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LARGE DEVIATIONS FOR DIFFUSIONS: DONSKER-VARADHAN MEET FREIDLIN-WENTZELL

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ABSTRACT. We consider a diffusion process on $\mathbb{R}^n$ and prove a large deviation principle for the empirical process in the joint limit in which the time window diverges and the noise vanishes. The corresponding rate function is given by the expectation of the Freidlin-Wentzell functional per unit of time. As an application of this result, we obtain a variational representation of the rate function for the Gallavotti-Cohen observable in the small noise and large time limits.

1. INTRODUCTION

A diffusion processes on $\mathbb{R}^n$ can be realized as the solution to the stochastic differential equation

$$\begin{cases} d\xi_t^\varepsilon = b(\xi_t^\varepsilon)dt + \sqrt{2\varepsilon}\,\sigma(\xi_t^\varepsilon)dw_t \\ \xi_0^\varepsilon = x \end{cases} \tag{1.1}$$

where b is a smooth vector field, w is a standard m-dimensional Brownian, σ is a $n \times m$ matrix field, and the parameter $\varepsilon > 0$, that can be interpreted as the temperature of the environment, will eventually vanish. We shall impose conditions on b and σ which ensure the ergodicity of the process ξ^ε.

An *additive functional* $\{A_T\}_{T\geq 0}$ of ξ^ε is a real-valued, progressively measurable, functional of ξ^ε vanishing at $T = 0$ and such that $A_{T+S} = A_T + A_S \circ \theta_T$, where θ_T denotes the translation by T. Readily, functions of the occupation measure, i.e. functional of the form

$$A_T = \int_0^T dt\, f(\xi_t^\varepsilon), \qquad f\colon \mathbb{R}^n \to \mathbb{R}, \tag{1.2}$$

are examples of additive functionals. The basic question that we here address is the behavior of additive functionals in the joint limit in which the time window $[0, T]$ diverges and the noise ε vanishes. More precisely, we establish a large deviation principle in such joint limit.

According to the Donsker-Varadhan ideology [11], rather than focusing on a single additive functional, the large deviation principle is better formulated for a whole family of additive functionals. This is formally realized by analyzing the asymptotics of the *empirical process* and the corresponding large deviations are usually called at *level three*. Of course, the rate function for a specific additive functional can then be obtained by projecting the level three rate function.

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To purse the joint limit $T \to \infty$ and $\varepsilon \to 0$ there are two simple alternatives. (i) By taking first the limit $\varepsilon \to 0$ the large deviations of the empirical process can be obtained by lifting the Freidlin-Wentzell asymptotic [12] to the set of translation invariant probabilities on the path space. The limit as $T \to \infty$ is then achieved by analyzing the variational convergence of the corresponding, T-dependent, rate function. (ii) By taking first the limit $T \to \infty$ the large deviations of the empirical process are directly given by the level three Donsker-Varadhan asymptotic [11]. The limit as $\varepsilon \to 0$ is then achieved by analyzing the variational convergence of the corresponding, ε-dependent, rate function. We here follow both these alternative and show they lead to the same conclusion, the resulting rate function being particularly simple to describe: it is the expectation of the Freidlin-Wentzell rate function per unit of time.

In the context of non-equilibrium statistical mechanics, a relevant additive functional not of the form (1.2) is the Gallavotti-Cohen observable [13,18,21,22]. As we here discuss, its large deviations in joint limit in which the time window diverges and the noise vanishes can be obtained by projection.

The analysis here performed shares common features with the one carried out in [4] for the weakly asymmetric exclusion process in the hydrodynamic scaling limit. The present setting avoids the technicalities involved in hydrodynamic limits and the core of the argument is more transparent. On the other hand, the non-compactness of the state space requires additional estimates.

2. Notation and main result

We denote by $\cdot$ the canonical inner product in $\mathbb{R}^n$ and by $|\cdot|$ the corresponding Euclidean norm. For $\varepsilon > 0$ we consider the diffusion process on $\mathbb{R}^n$ with generator L_ε defined on C^2 functions on $\mathbb{R}^n$ with compact support by

$$L_\varepsilon f = \varepsilon \operatorname{Tr}(aD^2 f) + b \cdot \nabla f \tag{2.1}$$

where $D^2 f$, respectively ∇f, denotes the Hessian, respectively the gradient, of f and $a = \{a_{i,j}(\cdot),\, i,j = 1,\dots,n\}$, respectively $b = \{b_i(\cdot),\, i = 1,\dots,n\}$, are the diffusion matrix and the drift. We suppose that the vector field b admits the decomposition

$$b = -a\nabla V + c. \tag{2.2}$$

Hereafter, we assume without further mention that a, V, c meet the following conditions in which we denote by $\mathbb{M}_n$ the set of symmetric $n \times n$ matrices.

Assumption 2.1.

(i) V belongs to $C^2(\mathbb{R}^n)$, $V \ge 0$, $\lim_{|x|\to\infty} \nabla V(x) \cdot \frac{x}{|x|} = +\infty$, and there exists $\varepsilon_0 > 0$ such that $\lim_{|x|\to\infty} \Big[\nabla V(x) \cdot a(x) \nabla V(x) - \varepsilon_0 \operatorname{Tr}\big(a(x) D^2 V(x)\big)\Big] = +\infty$;

(ii) c belongs to $C^1(\mathbb{R}^n; \mathbb{R}^n)$ and it is bounded with bounded derivatives;

(iii) a belongs to $C^2(\mathbb{R}^n; \mathbb{M}_n)$, it is bounded with bounded derivatives, and it is uniformly elliptic, i.e., there is constant $C > 0$ such that $v \cdot a(x) v \ge C^{-1}|v|^2$ for any $x, v \in \mathbb{R}^n$.

The process generated by L_ε and initial condition $x \in \mathbb{R}^n$ can be realized as the solution to the stochastic differential equation (1.1) choosing σ a globally Lipschitz matrix field satisfying $a = \sigma\sigma^\dagger$. In the present context, the vector field b is not necessary globally Lipschitz; however Assumption 2.1 implies there exists a unique

strong solution to (1.1), see e.g. [17, Thm. 3.5]. We shall denote the law of ξ^ε by $\mathbb{P}^\varepsilon_x$ that, given $T>0$, we regard as a probability on $C([0,T];\mathbb{R}^n)$.

We denote by $D(\mathbb{R};\mathbb{R}^n)$ the space of càdlàg paths with values on $\mathbb{R}^n$ that we consider endowed with the Skorokhod topology on bounded intervals and the associated Borel σ-algebra. Given $T>0$ and a path $X\in C([0,T];\mathbb{R}^n)$ we denote by $X^T\in D(\mathbb{R};\mathbb{R}^n)$ its *T-periodization*, i.e.,

$$(X^T)_t := X_{t-\lfloor t/T\rfloor T},\qquad t\in\mathbb{R}.$$

Observe that X^T is T-periodic and continuous except at the times kT, $k\in\mathbb{Z}$ where it has the jump of size X_0-X_T. For $t\in\mathbb{R}$ we denote by $\theta_t\colon D(\mathbb{R};\mathbb{R}^n)\to D(\mathbb{R};\mathbb{R}^n)$ the translation by t namely, $(\theta_t X)_s := X_{s-t}$, $s\in\mathbb{R}$. We finally denote by $\mathcal{P}_\theta$ the set of translation invariant probabilities on $D(\mathbb{R};\mathbb{R}^n)$, i.e. the set of Borel probabilities P satisfying $P\circ\theta_t^{-1}=P$ for any $t\in\mathbb{R}$. We consider $\mathcal{P}_\theta$ endowed with the topology induced by weak convergence and the associated Borel σ-algebra.

Given $T>0$, the *empirical process* is the map $R_T\colon C([0,T];\mathbb{R}^n)\to\mathcal{P}_\theta$ defined by

$$R_T(X) := \frac{1}{T}\int_0^T dt\,\delta_{\theta_t X^T}. \tag{2.3}$$

Note indeed that, by the T-periodicity of X^T, the right hand side defines a translation invariant probability on $D(\mathbb{R};\mathbb{R}^n)$.

Our main result establishes the large deviation principle for the family of probabilities on $\mathcal{P}_\theta$ given by $\{\mathbb{P}^\varepsilon_x\circ R_T^{-1}\}$ in the joint limit $\varepsilon\to0$ and $T\to\infty$. Let us first recall the Freidlin-Wentzell functional associated to (1.1). Given $T>0$, denote by $H_1([0,T])$ the set of absolutely continuous paths $X\colon[0,T]\to\mathbb{R}^n$ such that $\int_0^T dt\,|\dot X_t|^2<+\infty$ and let $I_{[0,T]}\colon C([0,T];\mathbb{R}^n)\to[0,+\infty]$ be the functional defined by

$$I_{[0,T]}(X) := \begin{cases} \dfrac14\displaystyle\int_0^T dt\big[\dot X_t - b(X_t)\big]\cdot a^{-1}(X_t)\big[\dot X_t - b(X_t)\big] & \text{if } X\in H_1([0,T]),\\ +\infty & \text{otherwise.}\end{cases} \tag{2.4}$$

We regard $I_{[0,T]}$ also as a functional on $D(\mathbb{R};\mathbb{R}^n)$ understating that $I_{[0,T]}(X)$ is infinite if the restriction of X to $[0,T]$ does not belong to $C([0,T];\mathbb{R}^n)$. We then let $\mathcal{I}\colon\mathcal{P}_\theta\to[0,+\infty]$ be the functional defined by

$$\mathcal{I}(P) := \int dP(X)\, I_{[0,1]}(X). \tag{2.5}$$

Observe that $\mathcal{I}$ is affine and, by the translation invariance of P, if $\mathcal{I}(P)<+\infty$ then P-a.s. $t\mapsto X_t$ is absolutely continuous. In the next statement we use the shorthand notation $\overline{\lim}_{T,\varepsilon}$ for either $\overline{\lim}_{\varepsilon\to0}\overline{\lim}_{T\to\infty}$ or $\overline{\lim}_{T\to\infty}\overline{\lim}_{\varepsilon\to0}$. Analogously, $\underline{\lim}_{T,\varepsilon}$ stands for either $\underline{\lim}_{\varepsilon\to0}\underline{\lim}_{T\to\infty}$ or $\underline{\lim}_{T\to\infty}\underline{\lim}_{\varepsilon\to0}$.

