear wave equations

Juliette Chabassier

Space time convergence of implicit discretization strategies for the mixed formulation of linear wave equations.

Juliette Chabassier

RESEARCH REPORT
N° 9529
November 2023
Project-Team MAKUTU

ISSN 0249-6399 ISRN INRIA/RR--9529--FR+ENG

Space time convergence of implicit discretization strategies for the mixed formulation of linear wave equations.

Juliette Chabassier

Project-Team MAKUTU

Research Report n° 9529 — November 2023 — 22 pages

Abstract: This report presents the space-time convergence analysis of two implicit discretization strategies for the mixed formulation of wave equations, based for the first on the Störmer-Verlet scheme and for the second on the Crank-Nicolson scheme. This analysis is uniform with respect to the space-time convergence ratio. For each discretization strategy, after obtaining an exploitable energy identity, a uniform stability result is achieved through a technique of projection onto well-chosen eigensubspaces of a certain symmetric real matrix appearing in the construction of the schemes. Uniform convergence is then demonstrated using increasingly refined techniques to reduce the regularity assumptions on the semi-discrete solution.

Key-words: numerical analysis, space time convergence, implicit discretization, mixed formulation

RESEARCH CENTRE
BORDEAUX – SUD-OUEST

200 avenue de la Vieille Tour
33405 Talence Cedex

Analyse de convergence espace temps de stratégies de discrétisation implicites de la formulation mixte des équations d'ondes linéaires.

Résumé : Ce rapport présente l'analyse de convergence espace temps de deux stratégies implicites de discrétisation de la formulation mixte des équations d'onde, basées pour la première sur le schéma de Störmer-Verlet et pour la seconde sur le schéma de Crank-Nicolson. Cette analyse est uniforme par rapport au ratio espace temps de convergence. Pour chaque stratégie de discrétisation, après avoir obtenu une identité d'énergie exploitable, un résultat de stabilité uniforme est atteint grâce à une technique de projection sur des sous espaces propres bien choisis d'une certaine matrice réelle symétrique apparaissant dans la construction des schémas. La convergence uniforme est ensuite montrée en utilisant des techniques de plus en plus raffinées afin de réduire les hypothèses de régularité sur la solution semi discrète.

Mots-clés : analyse numérique, convergence spatio-temporelle, discrétisation implicite, formulation mixte

Contents

1 Introduction

1.1 General model

The mixed formulation of linear wave equations is one possible modeling of wave propagation phenomena. It is chosen over its second order formulation counterpart in situations where the unknowns are more relevant to the physical context [Makridakis and Monk, 1995, Lanteri and Scheid, 2013], or where these unknowns are more natural for modeling purposes (as for instance coupling with other parts [Banjai et al., 2015]), or even where it is not possible to formulate the equations as a second order equation (as for instance in presence of intricate dissipative [Bilbao and Harrison, 2016] or nonlinear phenomena [He and Sun, 2020]).
On an abstract level, this system reads, for $0 \leq t \leq T$,

$$\begin{cases} p(0) = p_0 & \text{(1.1a)} \\ v(0) = v_0 & \text{(1.1b)} \\ \dot{p} + \mathcal{B}^* v = f & \text{(1.1c)} \\ \dot{v} - \mathcal{B} p = \dot{g} & \text{(1.1d)} \end{cases}$$

where $\mathcal{B} : H^1(\mathcal{B}) \subset P \to D$ is an operator and $\mathcal{B}^*$ its adjoint. The source terms and the initial conditions are supposed regular enough so that the following hypothesis holds, in three Hilbert spaces P, $H^1(\mathcal{B}) = \{p \in P \,|\mathcal{B}p \in D\} \subset P$ and D :
Hypothesis 1.1 (Stability of the continuous system). *The source terms and the initial conditions are regular enough such that there exists a constant $C > 0$ such that*

$$\|p\|_{\mathcal{C}^3(0,T;P)} + \|p\|_{\mathcal{C}^2(0,T;H^1(\mathcal{B})} + \|v\|_{\mathcal{C}^2(0,T;D)} \leq C \tag{1.2}$$

1.2 Space discretisation

Numerical methods to solve this system are numerous and can rely on several analysis tools. In this work, we want to introduce and compare two parametrized time discretisation strategies, and we therefore suppose that the spatial discretisation is done with usual methods such as Finite Differences [Zuazua, 2005], Finite Elements [Brezzi and Fortin, 1991], Finite Volumes [Eymard et al., 2000], or any other method that provides the following semi discrete system, along with some necessary bounds on the space discretisation error. More precisely, we assume that, after introducing a small space discretization parameter h, and following the steps that lead to the semi discrete system, the semi discrete solution (p_h, v_h) is sought in finite dimensional spaces $U_h \subset H^1(\mathcal{B}) \subset P$ and $D_h \subset D$, where P and D are two Hilbert spaces naturally prompted by the equation as the variational spaces for both variables:

$$\begin{cases} p_h(0) = p_{h,0} & \text{(1.3a)} \\ v_h(0) = v_{h,0} & \text{(1.3b)} \\ \dot{p}_h + B_h^* v_h = f_h & \text{(1.3c)} \\ \dot{v}_h - B_h p_h = \dot{g}_h & \text{(1.3d)} \end{cases}$$

where $B_h : U_h \to D_h$ is a discrete approximation of the operator $\mathcal{B}$, and $B_h^* : D_h \to U_h$ is its adjoint, and $p_{h,0}$, $v_{h,0}$, f_h and g_h are discrete representations of p_0, v_0, f and g in U_h and D_h. We also denote I_h the identity operator of U_h.

For the sequel, we will suppose that the spatial discretisation satisfies the
Hypothesis 1.2 (Stability of the semi discrete system). *The spatial discretisation is such that there exists a constant $C > 0$ independent of h such that*

$$\|p_h\|_{\mathcal{C}^3(0,T;P)} + \|p_h\|_{\mathcal{C}^2(0,T;H^1(\mathcal{B}))} + \|v_h\|_{\mathcal{C}^2(0,T;D)} \leq C \tag{1.4}$$

Additional hypothesis of stability of the semi discrete system will be assumed when necessary in the analysis of the two considered discretisation (see Hyp. 2.1, Hyp. 2.2 and Hyp. 3.1).
Hypothesis 1.3 (Convergence of the semi discrete system). *The spatial discretisation is such that there exists a function $\delta : \mathbb{R}_*^+ \to \mathbb{R}^+$, such that*

$$\|p - p_h\|_{\mathcal{C}^0(0,T;P)} + \|v - v_h\|_{\mathcal{C}^0(0,T;D)} \leq \delta(h), \qquad \text{with } \delta(h) \underset{h\to 0}{\longrightarrow} 0 \tag{1.5}$$

1.3 Implicit parametrized discretizations

The following sections aim at analyzing the two parametrized implicit discretization proposed in [Chabassier, 2023b]. They are two numerical schemes based on the mixed form (1.3). Let $\theta \geq 0$ be a real number and $\Delta t > 0$ a time step, which will be used in both discretizations. The first one is a generalization of the Crank-Nicolson scheme called θ-CN, where both unknowns are sought on the same time grid. The choice $\theta = 1/4$ amounts to the usual Crank-Nicolson scheme. The second one is a generalization of the Störmer-Verlet scheme called θ-SV, where both unknowns are sought on interleaved time grids. The choice $\theta = 0$ amounts to the usual Störmer-Verlet scheme.
The first scheme θ-CN is considered in section 2, while the second scheme θ-SV is considered in section 3. Both sections follow the same analysis strategy: first the scheme is presented along with an energy identity, then the stability is shown, and finally the convergence is shown relying on some regularity assumptions on the semi-discrete solution of (1.3).
A first work [Chabassier, 2023a] has focused on obtaining uniform space/time convergence of the 0-SV scheme, in the sense of

- obtaining convergence between the fully discrete solution and the semi discrete solution that do not depend on the space discretization parameter h,
- ensuring that these results are uniform with respect to the space/time convergence ratio.

The aim of the present work is to generalize this to the θ-SV and the θ-CN schemes, which are parametrized implicit generalizations of the Störmer-Verlet and Crank-Nicolson schemes. The aim here was to obtain stability and convergence results that are uniform with the space/time convergence ratio, especially when $\theta < 1/4$, in which case the schemes are conditionally stable. Note that the usual Crank-Nicolson scheme amounts to $\theta = 1/4$, hence showing uniform space/time convergence does not need such refinements in the stability and convergence proofs.

2 Analysis of the parametrized implicit discretization based on the Crank-Nicolson scheme

In this section, a parametrized implicit discretization, introduced in [Chabassier, 2023b], based on the Crank-Nicolson scheme is analyzed. The unknowns of the scheme are sought on the regular time grid $\{t^n = n\Delta t\}_{0\leq n\leq N}$. Let us define the discrete operators δ and μ as

$$\delta w_h^{n+\frac{1}{2}} = \frac{w_h^{n+1} - w_h^n}{\Delta t}, \quad \mu w_h^{n+\frac{1}{2}} = \frac{w_h^{n+1} + w_h^n}{2}, \quad \delta q_h^{n+\frac{1}{2}} = \frac{q_h^{n+1} - q_h^n}{\Delta t}, \quad \mu q_h^{n+\frac{1}{2}} = \frac{q_h^{n+1} + q_h^n}{2} \tag{2.1}$$

They satisfy the useful following properties, for X a general sequence,

$$\|\delta X\| \leq \frac{2}{\Delta t}\mu \, \|X\| \tag{2.2a}$$

$$\delta X^{n+\frac{1}{2}} \mu X^{n+\frac{1}{2}} = \frac{1}{2}\delta (X^2)^{n+\frac{1}{2}} \tag{2.2b}$$

In order to reduce the technicality of the proofs, the appendix B shows useful general results on a template system of discrete equations. The same general results will be used to show the stability and the convergence of the considered scheme.

2.1 Parametrized Crank-Nicolson (θ-CN) and energy identities

The numerical scheme that is considered to solve (1.3) is to seek two sequences $\{w_h^n\}_{0\leq n\leq N}$ and $\{q_h^n\}_{0\leq n\leq N}$ such that

$$\begin{cases} w_h^0 = v_h(0) & \text{(2.3a)} \\ q_h^0 = p_h(0) & \text{(2.3b)} \\ \left(I_h + (\theta - \frac{1}{4})\Delta t^2 B_h^* B_h\right) \delta q_h^{n+\frac{1}{2}} + B_h^* \mu w_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}, \qquad 0 \leq n \leq N & \text{(2.3c)} \\ \delta w_h^{n+\frac{1}{2}} - B_h \mu q_h^{n+\frac{1}{2}} = \delta g_h^{n+\frac{1}{2}}, \qquad 0 \leq n \leq N & \text{(2.3d)} \end{cases}$$

DEFINITION 2.1 (Modified mass matrix). *In the following, we denote* $\widetilde{I}_h = I_h + (\theta - \frac{1}{4})\Delta t^2 B_h^* B_h$ *the modified mass matrix of the* θ*-CN scheme that we analyze.*

PROPOSITION 2.1 (Energy identity). *Any solution to* (2.3) *satisfies, for* $n \in [0, N]$,

$$\delta\mathcal{E}_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}} \cdot \mu p_h^{n+\frac{1}{2}} + \delta g_h^{n+\frac{1}{2}} \cdot \mu w_h^{n+\frac{1}{2}} \quad \text{where} \quad \mathcal{E}_h^n = \frac{1}{2}\|q_h^n\|^2_{\widetilde{I}_h} + \frac{1}{2}\|w_h^n\|^2_D \tag{2.4}$$

Proof. Take the scalar product of (2.3c) with $\mu q_h^{n+\frac{1}{2}}$ and of (2.3d) with $\mu w_h^{n+\frac{1}{2}}$ and use (2.2b), and denote

$$\|q_h^n\|^2_{\widetilde{I}_h} = (\widetilde{I}_h q_h^n, q_h^n)_P \tag{2.5}$$

□

PROPOSITION 2.2 (Stability condition). *If* $\theta \geq \frac{1}{4}$, *the discrete energy* $\mathcal{E}_h^n$ *is always positive. If* $\theta < \frac{1}{4}$, *the discrete energy is positive if*

$$\eta \leq 1, \qquad \text{where} \quad \eta = \frac{\Delta t}{2}\sqrt{1-4\theta}\sqrt{\rho(B_h^* B_h)} \tag{2.6}$$

Proof. This directly follows from the definition of $\widetilde{I}_h$. □

DEFINITION 2.2. *In the sequel, we will write that the "stability condition holds" if the choices of spatial discretisation and time step ensure that the discrete energy* $\mathcal{E}_h^n$ *is positive, which is specified in the previous proposition. This will be true either if* $\theta \geq 1/4$, *either if Eq.* (2.6) *holds.*

The following proposition is trivial to obtain if $\theta \geq 1/4$ since in this case, the modified mass matrix is definite positive and defines an equivalent norm to the canonic norm. The difficulty here was to obtain a result that is uniform as $\eta \to 1$ when $\theta < 1/4$.

