



**HAL**  
open science

## Invisible floating objects

Lucas Chesnel, Mahran Rihani

► **To cite this version:**

Lucas Chesnel, Mahran Rihani. Invisible floating objects. 14th International Conference on Mathematical and Numerical Aspects of Wave Propagation, Vienna University of Technology, Aug 2019, Vienna, France. 10.34726/waves2019 . hal-04285523

**HAL Id: hal-04285523**

**<https://hal.science/hal-04285523>**

Submitted on 14 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Invisible floating objects

Lucas Chesnel<sup>1</sup>, Mahran Rihani<sup>2,\*</sup>

<sup>1</sup> INRIA/CMAP, Ecole Polytechnique, Université Paris-Saclay, Palaiseau, France

<sup>2</sup>POEMS, ENSTA-ParisTech, Palaiseau, France

\*Email: mahran.rihani@ensta-paristech.fr

### Abstract

We consider a time-harmonic water waves problem in a 2D waveguide. The geometry is symmetric with respect to an axis orthogonal to the direction of propagation of waves. Moreover, the waveguide contains two floating obstacles separated by a distance  $L$ . We study the behaviours of  $R$  (the reflection coefficient) and  $T$  (the transmission coefficient) as  $L$  tends to  $+\infty$ . From this analysis, we exhibit situations of non reflectivity ( $R = 0$ ,  $|T| = 1$ ) or perfect invisibility ( $R = 0$ ,  $T = 1$ ).

**Keywords:** water waves, waveguides, invisibility, asymptotic analysis

### Introduction

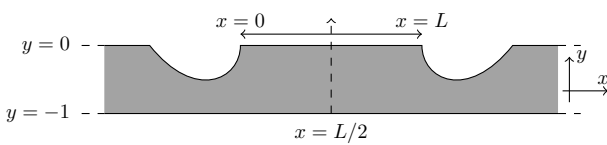


Figure 1: The waveguide  $\Omega_L$ .

We consider the propagation of water waves in a 2D waveguide  $\Omega_L$  which coincides with  $\mathbb{R} \times (-1; 0)$  outside of a bounded domain. We assume that  $\Omega_L$  is symmetric with respect to the vertical line  $x = L/2$  and contains two floating objects separated by a distance  $L$  (see Figure 1). This leads us to study the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_L \\ \partial_n u = \lambda u & \text{on } \Gamma_0 := \mathbb{R} \times \{0\} \\ \partial_n u = 0 & \text{on } \partial\Omega_L \setminus \Gamma_0 \end{cases} \quad (1)$$

where  $\lambda > 0$  is proportional to the square of the frequency and  $\partial_n$  stands for the outward normal derivative. Define  $\beta > 0$  such that  $\beta \tan(\beta) = \lambda$ . Only two waves  $w^\pm(x, y) = e^{\pm i\beta x} \cosh(\beta(y+1))$  can propagate in  $\Omega_L$ . And (1) admits the scattering solution

$$u = \begin{cases} w^+ + R w^- + \dots & \text{for } x < 0 \\ T w^+ + \dots & \text{for } x > 0, \end{cases}$$

where  $R, T \in \mathbb{C}$  and where the dots are terms which are exponentially decaying at  $\pm\infty$ . Due

to conservation of energy, the reflection and transmission coefficients satisfy  $|R|^2 + |T|^2 = 1$ . Thus, perfect invisibility implies non reflectivity but the converse is not true in general. Working as in [1], we compute an explicit asymptotic expansion of  $R$  and  $T$  as  $L \rightarrow +\infty$ . This allows us to exhibit non reflecting geometries. Then playing with an additional parameter, we explain how to get not only  $|T| = 1$  but also  $T = 1$  (perfect invisibility).

### Half waveguide problems

In this section, we use the symmetry of the problem to decompose it into two half waveguide problems. Set  $\omega_L := \{(x, y) \in \Omega_L, x < L/2\}$  and  $\Sigma := \{0\} \times (-1; 0)$  (see Figure 2).