Theorem 2.2. *As $\varepsilon\to0$ and $T\to\infty$, the family $\{\mathbb{P}^\varepsilon_x\circ R_T^{-1},\,T>0,\,\varepsilon>0\}$ satisfies, uniformly for x in compacts, a large deviation principle with speed $\varepsilon^{-1}T$ and rate function $\mathcal{I}$. Namely, for each compact set $K\subset\subset\mathbb{R}^n$, each closed set $C\subset\mathcal{P}_\theta$, and each open set $A\subset\mathcal{P}_\theta$*

$$\overline{\lim_{T,\varepsilon}}\ \sup_{x\in K}\frac{\varepsilon}{T}\log\mathbb{P}^\varepsilon_x\big(R_T\in C\big)\le-\inf_{P\in C}\mathcal{I}(P)\,,$$

$$\underline{\lim_{T,\varepsilon}}\ \inf_{x\in K}\frac{\varepsilon}{T}\log\mathbb{P}^\varepsilon_x\big(R_T\in A\big)\ge-\inf_{P\in A}\mathcal{I}(P)\,.$$

Moreover, the functional $\mathfrak{I}$ is good and affine.

Referring to Section 5 for an application of this result to the asymptotics of the Gallavotti-Cohen observable, we next mention some of its possible developments. While Theorem 2.2 suggests that the large deviations hold whenever $(\varepsilon, T) \to (0, +\infty)$, the proof relies in computing first the limit as $T \to \infty$ and then $\varepsilon \to 0$ or the converse. It thus appears that a truly joint limit requires new methods. In the case in which the limiting deterministic dynamical system obtained by setting $\varepsilon = 0$ in (1.1) has more than a single stationary probability, as it is the case for metastable processes, the zero level set of the functional $\mathfrak{I}$ is not a singleton. In the spirit of the so-called development by Γ-convergence, see e.g. [7, § 1.10], it is then possible to investigate higher order large deviations asymptotics. In the case of reversible diffusions, this development for the Fisher information, i.e. the Donsker-Varadhan level two rate function for the occupation measure, has been achieved in [10]. The corresponding analysis for finite state Markov chains has been carried out in [5,20]. We emphasize that the limits as $T \to \infty$ and $\varepsilon \to 0$ do not commute for the higher order large deviations. While the present analysis is carried out for non-degenerate diffusion processes, the problem of computing the small noise limit of the level three Donsker-Varadhan functional can be formulated for general Markov processes.

3. Large time limit after small noise limit

Recalling (2.4), for $T > 0$ and $x \in \mathbb{R}^n$ let $I^x_{[0,T]} \colon C([0,T];\mathbb{R}^n) \to [0,+\infty]$ be the functional defined by

$$I^x_{[0,T]}(X) := \begin{cases} I_{[0,T]}(X) & \text{if } X_0 = x, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.1}$$

Let also $\mathfrak{I}^x_{[0,T]} \colon \mathcal{P}_\theta \to [0,+\infty]$ be defined by

$$\mathfrak{I}^x_{[0,T]}(P) := \inf \left\{ I^x_{[0,T]}(X),\ R_T(X) = P \right\}, \tag{3.2}$$

where we adopt the standard convention that the infimum over the empty set is $+\infty$. Note that, for $X \in C([0,T];\mathbb{R}^n)$, if $X(0) \neq X(T)$ or if $X(0) = X(T) = x$, then $\mathfrak{I}^x_{[0,T]}(P) = I^x_{[0,T]}(X)$. In contrast, if $X(0) = X(T)$ and $X(0) \neq x$, $I^x_{[0,T]}(X) = +\infty$ and $\mathfrak{I}^x_{[0,T]}(P)$ may be finite if $X(t) = x$ for some $0 \le t \le T$. In view of the continuity of the map $C([0,T];\mathbb{R}^n) \ni X \mapsto R_T(X) \in \mathcal{P}_\theta$, the following statement follows directly, by the contraction principle, from the Freidlin-Wentzell asymptotics [12]. The present case of an unbounded vector field b is covered by [1, Thm. III.2.13].

Lemma 3.1. *Fix $T > 0$. As $\varepsilon \to 0$ the family $\left\{ \mathbb{P}^\varepsilon_x \circ R_T^{-1},\ \varepsilon > 0 \right\}$ satisfies, uniformly for x in compacts, a large deviation principle with speed ε^{-1} and good rate function $\mathfrak{I}^x_{[0,T]}$. Namely, for each $x \in \mathbb{R}^n$, each sequence $x_\varepsilon \to x$, each closed set $C \subset \mathcal{P}_\theta$, and each open set $A \subset \mathcal{P}_\theta$*

$$\begin{aligned} \overline{\lim_{\varepsilon \to 0}}\ \varepsilon \log \mathbb{P}^\varepsilon_{x_\varepsilon}\big(R_T \in C\big) &\le - \inf_{P \in C} \mathfrak{I}^x_{[0,T]}(P) \\ \underline{\lim_{\varepsilon \to 0}}\ \varepsilon \log \mathbb{P}^\varepsilon_{x_\varepsilon}\big(R_T \in A\big) &\ge - \inf_{P \in A} \mathfrak{I}^x_{[0,T]}(P). \end{aligned}$$

In order to achieve the proof of Theorem 2.2 we next analyze the variational convergence of the family of functionals $\left\{ T^{-1}\, \mathfrak{I}^x_{[0,T]} \right\}$ as $T \to \infty$. With respect to the standard framework of Γ-convergence, see e.g. [7], in the present setting there

is the additional dependence on the parameter x, for which we need uniformity on compacts.

Theorem 3.2. *Fix a compact set $K \subset\subset \mathbb{R}^n$.*

(i) *If a sequence $\{P_T\} \subset \mathcal{P}_\theta$ satisfies $\underline{\lim}_T T^{-1}\, \mathcal{I}^{x_T}_{[0,T]}(P_T) < +\infty$ for some $\{x_T\} \subset K$ then $\{P_T\}$ has a pre-compact sub-sequence.*

(ii) *For any $P \in \mathcal{P}_\theta$, any sequence $\{x_T\} \subset K$, and any sequence $P_T \to P$*

$$\varliminf_{T\to\infty} \frac{1}{T}\mathcal{I}^{x_T}_{[0,T]}(P_T) \geq \mathcal{I}(P).$$

(iii) *For any $P \in \mathcal{P}_\theta$ and any sequence $\{x_T\} \subset K$ there exists a sequence $P_T \to P$ such that*

$$\varlimsup_{T\to\infty} \frac{1}{T}\mathcal{I}^{x_T}_{[0,T]}(P_T) \leq \mathcal{I}(P).$$

Assuming the above result, we first show that it implies the large deviations of the empirical process in the limit in which first the noise vanishes and then the time interval diverges.

Proof of Theorem 2.2 ($T \to \infty$ after $\varepsilon \to 0$). We start by showing the goodness of the rate function. Since $I_{[0,T]}$ is lower semi-continuous, by Portmanteau theorem, $\mathcal{I}$ is also lower semi-continuous. It thus suffices to show that $\mathcal{I}$ has pre-compact sublevel sets. In view of the conditions in Assumption 2.1, by expanding the square in (2.4) we deduce there are constants $\gamma, C > 0$ depending only on V, c, a such that for any $X \in C([0,T];\mathbb{R}^n)$

$$I_{[0,T]}(X) \geq \frac{1}{2}\big[V(X_T) - V(X_0)\big] + \gamma \int_0^T dt\, \big[|\dot{X}_t|^2 + |\nabla V(X_t)|^2\big] - CT. \tag{3.3}$$

Take expectation with respect to P. The translation invariance of P and the bound $I_{[0,T+S]}(X) \leq I_{[0,T]}(X) + I_{[0,S]}(\theta_{-T}X)$ yields that for each bounded interval $[T_1, T_2]$

$$\int dP(X) \left[|\nabla V(X_0)|^2 + \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} dt\, |\dot{X}_t|^2 \right] \leq C\big[1 + \mathcal{I}(P)\big] \tag{3.4}$$

for a new constant C. By the assumptions on V and standard criterion, see e.g. [6, Thm. 8.2], $\mathcal{I}$ has pre-compact sublevel sets, as claimed.

To prove the upper bound, we first observe that the Feller property of the semigroup generated by L_ε and the continuity of R_T imply that for each closed set $C \subset \mathcal{P}_\theta$ the map $x \mapsto \mathbb{P}^\varepsilon_x(R_T \in C)$ is upper semi-continuous. Therefore, given a compact set $K \subset\subset \mathbb{R}^n$, there exists a sequence $\{x_{T,\varepsilon}\} \subset K$ such that

$$\sup_{x\in K} \mathbb{P}^\varepsilon_x(R_T \in C) = \mathbb{P}^\varepsilon_{x_{T,\varepsilon}}(R_T \in C).$$

By passing to a not relabeled sub-sequence we may assume that the sequence $\{x_{T,\varepsilon}\}_{\varepsilon>0}$ converges to some $x_T \in K$. From Lemma 3.1 we then deduce

$$\varlimsup_{\varepsilon\to 0} \sup_{x\in K} \varepsilon \log \mathbb{P}^\varepsilon_x\big(R_T \in C\big) \leq -\inf_{P\in C} \mathcal{I}^{x_T}_{[0,T]}(P)$$

so that

$$\varlimsup_{T\to\infty} \varlimsup_{\varepsilon\to 0} \sup_{x\in K} \frac{\varepsilon}{T} \log \mathbb{P}^\varepsilon_x\big(R_T \in C\big) \leq -\varliminf_{T\to\infty} \inf_{P\in C} \frac{1}{T}\mathcal{I}^{x_T}_{[0,T]}(P).$$

If $\underline{\lim}_T \inf_{P\in C} T^{-1}\mathcal{J}^{x_T}_{[0,T]}(P) = +\infty$ the right-hand side above is trivially bounded above by $-\inf_{P\in C}\mathcal{I}(P)$. If conversely $\underline{\lim}_T \inf_{P\in C} T^{-1}\mathcal{J}^{x_T}_{[0,T]}(P) < +\infty$, there exist sequences $T_k\to\infty$ and $\{P_k\}\subset C$ such that

$$\underline{\lim_{T\to\infty}} \inf_{P\in C} \frac{1}{T}\mathcal{J}^{x_T}_{[0,T]}(P) = \lim_{k\to\infty} \frac{1}{T_k}\mathcal{J}^{x_{T_k}}_{[0,T_k]}(P_k)\;.$$

By item (i) in Theorem 3.2, there exists P^* and a further sub-sequence, still denoted by $\{P_k\}\subset C$, converging to P^*. By the goodness of the rate function $\mathcal{J}^x_{[0,T]}$, $P^*\in C$, and, by item (ii) in Theorem 3.2,

$$\lim_{k\to\infty} \frac{1}{T_k}\mathcal{J}^{x_{T_k}}_{[0,T_k]}(P_k) \ge \mathcal{I}(P^*) \ge \inf_{P\in C}\mathcal{I}(P)$$

which concludes the proof of the upper bound.