PROPOSITION 2.3. *Suppose that the stability condition holds. Then, there exists* $\gamma > 0$, *independant of* h, Δt *and* η, *such that for all* $n \geq 0$,

$$\|\mu q_h^{n+\frac{1}{2}}\|_P \leq 2\gamma\mu\sqrt{2\mathcal{E}_h}^{n+\frac{1}{2}} + \gamma\left(\|g_h^{n+1}\|_D + \|g_h^n\|_D\right), \quad \|w_h^n\|_D \leq \sqrt{2\mathcal{E}_h}^n, \quad \|\mu w_h^{n+\frac{1}{2}}\|_D \leq \mu\sqrt{2\mathcal{E}_h}^{n+\frac{1}{2}} \tag{2.7}$$

where γ *only depends on the choice of* θ, *and is independant of* h, Δt *and* η.

Proof. Apply Prop. B.1 to the system (2.3), with $a_h^n = q_h^n$, $b_h^n = w_h^n$, $\eta_1^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}$, $\eta_2^{n+\frac{1}{2}} = 0$, $\eta_3^{n+\frac{1}{2}} = 0$ and $\eta_4^n = g_h^n$. The constant γ is defined in (B.12). □

2.2 Stability analysis

PROPOSITION 2.4. *Suppose that the stability condition holds. The discrete energy* $\{\mathcal{E}_h^n\}_n$ *satisfies*

$$\sqrt{\mathcal{E}_h^n} \leq \sqrt{3\mathcal{E}_h^0} + \sqrt{2}\gamma\left(\sqrt{T} + 4t^n\right)\|f_h\|_{\mathcal{C}^0(0,T;P)} + \sqrt{2}(2\sqrt{T} + t^n)\|g_h\|_{\mathcal{C}^1(0,T;D)} \tag{2.8}$$

where γ *only depends on the choice of* θ, *and is independant of* h, Δt *and* η.

Proof. From (2.4) we see that we can apply Prop. B.3 to our problem, with $a_h^n = q_h^n$, $b_h^n = w_h^n$, $\eta_1^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}$, $\eta_2^{n+\frac{1}{2}} = 0$, $\eta_3^{n+\frac{1}{2}} = 0$ and $\eta_4^n = g_h^n$. This prompts

$$\sqrt{\mathcal{E}_h^n} \leq A + 2t^n B \tag{2.9}$$

with

$$A = \sqrt{2}\left[\sqrt{T}\sup_{k=0}^{N}\left(\gamma\|f_h^{k+\frac{1}{2}}\| + 2\mu\|g_h\|^{k+\frac{1}{2}}\right) + \sqrt{3\mathcal{E}_h^0}\right] \tag{2.10}$$

$$B = \sqrt{2}\sup_{k=0}^{N}\left(2\gamma\|f_h^{k+\frac{1}{2}}\| + \|\delta g_h^{k+\frac{1}{2}}\|\right) \tag{2.11}$$

The term $\|\delta g_h^{k+\frac{1}{2}}\|$ can be upper bounded by the $\mathcal{C}^1(0,T;D)$ norm of g_h, hence the expected result. □

THEOREM 2.1 (Stability of the θ-CN scheme). *Suppose that the stability condition holds. Any solution to the θ-CN scheme* (2.3) *satisfies for all $n \geq 0$*

$$\|w_h^n\|_D \leq \sqrt{6\mathcal{E}_h^0} + 2\gamma\left(\sqrt{T} + 4t^n\right)\|f_h\|_{\mathcal{C}^0(0,T;P)} + 2(2\sqrt{T} + t^n)\|g_h\|_{\mathcal{C}^1(0,T;D)} \tag{2.12}$$

and

$$\|\mu q_h^{n+\frac{1}{2}}\|_P \leq 2\gamma\sqrt{6\mathcal{E}_h^0} + 4\gamma^2\left(\sqrt{T} + 4t^n\right)\|f_h\|_{\mathcal{C}^0(0,T;P)} + 4\gamma(2\sqrt{T} + t^n + \frac{1}{2})\|g_h\|_{\mathcal{C}^1(0,T;D)} \tag{2.13}$$

where γ only depends on the choice of θ, and is independant of h, Δt and η.

Proof. Using again the results of Prop. B.3 with $a_h^n = q_h^n$, $b_h^n = w_h^n$, $\eta_1^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}$, $\eta_2^{n+\frac{1}{2}} = 0$, $\eta_3^{n+\frac{1}{2}} = 0$ and $\eta_4^n = g_h^n$, we get

$$\|\mu q_h^{n+\frac{1}{2}}\| \leq 2\sqrt{2}\gamma\left(A + 2t^n B\right) + \gamma C \tag{2.14}$$

$$\|w_h^n\| \leq \sqrt{2}\left(A + 2t^n B\right) \tag{2.15}$$

where A and B are defined in (2.10) and (2.11), and

$$C = 2\sup_{k=0}^{N} \mu\|g_h\|^{n+\frac{1}{2}} \tag{2.16}$$

Hence,

$$\|w_h^n\| \leq \sqrt{6\mathcal{E}_h^0} + 2\gamma\left(\sqrt{T} + 4t^n\right)\|f_h\|_{\mathcal{C}^0(0,T;P)} + 2(2\sqrt{T} + t^n)\|g_h\|_{\mathcal{C}^1(0,T;D)} \tag{2.17}$$

$$\|\mu q_h^{n+\frac{1}{2}}\| \leq 2\gamma\sqrt{6\mathcal{E}_h^0} + 4\gamma^2\left(\sqrt{T} + 4t^n\right)\|f_h\|_{\mathcal{C}^0(0,T;P)} + 4\gamma(2\sqrt{T} + t^n)\|g_h\|_{\mathcal{C}^1(0,T;D)} + 2\gamma\|g_h\|_{\mathcal{C}^0(0,T;D)} \tag{2.18}$$

Hence the expected result. □

2.3 A first space/time uniform convergence result for smooth enough solutions

HYPOTHESIS 2.1. *Let (v_h, p_h) be the solution to* (1.3). *We suppose that there exist $C_{k,\ell} > 0$ independent of h such that*

$$\|p_h\|_{\mathcal{C}^\ell(0,T;H^k(\mathcal{B}))} \leq C_{k,\ell}, \quad \forall\ (k,\ell) = \{(0,3),(1,2)\} \tag{2.19}$$

$$\|v_h\|_{\mathcal{C}^\ell(0,T;H^k(\mathcal{B}^*)} \leq \tilde{C}_{k,\ell}, \quad \forall (k,\ell) = \{(0,3),(1,2)\} \tag{2.20}$$

$$\|g_h\|_{\mathcal{C}^\ell(0,T;H^k(\mathcal{B}^*)} \leq G_{k,\ell}, \quad \forall (k,\ell) = \{(0,3),(1,2)\} \tag{2.21}$$

where implicitly, $H^0(\mathcal{B}) = P$ and $H^0(\mathcal{B}^*) = D$.

THEOREM 2.2 (Convergence of the θ-CN scheme for a smooth enough discrete solution). *Suppose that the stability condition holds and that hypothesis 2.1 is satisfied by the semi discrete solution (v_h, p_h) of* (1.3). *The solution (w_h^n, q_h^n) to the θ-CN scheme* (2.3) *satisfies*

$$\|\bar{p}_h^n - \mu q_h^n\|_P \leq c\Delta t^2\left[C_{0,3} + |\theta - \frac{1}{4}|(\tilde{C}_{1,2} + G_{1,2}) + G_{0,3} + \tilde{C}_{0,3} + C_{1,2})\right] \tag{2.22}$$

$$\|\bar{v}_h^n - v_h^n\|_D \leq c\Delta t^2\left[C_{0,3} + |\theta - \frac{1}{4}|(\tilde{C}_{1,2} + G_{1,2}) + G_{0,3} + \tilde{C}_{0,3} + C_{1,2})\right] \tag{2.23}$$

where $c > 0$ depends on T and γ but is independent of h, η and Δt.

Proof. Define e_p and e_v the error of the θ-CN scheme (2.3) to the semi discrete system (1.3) as

$$e_p^n = \bar{p}_h^n - q_h^n, \qquad e_v^n = \bar{v}_h^n - w_h^n \tag{2.24}$$

These errors satisfy

$$\begin{aligned}\widetilde{I}_h\delta e_p^{n+\frac{1}{2}} + B_h^*\mu e_v^{n+\frac{1}{2}} &= \widetilde{I}_h\delta \bar{p}_h^{n+\frac{1}{2}} + B_h^*\mu \bar{v}_h^{n+\frac{1}{2}} - \widetilde{I}_h\delta q_h^{n+\frac{1}{2}} - B_h^*\mu w_h^{n+\frac{1}{2}} \\ &= \dot{\overline{p}}_h^{n+\frac{1}{2}} + \dot{\varepsilon}_p^{n+\frac{1}{2}} + \Delta t^2(\theta-\frac{1}{4})B_h^*B_h\delta\bar{p}_h^{n+\frac{1}{2}} + B_h^*(\overline{v}_h^{n+\frac{1}{2}} + \eta_v^{n+\frac{1}{2}}) - f_h^{n+\frac{1}{2}} \\ &= \overline{f}_h^{n+\frac{1}{2}} + \dot{\varepsilon}_p^{n+\frac{1}{2}} + B_h^*\xi_p^{n+\frac{1}{2}} + B_h^*\eta_v^{n+\frac{1}{2}} - f_h^{n+\frac{1}{2}} \\ &= \dot{\varepsilon}_p^{n+\frac{1}{2}} + B_h^*\xi_p^{n+\frac{1}{2}} + B_h^*\eta_v^{n+\frac{1}{2}} \end{aligned} \tag{2.25a}$$

and

$$\begin{aligned}\delta e_v^{n+\frac{1}{2}} - B_h\mu e_p^{n+\frac{1}{2}} &= \delta\bar{v}_h^{n+\frac{1}{2}} - B_h\mu\bar{p}_h^{n+\frac{1}{2}} - (\delta w_h^{n+\frac{1}{2}} - B_h\mu q_h^{n+\frac{1}{2}}) \\ &= \dot{\overline{v}}_h^{n+\frac{1}{2}} + \dot{\varepsilon}_v^{n+\frac{1}{2}} - B_h\bar{p}_h^{n+\frac{1}{2}} - B_h\eta_p^{n+\frac{1}{2}} - \delta g_h^{n+\frac{1}{2}} \\ &= \dot{\overline{g}}_h^{n+\frac{1}{2}} + \dot{\varepsilon}_v^{n+\frac{1}{2}} - B_h\eta_p^{n+\frac{1}{2}} - \delta g_h^{n+\frac{1}{2}} \\ &= \varepsilon_g^{n+\frac{1}{2}} + \dot{\varepsilon}_v^{n+\frac{1}{2}} - B_h\eta_p^{n+\frac{1}{2}} \end{aligned} \tag{2.25b}$$