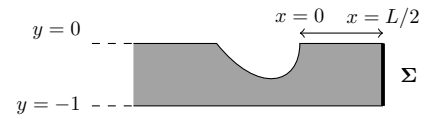


Figure 2: The half waveguide  $\omega_L$ .

Consider the two problems

$$\begin{cases} \Delta u_D = 0 & \text{in } \omega_L \\ \partial_n u_D = \lambda u_D & \text{in } \Gamma_0 \\ \partial_n u_D = 0 & \text{in } \partial\omega_L \setminus \Gamma_0 \\ u_D = 0 & \text{in } \Sigma \end{cases} \quad \begin{cases} \Delta u_N = 0 & \text{in } \omega_L \\ \partial_n u_N = \lambda u_N & \text{in } \Gamma_0 \\ \partial_n u_N = 0 & \text{in } \partial\omega_L \setminus \Gamma_0 \\ \partial_n u_N = 0 & \text{in } \Sigma. \end{cases}$$

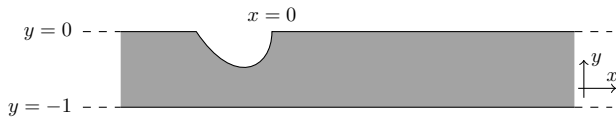
They admit solutions such that  $u_D = w^+ + R_D w^- + \dots$  and  $u_N = w^+ + R_N w^- + \dots$ . Here  $R_D, R_N \in \mathbb{C}$  and the dots are terms which are exponentially decaying at  $-\infty$ . Due to conservation of energy, we have  $|R_D| = |R_N| = 1$ . We can show that

$$R = \frac{R_N + R_D}{2}; \quad T = \frac{(R_N - R_D)}{2} e^{-i\beta L}. \quad (2)$$

Therefore, to obtain an asymptotic expansion of  $R, T$  as  $L \rightarrow +\infty$ , it is enough to work on  $R_D, R_N$ .

When  $L \rightarrow +\infty$ , the limit geometry of  $\omega_L$  is the domain  $\omega_\infty$  with just one floating object pictured in Figure 3. We denote by

$$s = \begin{pmatrix} r & t \\ t & \tilde{r} \end{pmatrix} \in \mathbb{C}^{2 \times 2}$$


 Figure 3: The limit geometry  $\omega_\infty$ .

the unitary scattering matrix of the propagation problem set in  $\omega_\infty$ . When  $rt \neq 0$ , as  $L \rightarrow +\infty$ , we can prove the expansions  $R_D = R_D^{asy} + \dots$  and  $R_N = R_N^{asy} + \dots$  with

$$R_D^{asy} = r - \frac{t^2}{e^{-i\beta L} + \tilde{r}}; \quad R_N^{asy} = r - \frac{t^2}{-e^{-i\beta L} + \tilde{r}}.$$

One can check that  $|R_D^{asy}| = |R_N^{asy}| = 1$ . Introduce  $\theta_D, \theta_N, \theta_D^{asy}, \theta_N^{asy} \in \mathbb{R}/(2\pi\mathbb{Z})$  the phases of  $R_D, R_N, R_D^{asy}, R_N^{asy}$ . Finally, define

$$e := \theta_D - \theta_N - \pi; \quad e^{asy} := \theta_D^{asy} - \theta_N^{asy} - \pi.$$

### Non reflectivity

From (2), we see that we have  $R(L) = 0$  if and only if  $e(L) = 0$ . To prove that this occurs for certain  $L$ , first we show that there is  $L^*$  such that  $e^{asy}(L^*) = 0$  and  $\partial_L e^{asy}(L^*) \neq 0$ . Then using the exponential convergence of  $e(L)$  to  $e^{asy}(L)$  when  $L \rightarrow +\infty$ , as well as the continuous dependence of  $e(L)$  with respect to  $L$ , we can obtain the desired result.