To prove the lower bound, observe that, again by the Feller property of the semigroup generated by L_ε and the continuity of R_T, for each open set $A\subset\mathcal{P}_\theta$ the map $x\mapsto \mathbb{P}^\varepsilon_x(R_T\in A)$ is lower semi-continuous. Therefore, given a compact set $K\subset\subset\mathbb{R}^n$, there exists a sequence $\{x_{T,\varepsilon}\}\subset K$ such that

$$\inf_{x\in K}\mathbb{P}^\varepsilon_x(R_T\in A) = \mathbb{P}^\varepsilon_{x_{T,\varepsilon}}(R_T\in A).$$

By passing to a not relabeled sub-sequence we may assume that the sequence $\{x_{T,\varepsilon}\}_{\varepsilon>0}$ converges to some $x_T\in K$. From Lemma 3.1 we then deduce

$$\underline{\lim_{\varepsilon\to 0}} \inf_{x\in K} \varepsilon\log\mathbb{P}^\varepsilon_x\big(R_T\in A\big) \ge -\inf_{P\in A}\mathcal{J}^{x_T}_{[0,T]}(P).$$

If $\inf_{P\in A}\mathcal{I}(P)=+\infty$, the right-hand side is bounded below by $-\inf_{P\in A}\mathcal{I}(P)$, and the lower bound of Theorem 2.2 is proved. Conversely, assume that $\inf_{P\in A}\mathcal{I}(P)<+\infty$. In this case, given $\delta>0$, let $P^*\in A$ be such that $\inf_{P\in A}\mathcal{I}(P)\ge\mathcal{I}(P^*)-\delta$. By item (iii) in Theorem 3.2, for $\{x_T\}\subset K$ as above there exists a sequence $\{P_T\}$ converging to P^* and such that $\overline{\lim}_T T^{-1}\mathcal{J}^{x_T}_{[0,T]}(P_T)\le\mathcal{I}(P^*)$. Since $P^*\in A$, $P_T\to P^*$ and A is an open set, $P_T\in A$ for T large enough. Therefore,

$$\begin{aligned}\underline{\lim_{T\to\infty}}\,\underline{\lim_{\varepsilon\to 0}}\inf_{x\in K}\frac{\varepsilon}{T}\log\mathbb{P}^\varepsilon_x\big(R_T\in A\big) &\ge -\overline{\lim_{T\to\infty}}\frac{1}{T}\inf_{P\in A}\mathcal{J}^{x_T}_{[0,T]}(P)\\ &\ge -\overline{\lim_{T\to\infty}}\frac{1}{T}\mathcal{J}^{x_T}_{[0,T]}(P_T)\ge -\mathcal{I}(P^*)\ge -\inf_{P\in A}\mathcal{I}(P)-\delta,\end{aligned}$$

which, by taking the limit $\delta\to 0$, concludes the proof. □

To prove Theorem 3.2, we premise a density result on set of translation invariant probability measures on $D(\mathbb{R};\mathbb{R}^n)$. An element P in $\mathcal{P}_\theta$ is said to be *S-holonomic* if there exists a S-periodic path $Y\in C(\mathbb{R};\mathbb{R}^n)$ such that

$$P = \frac{1}{S}\int_0^S dt\,\delta_{\theta_t Y}, \tag{3.5}$$

where we emphasize that we require Y to satisfy $Y_S = Y_0$. An element of $\mathcal{P}_\theta$ is *holonomic* if it is S-holonomic for some $S>0$; it is *smooth holonomic* when the path Y in (3.5) belongs to $C^1(\mathbb{R};\mathbb{R}^n)$.

Lemma 3.3. *Fix $P\in\mathcal{P}_\theta$ satisfying $\mathcal{I}(P)<+\infty$. There exist a triangular array $\{\alpha^n_i,\, n\in\mathbb{N},\, i=1,\dots,n\}$ with $\alpha^n_i\ge 0$, $\sum_i\alpha^n_i=1$ and a triangular array $\{P^n_i\,,\, n\in\mathbb{N}, i=1,\dots,n\}$ of smooth holonomic probability measures such that by setting $P^n := \sum_i\alpha^n_i P^n_i$ we have $P^n\to P$ and $\mathcal{I}(P^n)\to\mathcal{I}(P)$.*

Proof. We follow the argument in [4, Thm. 4.10], see also [2] for similar results.

The proof is achieved, by a diagonal argument, from the following claims. Recall that $P \in \mathcal{P}_\theta$ is *ergodic* when the tail σ-algebra is P-trivial.

Claim 1. Let $P \in \mathcal{P}_\theta$ be such that $\mathcal{I}(P) < +\infty$. There exist a triangular array $\{\alpha_i^n,\, n \in \mathbb{N},\, i = 1, \dots, n\}$ with $\alpha_i^n \geq 0$, $\sum_{i=1}^n \alpha_i^n = 1$ and a triangular array $\{P_i^n\,,\, n \in \mathbb{N}, i = 1, \dots, n\}$ of ergodic probability measures such that $\sum_{i=1}^n \alpha_i^n P_i^n \to P$ and $\sum_{i=1}^n \alpha_i^n \mathcal{I}(P_i^n) \to \mathcal{I}(P)$.

This follows directly from the fact that the ergodic probabilities are extremal in $\mathcal{P}_\theta$ and $\mathcal{I}$ is affine.

Claim 2. Let $P \in \mathcal{P}_\theta$ be ergodic and such that $\mathcal{I}(P) < +\infty$. Then there exists a sequence $P^n \to P$ such that $\mathcal{I}(P^n) \to \mathcal{I}(P)$ and for each n the probability P^n is holonomic.

Recalling (2.3), to construct the required sequence set

$$\mathcal{A}_P := \Big\{ X \in D(\mathbb{R}; \mathbb{R}^n) \colon \lim_{T\to\infty} R_T(X) = P, \text{ and } \lim_{T\to\infty} \frac{1}{T} I_{[0,T]}(X) = \mathcal{I}(P) \Big\}.$$

Since $\mathcal{I}(P) < +\infty$ then $I_{[0,1]} \in L_1(dP)$. The Birkhoff's ergodic theorem then implies $P(\mathcal{A}_P) = 1$. Pick an element $Y \in \mathcal{A}_P$. By definition, the T-holonomic probability associated to the T-periodization of Y converges to P but, in general, its rate function does not since when T-periodizing paths we may insert jumps. This issue is easily solved by modifying the path Y in the time interval $[T-1, T]$ in such a way that $Y_T = Y_0$ and $T^{-1} I_{[T-1,T]}(Y) \to 0$.

Claim 3. Let $P \in \mathcal{P}_\theta$ be holonomic and such that $\mathcal{I}(P) < +\infty$. Then there exists a sequence of C^1 holonomic probability measures $P_n \in \mathcal{P}_\theta$ such that $P^n \to P$ and $\mathcal{I}(P^n) \to \mathcal{I}(P)$.

The required sequence is constructed by taking the convolution $\imath_n * X$ where $\imath_n$ is a smooth approximation of the identity and X is the continuous periodic path associated to the measure P. □

Proof of Theorem 3.2.
Item (i). By assumption, there exist a finite constant C_0 and sequences $T_j \to \infty$, $x_j \in K$, and $P_j \in \mathcal{P}_\theta$ such that $\mathcal{I}^{x_j}_{[0,T_j]}(P_j) \leq C_0 T_j$. Fix j. By definition of $\mathcal{I}^{x_j}_{[0,T_j]}(P_j)$, there exists $Y \in C([0,T_j]; \mathbb{R}^n)$ satisfying $R_{T_j}(Y) = P_j$, $I^{x_j}_{[0,T_j]}(Y) \leq \mathcal{I}^{x_j}_{[0,T_j]}(P_j) + 1$. As the rate function is finite, $Y(0) = x_j$. By (3.3) and since $V \geq 0$,

$$\begin{aligned} \int dP_j(X)\, |\nabla V(X_0)|^2 &= \frac{1}{T_j} \int_0^{T_j} dt |\nabla V(X_t)|^2 \\ &\leq \frac{1}{2T_j} V(x_j) + \frac{1}{\gamma} \Big[C + \frac{1}{T_j} I^{x_j}_{[0,T_j]}(P_j) \Big]. \end{aligned} \tag{3.6}$$

Since $I^{x_j}_{[0,T_j]}(Y) \leq \mathcal{I}^{x_j}_{[0,T_j]}(P_j) + 1$ and x_j belongs to a compact, the right-hand side is bounded by a finite constant, uniformly in j.

The bound on the continuity modulus is somewhat more delicate as the T-periodization introduces, in general, jumps. On the other hand, given $T_1 < T_2$ and $P = R_T(Y)$ for some $Y \in C([0,T]; \mathbb{R}^n)$, the P probability of observing a jump in the time window $[T_1, T_2]$ is at most $(T_2 - T_1)/T$. For $\delta > 0$, $T_1 < T_2$, and

$X \in D(\mathbb{R};\mathbb{R}^n)$, introduce the continuity modulus

$$\omega^\delta_{[T_1,T_2]}(X) := \sup_{\substack{t,s\in[T_1,T_2]\\ |t-s|<\delta}} |X_t - X_s|.$$

By Cauchy-Schwarz, if the restriction of X to $[T_1,T_2]$ belongs to $H^1([T_1,T_2])$ then

$$\omega^\delta_{[T_1,T_2]}(X)^2 \le \delta \int_{T_1}^{T_2} dt\, |\dot X|^2.$$

By the previous observations, if $P = R_T(Y)$ for some Y satisfying $I^x_{[0,T]}(Y) < +\infty$ for some $x \in K$, from Chebyshev inequality we deduce that for each $\zeta > 0$

$$\begin{aligned} P\big(\omega^\delta_{[T_1,T_2]} > \zeta\big) &\le \frac{T_2-T_1}{T} + \frac{(T_2-T_1)\delta}{\zeta^2}\frac{1}{T}\int_0^T dt\,|\dot Y|^2 \\ &\le \frac{T_2-T_1}{T} + \frac{(T_2-T_1)\delta}{\gamma\zeta^2}\Big[\frac{1}{2T}\sup_{y\in K} V(y) + C + \frac{1}{T}\mathcal{I}^x_{[0,T]}(P)\Big]. \end{aligned}$$

where we used (3.3) in the second step.