where there exist $t^n \le t_1, t_2, t_3, t_4, t_5, t_6 \le t^{n+1}$ such that

$$\begin{aligned}\dot{\varepsilon}_p^{n+\frac{1}{2}} = \frac{\Delta t^2}{24}p_h^{(3)}(t_1), \quad \eta_v^{n+\frac{1}{2}} = \frac{\Delta t^2}{8}v_h^{(2)}(t_2), \\ \xi_p^{n+\frac{1}{2}} = \Delta t^2(\theta-\frac{1}{4})B_h p_h^{(1)}(t_3) \equiv_{\frac{d}{dt}(1.3)} \Delta t^2(\theta-\frac{1}{4})\left[v_h^{(2)}(t_3) - g_h^{(2)}(t_3)\right] \\ \varepsilon_g^{n+\frac{1}{2}} = \frac{\Delta t^2}{24}g_h^{(3)}(t_4), \quad \dot{\varepsilon}_v^{n+\frac{1}{2}} = \frac{\Delta t^2}{24}v_h^{(3)}(t_5), \quad \eta_p^{n+\frac{1}{2}} = \frac{\Delta t^2}{8}p_h^{(2)}(t_6)\end{aligned} \tag{2.26}$$

and

$$e_p^0 = 0, \quad e_v^0 = 0 \tag{2.27}$$

We can apply the results of Prop. B.3 with

$$\begin{aligned}a_h^n = e_p^n, \quad b_h^n = e_v^n, \quad \eta_1^{n+\frac{1}{2}} = \dot{\varepsilon}_p^{n+\frac{1}{2}} + B_h^*\xi_p^{n+\frac{1}{2}} + B_h^*\eta_v^{n+\frac{1}{2}}, \quad \eta_2^{n+\frac{1}{2}} = \varepsilon_g^{n+\frac{1}{2}} + \dot{\varepsilon}_v^{n+\frac{1}{2}} - B_h\eta_p^{n+\frac{1}{2}}, \\ \eta_3^{n+\frac{1}{2}} = 0, \quad \eta_4^n = 0, \quad \mathcal{E}_h^n \equiv \mathcal{E}_e^n = \frac{1}{2}\|e_p^n\|^2_{\widetilde{I}_h} + \frac{1}{2}\|e_v^n\|^2\end{aligned} \tag{2.28}$$

where, from the hypothesis 2.1, we know that there exists $c > 0$ such that

$$H_1^N := \sup_{k=0}^{N}\|\eta_1^{k+\frac{1}{2}}\| \le c\Delta t^2\left[\left(C_{0,3} + |\theta-\frac{1}{4}|(\tilde{C}_{1,2} + G_{1,2})\right)\right] \tag{2.29}$$

$$H_2^N := \sup_{k=0}^{N}\|\eta_2^{k+\frac{1}{2}}\| \le c\Delta t^2\left[G_{0,3} + \tilde{C}_{0,3} + C_{1,2})\right] \tag{2.30}$$

Hence,

$$\|\mu e_p^{n+\frac{1}{2}}\| \le 2\sqrt{2}\gamma\left(A + 2t^nB\right) + \gamma C \tag{2.31}$$

$$\|e_v^n\| \le \sqrt{2}\left(A + 2t^nB\right) \tag{2.32}$$

where

$$A = \sqrt{2}\left[\sqrt{T}\left[\gamma H_1^N + (1+\Delta t)H_2^N\right] + \sqrt{3\mathcal{E}_e^0}\right] \tag{2.33}$$

$$B = \sqrt{2}\left(2\gamma H_1^N + H_2^N\right) \tag{2.34}$$

$$C = \Delta t H_2^N \tag{2.35}$$

We know that $\mathcal{E}_e^0 = 0$ from the initial conditions. Moreover, $\Delta t \le t^n \le T$. Hence,

$$A + 2t^nB \le \sqrt{2}\sqrt{T}\left[\gamma H_1^N + (1+T)H_2^N\right] + 2T\sqrt{2}\left(2\gamma H_1^N + H_2^N\right) \tag{2.36}$$

$$\le \sqrt{2}\gamma\left[\sqrt{T} + 4T\right]H_1^N + \sqrt{2}\left[(1+T)\sqrt{T} + 2T\right]H_2^N \tag{2.37}$$

$$\|\mu e_p^{n+\frac{1}{2}}\| \leq 2\sqrt{2}\gamma\,(A + 2t^n B) + \gamma T H_2^N \tag{2.38}$$

$$\leq 4\gamma^2 \left[\sqrt{T} + 4T\right] H_1^N + 4\gamma \left[(1+T)\sqrt{T} + \frac{9T}{4}\right] H_2^N \tag{2.39}$$

$$\leq c(H_1^N + H_2^N) \tag{2.40}$$

and

$$\|e_v^n\| \leq 2\gamma \left[\sqrt{T} + 4T\right] H_1^N + 2\left[(1+T)\sqrt{T} + 2T\right] H_2^N \tag{2.41}$$

$$\leq c(H_1^N + H_2^N) \tag{2.42}$$

where $c > 0$ is a constant that can change from line to line, and only depends on γ and T. Finally,

$$\|\bar{p}_h^{n+\frac{1}{2}} - \mu q_h^{n+\frac{1}{2}}\| \leq \|\bar{p}_h^{n+\frac{1}{2}} - \mu \bar{p}_h^{n+\frac{1}{2}}\| + \|\mu(\bar{p}_h^{n+\frac{1}{2}} - q_h^{n+\frac{1}{2}})\| \tag{2.43}$$

$$\leq \|\eta_p^{n+\frac{1}{2}}\| + \|\mu e_p^{n+\frac{1}{2}}\| \tag{2.44}$$

$$\leq c\Delta t^2 \left[C_{02} + C_{0,3} + |\theta - \frac{1}{4}|(\tilde{C}_{1,2} + G_{1,2}) + G_{0,3} + \tilde{C}_{0,3} + C_{1,2})\right] \tag{2.45}$$

and

$$\|\bar{v}_h^n - v_h^n\| \leq c\Delta t^2 \left[C_{0,3} + |\theta - \frac{1}{4}|(\tilde{C}_{1,2} + G_{1,2}) + G_{0,3} + \tilde{C}_{0,3} + C_{1,2})\right] \tag{2.46}$$

□

Remark 2.1. Notice that the specific choice $\theta = \frac{1}{4}$, which leads to the usual Crank-Nicolson scheme, removes the terms in $(\tilde{C}_{1,2} + G_{1,2})$ in the error constant.

Remark 2.2. This result is not optimal, it is indeed possible to reduce the regularity assumptions on the semi-discrete solution, as done in the following paragraph.

2.4 A second space/time uniform convergence result for less regular solution

HYPOTHESIS 2.2. *Let (v_h, p_h) be the solution to* (1.3)*. We suppose that there exist constants $C_{k,\ell} > 0$ independent of h such that*

$$\|p_h\|_{\mathcal{C}^\ell(0,T;H^k(\mathcal{B}))} \leq C_{k,\ell}, \quad \forall\ (k,\ell) = \{(0,3),(1,2)\} \tag{2.47}$$

$$\|v_h\|_{\mathcal{C}^\ell(0,T;H^k(\mathcal{B}^*)} \leq \tilde{C}_{k,\ell}, \quad \forall (k,\ell) = \{(0,3)\} \tag{2.48}$$

$$\|g_h\|_{\mathcal{C}^\ell(0,T;H^k(\mathcal{B}^*)} \leq G_{k,\ell}, \quad \forall (k,\ell) = \{(0,3)\} \tag{2.49}$$

where implicitly, $H^0(\mathcal{B}) = P$ and $H^0(\mathcal{B}^*) = D$.

THEOREM 2.3 (Convergence of the θ-CN scheme for a less regular discrete solution). *Suppose that the stability condition holds and that hypothesis 2.2 is satisfied by the semi discrete solution (v_h, p_h) of* (1.3)*. The solution (w_h^n, q_h^n) to the θ-CN scheme* (2.3) *satisfies*

$$\|\bar{p}_h^n - \mu q_h^n\|_P \leq c\Delta t^2 \left[\|p_h\|_{\mathcal{C}^3(0,T;P)} + \|p_h\|_{\mathcal{C}^1(0,T;H^1(\mathcal{B}))} + \|v_h\|_{\mathcal{C}^3(0,T;D)} + \|g_h\|_{\mathcal{C}^3(0,T;D)}\right] \tag{2.50}$$

$$\|\bar{v}_h^n - w_h^n\|_D \leq c\Delta t^2 \left[\|p_h\|_{\mathcal{C}^3(0,T;P)} + \|p_h\|_{\mathcal{C}^1(0,T;H^1(\mathcal{B}))} + \|v_h\|_{\mathcal{C}^3(0,T;D)} + \|g_h\|_{\mathcal{C}^3(0,T;D)}\right] \tag{2.51}$$

where c depends on T and γ but is independent of h, η and Δt.

Proof of Thm 2.3 with usual manipulation. The proof begins exactly as Proof 2.3, to write the scheme satisfied by the error terms. However, we choose to apply the results of Prop. B.3 with another interpretation of the RHS terms of (2.25):

$$a_h^n = e_p^n, \quad b_h^n = e_v^n, \quad \eta_1^{n+\frac{1}{2}} = \dot{\varepsilon}_p^{n+\frac{1}{2}}, \quad \eta_2^{n+\frac{1}{2}} = \varepsilon_g^{n+\frac{1}{2}} + \dot{\varepsilon}_v^{n+\frac{1}{2}} - B_h \eta_p^{n+\frac{1}{2}},$$

$$\eta_3^{n+\frac{1}{2}} = \xi_p^{n+\frac{1}{2}} + \eta_v^{n+\frac{1}{2}}, \quad \eta_4^n = 0, \quad \mathcal{E}_h^n \equiv \mathcal{E}_e^n = \frac{1}{2}\|e_p^n\|_{\tilde{I}_h}^2 + \frac{1}{2}\|e_v^n\|^2 \tag{2.52}$$

When η_3, the proof of Prop. B.3 makes use of an Abel summation formula to "exchange" spatial differentiation with B_h^* with discrete time differentiation. Hence,

$$\|\mu e_p^{n+\frac{1}{2}}\| \leq 2\sqrt{2}\gamma\left(A + 2t^n B\right) + \gamma C \tag{2.53}$$

$$\|e_v^n\| \leq \sqrt{2}\left(A + 2t^n B\right) \tag{2.54}$$

where

$$A = \sqrt{2}\left[\sqrt{T}\left[\gamma H_1^N + (1+\Delta t)H_2^N + H_3^N\right] + 2\sqrt{2}H_3^N\right] \tag{2.55}$$

$$B = \sqrt{2}\left(2\gamma H_1^N + H_2^N + 2{H_3'}^N\right) \tag{2.56}$$

$$C = \Delta t H_2^N \tag{2.57}$$

where, from the hypothesis 2.1, we know that there exists $c > 0$ such that

$$H_1^N := \sup_{k=0}^{N} \|\eta_1^{k+\frac{1}{2}}\| \leq c\Delta t^2 C_{0,3} \tag{2.58}$$

$$H_2^N := \sup_{k=0}^{N} \|\eta_2^{k+\frac{1}{2}}\| \leq c\Delta t^2\left[G_{0,3} + \tilde{C}_{0,3} + C_{1,2})\right] \tag{2.59}$$

$$H_3^N := \sup_{k=0}^{N} \|\eta_3^{k+\frac{1}{2}}\| \leq c\Delta t^2\left[\left|\theta - \frac{1}{4}\right| C_{1,1} + \tilde{C}_{0,2}\right] \tag{2.60}$$

$${H_3'}^N := \sup_{k=0}^{N} \|\delta\eta_3^{k+\frac{1}{2}}\| \leq c\Delta t^2\left[\left|\theta - \frac{1}{4}\right| C_{1,2} + \tilde{C}_{0,3}\right] \tag{2.61}$$

Hence, there exists a constant $c > 0$ that only depends on γ and T such that

$$\|\mu e_p^{n+\frac{1}{2}}\| \leq c\Delta t^2\left[C_{0,3} + G_{0,3} + \tilde{C}_{0,3} + C_{1,2}\right] \tag{2.62}$$

$$\|e_v^n\| \leq c\Delta t^2\left[C_{0,3} + G_{0,3} + \tilde{C}_{0,3} + C_{1,2}\right] \tag{2.63}$$

The rest of the proof follows the end of Proof 2.3.