**Theorem 1** *Assume that  $rt \neq 0$ . Then there is a sequence  $(L_n)$  of values of  $L$  such that  $R(L_n) = 0$ . And we have  $L_{n+1} - L_n \underset{n \rightarrow +\infty}{\sim} \pi/\beta$ .*

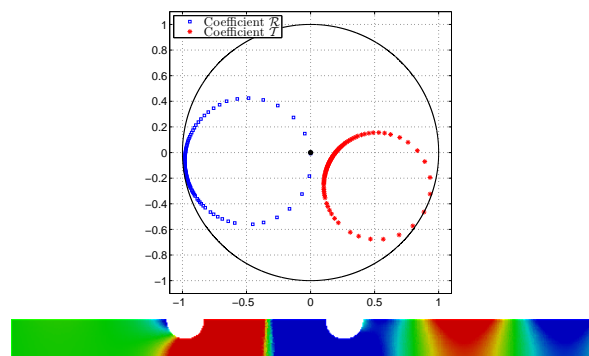


Figure 4:  $R(L), T(L)$  for  $L \in (0.5; 7)$  (top). Real part of the scattered field  $u_s = u - w^+$  for  $L = 3.15$  (bottom). In this case,  $R(L) \approx 0$  and  $u_s$  is exponentially decaying in the left branch.

We illustrate this theorem in Figure 4. The floating objects are half disks of radius 0.5. We approximate numerically (FEM) the scattering coefficients  $R(L), T(L)$  for  $L \in (0.5; 7)$ . And we

observe that the curve  $L \mapsto R(L)$  indeed passes through zero. An example of non reflecting geometry is given in Figure 4.

### Perfect transmission

Now we study the problem of imposing  $T = 1$ . Using (2) and the expansions of  $R_D, R_N$ , we get

$$T = T^{asy} + \dots \quad \text{with} \quad T^{asy} = \frac{t^2 e^{-2i\beta L}}{e^{-2i\beta L} - \tilde{r}^2}.$$

Then a direct calculus shows that the curve  $L \mapsto T^{asy}(L)$  passes through 1 if and only if  $t \in \mathbb{R}$ . This occurs for example when the floating object is a demi ellipse of semi minor axe 0.3 and semi major axe  $l$  for  $l = l^* \approx 3.6$  (see Figure 5). With this notation, with the inverse function theorem, we can prove the following result.

**Theorem 2** *Assume that  $r(l^*)t(l^*) \neq 0$ . Then there is a sequence  $(l_n, L_n)$  converging to  $(l^*, +\infty)$  such that  $T(l_n, L_n) = 1$ . And we have  $L_{n+1} - L_n \underset{n \rightarrow +\infty}{\sim} \pi/\beta$ .*

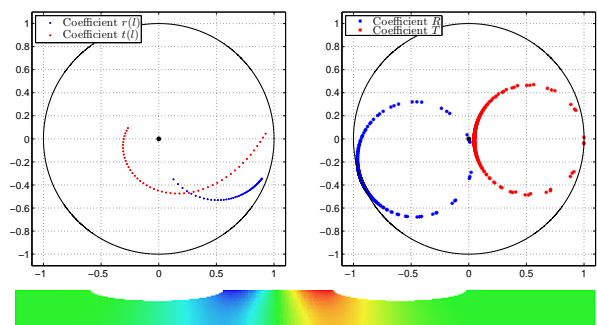


Figure 5: Coefficients  $r(l), t(l)$  for  $l \in (0.5; 3.9)$  (top left). Coefficients  $R(l^*, L), T(l^*, L)$  for  $L \in (0.5; 10)$  (top right). Real part of the scattered field  $u_s = u - w^+$  for  $L = 2.9$  and  $l = 3.6$ . In this case,  $T(l, L) \approx 0$  and  $u_s$  is almost exponentially decaying in the two branches.

### References

- [1] L. Chesnel, S.A. Nazarov, V. Pagneux, Invisibility and perfect reflectivity in waveguides with finite length branches, *SIAM J. Appl. Math.*, vol. 78, 4:2176-2199, 2018.