By standard criterion on the tightness of probability measures on $D(\mathbb{R};\mathbb{R}^n)$, see e.g. [6, Thm. 15.5], the previous displayed bound together with (3.6) yield the statement.

Item (ii). If $\mathcal{I}^x_{[0,T]}(P) < +\infty$ then there exists $Y \in C([0,T];\mathbb{R}^n)$ such that $P = R_T(Y)$ and for $T \ge 1$

$$\begin{aligned} \mathcal{I}^x_{[0,T]}(P) = I^x_{[0,T]}(Y) &\ge I_{[0,T]}(Y) \ge \int_0^{T-1} dt\, I_{[0,1]}(\theta_{-t}Y) \\ &= (T-1)\int d\tilde P(X)\, I_{[0,1]}(X) \end{aligned}$$

where we used (3.1) in the second step and we have set

$$\tilde P := \frac{1}{T-1}\int_0^{T-1} dt\,\delta_{\theta_t Y^T} = \frac{T}{T-1}\,P - \frac{1}{T-1}\int_{T-1}^{T} dt\,\delta_{\theta_t Y^T}. \tag{3.7}$$

Consider now $P \in \mathcal{P}_\theta$ and sequences $\{x_T\}$, $P_T \to P$ as in the statement. By passing to a not relabeled sub-sequence we may assume that $P_T = R_T(Y)$ for some $Y = Y(T) \in C([0,T];\mathbb{R}^n)$. Letting $\tilde P_T$ be defined as in (3.7) we then have $\tilde P_T \to P$ and

$$\varliminf_{T\to\infty} \frac{1}{T}\mathcal{I}^{x_T}_{[0,T]}(P_T) \ge \varliminf_{T\to\infty} \frac{T-1}{T}\int d\tilde P_T(X)\, I_{[0,1]}(X) \ge \int dP(X)\, I_{[0,1]}(X) = \mathcal{I}(P)$$

where we have used the lower semi-continuity of $I_{[0,1]}$.

Item (iii). In view of Lemma 3.3, it suffices to consider the case in which P is smooth holonomic, i.e. $P = S^{-1}\int_0^S ds\,\delta_{\theta_s Y}$ for some $S > 0$ and some S-periodic path $Y \in C^1(\mathbb{R},\mathbb{R}^n)$. In particular, $\mathcal{I}(P) = S^{-1} I_{[0,S]}(Y)$.

Given $x, y \in \mathbb{R}^n$ let $\bar Y^{x,y} \in C([0,1];\mathbb{R}^n)$ be the affine interpolation between x and y, i.e. $\bar Y^{x,y}_t = x(1-t) + yt$, $t \in [0,1]$. By a direct computation there exist $C(|x|,|y|) > 0$ depending on V, c, a such that

$$I_{[0,1]}(\bar Y^{x,y}) \le C(|x|,|y|).$$

For $T>0$ and a sequence $\{x_T\}\subset K$ as in the statement, let $\tilde Y\in C([0,+\infty);\mathbb{R}^n)$ be the path defined by

$$\tilde Y_t := \begin{cases} \bar Y_t^{x_T,Y_0} & \text{if } t\in[0,1],\\ Y_{t-1} & \text{if } t>1,\end{cases}$$

and set $P_T := R_T(\tilde Y)$. Then $P_T\to P$ and for $T\geq 1$

$$\mathcal{I}^{x_T}_{[0,T]}(P_T) = I^{x_T}_{[0,T]}(\tilde Y) = I_{[0,1]}\big(\bar Y^{x_T,Y_0}\big) + I_{[0,T-1]}(Y)$$

so that

$$\varlimsup_{T\to\infty}\frac{1}{T}\mathcal{I}^{x_T}_{[0,T]}(P_T) \leq \varlimsup_{T\to\infty}\Big[\frac{1}{T}\sup_{x\in K}C(|x|,|Y_0|)+\frac{1}{T}I_{[0,T-1]}(Y)\Big] = \mathcal{I}(P)$$

by the S-periodicity of Y. □

4. Small noise limit after large time limit

By Assumption 2.1 and standard criteria, see e.g. [17, Thm. 3.7 and Cor. 4.4], for each $\varepsilon>0$ the process ξ^ε that solves (1.1) admits a unique invariant probability π^ε. We denote by $\mathbb{P}^\varepsilon_{\pi^\varepsilon}$ the corresponding stationary process, that we regard as a probability on $D(\mathbb{R};\mathbb{R}^n)$. For fixed $\varepsilon>0$, the Donsker-Varadhan theorem [9,11,26] states the large deviation principle as $T\to\infty$ for the family $\{\mathbb{P}^\varepsilon_x\circ R_T^{-1}\}_{T>0}$ with rate function given by the relative entropy per unit of time with respect to $\mathbb{P}^\varepsilon_{\pi^\varepsilon}$.

We first introduce such rate function by a variational representation. For $T>0$, let $\mathcal{H}^\varepsilon(T,\cdot)\colon \mathcal{P}_\theta\to[0,+\infty]$ be the functional defined by

$$\mathcal{H}^\varepsilon(T,P) := \sup_\Phi \int dP(X)\Big[\Phi(X)-\log\mathbb{E}^\varepsilon_{X_0}\big(e^\Phi\big)\Big], \tag{4.1}$$

where $\mathbb{E}^\varepsilon_x$ denotes the expectation with respect to $\mathbb{P}^\varepsilon_x$, $x\in\mathbb{R}^n$ and the supremum is carried over the bounded and continuous functions Φ on $D(\mathbb{R},\mathbb{R}^n)$ that are measurable with respect to $\sigma\{X_s,\ s\in[0,T]\}$. Let then $\mathcal{H}^\varepsilon\colon\mathcal{P}_\theta\to[0,+\infty]$ be the functional defined by

$$\mathcal{H}^\varepsilon(P) := \sup_{T>0}\frac{1}{T}\mathcal{H}^\varepsilon(T,P) = \lim_{T\to\infty}\frac{1}{T}\mathcal{H}^\varepsilon(T,P), \tag{4.2}$$

where the second identity follows from the inequality before [26, Thm. 10.9]. By [26, Thm.s 10.6 and 10.8], the functional $\mathcal{H}_\varepsilon$ is good and affine.

We next characterize $\mathcal{H}^\varepsilon$ as the relative entropy per unit of time with respect $\mathbb{P}^\varepsilon_{\pi^\varepsilon}$. Given $T_1<T_2$, denote by $\imath_{T_1,T_2}\colon D(\mathbb{R},\mathbb{R}^n)\to D([T_1,T_2],\mathbb{R}^n)$ the canonical projection. Given two probability measures P^1,P^2 on $D(\mathbb{R},\mathbb{R}^n)$, let $\mathcal{H}_{[T_1,T_2]}(\cdot|\cdot)$ be the relative entropy of the marginal of P^2 on the time interval $[T_1,T_2]$ with respect to the marginal of P^1 on the same interval, i.e.,

$$\mathcal{H}_{[T_1,T_2]}(P^2|P^1) = \mathrm{Ent}\big(P^2_{[T_1,T_2]}\big|P^1_{[T_1,T_2]}\big) := \int dP^2_{[T_1,T_2]}\,\log\frac{dP^2_{[T_1,T_2]}}{dP^1_{[T_1,T_2]}} \tag{4.3}$$

where $P^j_{[T_1,T_2]} = P^j\circ\imath^{-1}_{T_1,T_2}$, $j=1,2$. By [9, Thm. 5.4.27], for each $P\in\mathcal{P}_\theta$

$$\mathcal{H}^\varepsilon(P) = \lim_{T\to\infty}\frac{1}{T}\mathcal{H}_{[0,T]}\big(P\big|\mathbb{P}^\varepsilon_{\pi^\varepsilon}\big) = \sup_{T>0}\frac{1}{T}\mathcal{H}_{[0,T]}\big(P\big|\mathbb{P}^\varepsilon_{\pi^\varepsilon}\big), \tag{4.4}$$

where the second identity follows by a super-additivity argument which stems from [26, Lemma 10.3].

Recalling that $\varepsilon_0 > 0$ is the constant appearing in item (i) of Assumption 2.1, the large deviation principle in the limit $T \to \infty$ is then stated as follows.

Lemma 4.1. *Fix $\varepsilon \in (0, \varepsilon_0)$. As $T \to \infty$ the family $\{\mathbb{P}^\varepsilon_x \circ R_T^{-1},\, T > 0\}$ satisfies, uniformly for x in compacts, a large deviation principle with speed T and good affine rate function $\mathcal{H}^\varepsilon$. Namely, for each compact set $K \subset\subset \mathbb{R}^n$, each closed set $C \subset \mathcal{P}_\theta$, and each open set $A \subset \mathcal{P}_\theta$*

$$\overline{\lim_{T\to\infty}}\ \frac{1}{T} \sup_{x\in K} \log \mathbb{P}^\varepsilon_x\big(R_T \in C\big) \le -\inf_{P\in C} \mathcal{H}^\varepsilon(P)$$

$$\underline{\lim_{T\to\infty}}\ \frac{1}{T} \inf_{x\in K} \log \mathbb{P}^\varepsilon_x\big(R_T \in A\big) \ge -\inf_{P\in A} \mathcal{H}^\varepsilon(P).$$

Proof. The statement follows from [26, Thm.s 11.6 and 12.5], we only need to check that the hypotheses of those theorems are met.

Regarding the upper bound, given $\gamma \in (0,1)$ set

$$u_\varepsilon(x) := \exp\Big\{\frac{\gamma}{\varepsilon} V(x)\Big\}, \qquad x \in \mathbb{R}^n. \tag{4.5}$$

We claim that Assumption 2.1 implies that u_ε meets conditions (1)–(5) in [26, Pag. 34] for any $\varepsilon \in (0, \varepsilon_0)$ and a suitable $\gamma \in (0,1)$. Indeed, $u_\varepsilon \ge 1$ and u_ε is bounded on compacts. Moreover, by a direct computation,

$$W_\varepsilon := -\frac{L_\varepsilon u}{u} = \frac{\gamma}{\varepsilon}\Big[(1-\gamma)\nabla V \cdot a \nabla V - c\cdot \nabla V - \varepsilon \operatorname{Tr}(aD^2 V)\Big] \tag{4.6}$$

satisfies $\inf_x W_\varepsilon(x) > -\infty$ and $\lim_{|x|\to\infty} W_\varepsilon(x) = +\infty$ for γ small enough. Even if u_ε does not really belong to the domain of the generator L_ε, it is straightforward to introduce a cutoff function $\phi_n \colon \mathbb{R}^n \to (0, +\infty)$ such that $u_{\varepsilon,n} := u_\varepsilon\, \phi_n$ belongs to the domain of L_ε for each $n \in \mathbb{N}$ and the sequence $\{u_{\varepsilon,n},\, n \in \mathbb{N}\}$ satisfies conditions (1)–(5) in [26, Pag. 34].