□

Proof of Thm 2.3 based on the field. Let us prove the same theorem with another approach. Let $u_h \in U_h$ be the field defined as

$$\dot{u}_h = p_h \tag{2.64}$$

with $u_h(0)$ chosen such that $B_h u_h(0) = v_h(0)$. We also define a discrete field $\{u_h^{n+\frac{1}{2}}\}_{0\leq n\leq N-1}$ as

$$\delta u_h^n = q_h^n \tag{2.65}$$

with u_h^0 chosen such that $B_h u_h^0 = w_h^0$. Then, the semi discrete and discrete unknowns are solution to

$$\begin{cases} \ddot{u}_h + B_h^* v_h = f_h & \text{(2.66a)} \\ v_h - B_h u_h = g_h & \text{(2.66b)} \\ \dot{u}_h = p_h & \text{(2.66c)} \end{cases} \qquad \begin{cases} \widetilde{I}_h \delta^2 u_h^{n+\frac{1}{2}} + B_h^* \mu w_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}, \quad 0 \leq n \leq N & \text{(2.67a)} \\ w_h^n - B_h \mu u_h^n = g_h^n, \quad 0 \leq n \leq N & \text{(2.67b)} \\ \delta u_h^n = q_h^n, \quad 0 \leq n \leq N & \text{(2.67c)} \end{cases}$$

Therefore, letting e_u, e_v and e_p be defined as

$$e_u^{n+\frac{1}{2}} = \bar{u}_h^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}, \quad e_v^n = \bar{v}_h^n - w_h^n, \quad e_p^n = \bar{p}_h^n - q_h^n \tag{2.68}$$

we get that

$$\begin{aligned}
\widetilde{I}_h\delta^2 e_u^{n+\frac12} + B_h^*\mu e_v^{n+\frac12} &= \widetilde{I}_h\delta^2\bar{u}_h^{n+\frac12} - \widetilde{I}_h\delta^2 u_h^{n+\frac12} + B_h^*\mu\bar{v}_h^n - B_h^*\mu w_h^n \\
&= \delta^2\bar{u}_h^{n+\frac12} + B_h^*\mu\bar{v}_h^n - \widetilde{I}_h\delta^2 u_h^{n+\frac12} - B_h^*\mu w_h^n + \Delta t^2(\theta-\frac14)B_h^*B_h\delta^2\bar{u}_h^{n+\frac12} \\
&= \bar{u}_h^{(2)}(t^{n+\frac12}) + \ddot{\varepsilon}_u^{n+\frac12} + B_h^*\left[\bar{v}_h(t^{n+\frac12}) + \eta_v^{n+\frac12}\right] - f_h^{n+\frac12} + \Delta t^2(\theta-\frac14)B_h^*B_h\delta^2\bar{u}_h^{n+\frac12} \\
&= \ddot{\varepsilon}_u^{n+\frac12} + B_h^*\eta_v^{n+\frac12} + (\theta-\frac14)B_h^*\eta_u^{n+\frac12} \qquad (2.69)\\
e_v^n - B_h\mu e_u^n &= -B_h\eta_u^n, \qquad (2.70)\\
\delta e_u^n - e_p^n &= \dot{\varepsilon}_u^n \qquad (2.71)
\end{aligned}$$

where there exist $t^{n-1} \le t_1, t_2, t_3, t_4^{n+\frac12} \le t^{n+1}, t_5$ such that

$$\ddot{\varepsilon}_u^{n+\frac12} = \delta^2\bar{u}_h^{n+\frac12} - u_h^{(2)}(t^{n+\frac12}) = \frac{\Delta t^2}{12}u_h^{(4)}(t_1) = \frac{\Delta t^2}{12}p_h^{(3)}(t_1) \qquad (2.72)$$

$$\eta_v^{n+\frac12} = \mu\bar{v}_h^{n+\frac12} - v_h(t^{n+\frac12}) = \frac{\Delta t^2}{8}v_h^{(2)}(t_2), \qquad (2.73)$$

$$\dot{\varepsilon}_u^n = \delta\bar{u}_h^n - u_h^{(1)}(t^n) = \frac{\Delta t^2}{3}u_h^{(3)}(t_3) = \frac{\Delta t^2}{3}p_h^{(2)}(t_3) \qquad (2.74)$$

$$\eta_u^n = \mu\bar{u}_h^n - u_h(t^n) = \frac{\Delta t^2}{8}u_h^{(2)}(t_5) \qquad (2.75)$$

$$\nu_u^{n+\frac12} = \Delta t^2 B_h\delta^2\bar{u}_h^{n+\frac12} = \Delta t^2 B_h\ddot{u}_h(t_4^{n+\frac12}) = \Delta t^2 B_h\dot{p}_h(t_4^{n+\frac12}) \qquad (2.76)$$

At this point, we introduce the sequence $\{\mathfrak{e}_p^n\}_{0\le n\le N}$ as

$$\mathfrak{e}_p^n = e_p^n + \dot{\varepsilon}_u^n \quad \text{such that} \quad \delta e_u^n = \mathfrak{e}_p^n \qquad (2.77)$$

The first order system on $(\mathfrak{e}_p^n, e_v^n)$ reads

$$\begin{cases}
\widetilde{I}_h\delta\mathfrak{e}_p^n + B_h^*\mu e_v^{n+\frac12} = \ddot{\varepsilon}_u^{n+\frac12} + B_h^*\eta_v^{n+\frac12} + (\theta-\frac14)B_h^*\nu_u^{n+\frac12} & (2.78a)\\
\delta e_v^{n+\frac12} - B_h\mu\mathfrak{e}_p^{n+\frac12} = -B_h\delta\eta_u^{n+\frac12} & (2.78b)\\
\mathfrak{e}_p^0 = \dot{\varepsilon}_u^0 & (2.78c)\\
e_v^0 = 0 & (2.78d)
\end{cases}$$

Hence, we can apply the results of Prop. B.3 with (note in blue between parenthesis the upper bounds of each corresponding term)

$$\begin{aligned}
a_h^n = \mathfrak{e}_p^n, \quad b_h^n = e_v^n, \quad \eta_1^{n+\frac12} = \ddot{\varepsilon}_u^{n+\frac12}(C_{0,3}), \quad \eta_2^{n+\frac12} = 0, \\
\eta_3^{n+\frac12} = \eta_v^{n+\frac12}(\tilde{C}_{0,2}) + (\theta-\frac14)\nu_u^{n+\frac12}(C_{1,1}), \quad \eta_4^n = -B_h\eta_u^n(C_{1,1}) \qquad (2.79)
\end{aligned}$$

Hence,

$$\|\mu\mathfrak{e}_p^{n+\frac12}\| \le 2\sqrt{2}\gamma\,(A + 2t^n B) + \gamma C \qquad (2.80)$$

$$\|e_v^n\| \le \sqrt{2}\,(A + 2t^n B) \qquad (2.81)$$

where

$$A = \sqrt{2}\left[\sqrt{T}\sup_{k=0}^{N}\left[\gamma\|\eta_1^{k+\frac12}\| + 2\mu\|\eta_4\|^{k+\frac12} + \|\eta_3^{k+\frac12}\|+\right] + \sqrt{2}\|\eta_3^N\| + \sqrt{2}\|\eta_3^0\| + \sqrt{3\mathcal{E}_h^0}\right] \qquad (2.82)$$

$$B = \sqrt{2}\sup_{k=0}^{N}\left(2\gamma\|\eta_1^{k+\frac12}\| + \|\delta\eta_4^{k+\frac12}\| + 2\|\delta\eta_3^{k+\frac12}\|\right) \qquad (2.83)$$

$$C = 2\sup_{k=0}^{N}\|\eta_4\|^{n+\frac12} \qquad (2.84)$$

Hence there exists $c > 0$ such that

$$\|\mu\mathfrak{e}_p^{n+\frac{1}{2}}\| \leq c\Delta t^2 \left[C_{0,3} + C_{0,1} + \tilde{C}_{0,2} + \left|\theta - \frac{1}{4}\right| (C_{1,1} + C_{1,2}) + C_{0,2} + C_{1,2} + \tilde{C}_{0,3}\right] \tag{2.85}$$

$$\leq c\Delta t^2 \left[C_{0,3} + C_{1,2} + \tilde{C}_{0,3}\right] \tag{2.86}$$

$$\|e_v^n\| \leq c\Delta t^2 \left[C_{0,3} + C_{1,2} + \tilde{C}_{0,3}\right] \tag{2.87}$$

The rest of the proof follows the end of the previous one. □

3 Analysis of the parametrized implicit discretization based on Störmer-Verlet scheme

In this section, a parametrized implicit discretization, introduced in [Chabassier, 2023b], based on the Crank-Nicolson scheme, is analyzed. The unknowns of the scheme are sought on interleaved time grids : v_h is discretized on the regular time grid $\{t^n = n\Delta t\}_{0\leq n\leq N}$ while p_h is discretized on the interleaved time grid $\{t^{n+\frac{1}{2}} = (n + \frac{1}{2})\Delta t\}_{0\leq n\leq N}$. Let us define the discrete operators δ and μ as

$$\delta v_h^{n+\frac{1}{2}} = \frac{v_h^{n+1} - v_h^n}{\Delta t}, \quad \mu v_h^{n+\frac{1}{2}} = \frac{v_h^{n+1} + v_h^n}{2}, \quad \delta p_h^n = \frac{p_h^{n+\frac{1}{2}} - p_h^{n-\frac{1}{2}}}{\Delta t}, \quad \mu p_h^n = \frac{p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}}{2} \tag{3.1}$$

A new useful property is the following one:

$$v_h^{n-\frac{1}{2}\pm\frac{1}{2}} = \mu v_h^{n-\frac{1}{2}} \pm \frac{\Delta t}{2}\delta v_h^{n-\frac{1}{2}} \tag{3.2}$$

3.1 Parametrized Störmer-Verlet (θ-SV) and energy identities

The numerical scheme that is considered to solve (1.3) is to seek two sequences $\{v_h^n\}_{0\leq n\leq N}$ and $\{p_h^{n+\frac{1}{2}}\}_{0\leq n\leq N}$ such that

$$\begin{cases} v_h^0 = v_h(0) & \text{(3.3a)} \\ p_h^{\frac{1}{2}} = p_h(0) + \dfrac{\Delta t}{2}\left[f_h^0 + B_h^* v_h(0)\right] & \text{(3.3b)} \\ (I_h + \theta\Delta t^2 B_h^* B_h)\delta p_h^n + B_h^* v_h^n = f_h^n, \qquad 1 \leq n \leq N & \text{(3.3c)} \\ \delta v_h^{n+\frac{1}{2}} - B_h p_h^{n+\frac{1}{2}} = \delta g_h^{n+\frac{1}{2}}, \qquad 0 \leq n \leq N & \text{(3.3d)} \end{cases}$$

The case $\theta = 0$ corresponds to the usual Störmer-Verlet scheme. Its space/time convergence was showed in [Chabassier, 2023a]. The obtained results are here generalized to the θ-SV scheme in the following propositions and theorems.

Proposition 3.1 (Naïve energy identity). *Any solution to* (3.3) *satisfies, for* $n \in [1, N]$,

$$\delta\mathcal{E}_{pv,h}^n = (f_h^n, \mu p_h^n)_P + (\mu\delta g_h^n, v_h^n)_D \quad \textit{with} \quad \mathcal{E}_{pv,h}^{n+\frac{1}{2}} = \frac{1}{2}((I_h + \theta\Delta t^2 B_h^* B_h)p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2}(v_h^{n+1}, v_h^n)_D \tag{3.4}$$

Proof. Take the scalar product of (3.3c) with μp_h^n and of μ(3.3d) with $v_h^{n+\frac{1}{2}}$ □

Proposition 3.2.

$$\delta\mathcal{E}_h^n = (f_h^n, \mu p_h^n)_P - (B_h^* g_h^n, \mu p_h^n)_P$$
$$\textit{with} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2}((I_h + \theta\Delta t^2 B_h^* B_h)p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2}(v_h^{n+1} - g_h^{n+1}, v_h^n - g_h^n)_D \tag{3.5}$$

Proof. This is obtained by subtracting and adding the term $B_h^* g_h^n$ to the first line and taking the scalar product of μ (3.3d) with $v_h^{n+\frac{1}{2}} - g_h^{n+\frac{1}{2}}$, and of (3.3c) with μp_h^n. □

PROPOSITION 3.3 (Reformulation of the energy).

$$\mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_{\widetilde{I}_h} + \frac{1}{2}(\mu x_h^{n+\frac{1}{2}}, \mu x_h^{n+\frac{1}{2}})_D \tag{3.6}$$

where the sequence x_h^n is defined as

$$x_h^n = v_h^n - g_h^n \tag{3.7}$$

and the modified mass matrix $\widetilde{I}_h$ is defined as

$$\widetilde{I}_h = I_h + (\theta - \frac{1}{4})\Delta t^2 B_h^* B_h \tag{3.8}$$

Proof. This is obtained by applying formula (3.2) on the sequence $\{x_h^n\}$ and using Eq. (3.3d) to replace the occurences of $\delta x_h^{n+\frac{1}{2}}$. □

PROPOSITION 3.4 (Stability condition). *If $\theta \geq \frac{1}{4}$, the discrete energy $\mathcal{E}_h^n$ is always positive. If $\theta < \frac{1}{4}$, the discrete energy is positive if*

$$\eta \leq 1, \qquad \textit{where} \quad \eta = \frac{\Delta t}{2}\sqrt{1-4\theta}\sqrt{\rho(B_h^* B_h)} \tag{3.9}$$

DEFINITION 3.1. *In the sequel, we will write that the "stability condition holds" if the choices of spatial discretisation and time step ensure that the discrete energy $\mathcal{E}_h^{n+\frac{1}{2}}$ is positive, which is specified in the previous proposition. This will be true either if $\theta \geq 1/4$, either if Eq. (3.9) holds.*

Remark 3.1. Note that the stability condition is exactly the same as the θ-CN scheme, which is consistant with the fact that, up to the initial conditions, the two schemes can be seen as two algorithmic strategies to compute the same quantities (see [Chabassier, 2023b]).

3.2 Stability analysis

The stability analysis of this parametrized implicit scheme is very similar to the one in [Chabassier, 2023a], so we only provide here the main results.

PROPOSITION 3.5. *Any $\{p_h^{n+\frac{1}{2}}\}_n \in U_h$ solution to (3.3) satisfies*

$$\begin{cases} \|\Pi_k \mu p_h^n\|_P \leq \dfrac{C_k^{-1/2}}{2}\left(\sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} + \sqrt{2\mathcal{E}_h^{n-\frac{1}{2}}}\right) & \text{(3.10a)} \\ \|\Pi_p \mu p_h^n\|_P \leq C_p^{-1/2}\left(\sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} + \sqrt{2\mathcal{E}_h^{n-\frac{1}{2}}}\right) & \text{(3.10b)} \end{cases}$$

where C_k and C_p are defined in Appendix A.

Proof. See the results of Appendix A and proof of Prop. 2.7 of [Chabassier, 2023a]. □

PROPOSITION 3.6 (Majoration of the averaged unknowns).

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} + \mu\|g_h\|_D^{n+\frac{1}{2}} \quad \|\mu p_h^n\|_P \leq \gamma\left(\sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} + \sqrt{2\mathcal{E}_h^{n-\frac{1}{2}}}\right) \tag{3.11}$$

where

$$\gamma = \frac{C_k^{-1/2}}{2} + C_p^{-1/2} \tag{3.12}$$

Proof. See the results of Appendix A and proof of Prop. 2.8 of [Chabassier, 2023a]. We get

$$\|\mu x_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}}, \qquad \|\mu p_h^n\|_P \leq \gamma\left(\sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} + \sqrt{2\mathcal{E}_h^{n-\frac{1}{2}}}\right) \tag{3.13}$$

So,

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \|\mu x_h^{n+\frac{1}{2}}\|_D + \|\mu g_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} + \mu\|g_h\|_D^{n+\frac{1}{2}} \tag{3.14}$$

□

PROPOSITION 3.7 (Stability of the energy). *The energy satisfies*

$$\sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^{n} \left[\|f_h^k\|_P + \|B_h^* g_h^k\|_P\right] \tag{3.15}$$

where

$$\sqrt{\mathcal{E}_h^{\frac{1}{2}}} \leq \frac{1+\sqrt{2}}{2}\|p_h(0)\|_P + \frac{1+\sqrt{2}}{2}\frac{\Delta t}{2}\|f_h^0\|_P + \frac{1+2\sqrt{2}}{2}\|v_h(0)\|_D + \frac{1}{\sqrt{2}}\|g_h^0\|_D \tag{3.16}$$

Proof. See proof of Prop. 2.9 of [Chabassier, 2023a]. □

3.3 Space/time uniform convergence

HYPOTHESIS 3.1. *Let (v_h, p_h) be the solution to (1.3). We suppose that there exist constants $C_{k,\ell} > 0$ independent of h such that*

$$\|p_h\|_{\mathcal{C}^\ell(0,T;H^k(\mathcal{B}))} \leq C_{k,\ell}, \quad \forall\ (k,\ell) = \{(0,3),(1,2)\} \tag{3.17}$$

$$\|v_h\|_{\mathcal{C}^\ell(0,T;H^k(\mathcal{B}^*)} \leq \tilde{C}_{k,\ell}, \quad \forall (k,\ell) = \{(0,2)\} \tag{3.18}$$

where implicitly, $H^0(\mathcal{B}) = P$ and $H^0(\mathcal{B}^*) = D$.

The following theorem extends Thm 3.1 of [Chabassier, 2023a] which was only valid for $\theta = 0$. It generalizes the proof technique in order to encompass the resulting additional terms of the θ-SV scheme analysis, but nicely reduces to the expected result in the case $\theta = 0$.

THEOREM 3.1. *Suppose that the stability condition holds. Then there exists a constant $C > 0$ independent of h and η, such that*

$$\|\bar{p}_h^n - \mu p_h^n\|_P \leq C\Delta t^2 \left(\|p_h\|_{\mathcal{C}^3(0,T,P)} + \sqrt{\theta}\|p_h\|_{\mathcal{C}^2(0,T,H^1(\mathcal{B}))}\right) \tag{3.19}$$

$$\|\bar{v}_h^{n+\frac{1}{2}} - \mu v_h^{n+\frac{1}{2}}\|_D \leq C\Delta t^2 \left(\|v_h\|_{\mathcal{C}^2(0,T;D)} + \|p_h\|_{\mathcal{C}^3(0,T,P)} + \sqrt{\theta}\|p_h\|_{\mathcal{C}^2(0,T,H^1(\mathcal{B}))}\right) \tag{3.20}$$

Proof. The proof of this theorem follows the one of Thm 3.1 in [Chabassier, 2023a] but adds a manipulation based on the Abel discrete summation recalled in Lemma B.2. Let $u_h \in U_h$ be the field defined as

$$\dot{u}_h = p_h \tag{3.21}$$

with $u_h(0)$ chosen such that $B_h u_h(0) = v_h(0)$. We also define a discrete field $\{u_h^n\}_{0\leq n\leq N-1}$ as

$$\delta u_h^{n+\frac{1}{2}} = p_h^{n+\frac{1}{2}} \tag{3.22}$$

with u_h^0 chosen such that $B_h u_h^0 = v_h^0$. Then, the semi discrete and discrete unknowns are solution to

$$\begin{cases} \ddot{u}_h + B_h^* v_h = f_h & \text{(3.23a)} \\ v_h - B_h u_h = g_h & \text{(3.23b)} \\ \dot{u}_h = p_h & \text{(3.23c)} \end{cases} \qquad \begin{cases} (I_h + \theta\Delta t^2 B_h^* B_h)\delta^2 u_h^n + B_h^* v_h^n = f_h^n, \quad 1 \leq n \leq N & \text{(3.24a)} \\ v_h^n - B_h u_h^n = g_h^n, \quad 0 \leq n \leq N & \text{(3.24b)} \\ \delta u_h^{n+\frac{1}{2}} = p_h^{n+\frac{1}{2}}, \quad 0 \leq n \leq N & \text{(3.24c)} \end{cases}$$

Therefore, letting e_u, e_v and e_p be defined as

$$e_u^{n+\frac{1}{2}} = \bar{u}_h^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}, \quad e_v^n = \bar{v}_h^n - v_h^n, \quad e_p^{n+\frac{1}{2}} = \bar{p}_h^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}} \tag{3.25}$$

we get that

$$\begin{aligned} (I_h + \theta\Delta t^2 B_h^* B_h)\delta^2 e_u^n + B_h^* e_v^n &= (I_h + \theta\Delta t^2 B_h^* B_h)\delta^2 \bar{u}_h^n + B_h^* \bar{v}_h^n - (I_h + \theta\Delta t^2 B_h^* B_h)\delta^2 u_h^n - B_h^* v_h^n \\ &= \ddot{\varepsilon}_u^n + \theta\Delta t^2 B_h^* B_h \delta^2 \bar{u}_h^n \qquad (3.26) \\ e_v^n - B_h e_u^n &= 0 \qquad (3.27) \\ \delta e_u^{n+\frac{1}{2}} - e_p^{n+\frac{1}{2}} &= \delta \bar{u}_h^{n+\frac{1}{2}} - \delta u_h^{n+\frac{1}{2}} - \bar{p}_h^{n+\frac{1}{2}} + p_h^{n+\frac{1}{2}} \\ &= \dot{\varepsilon}_u^{n+\frac{1}{2}} \qquad (3.28) \end{aligned}$$

where there exists $t^{n-1} \leq t_1, t_2 \leq t^{n+1}$ such that

$$\ddot{\varepsilon}_u^n = \delta^2 \bar{u}_h^n - u_h^{(2)}(t^n) = \frac{\Delta t^2}{12} u_h^{(4)}(t_1), \quad \dot{\varepsilon}_u^{n+\frac{1}{2}} = \delta \bar{u}_h^{n+\frac{1}{2}} - \dot{u}_h(t^{n+\frac{1}{2}}) = \frac{\Delta t^2}{3} u_h^{(3)}(t_2) \tag{3.29}$$

Let us introduce the field $\{\mathfrak{e}_p^{n+\frac{1}{2}}\}$ as

$$\mathfrak{e}_p^{n+\frac{1}{2}} = e_p^{n+\frac{1}{2}} + \dot{\varepsilon}_u^{n+\frac{1}{2}} \qquad \Rightarrow \quad \delta e_u^{n+\frac{1}{2}} = \mathfrak{e}_p^{n+\frac{1}{2}} \tag{3.30}$$

such that $(e_v^n, \mathfrak{e}_p^{n+\frac{1}{2}})$ is solution to

$$\begin{cases} (I_h + \theta \Delta t^2 B_h^* B_h) \delta \mathfrak{e}_p^n + B_h^* e_v^n = \ddot{\varepsilon}_u^n + \theta \Delta t^2 B_h^* B_h \delta^2 \bar{u}_h^n & \text{(3.31a)} \\ \delta e_v^{n+\frac{1}{2}} - B_h \mathfrak{e}_p^{n+\frac{1}{2}} = 0 & \text{(3.31b)} \end{cases}$$