Regarding the lower bound, denote by $p^\varepsilon(t,x,\cdot)$, $t \ge 0$, $x \in \mathbb{R}^n$ the transition probability of the Markov process ξ^ε and by α the Lebesgue measure on $\mathbb{R}^n$. By standard parabolic regularity, $p^\varepsilon(1,x,\cdot)$ satisfies conditions I–II in [26, Pag. 34]. □

In view of the argument presented in the previous section, the proof of Theorem 2.2 is completed by the variational convergence of $\varepsilon\,\mathcal{H}^\varepsilon$ to $\mathcal{I}$. As the x-dependence has disappeared in the limit $T \to \infty$, the following statement amounts to the standard Γ-convergence of the sequence $\{\varepsilon\,\mathcal{H}^\varepsilon\}$, see e.g. [7], together with the pre-compactness of sequences $\{P_\varepsilon\}$ with equi-bounded rate function.

Theorem 4.2.

(i) *If a sequence $\{P_\varepsilon\} \subset \mathcal{P}_\theta$ satisfies $\underline{\lim}_\varepsilon\, \varepsilon\,\mathcal{H}^\varepsilon(P_\varepsilon) < +\infty$ then it has a pre-compact sub-sequence.*

(ii) *For any $P \in \mathcal{P}_\theta$ and any sequence $P_\varepsilon \to P$*

$$\underline{\lim_{\varepsilon\to 0}}\ \varepsilon\,\mathcal{H}^\varepsilon(P_\varepsilon) \ge \mathcal{I}(P).$$

(iii) *For any $P \in \mathcal{P}_\theta$ there exists a sequence $P_\varepsilon \to P$ such that*

$$\overline{\lim_{\varepsilon\to 0}}\ \varepsilon\,\mathcal{H}^\varepsilon(P_\varepsilon) \le \mathcal{I}(P).$$

We next prove separately the three statements, each one having a preliminary lemma.

Lemma 4.3. *The sequence $\{\mathbb{P}^\varepsilon_{\pi^\varepsilon}\} \subset \mathcal{P}_\theta$ is exponentially tight, i.e., there exists a sequence of compact sets $\mathcal{K}_\ell \subset\subset D(\mathbb{R}, \mathbb{R}^n)$ such that*

$$\lim_{\ell\to\infty} \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbb{P}^\varepsilon_{\pi^\varepsilon}\big(\mathcal{K}_\ell^c\big) = -\infty.$$

Proof. We first show that $\{\pi^\varepsilon\}$ is an exponentially tight family of probabilities on $\mathbb{R}^n$. Observe that, by ergodicity, $\pi^\varepsilon = \lim_{T\to\infty} T^{-1}\int_0^T dt\, \mathbb{P}^\varepsilon_0(X_t \in \cdot)$. Recalling (4.5), for $R > 0$ let $u^R_\varepsilon \colon \mathbb{R}^n \to [1, +\infty)$ be a smooth function such that

$$u^R_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & \text{if } |x| \geq R+1, \\ 1 & \text{if } |x| \leq R. \end{cases}$$

In view of Assumption 2.1 and (4.6), there are $R, \alpha > 0$ such that for any ε small enough $L_\varepsilon u^R_\varepsilon \leq -\alpha u^R_\varepsilon$ so that

$$\mathbb{E}^\varepsilon_0\big(u^R_\varepsilon(X_t)\big) \leq 1 - \alpha \int_0^t ds\, \mathbb{E}^\varepsilon_0\big(u^R_\varepsilon(X_s)\big).$$

Whence, by Gronwall's lemma, $\sup_t \mathbb{E}^\varepsilon_0\big(u^R_\varepsilon(t)\big) \leq 1$. By changing the value of the parameter $\gamma \in (0,1)$ in (4.5), this bound provides the uniform integrability of u^R_ε with respect to $\big\{T^{-1}\int_0^T dt\, \mathbb{P}^\varepsilon_0(X_t \in \cdot)\big\}_{T>0}$. Therefore, by ergodicity,

$$\int d\pi^\varepsilon(x)\, u^R_\varepsilon(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, \mathbb{E}^\varepsilon_0\big(u^R_\varepsilon(X_t)\big) \leq 1$$

which, by Chebyshev inequality, yields the exponential tightness of $\{\pi^\varepsilon\}$.

We now observe that the Freidlin-Wentzell asymptotics implies that for each $T > 0$ the family $\{\mathbb{P}^\varepsilon_x\}_{\varepsilon>0}$ is exponentially tight on $C([0,T];\mathbb{R}^n)$ uniformly for x in compacts. Since $\mathbb{P}^\varepsilon_{\pi^\varepsilon} = \int d\pi^\varepsilon(x)\, \mathbb{P}^\varepsilon_x$, the statement follows. □

Proof of Theorem 4.2, item (i). Fix $T_1 < T_2$. By the basic entropy inequality, see e.g. [16, Prop. A1.8.2], and (4.4), for any $P \in \mathcal{P}_\theta$ and any event B on $D([T_1, T_2];\mathbb{R}^n)$

$$P(B) \leq \frac{\log 2 + (T_2 - T_1)\, \mathcal{H}^\varepsilon(P)}{\log\Big(1 + \big[\mathbb{P}^\varepsilon_{\pi^\varepsilon}(B)\big]^{-1}\Big)}.$$

The statement now follows from Lemma 4.3. □

As just proven, sequences $\{P_\varepsilon\}$ with equi-bounded rate function admit cluster points. We next show they enjoy some regularity.

Lemma 4.4. *There is a constant $C > 0$ depending on V, c, a such that the following holds. If $\{P_\varepsilon\} \subset \mathcal{P}_\theta$ is a sequence converging to P then for any $T_1 < T_2$*

$$\int dP(X)\bigg[\big|\nabla V(X_0)\big|^2 + \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} dt\, |\dot X_t|^2\bigg] \leq C\Big[1 + \varliminf_{\varepsilon\to 0}\, \varepsilon\, \mathcal{H}^\varepsilon(P_\varepsilon)\Big].$$

Proof. In order to obtain the estimate on $\int dP(X)\, |\nabla V(X_0)|^2$, we first prove the following bound. There are constants $\gamma, C > 0$ such that for any $T > 0$

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbb{E}^\varepsilon_{\pi^\varepsilon}\Big(\exp\Big\{\frac{\gamma}{\varepsilon}\int_0^T dt\, \nabla V(X_t)\cdot a(X_t)\nabla V(X_t)\Big\}\Big) \leq C(1+T). \tag{4.7}$$

For $\lambda \in (0,1)$ to be chosen later, let M^λ be the $\mathbb{P}^\varepsilon_x$ martingale given by

$$M^\lambda_t := \frac{\lambda}{\varepsilon}\int_0^t \nabla V(X_s)\cdot\big(dX_s - b(X_s)ds\big)$$

where we understand the Itô integral. Its quadratic variation is

$$\langle M^\lambda\rangle_t := \frac{2\lambda^2}{\varepsilon}\int_0^t ds\,\nabla V(X_s)\cdot a(X_s)\nabla V(X_s).$$

Setting $\Phi_T^\lambda := M_T^\lambda - (1/2)\langle M^\lambda\rangle_T$ and recalling that $b = -a\nabla V + c$, from Itô's formula we get

$$\Phi_T^\lambda = \frac{\lambda}{\varepsilon}\Big\{V(X_T) - V(X_0) + \int_0^T dt\,\Big[(1-\lambda)\nabla V(X_t)\cdot a(X_t)\nabla V(X_t) \\ - \varepsilon\,\mathrm{Tr}\,\big(a(X_t)D^2V(X_t)\big) - \nabla V(X_t)\cdot c(X_t)\Big]\Big\}$$

Assumption 2.1 implies that for each $\sigma\in(0,1-\lambda)$ there is a constant C_σ such that for any ε small enough

$$\Phi_T^\lambda \ge \frac{\lambda}{\varepsilon}\Big\{ - V(X_0) - C_\sigma T + (1-\lambda-\sigma)\int_0^T dt\,\nabla V(X_t)\cdot a(X_t)\nabla V(X_t)\Big\}.$$

Hence, setting $\gamma := \lambda\,(1-\lambda-\sigma)/2$,

$$\frac{\gamma}{\varepsilon}\int_0^T dt\,\nabla V(X_t)\cdot a(X_t)\nabla V(X_t) \le \frac{1}{2}\Phi_T^\lambda + \frac{\lambda}{2\varepsilon}\big[C_\sigma T + V(X_0)\big]$$

so that, by Cauchy-Schwarz,

$$\Big[\mathbb{E}^\varepsilon_{\pi^\varepsilon}\Big(e^{\frac{\gamma}{\varepsilon}\int_0^T dt\,\nabla V(X_t)\cdot a(X_t)\nabla V(X_t)}\Big)\Big]^2 \le e^{\frac{\lambda C_\sigma T}{\varepsilon}}\,\mathbb{E}^\varepsilon_{\pi^\varepsilon}\big(e^{\Phi_T^\lambda}\big)\int d\pi^\varepsilon\,e^{\frac{\lambda}{\varepsilon}V}.$$

We deduce the bound (4.7) by observing that $\mathbb{E}^\varepsilon_{\pi^\varepsilon}\big(e^{\Phi_T^\lambda}\big) = 1$ and, as follows from the proof of Lemma 4.3, that there exists $\lambda\in(0,1)$ for which

$$\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\int d\pi^\varepsilon\,e^{\frac{\lambda}{\varepsilon}V} < +\infty.$$

By the variational characterization of the relative entropy, for any $P_\varepsilon\in\mathcal{P}_\theta$

$$\int dP_\varepsilon(X)\int_0^T dt\,\nabla V(X_t)\cdot a(X_t)\nabla V(X_t) \\ \le \frac{\varepsilon}{\gamma}\log\mathbb{E}^\varepsilon_{\pi^\varepsilon}\Big(e^{\frac{\gamma}{\varepsilon}\int_0^T dt\,\nabla V(X_t)\cdot a(X_t)\nabla V(X_t)}\Big) + \frac{\varepsilon}{\gamma}\mathcal{H}_{[0,T]}\big(P_\varepsilon\big|\mathbb{P}^\varepsilon_{\pi^\varepsilon}\big).$$

If $P_\varepsilon\to P$, by the translation invariance of P, Fatou's lemma, the previous bound, (4.4) and (4.7),

$$\int dP(X)\,\nabla V(X_0)\cdot a(X_0)\nabla V(X_0) = \int dP(X)\,\frac{1}{T}\int_0^T dt\,\nabla V(X_t)\cdot a(X_t)\nabla V(X_t) \\ \le \varliminf_{\varepsilon\to 0}\int dP_\varepsilon(X)\,\frac{1}{T}\int_0^T dt\,\nabla V(X_t)\cdot a(X_t)\nabla V(X_t) \\ \le \frac{C}{\gamma}\big(1+\frac{1}{T}\big) + \frac{1}{\gamma}\varliminf_{\varepsilon\to 0}\varepsilon\,\mathcal{H}^\varepsilon(P_\varepsilon)\,.$$

As the left-hand side does not depend on ε, we may choose at the beginning a sequence ε_k which achieves the lim inf on the right-hand side. Since a is uniformly elliptic, the first assertion of the Lemma is proved.

In order to obtain the estimate on the derivative, we next prove the following bound. There are constants $\gamma_1, \gamma_2, C > 0$ such that for any $T > 0$ and any $v \in C^1([0,T];\mathbb{R}^n)$ with support in $(0,T)$

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbb{E}^\varepsilon_{\pi^\varepsilon}\left(\exp\Big\{ \frac{\gamma_1}{\varepsilon} \int_0^T dt \big[\dot v_t \cdot X_t - \gamma_2 |v_t|^2 \big] \Big\} \right) \le C(1+T). \tag{4.8}$$

For $\lambda > 0$ to be chosen later, let M^λ be the $\mathbb{P}^\varepsilon_x$ martingale given by

$$M^\lambda_t := -\frac{2\lambda}{\varepsilon} \int_0^t v_s \cdot \big(dX_s - b(X_s) ds \big)$$

whose quadratic variation is

$$\langle M^\lambda \rangle_t := \frac{8\lambda^2}{\varepsilon} \int_0^t ds\, v_s \cdot a(X_s) v_s.$$

Set $\Phi^\lambda_T := M^\lambda_T - (1/2)\langle M^\lambda \rangle_T$ and recall $v_0 = v_T = 0$. Integrating by parts and using Assumption 2.1 we deduce there are constants $\gamma_2, C > 0$ such that

$$\frac{\lambda}{\varepsilon} \int_0^T dt \big[\dot v_t \cdot X_t - \gamma_2 |v_t|^2 \big] \le \frac{1}{2} \Phi^\lambda_T + \frac{C\lambda}{2\varepsilon} \Big\{ T + \int_0^T dt\, \nabla V(X_t) \cdot a(X_t) \nabla V(X_t) \Big\}.$$

By choosing λ small enough and using $\mathbb{E}^\varepsilon_{\pi^\varepsilon}\big(e^{\Phi^\lambda_T}\big) = 1$ together with (4.7) we thus achieve the proof of (4.8) by Cauchy-Schwarz.

Pick a family $\{v^k\}$ of paths in $C^1((0,T);\mathbb{R}^n)$ with compact support and dense in $L^2((0,T);\mathbb{R}^n)$. Assume that $v^1 = 0$. In view of (4.8), the variational characterization of the relative entropy, and a classical argument which allows to bound a maximum over a finite set in exponential estimates, there exists a constant $C > 0$ such that for any $N \in \mathbb{N}$,

$$\varlimsup_{\varepsilon\to 0} \int dP_\varepsilon(X) \max_{k\in\{1,\dots,N\}} \int_0^T dt \big[\dot v^k_t \cdot X_t - \gamma_2 \big| v^k_t \big|^2 \big] \le C(1+T) \Big[1 + \varlimsup_{\varepsilon\to 0} \varepsilon \mathcal{H}^\varepsilon(P_\varepsilon) \Big].$$

Since $P_\varepsilon \to P$ and $v^1 = 0$, from Fatou's lemma we deduce

$$\int dP(X) \max_{k\in\{1,\dots,N\}} \int_0^T dt \big[\dot v^k_t \cdot X_t - \gamma_2 \big| v^k_t \big|^2 \big] \le C(1+T) \Big[1 + \varlimsup_{\varepsilon\to 0} \varepsilon \mathcal{H}^\varepsilon(P_\varepsilon) \Big]$$

whence, by monotone convergence,

$$\int dP(X) \sup_{k\in\mathbb{N}} \int_0^T dt \big[\dot v^k_t \cdot X_t - \gamma_2 \big| v^k_t \big|^2 \big] \le C_0(1+T) \Big[1 + \varlimsup_{\varepsilon\to 0} \varepsilon \mathcal{H}^\varepsilon(P_\varepsilon) \Big].$$

Since the family $\{v^k\}$ is dense in $L^2((0,T);\mathbb{R}^n)$ this estimate implies that P-a.s. X belongs to $H_1([0,T])$ and, by the translation invariance of P, the second part of the bound in the statement. □

Proof of Theorem 4.2, item (ii). For $\delta > 0$ let $\imath_\delta$ be a smooth probability density on $\mathbb{R}$ with support contained in $(0,\delta)$. For $X \in D(\mathbb{R};\mathbb{R}^n)$ let $\imath_\delta * X \in C^\infty(\mathbb{R};\mathbb{R}^n)$ be defined by

$$(\imath_\delta * X)_t := \int ds\, \imath_\delta(t-s) X_s,$$

where, by the support property of $\imath_\delta$, we can restrict the integral to $(t-\delta, t)$. In particular, $\frac{d}{dt}\imath_\delta * X = \imath'_\delta * X$. Given $w \in C\big(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n\big)$ bounded, let W_δ be the $\mathbb{R}^n$-valued function on $\mathbb{R} \times D(\mathbb{R};\mathbb{R}^n)$ defined by

$$W_\delta(t, X) := \chi_\delta(t)\, w\big((\imath_\delta * X)_t, (\imath'_\delta * X)_t \big),$$

where $\chi_\delta\colon \mathbb{R} \to [0,1]$ is a smooth function satisfying $\chi_\delta(t) = 0$ for $t \le \delta$ and $\chi_\delta(t) = 1$ for $t \ge 2\delta$. Note that, by construction, $W_\delta(t,\cdot)$ is continuous, bounded, and measurable with respect to the σ-algebra generated by $\{X_s,\, s \in [0,t]\}$.

Consider now the $\mathbb{P}^\varepsilon_x$–martingale $M^{\delta,\varepsilon}$ defined by

$$M^{\delta,\varepsilon}_t := \frac{1}{\varepsilon}\int_0^t W_\delta(s,X)\cdot\big(dX_s - b(X_s)ds\big)$$

whose quadratic variation is

$$\langle M^{\delta,\varepsilon}\rangle_t = \frac{2}{\varepsilon}\int_0^t ds\, W_\delta(s,X)\cdot a(X_s) W_\delta(s,X).$$

Let finally $\Phi_{\delta,\varepsilon}\colon D(\mathbb{R};\mathbb{R}^n) \to \mathbb{R}$ be the $\sigma\{X_s,\, s\in[0,1]\}$ measurable function defined by

$$\Phi_{\delta,\varepsilon} := M^{\delta,\varepsilon}_1 - \frac{1}{2}\langle M^{\delta,\varepsilon}\rangle_1$$

and observe that $\mathbb{E}^\varepsilon_x\big(e^{\Phi_{\delta,\varepsilon}}\big) = 1$, $x\in\mathbb{R}^n$.

Even if $\Phi_{\delta,\varepsilon}$ is neither continuous nor bounded, by a truncation procedure whose details are omitted, see e.g. [26, Lemma 6.2] for a similar argument, we can take $\Phi = \Phi_{\delta,\varepsilon}$ in the variational representation (4.1). If $\{P_\varepsilon\} \subset \mathcal{P}_\theta$ is a sequence converging to P, by (4.2) and the regularity of P in Lemma 4.4 we deduce that

$$\varliminf_{\varepsilon\to 0}\, \varepsilon\, \mathcal{H}^\varepsilon(P_\varepsilon) \ge \int dP(X)\int_0^1 dt\Big[W_\delta(t,X)\cdot\big(\dot X_t - b(X_t)\big) - W_\delta(t,X)\cdot a(X_t) W_\delta(t,X)\Big].$$

In view of Lemma 4.4 and dominated convergence, we can take the limit as $\delta\to 0$ inside the integrals on the right hand side above. We thus infer that for any bounded $w \in C(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$

$$\varliminf_{\varepsilon\to 0}\, \varepsilon\mathcal{H}^\varepsilon(P_\varepsilon) \ge \int dP(X)\int_0^1 dt\Big[w(X_t,\dot X_t)\cdot\big(\dot X_t - b(X_t)\big) - w(X_t,\dot X_t)\cdot a(X_t) w(X_t,\dot X_t)\Big].$$

Recalling (2.4) and (2.5) we conclude, using again Lemma 4.4 and dominated convergence, by considering a suitable sequence $\{w_n\}$ with w_n bounded for each n and converging pointwise to w^* with $w^*(x,y) = (1/2)\, a(x)^{-1}[y - b(x)]$. □

In view of density result proven in Lemma 3.3, in order to construct the recovery sequence in item (iii) of Theorem 3.2 it suffices to consider the case in which P is smooth holonomic, i.e. $P = S^{-1}\int_0^S ds\, \delta_{\theta_s Y}$ for some $S>0$ and some S-periodic path $Y\in C^1(\mathbb{R},\mathbb{R}^n)$. To construct the sequence $\{P_\varepsilon\}$ for such P, pick first $U\colon \mathbb{R}^n\to\mathbb{R}$ such that: $U\in C^2(\mathbb{R}^n)$, the minimum of U is uniquely attained at $x=0$, the Hessian $D^2U(0)$ is strictly positive definite, and $U = V$ outside some compact set $K\subset\subset\mathbb{R}^n$. Consider now the non-autonomous stochastic differential equation

$$\begin{cases} d\eta^\varepsilon_t = \tilde b_\varepsilon(t,\eta^\varepsilon_t)dt + \sqrt{2\varepsilon}\,\sigma(\eta^\varepsilon_t - Y_t)dw_t \\ \eta^\varepsilon_0 = x \end{cases} \tag{4.9}$$

where

$$\tilde b_\varepsilon(t,x) := -a(x-Y_t)\nabla U\big(x-Y_t\big) + \varepsilon\,\nabla\cdot a(x-Y_t) + \dot Y_t, \qquad (t,x)\in[0,+\infty)\times\mathbb{R}^n \tag{4.10}$$

in which $\nabla\cdot a$ is the vector field given by the divergence of a, i.e. $(\nabla\cdot a)_i = \sum_j \partial_j a_{j,i}$. Note that $\tilde b_\varepsilon$ is S-periodic in the first variable. Denote the law of η^ε by $\mathbb{Q}^\varepsilon_x$ and

let μ^ε be the probability on $\mathbb{R}^n$ whose density is proportional to $\exp\{-U/\varepsilon\}$. Set finally $\nu^\varepsilon := \mu^\varepsilon(Y_0 + \cdot)$ and $\mathbb{Q}^\varepsilon_{\nu^\varepsilon} := \int d\nu^\varepsilon(x)\, \mathbb{Q}^\varepsilon_x$.