Taking the scalar product of the first line of this system with $\mu \mathfrak{e}_p^n$ and of μ operator applied to the second line with e_v^n yields

$$\delta \mathcal{E}_e^{n+\frac{1}{2}} = (\eta_1^n, \mu \mathfrak{e}_p^n) + \theta (\eta_2^n, B_h \mu \mathfrak{e}_p^n), \quad \mathcal{E}_e^{n+\frac{1}{2}} = \frac{1}{2} \|\mathfrak{e}_p^{n+\frac{1}{2}}\|_{\tilde{I}_h}^2 + \frac{1}{2} \|\mu e_v^{n+\frac{1}{2}}\|_D^2 \tag{3.32}$$

where

$$\eta_1^n = \ddot{\varepsilon}_u^n, \quad \eta_2^n = \Delta t^2 B_h \delta^2 \bar{u}_h^n \tag{3.33}$$

We apply the μ operator on (3.31b) to replace $B_h \mu \mathfrak{e}_p^n$ in the previous identity, and we sum from 1 to n to get

$$\Delta t \sum_{k=1}^n \delta \mathcal{E}_e^{k+\frac{1}{2}} = \Delta t \sum_{k=1}^n (\eta_1^k, \mu \mathfrak{e}_p^k) + \theta \Delta t \sum_{k=1}^n (\eta_2^k, \delta \mu e_v^k) \tag{3.34}$$

We can now apply the Abel discrete summation lemma (see Eq (46) from [Chabassier and Imperiale, 2021]) to get

$$\mathcal{E}_e^{n+\frac{1}{2}} - \mathcal{E}_e^{\frac{1}{2}} = \Delta t \sum_{k=1}^n (\eta_1^k, \mu \mathfrak{e}_p^k) + \theta \left[-\Delta t \sum_{k=1}^{n-1} (\delta \eta_2^{k+\frac{1}{2}}, \mu e_v^{k+\frac{1}{2}}) + (\eta_2^n, \mu e_v^{n+\frac{1}{2}}) - (\eta_2^1, \mu e_v^{\frac{1}{2}}) \right] \tag{3.35}$$

Using the results of Appendix A, we can show as was done in Prop. 3.6 that

$$\|\mu e_v^{k+\frac{1}{2}}\| \leq \sqrt{2 \mathcal{E}_e^{k+\frac{1}{2}}}, \quad \|\mu \mathfrak{e}_p^k\| \leq \gamma \left[\sqrt{2 \mathcal{E}_e^{k+\frac{1}{2}}} + \sqrt{2 \mathcal{E}_e^{k-\frac{1}{2}}} \right] \tag{3.36}$$

Let us call

$$H_1^N = \sup_{k=0}^N \|\eta_1^k\| \leq C \Delta t^2 C_{0,3}, \quad H_2^N = \sup_{k=0}^N \|\eta_2^k\| \leq C \Delta t^2 C_{1,1}, \quad (H_2')^N = \sup_{k=0}^n \|\delta \eta_2^{k+\frac{1}{2}}\| \leq C \Delta t^2 C_{1,2} \tag{3.37}$$

Hence

$$\mathcal{E}_e^{n+\frac{1}{2}} \leq \mathcal{E}_e^{\frac{1}{2}} + \Delta t H_1^N \gamma \sum_{k=1}^n \left(\sqrt{2 \mathcal{E}_e^{k+\frac{1}{2}}} + \sqrt{2 \mathcal{E}_e^{k-\frac{1}{2}}} \right) + \theta \left[\Delta t (H_2')^N \sum_{k=1}^n \sqrt{2 \mathcal{E}_e^{k+\frac{1}{2}}} + H_2^N \sqrt{2 \mathcal{E}_e^{n+\frac{1}{2}}} + H_2^1 \sqrt{2 \mathcal{E}_e^{\frac{1}{2}}} \right] \tag{3.38}$$

We perform a Young's inequality on the two last terms and we get (note that $H_2^1 \leq H_2^N$)

$$\frac{1}{2} \mathcal{E}_e^{n+\frac{1}{2}} \leq \frac{3}{2} \mathcal{E}_e^{\frac{1}{2}} + 2\theta^2 (H_2^N)^2 + \Delta t \gamma H_1^N \left(\sum_{k=1}^n \sqrt{2 \mathcal{E}_e^{k+\frac{1}{2}}} + \sum_{k=0}^{n-1} \sqrt{2 \mathcal{E}_e^{k+\frac{1}{2}}} \right) + \theta \Delta t (H_2')^N \sum_{k=1}^{n-1} \sqrt{2 \mathcal{E}_e^{k+\frac{1}{2}}} \tag{3.39}$$

$$\mathcal{E}_e^{n+\frac{1}{2}} \leq 3 \mathcal{E}_e^{\frac{1}{2}} + 4\theta^2 (H_2^N)^2 + 2\sqrt{2} \Delta t \left(2\gamma H_1^N + \theta (H_2')^N \right) \sum_{k=0}^n \sqrt{\mathcal{E}_e^{k+\frac{1}{2}}} \tag{3.40}$$

This inequality has the form of the discrete Grönwall Lemma B.1 with

$$a = 3\mathcal{E}_e^{\frac{1}{2}} + 4\theta^2 (H_2^N)^2, \quad b = 2\sqrt{2}\Delta t \left(2\gamma H_1^N + \theta (H_2')^N\right) \tag{3.41}$$

Hence,

$$\sqrt{\mathcal{E}_e^{n+\frac{1}{2}}} \leq A + 2TB \tag{3.42}$$

with

$$A = \sqrt{2}\left(\sqrt{3\mathcal{E}_e^{\frac{1}{2}}} + 2\theta H_2^N\right), \quad B = 2\sqrt{2}\left(2\gamma H_1^N + \theta (H_2')^N\right) \tag{3.43}$$

and we get thus

$$\|\mu e_v^{k+\frac{1}{2}}\| \leq \sqrt{2}\ (A + 2TB), \quad \|\mu \mathfrak{e}_p^k\| \leq 2\sqrt{2}\gamma (A + 2TB) \tag{3.44}$$

To finish, we must evaluate $\mathcal{E}_e^{\frac{1}{2}}$ by writing the energy at time $t^{\frac{1}{2}}$ in the naïve formulation:

$$\mathcal{E}_e^{\frac{1}{2}} = \frac{1}{2}((I_h + \theta \Delta t^2 B_h^* B_h)\mathfrak{e}_p^{\frac{1}{2}}, \mathfrak{e}_p^{\frac{1}{2}}) + \frac{1}{2}(e_v^0, e_v^1) \tag{3.45}$$

Since $e_v^0 = 0$ only the first term remains, where

$$\mathfrak{e}_p^{\frac{1}{2}} = e_p^{\frac{1}{2}} + \dot{\varepsilon}_u^{\frac{1}{2}} = e_p^{\frac{1}{2}} = \frac{\Delta t^2}{8} p_h^{(2)}(t_5) + \frac{\Delta t^2}{3} u_h^{(3)}(t_2) \tag{3.46}$$

Hence

$$\sqrt{\mathcal{E}_e^{\frac{1}{2}}} \leq \frac{1}{\sqrt{2}} \|e_p^{\frac{1}{2}} + \dot{\varepsilon}_u^{\frac{1}{2}}\|_P + \sqrt{\theta}\frac{\Delta t}{\sqrt{2}} \|B_h \left(e_p^{\frac{1}{2}} + \dot{\varepsilon}_u^{\frac{1}{2}}\right)\|_D \leq C\Delta t^2 C_{0,2} + \sqrt{\theta} C \Delta t^2 C_{1,2} \tag{3.47}$$

Then,

$$\|\mu e_p^n\| \leq \|\mu \mathfrak{e}_p^n\| + \|\mu \dot{\varepsilon}_u^n\| \leq C\left[\sqrt{\mathcal{E}_e^{\frac{1}{2}}} + H_1^N + \theta (H_2^N + (H_2')^N)\right] \tag{3.48}$$

Finally,

$$\|\bar{p}_h^n - \mu p_h^n\| \leq \|\bar{p}_h^n - \mu \bar{p}_h^n\| + \|\mu e_p^n\| \tag{3.49}$$

$$\leq C\Delta t^2 \left[C_{02} + C_{0,2} + \sqrt{\theta} C_{1,2} + C_{0,3} + \theta (C_{1,1} + C_{1,2})\right] \tag{3.50}$$

$$\leq C\Delta t^2 \left[C_{0,3} + \sqrt{\theta} C_{1,2}\right] \tag{3.51}$$

and

$$\|\bar{v}_h^{n+\frac{1}{2}} - \mu v_h^{n+\frac{1}{2}}\| \leq \|\bar{v}_h^{n+\frac{1}{2}} - \mu \bar{v}_h^{n+\frac{1}{2}}\| + \|\mu e_v^{n+\frac{1}{2}}\| \tag{3.52}$$

$$\leq C\Delta t^2 \left[\tilde{C}_{0,2} + C_{0,2} + \sqrt{\theta} C_{1,2} + C_{0,3} + \theta (C_{1,1} + C_{1,2})\right] \tag{3.53}$$

$$\leq C\Delta t^2 \left[\ \tilde{C}_{0,2} + C_{0,3} + \sqrt{\theta} C_{1,2}\right] \tag{3.54}$$

□

4 Conclusion

In this work, we have obtained uniform space/time convergence of the θ-SV and the θ-CN schemes, which are parametrized implicit generalizations of the Störmer-Verlet and Crank-Nicolson schemes. The aim here was to obtain stability and convergence results that are uniform with the space/time convergence ratio, especially when $\theta < 1/4$, in which case the schemes are conditionally stable. The obtained results generalize the ones known for the usual 0-SV and 1/4-CN. We note that our results for θ-SV (Thm 3.1) and for θ-CN (Thm 2.3) do not require the same regularity on the semi-discrete solution. More precisely, the requirements are stronger for the θ-CN scheme. However, one intriguing fact is that in [Chabassier, 2023b] the two schemes are shown to be equivalent (providing that the initial data are well chosen), in the sense that the time series of θ-CN can be reconstructed from the time series of θ-SV, and reciprocally. Therefore an open question remains to achieve uniform space time convergence results for both schemes and based on the same regularity assumptions.

A Spectral decomposition of $B_h^* B_h$ and projectors identities

PROPOSITION A.1 (Spectral decomposition of $B_h^* B_h$). *The operator $B_h^* B_h : U_h \to U_h$ is diagonalizable in $\mathbb{R}$. We call $(\lambda_{h,i}, e_{h,i})$ its eigenpairs which are chosen orthonormal in P.*

Proof. In finite dimensional spaces, any symmetric real operator is diagonalizable in an orthonormal basis. □

Following [Chabassier and Imperiale, 2017], we introduce the polynomial

$$P_k(x) = 1 + (\theta - \frac{1}{4})x \tag{A.1}$$

which is non-negative on the interval $[0, 4/(1-4\theta)]$ for $0 \leq \theta < 1/4$ and on $[0, +\infty[$ for $\theta \geq 1/4$. Then $\widetilde{\mathcal{I}}_h$ can be expressed as a polynomial of the operator $\Delta t^2 B_h^* B_h$:

$$\widetilde{\mathcal{I}}_h = \mathcal{I}_h + \Delta t^2 (\theta - \frac{1}{4}) B_h^* B_h = P_k(\Delta t^2 B_h^* B_h) \tag{A.2}$$

PROPOSITION A.2 (Partitionning). *The interval of non-negativity can be partitionned as $J_k \cup J_p$ with $J_k \cap J_p = \emptyset$ such that there exist $C_k > 0$ and $C_p > 0$ such that for all $x \in J_k$, $P_k(x) \geq C_k$ and for all $x \in J_p$, $x \geq C_p$.*

Proof. See [Chabassier and Imperiale, 2017, appendix] for the scheme called "TS". □

PROPOSITION A.3. *Let us define the two projectors Π_k and Π_p such that for all $u_h \in U_h$*

$$\Pi_k u_h = \sum_{\substack{\Delta t^2 \lambda_{h,i} \in J_k \\ \lambda_{h,i} \in Sp(\mathcal{K}_h)}} (u_h, e_{h,i})_P \, e_{h,i}, \qquad \Pi_p u_h = \sum_{\substack{\Delta t^2 \lambda_{h,i} \in J_p \\ \lambda_{h,i} \in Sp(\mathcal{K}_h)}} (u_h, e_{h,i})_P \, e_{h,i} \tag{A.3}$$