Lemma 4.5. *The probability $\mathbb{Q}^\varepsilon_{\nu^\varepsilon}$ is invariant with respect to θ_S. Furthermore $\mathbb{Q}^\varepsilon_{\nu^\varepsilon} \to \delta_Y$ as $\varepsilon \to 0$ and for each $\varepsilon \in (0, \varepsilon_0)$ there exist a constant C_ε such that for any $n \in \mathbb{N}$ and $s \in [0, S]$*

$$\frac{\varepsilon}{n}\, \mathcal{H}^\varepsilon_{[0,nS]}\big(\mathbb{Q}^\varepsilon_{\nu^\varepsilon} \circ \theta_s^{-1}\big) \le \frac{C_\varepsilon}{n} + \frac{1}{4}\int d\mathbb{Q}^\varepsilon_{\nu^\varepsilon}(X) \int_0^S dt\, \big[\tilde b_\varepsilon(t, X_t) - b(X_t)\big] \cdot a^{-1}(X_t)\big[\tilde b_\varepsilon(t, X_t) - b(X_t)\big].$$

Proof. By direct computation $\mathbb{Q}^\varepsilon_{\nu^\varepsilon}$ is the law of $Y + \zeta^\varepsilon$ where ζ^ε is the stationary process associated to the autonomous stochastic differential equation

$$d\zeta^\varepsilon_t = \big[- a(\zeta^\varepsilon_t)\nabla U\big(\zeta^\varepsilon_t\big) + \varepsilon\, \nabla \cdot a(\zeta^\varepsilon_t)\big]dt + \sqrt{2\varepsilon}\, \sigma(\zeta^\varepsilon_t) dw_t.$$

Observe indeed that ζ^ε is reversible with respect to μ^ε. Since Y is S-periodic and the law of ζ^ε is translation invariant we deduce that $\mathbb{Q}^\varepsilon_{\nu^\varepsilon}$ is invariant with respect to θ_S. By the properties of U, we readily conclude that ζ^ε converges to 0 in probability and therefore that $\mathbb{Q}^\varepsilon_{\nu^\varepsilon} \to \delta_Y$.

For notation simplicity, we prove the entropy bound only when $s = 0$. Let M^ε be the $\mathbb{P}^\varepsilon_x$ martingale given by

$$M^\varepsilon_t := \frac{1}{2\varepsilon}\int_0^t a^{-1}(X_s)\big[\tilde b_\varepsilon(s, X_s) - b(X_s)\big] \cdot \big(dX_s - b(X_s)ds\big)$$

whose quadratic variation is

$$\langle M^\varepsilon\rangle_t := \frac{1}{2\varepsilon}\int_0^t ds\, \big[\tilde b_\varepsilon(s, X_s) - b(X_s)\big] \cdot a^{-1}(X_s)\big[\tilde b_\varepsilon(s, X_s) - b(X_s)\big].$$

By Girsanov formula, for each $T > 0$

$$\frac{d\big(\mathbb{Q}^\varepsilon_x\big)_{[0,T]}}{d\big(\mathbb{P}^\varepsilon_x\big)_{[0,T]}} = \exp\Big\{M^\varepsilon_T - \frac{1}{2}\langle M^\varepsilon\rangle_T\Big\}.$$

Using [24, Thm. VIII.1.7] we deduce

$$\mathrm{Ent}\Big(\big(\mathbb{Q}^\varepsilon_x\big)_{[0,T]}\big|\big(\mathbb{P}^\varepsilon_x\big)_{[0,T]}\Big) = \int d\mathbb{Q}^\varepsilon_x\Big[M^\varepsilon_T - \frac{1}{2}\langle M^\varepsilon\rangle_T\Big] = \frac{1}{2}\int d\mathbb{Q}^\varepsilon_x\, \langle M^\varepsilon\rangle_T$$

which yields

$$\mathcal{H}^\varepsilon_{[0,T]}\big(\mathbb{Q}^\varepsilon_{\nu^\varepsilon}\big) = \mathrm{Ent}(\nu^\varepsilon|\pi^\varepsilon) + \frac{1}{2}\int d\mathbb{Q}^\varepsilon_{\nu^\varepsilon}\, \langle M^\varepsilon\rangle_T.$$

In view of the θ_S invariance of $\mathbb{Q}^\varepsilon_{\nu^\varepsilon}$, setting $C_\varepsilon := \mathrm{Ent}(\nu^\varepsilon|\pi^\varepsilon)$, the stated bound follows once we show that C_ε is finite. To this end, we first obtain a lower bound on the tail of π^ε. Denote by ρ^ε the density of π^ε with respect to the Lebesgue measure, $d\pi^\varepsilon = \rho^\varepsilon\, dx$. By Assumption 2.1 and standard results, ρ^ε is smooth, strictly positive, and solves the stationary Fokker-Planck equation

$$\varepsilon \sum_{i,j=1}^n \partial_i\partial_j\big(a_{i,j}\rho^\varepsilon\big) - \sum_{i=1}^n \partial_i\big(b_i\rho^\varepsilon\big) = 0\,.$$

Set $v^\varepsilon := \rho^\varepsilon \exp\{\gamma V/\varepsilon\}$ for some $\gamma > 0$ to be chosen later; by direct computation it solves

$$A_\varepsilon v^\varepsilon + h v^\varepsilon = 0$$

where A_ε is the elliptic operator defined by

$$A_\varepsilon v := \varepsilon \,\mathrm{Tr}(aD^2 v) - \big(b + 2\gamma a\nabla V - 2\varepsilon \nabla\cdot a\big)\cdot\nabla v$$

and

$$h := \frac{\gamma}{\varepsilon}\big[b\cdot\nabla V + \gamma \nabla V\cdot a\nabla V\big] - \gamma\,\mathrm{Tr}(aD^2V) - 2\gamma(\nabla\cdot a)\cdot\nabla V - \nabla\cdot b + \varepsilon\partial_i\partial_j a_{i,j}.$$

As follows from Assumption 2.1, for each $\varepsilon\in(0,\varepsilon_0)$ there exist $\gamma, R>0$ such that $h(x)\ge 0$ for all $x\in\mathbb{R}^n$ such that $|x|\ge R$. Let now $m_\varepsilon := \inf\{v^\varepsilon(x),\, |x|=R\}>0$ and set $u^\varepsilon = m_\varepsilon - v^\varepsilon$. Then $u^\varepsilon(x)\le 0$ for $|x|=R$ and, by the positivity of v^ε, we have $u^\varepsilon(x)\le m_\varepsilon$ for any $x\in\mathbb{R}$. Finally, by the choices of γ and R, for $|x|>R$ the function u^ε solves

$$A_\varepsilon u^\varepsilon = A_\varepsilon(m_\varepsilon - v^\varepsilon) = -A_\varepsilon v^\varepsilon = h v^\varepsilon \ge 0.$$

From the Phragmèn-Lindelhöf maximum principle, see [23, Thm. 2.19], we then deduce $u^\varepsilon(x)\le 0$, for all $x\in\mathbb{R}^n$ such that $|x|>R$. Hence $\rho^\varepsilon(x)\ge m_\varepsilon \exp\{-\gamma V(x)/\varepsilon\}$ for $|x|\ge R$. As $\nu^\varepsilon(dx) = Z_\varepsilon^{-1}\exp\{-U(x-Y_0)/\varepsilon\}dx$ with Z_ε the appropriate normalization, we get

$$\begin{aligned}
\mathrm{Ent}(\nu^\varepsilon|\pi^\varepsilon) &= \int d\nu^\varepsilon(x)\,\log\frac{e^{-U(x-Y_0)/\varepsilon}}{Z_\varepsilon\rho^\varepsilon(x)}\\
&\le \int_{|x|\le R} d\nu^\varepsilon(x)\,\log\frac{e^{-U(x-Y_0)/\varepsilon}}{Z_\varepsilon\rho^\varepsilon(x)}\\
&\quad + \int_{|x|>R} d\nu^\varepsilon(x)\,\Big[\log\frac{1}{Z_\varepsilon m_\varepsilon} - \frac{1}{\varepsilon}U(x-Y_0) + \frac{\gamma}{\varepsilon}V(x)\Big]
\end{aligned}$$

which is bounded as V has super-linear growth as $|x|\to\infty$ and $U=V$ outside a compact. □

Proof of Theorem 4.2, item (iii). By Lemma 3.3 it suffices to consider the case in which P is smooth holonomic. For P and $\mathbb{Q}^\varepsilon_{\nu^\varepsilon}$ as introduced before Lemma 4.5, set

$$P_\varepsilon := \frac{1}{S}\int_0^S ds\,\mathbb{Q}^\varepsilon_{\nu^\varepsilon}\circ\theta_s^{-1}$$

that is translation invariant by the θ_S invariance of $\mathbb{Q}^\varepsilon_{\nu^\varepsilon}$. By Lemma 4.5, the sequence $\{P_\varepsilon\}$ converges to P. Moreover, using also (4.4) and the convexity of the relative entropy,

$$\varepsilon\,\mathcal{H}^\varepsilon(P_\varepsilon) \le \frac{1}{4S}\int_0^S dt\int d\mathbb{Q}^\varepsilon_{\nu^\varepsilon}(X)\,\big[\tilde b_\varepsilon(t,X_t) - b(X_t)\big]\cdot a^{-1}(X_t)\big[\tilde b_\varepsilon(t,X_t) - b(X_t)\big].$$

Recalling (4.10), since $\mathbb{Q}^\varepsilon_{\nu^\varepsilon}\to\delta_Y$ then $\tilde b_\varepsilon(t,\cdot)$ converges in $\mathbb{Q}^\varepsilon_{\nu^\varepsilon}$-probability to $\dot Y_t$. As the marginal at time t of $\mathbb{Q}^\varepsilon_{\nu^\varepsilon}$ is equal to $\nu^\varepsilon_t := \mu^\varepsilon(Y_t+\cdot)$ and $U=V$ outside some compact, we obtain the needed uniform integrability to infer

$$\varlimsup_{\varepsilon\to 0}\varepsilon\,\mathcal{H}^\varepsilon(P_\varepsilon) \le \frac{1}{4S}\int_0^S dt\,\big[\dot Y_t - b(Y_t)\big]\cdot a^{-1}(Y_t)\big[\dot Y_t - b(Y_t)\big] = \mathcal{I}(P)\,,$$

which concludes the proof. □

5. Large deviations of the Gallavotti-Cohen observable

The Gallavotti-Cohen observable has been originally introduced in the context of chaotic deterministic dynamical systems as the expansion rate of the phase-space volume and it has been shown to satisfy the so-called fluctuation theorem [13]. The definition of this observable for stochastic dynamics has been originally discussed in [18] and in more generality in [21, 22]; we refer to [14] for a review and to [8] for an experimental check of the fluctuation theorem.