Then, for all $u_h \in U_h$

$$\begin{cases} \|\Pi_k u_h\|_P^2 \leq C_k^{-1} (\widetilde{\mathcal{I}}_h u_h, u_h)_P & \text{(A.4a)} \\ \|\Pi_p u_h\|_P^2 \leq \Delta t^2 C_p^{-1} (B_h u_h, B_h u_h)_D & \text{(A.4b)} \end{cases}$$

Proof. See [Chabassier and Imperiale, 2017]. □

B Some technical results for the θ-CN analysis

In the following of this appendix, we will prove some useful results on arbitrary sequences $\{a_h^n\}_n$ and $\{b_h^n\}_n$ which satisfy

$$\begin{cases} \tilde{I}_h \delta a_h^{n+\frac{1}{2}} + B_h^* \mu b_h^{n+\frac{1}{2}} = \eta_1^{n+\frac{1}{2}} + B_h^* \eta_3^{n+\frac{1}{2}} & \text{(B.1a)} \\ \delta b_h^{n+\frac{1}{2}} - B_h \mu a_h^{n+\frac{1}{2}} = \eta_2^{n+\frac{1}{2}} + \delta \eta_4^{n+\frac{1}{2}} & \text{(B.1b)} \end{cases}$$

and a sequence $\mathcal{E}_h^n$ defined by

$$\mathcal{E}_h^n = \frac{1}{2}\|a_h^n\|_{\tilde{I}_h}^2 + \frac{1}{2}\|b_h^n\|^2 \tag{B.2}$$

where the sequences $\eta_1^{n+\frac{1}{2}}$, $\eta_2^{n+\frac{1}{2}}$, $\eta_3^{n+\frac{1}{2}}$ and η_4^n are given.

Proposition B.1. *Let $\{a_h^n\}_n$ and $\{b_h^n\}_n$ satisfying* (B.1) *and $\mathcal{E}_h^n$ defined as* (B.2)*. Then, there exists $\gamma > 0$ such that*

$$\|b_h^n\| \leq \sqrt{2\mathcal{E}_h^n}, \quad \|\mu b_h^{n+\frac{1}{2}}\| \leq \mu\sqrt{2\mathcal{E}_h}^{n+\frac{1}{2}}, \quad \|\mu a_h^{n+\frac{1}{2}}\| \leq 2\gamma\left(\mu\sqrt{2\mathcal{E}_h}^{n+\frac{1}{2}} + \mu\|\eta_4\|^{n+\frac{1}{2}}\right) + \gamma\Delta t\|\eta_2^{n+\frac{1}{2}}\| \tag{B.3}$$

Proof. The identities

$$\|b_h^n\| \leq \sqrt{2\mathcal{E}_h}^n \quad \text{and} \quad \|\mu b_h^{n+\frac{1}{2}}\| \leq \mu\sqrt{2\mathcal{E}_h}^{n+\frac{1}{2}} \tag{B.4}$$

directly follow from the definition of $\mathcal{E}_h^n$. The rest of the proof relies on the spectral decomposition technique developed in [Chabassier and Imperiale, 2017] and used in [Chabassier and Imperiale, 2021]. Using the relations in Prop. A.3,

$$\begin{aligned} \|\Pi_p \mu a_h^{n+\frac{1}{2}}\| &\leq_{\text{(A.4b)}} \Delta t C_p^{-1/2}\|B_h \mu a_h^{n+\frac{1}{2}}\| && \text{(B.5)} \\ &\leq_{\text{(B.1a)}} \Delta t C_p^{-1/2}\|\delta\left(b_h - \eta_4\right)^{n+\frac{1}{2}} - \eta_2^{n+\frac{1}{2}}\| && \text{(B.6)} \\ &\leq_{\text{(2.2a)}} 2C_p^{-1/2}\mu\|b_h - \eta_4\|^{n+\frac{1}{2}} + \Delta t C_p^{-1/2}\|\eta_2^{n+\frac{1}{2}}\| && \text{(B.7)} \\ &\leq C_p^{-1/2}\left[\|b_h^{n+1}\| + \|b_h^n\| + \|\eta_4^{n+1}\| + \|\eta_4^n\| + \Delta t\|\eta_2^{n+\frac{1}{2}}\|\right] && \text{(B.8)} \\ &\leq C_p^{-1/2}\left[\sqrt{2\mathcal{E}_h^{n+1}} + \sqrt{2\mathcal{E}_h^n} + \|\eta_4^{n+1}\| + \|\eta_4^n\| + \Delta t\|\eta_2^{n+\frac{1}{2}}\|\right] && \text{(B.9)} \end{aligned}$$

Morerover,

$$\|\Pi_k a_h^n\| \leq_{\text{(A.4a)}} C_k^{-1/2}\ \|a_h^n\|_{\tilde{I}_h} \leq C_k^{-1/2}\sqrt{2\mathcal{E}_h^n} \quad \Rightarrow\ \|\Pi_k \mu a_h^{n+\frac{1}{2}}\| \leq C_k^{-1/2}\mu\sqrt{2\mathcal{E}_h}^{n+\frac{1}{2}} \tag{B.10}$$

Finally,

$$\|\mu a_h^{n+\frac{1}{2}}\| \leq \|\Pi_k \mu a_h^{n+\frac{1}{2}}\| + \|\Pi_p \mu a_h^{n+\frac{1}{2}}\| \tag{B.11}$$

This shows the expected result with

$$\gamma = C_k^{-1/2}/2 + C_p^{-1/2} \tag{B.12}$$

and noticing that $C_p^{-1/2} \leq \gamma$. □

Lemma B.1. *Let $a > 0$, $b > 0$, and $\mathcal{E}_h$ be a sequence that satisfies*

$$\mathcal{E}_h^n \leq a + \Delta t b\sum_{k=0}^{n}\sqrt{\mathcal{E}_h^k} \tag{B.13}$$

Then,

$$\sqrt{\mathcal{E}_h^n} \leq \sqrt{2a} + 2t^n b \tag{B.14}$$

Proof. First, let us apply Young's inequality on the last term of the sum, to get that

$$\frac{1}{2}\mathcal{E}_h^n \leq a + \frac{1}{2}\Delta t^2 b^2 + \Delta t b \sum_{k=0}^{n-1} \sqrt{\mathcal{E}_h^k} \tag{B.15}$$

Let

$$F^n = a + \frac{1}{2}\Delta t^2 b + \Delta t b \sum_{k=0}^{n} \sqrt{\mathcal{E}_h^k} \tag{B.16}$$

Notice that $F^0 = a + \frac{1}{2}\Delta t^2 b^2$ and that $\mathcal{E}_h^n \leq 2F^{n-1}$. Then, for all $n \geq 1$,

$$F^n - F^{n-1} = \Delta t b \sqrt{\mathcal{E}_h^n} \leq \Delta t b \sqrt{2F^{n-1}} \leq \sqrt{2}\Delta t b \left(\sqrt{F^n} + \sqrt{F^{n-1}}\right) \tag{B.17}$$

Adding this telescopic sum from 0 to $n-1$ yields

$$\sqrt{F^{n-1}} \leq \sqrt{F^0} + \sqrt{2} b \sum_{k=0}^{n-2} \Delta t \leq \sqrt{a + \frac{1}{2}\Delta t^2 b^2} + \sqrt{2} b t^{n-1} \tag{B.18}$$

Using the Grönwall assumption, we get

$$\sqrt{\mathcal{E}_h^n} \leq \sqrt{2F^{n-1}} \leq \sqrt{2a} + \Delta t b + 2bt^{n-1} \leq \sqrt{2a} + 2bt^n \tag{B.19}$$

□

Lemma B.2 (Discrete summation by part : Abel transformation).

$$\Delta t \sum_{k=0}^{n} b^k \delta a^{k+\frac{1}{2}} = -\Delta t \sum_{k=0}^{n-1} \delta b^{k+\frac{1}{2}} a^{k+1} + b^n a^{n+1} - b^0 a^0 \tag{B.20}$$

Proposition B.2. *Let $\{a_h^n\}_n$, $\{b_h^n\}_n$ and $\{\mathcal{E}_h^n\}_n$ be sequences satisfying*

$$\|\mu a_h^{n+\frac{1}{2}}\| \leq 2\gamma\mu \sqrt{2\mathcal{E}_h}^{n+\frac{1}{2}} + \gamma r_h^{n+\frac{1}{2}} \tag{B.21}$$

$$\|b_h^n\| \leq \sqrt{2\mathcal{E}_h^n} \tag{B.22}$$

$$\delta\mathcal{E}_h^{n+\frac{1}{2}} = (e_1^{n+\frac{1}{2}}, \mu a_h^{n+\frac{1}{2}}) + (e_2^{n+\frac{1}{2}}, \mu b_h^{n+\frac{1}{2}}) + (e_3^{n+\frac{1}{2}}, e_4^{n+\frac{1}{2}}) + (e_5^{n+\frac{1}{2}}, \delta b_h^{n+\frac{1}{2}}) \tag{B.23}$$

Then,

$$\sqrt{\mathcal{E}_h^n} \leq A + 2t^n B \tag{B.24}$$

where

$$A = \sqrt{2}\left[\sqrt{T}\sup_{k=0}^{N}\left[\gamma\|e_1^{k+\frac{1}{2}}\| + r_h^{k+\frac{1}{2}} + \|e_3^{k+\frac{1}{2}}\| + \|e_4^{k+\frac{1}{2}}\|\right] + \sqrt{2}\|e_5^N\| + \sqrt{2}\|e_5^0\| + \sqrt{3\mathcal{E}_h^0}\right] \tag{B.25}$$

$$B = \sqrt{2}\sup_{k=0}^{N}\left(2\gamma\|e_1^{k+\frac{1}{2}}\| + \|e_2^{k+\frac{1}{2}}\| + 2\|\delta e_5^{k+\frac{1}{2}}\|\right) \tag{B.26}$$

and therefore,

$$\|\mu a_h^{n+\frac{1}{2}}\| \leq 2\sqrt{2}\gamma\left(A + 2t^n B\right) + \gamma C \tag{B.27}$$

$$\|b_h^n\| \leq \sqrt{2}\left(A + 2t^n B\right) \tag{B.28}$$

where

$$C = \sup_{k=0}^{N} r_h^{k+\frac{1}{2}} \tag{B.29}$$

Proof. Let us multiply (B.23) by Δt and sum from 0 to n:

$$\Delta t \sum_{k=0}^{n} \delta \mathcal{E}_h^{n+\frac{1}{2}} = \Delta t \sum_{k=0}^{n} \left[(e_1^{k+\frac{1}{2}}, \mu a_h^{k+\frac{1}{2}}) + (e_2^{k+\frac{1}{2}}, \mu b_h^{k+\frac{1}{2}}) + (e_3^{k+\frac{1}{2}}, e_4^{k+\frac{1}{2}}) + (e_5^{k+\frac{1}{2}}, \delta b_h^{k+\frac{1}{2}}) \right] \tag{B.30}$$

Let us use Lemma B.2 on the last term and use Cauchy-Schwarz inequality to get

$$\begin{aligned} \mathcal{E}_h^{n+1} - \mathcal{E}_h^0 \leq \Delta t \sum_{k=0}^{n} \left[\|e_1^{k+\frac{1}{2}}\| \|\mu a_h^{k+\frac{1}{2}}\| + \|e_2^{k+\frac{1}{2}}\| \|\mu b_h^{k+\frac{1}{2}}\| + \|e_3^{k+\frac{1}{2}}\| \|e_4^{k+\frac{1}{2}}\| \right] \\ + \Delta t \sum_{k=0}^{n-1} \|\delta e_5^{k+\frac{1}{2}}\| \|b_h^{k+1}\| + \|e_5^n\| \|b_h^{n+1}\| + \|e_5^0\| \|b_h^0\| \end{aligned} \tag{B.31}$$

We now use the majoration hypothesis on a_h and b_h:

$$\begin{aligned} \mathcal{E}_h^{n+1} - \mathcal{E}_h^0 \leq \Delta t \sum_{k=0}^{n} \left[\gamma \|e_1^{k+\frac{1}{2}}\| \left(2\mu \sqrt{2\mathcal{E}_h}^{k+\frac{1}{2}} + r_h^{k+\frac{1}{2}} \right) + \|e_2^{k+\frac{1}{2}}\| \mu \sqrt{2\mathcal{E}_h}^{k+\frac{1}{2}} + \|e_3^{k+\frac{1}{2}}\| \|e_4^{k+\frac{1}{2}}\| \right] \\ + \Delta t \sum_{k=0}^{n-1} \|\delta e_5^{k+\frac{1}{2}}\| \sqrt{2\mathcal{E}_h^{k+1}} + \|e_5^n\| \sqrt{2\mathcal{E}_h^{n+1}} + \|e_5^0\| \sqrt{2\mathcal{E}_h^0} \end{aligned} \tag{B.32}$$

Using Young's inequality and grouping similar terms, we get

$$\begin{aligned} \mathcal{E}_h^{n+1} - \mathcal{E}_h^0 \leq \Delta t \sum_{k=0}^{n} \left[\frac{1}{2}\gamma^2 \|e_1^{k+\frac{1}{2}}\|^2 + \frac{1}{2}(r_h^{k+\frac{1}{2}})^2 + \left(2\gamma \|e_1^{k+\frac{1}{2}}\| + \|e_2^{k+\frac{1}{2}}\| \right) \mu \sqrt{2\mathcal{E}_h}^{k+\frac{1}{2}} + \frac{1}{2}\|e_3^{k+\frac{1}{2}}\|^2 + \frac{1}{2}\|e_4^{k+\frac{1}{2}}\|^2 \right] \\ + \Delta t \sum_{k=0}^{n-1} \|\delta e_5^{k+\frac{1}{2}}\| \sqrt{2\mathcal{E}_h^{k+1}} + \|e_5^n\|^2 + \frac{1}{2}\mathcal{E}_h^{n+1} + \|e_5^0\|^2 + \frac{1}{2}\mathcal{E}_h^0 \end{aligned} \tag{B.33}$$

Let

$$\alpha = \frac{2T}{2} \sup_{k=0}^{N} \left[\gamma^2 \|e_1^{k+\frac{1}{2}}\|^2 + (r_h^{k+\frac{1}{2}})^2 + \|e_3^{k+\frac{1}{2}}\|^2 + \|e_4^{k+\frac{1}{2}}\|^2 \right] + 2\|e_5^N\|^2 + 2\|e_5^0\|^2 + 3\mathcal{E}_h^0 \tag{B.34}$$

Then

$$\frac{1}{2}\mathcal{E}_h^{n+1} \leq \Delta t \sum_{k=0}^{n} \left(2\gamma \|e_1^{k+\frac{1}{2}}\| + \|e_2^{k+\frac{1}{2}}\| \right) \frac{\sqrt{2\mathcal{E}_h^{k+1}} + \sqrt{2\mathcal{E}_h^k}}{2} + \Delta t \sum_{k=0}^{n-1} \|\delta e_5^{k+\frac{1}{2}}\| \sqrt{2\mathcal{E}_h^{k+1}} + \frac{\alpha}{2} \tag{B.35}$$

$$\leq \Delta t \sum_{k=0}^{n} \left(\gamma \|e_1^{k+\frac{1}{2}}\| + \frac{1}{2}\|e_2^{k+\frac{1}{2}}\| \right) \sqrt{2\mathcal{E}_h^k} + \Delta t \sum_{k=0}^{n-1} \left(\gamma \|e_1^{k+\frac{1}{2}}\| + \frac{1}{2}\|e_2^{k+\frac{1}{2}}\| + \|\delta e_5^{k+\frac{1}{2}}\| \right) \sqrt{2\mathcal{E}_h^{k+1}} + \frac{\alpha}{2} \tag{B.36}$$

With

$$\beta = \sqrt{2} \sup_{k=0}^{N} \left(2\gamma \|e_1^{k+\frac{1}{2}}\| + \|e_2^{k+\frac{1}{2}}\| + 2\|\delta e_5^{k+\frac{1}{2}}\| \right) \tag{B.37}$$

We get that

$$\mathcal{E}_h^{n+1} \leq \alpha + \Delta t \beta \sum_{k=0}^{n+1} \sqrt{\mathcal{E}_h^k} \tag{B.38}$$

We can use Lemma B.1 to get that

$$\sqrt{\mathcal{E}_h^{n+1}} \leq \sqrt{2\alpha} + 2t^{n+1}\beta \tag{B.39}$$

□

PROPOSITION B.3. *Let* $\{a_h^n\}_n$ *and* $\{b_h^n\}_n$ *satisfying* (B.1) *and* $\mathcal{E}_h^n$ *defined as* (B.2). *Then*

$$\sqrt{\mathcal{E}_h^n} \leq A + 2t^n B \tag{B.40}$$

$$\|\mu a_h^{n+\frac{1}{2}}\| \leq 2\sqrt{2}\gamma\,(A + 2t^n B) + \gamma C \tag{B.41}$$

$$\|b_h^n\| \leq \sqrt{2}\,(A + 2t^n B) \tag{B.42}$$

where

$$A = \sqrt{2}\left[\sqrt{T}\sup_{k=0}^{N}\left[\gamma\|\eta_1^{k+\frac{1}{2}}\| + 2\mu\|\eta_4\|^{k+\frac{1}{2}} + \Delta t\|\eta_2^{k+\frac{1}{2}}\| + \|\eta_3^{k+\frac{1}{2}}\| + \|\eta_2^{k+\frac{1}{2}}\|\right] + \sqrt{2}\|\eta_3^N\| + \sqrt{2}\|\eta_3^0\| + \sqrt{3\mathcal{E}_h^0}\right] \tag{B.43}$$

$$B = \sqrt{2}\sup_{k=0}^{N}\left(2\gamma\|\eta_1^{k+\frac{1}{2}}\| + \|\eta_2^{k+\frac{1}{2}}\| + \|\delta\eta_4^{k+\frac{1}{2}}\| + 2\|\delta\eta_3^{k+\frac{1}{2}}\|\right) \tag{B.44}$$

$$C = \sup_{k=0}^{N}\left(2\mu\|\eta_4\|^{n+\frac{1}{2}} + \Delta t\|\eta_2^{n+\frac{1}{2}}\|\right) \tag{B.45}$$

Proof. Taking the scalar product of (B.1a) with $\mu a_h^{n+\frac{1}{2}}$ and of (B.1b) with $\mu b_h^{n+\frac{1}{2}}$, we get that

$$\delta\mathcal{E}_h^n = (\eta_1^{n+\frac{1}{2}} + B_h^*\eta_3^{n+\frac{1}{2}}, \mu a_h^{n+\frac{1}{2}}) + (\eta_2^{n+\frac{1}{2}} + \delta\eta_4^{n+\frac{1}{2}}, \mu b_h^{n+\frac{1}{2}}) \tag{B.46}$$

Since from (B.1b)

$$B_h\mu a_h^{n+\frac{1}{2}} = \delta b_h^{n+\frac{1}{2}} - \eta_2^{n+\frac{1}{2}} - \delta\eta_4^{n+\frac{1}{2}} \tag{B.47}$$

The energy identity becomes

$$\delta\mathcal{E}_h^n = (\eta_1^{n+\frac{1}{2}}, \mu a_h^{n+\frac{1}{2}}) + (\eta_3^{n+\frac{1}{2}}, B_h\mu a_h^{n+\frac{1}{2}}) + (\eta_2^{n+\frac{1}{2}} + \delta\eta_4^{n+\frac{1}{2}}, \mu b_h^{n+\frac{1}{2}}) \tag{B.48}$$

$$= (\eta_1^{n+\frac{1}{2}}, \mu a_h^{n+\frac{1}{2}}) + (\eta_3^{n+\frac{1}{2}}, \delta b_h^{n+\frac{1}{2}}) - (\eta_3^{n+\frac{1}{2}}, \eta_2^{n+\frac{1}{2}} + \delta\eta_4^{n+\frac{1}{2}}) + (\eta_2^{n+\frac{1}{2}} + \delta\eta_4^{n+\frac{1}{2}}, \mu b_h^{n+\frac{1}{2}}) \tag{B.49}$$

Moreover from Prop. B.1, we get that the hypotheses of Prop. B.2 are satisfied with

$$r_h^{n+\frac{1}{2}} = 2\mu\|\eta_4\|^{n+\frac{1}{2}} + \Delta t\|\eta_2^{n+\frac{1}{2}}\| \tag{B.50}$$

$$e_1^{n+\frac{1}{2}} = \eta_1^{n+\frac{1}{2}} \tag{B.51}$$

$$e_2^{n+\frac{1}{2}} = \eta_2^{n+\frac{1}{2}} + \delta\eta_4^{n+\frac{1}{2}} \tag{B.52}$$

$$e_3^{n+\frac{1}{2}} = -\eta_3^{n+\frac{1}{2}} \tag{B.53}$$

$$e_4^{n+\frac{1}{2}} = e_2^{n+\frac{1}{2}} \tag{B.54}$$

$$e_5^{n+\frac{1}{2}} = \eta_3^{n+\frac{1}{2}} \tag{B.55}$$

$$\tag{B.56}$$

which yields the result. □

References

[Banjai et al., 2015] Banjai, L., Lubich, C., and Sayas, F.-J. (2015). Stable numerical coupling of exterior and interior problems for the wave equation. *Numerische Mathematik*, 129(4):611–646.

[Bilbao and Harrison, 2016] Bilbao, S. and Harrison, R. (2016). Passive time-domain numerical models of viscothermal wave propagation in acoustic tubes of variable cross section. *The Journal of the Acoustical Society of America*, 140(1):728–740.

[Brezzi and Fortin, 1991] Brezzi, F. and Fortin, M. (1991). *Mixed and Hybrid Finite Element Methods*, volume 2. New York: Springer-Verlag.

[Chabassier, 2023a] Chabassier, J. (2023a). Stability and space/time convergence of Störmer-Verlet time integration of the mixed formulation of linear wave equations. *preprint submitted to M2AN*.

[Chabassier, 2023b] Chabassier, J. (2023b). Two implicit parametrized discretization strategies for the mixed formulation of linear wave equations. Research Report 9527, RR Inria.

[Chabassier and Imperiale, 2017] Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathematiques*, 355(3):282–289.

[Chabassier and Imperiale, 2021] Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

[Eymard et al., 2000] Eymard, R., Gallouët, T., and Herbin, R. (2000). Finite volume methods. In *Handbook of Numerical Analysis*, volume 7 of *Solution of Equation in* $\mathbb{R}$ *(Part 3), Techniques of Scientific Computing (Part 3)*, pages 713–1018. Elsevier.

[He and Sun, 2020] He, M. and Sun, P. (2020). Energy-preserving finite element methods for a class of nonlinear wave equations. *Applied Numerical Mathematics*, 157:446–469.

[Lanteri and Scheid, 2013] Lanteri, S. and Scheid, C. (2013). Convergence of a discontinuous Galerkin scheme for the mixed time-domain Maxwell's equations in dispersive media. *IMA Journal of Numerical Analysis*, 33(2):432–459.

[Makridakis and Monk, 1995] Makridakis, Ch. G. and Monk, P. (1995). Time-discrete finite element schemes for Maxwell's equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 29(2):171–197.

[Zuazua, 2005] Zuazua, E. (2005). Propagation, Observation, and Control of Waves Approximated by Finite Difference Methods. *SIAM Review*, 47(2):197–243.

RESEARCH CENTRE
BORDEAUX – SUD-OUEST

200 avenue de la Vieille Tour
33405 Talence Cedex

Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
inria.fr

ISSN 0249-6399