In the present context of non-degenerate diffusion processes, introduce the time inversion as the involution $\Theta\colon C(\mathbb{R};\mathbb{R}^n) \to C(\mathbb{R};\mathbb{R}^n)$ given by $(\Theta X)_t := X_{-t}$. Recalling that $\mathbb{P}^\varepsilon_{\pi^\varepsilon}$ denotes the stationary process associated to (1.1), the Gallavotti-Cohen observable can then be seen as an empirical estimator for the relative entropy per unit of time of $\mathbb{P}^\varepsilon_{\pi^\varepsilon}$ with respect to $\mathbb{P}^\varepsilon_{\pi^\varepsilon}\circ\Theta^{-1}$, namely

$$\widehat{W}^\varepsilon_{[0,T]} := \frac{\varepsilon}{T}\log\frac{d\big(\mathbb{P}^\varepsilon_{\pi^\varepsilon}\big)_{[0,T]}}{d\big(\mathbb{P}^\varepsilon_{\pi^\varepsilon}\circ\Theta^{-1}\big)_{[0,T]}}$$

where the subscript $[0,T]$ denotes the restriction of the probability to that time interval. The factor ε has been inserted for notation convenience when discussing the small noise limit $\varepsilon\to 0$. Note indeed that $\mathbb{E}^\varepsilon_{\pi^\varepsilon}\big(\widehat{W}^\varepsilon_{[0,T]}\big) \geq 0$ and it is in fact proportional to the relative entropy per unit of time of $\mathbb{P}^\varepsilon_{\pi^\varepsilon}$ with respect to $\mathbb{P}^\varepsilon_{\pi^\varepsilon}\circ\Theta^{-1}$. The content of the fluctuation theorem is the following. Assume that the family of real random variables $\{\widehat{W}^\varepsilon_{[0,T]}\}_{T>0}$ satisfies a large deviation principle as $T\to\infty$ and denote by $s_\varepsilon\colon\mathbb{R}\to[0,+\infty]$ the rate function. Then the odd part of s_ε is linear, $s_\varepsilon(q)-s_\varepsilon(-q) = -q/\varepsilon$, where the factor ε is due to the choice of the normalization. The physical interpretation of the fluctuation theorem is that the ratio between the probability of the events $\{\widehat{W}^\varepsilon_{[0,T]}\approx q\}$ and $\{\widehat{W}^\varepsilon_{[0,T]}\approx -q\}$ becomes fixed, independently of the model, in the large time limit.

An informal computation based on the Girsanov formula shows that

$$\widehat{W}^\varepsilon_{[0,T]}(X) = \frac{1}{T}\int_0^T a(X_t)^{-1}b(X_t)\circ dX_t - \frac{\varepsilon}{T}\log\frac{\rho^\varepsilon(X_T)}{\rho^\varepsilon(X_0)} \tag{5.1}$$

where $\circ$ denotes the Stratonovich integral and ρ^ε is the density of the invariant measure π^ε. In the case of a compact state space, the standard route to obtain the large deviation principle for the family $\{\widehat{W}^\varepsilon_{[0,T]}\}_{T>0}$ is the following [21]. Neglect the second term on the right hand side of (5.1), which becomes irrelevant in the limit $T\to\infty$, and prove, by using Girsanov and Fenyman-Kac formulae together with the Perron-Frobenious theorem, that the limit

$$\Lambda_\varepsilon(\lambda) := \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}^\varepsilon_{\pi^\varepsilon}\Big(\exp\Big\{\frac{\lambda}{T}\int_0^T a(X_t)^{-1}b(X_t)\circ dX_t\Big\}\Big), \qquad \lambda\in\mathbb{R} \tag{5.2}$$

exists and it can be expressed as the maximal eigenvalue of a perturbed generator. An application of the Gartner-Ellis theorem then yields the large deviation principle while the fluctuation theorem follows from the symmetry $\Lambda_\varepsilon(\lambda) = \Lambda_\varepsilon(-\varepsilon^{-1}-\lambda)$.

As detailed in [15], the route sketched above in general fails in the present case of non-compact space state: it is neither possible to neglect the second term on the right hand side of (5.1) nor to prove the existence of the limit in (5.2) for any $\lambda\in\mathbb{R}$. Following [3, 25, 27] and recalling the decomposition (2.2), we here *define*

the Gallavotti-Cohen observable by

$$W_{[0,T]}(X) := \frac{1}{T}\int_0^T a(X_t)^{-1} c(X_t) \circ dX_t, \tag{5.3}$$

namely as the work done, in the metric defined by the diffusion matrix, by the non-conservative part of the drift. As shown in [3, 25, 27], for each $\varepsilon > 0$ the family of probabilities on $\mathbb{R}$ given by $\{\mathbb{P}^\varepsilon_{\pi^\varepsilon} \circ (W_{[0,T]})^{-1}\}_{T>0}$ satisfies a large deviation principle and the corresponding rate function s_ε satisfies the fluctuation theorem. The present purpose is to obtain a variational representation of this rate function in the small noise limit $\varepsilon \to 0$. This problem has been originally addressed heuristically in [19]. A mathematical analysis has been carried out in [3] when the limit $\varepsilon \to 0$ is taken before the limit $T \to \infty$ and the limiting rate function is then expressed in terms of the Freidlin-Wentzell rate functional. The analysis when the limits are taken in the converse order has been carried out in [25] by using tools from semi-classical analysis. It is a byproduct of the current analysis that the limiting rate function is, surprisingly, actually independent of the limiting procedure.

Before discussing the Gallavotti-Cohen observable, we note that the fluctuation theorem is already present at the level of the empirical process, i.e. for the rate functional $\mathcal{I}$ in (2.5).

Proposition 5.1. *For any $P \in \mathcal{P}_\theta$ such that $\mathcal{I}(P) < +\infty$*

$$\mathcal{I}\big(P \circ \Theta^{-1}\big) - \mathcal{I}\big(P\big) = \int dP(X)\, W_{[0,1]}(X) = \int dP(X) \int_0^1 dt\, a(X_t)^{-1} c(X_t) \cdot \dot{X}_t.$$

Proof. Recalling (3.4), that provides the needed integrability conditions, the proof is simply achieved by using the decomposition (2.2) and expanding the square in (2.4). Note indeed that the boundary term vanishes by translation invariance. □

In the next statement we employ the same convention on $\overline{\lim}_{\varepsilon,T}$ and $\underline{\lim}_{\varepsilon,T}$ as the one used in Theorem 2.2.

Theorem 5.2. *Assume that $|x| \le C\big(1 + \big|\nabla V(x)\big|^2\big)$, $x \in \mathbb{R}^n$, for some constant $C > 0$. Then, as $\varepsilon \to 0$ and $T \to \infty$, the family of probabilities on $\mathbb{R}$ given by $\big\{\mathbb{P}^\varepsilon_x \circ (W_{[0,T]})^{-1},\, T > 0,\, \varepsilon > 0\big\}$ satisfies, uniformly for x in compact sets, a large deviation principle with speed $\varepsilon^{-1}T$ and rate function $s\colon \mathbb{R} \to [0, +\infty]$ given by*

$$s(q) = \inf\Big\{\mathcal{I}(P),\, \int dP(X) \int_0^1 dt\, a(X_t)^{-1} c(X_t) \cdot \dot{X}_t = q\Big\}.$$

Namely, for each compact set $K \subset\subset \mathbb{R}^n$, each closed set $C \subset \mathbb{R}$, and each open set $A \subset \mathbb{R}$

$$\overline{\lim_{T,\varepsilon}} \sup_{x\in K} \frac{\varepsilon}{T} \log \mathbb{P}^\varepsilon_x\big(W_{[0,T]} \in C\big) \le -\inf_{q\in C} s(q)$$

$$\underline{\lim_{T,\varepsilon}} \inf_{x\in K} \frac{\varepsilon}{T} \log \mathbb{P}^\varepsilon_x\big(W_{[0,T]} \in A\big) \ge -\inf_{q\in A} s(q).$$

Moreover, the function s is good, convex, and satisfies the fluctuation theorem $s(-q) - s(q) = q$.

Since, as proven in Lemma 3.3, the family of probabilities $\{\pi^\varepsilon\}_{\varepsilon>0}$ is exponentially tight, the previous statement also holds when $\mathbb{P}^\varepsilon_x$ is replaced by the stationary process $\mathbb{P}^\varepsilon_{\pi^\varepsilon}$.

Proof. It is convenient to rewrite $W_{[0,T]}$ in (5.3) in terms of the Itô integral,

$$W_{[0,T]}(X) = \widetilde{W}_{[0,T]}(X) + \varepsilon\, Z^1_T(X)$$

where

$$\widetilde{W}_{[0,T]}(X) := \frac{1}{T}\int_0^T a(X_t)^{-1} c(X_t)\cdot dX_t$$

and, by Assumption 2.1, $Z^1_T(X)$ is bounded uniformly in T and X and therefore irrelevant for the large deviations. Recalling the definition of the empirical process in (2.3) we next observe that

$$\int dR_T(X)\, \widetilde{W}_{[0,1]}(X) = \widetilde{W}_{[0,T]}(X) + \frac{1}{T}\, Z^2_T(X) \tag{5.4}$$

where Z^2_T takes into account the jump inserted by the T-periodization,

$$Z^2_T(X) = a^{-1}(X_T)\, c(X_T)\cdot [X_0 - X_T].$$

As we assumed $|x| \le C(1+|\nabla V(x)|^2\big)$, the bounds provided by (3.4) and Lemma 4.4 imply that also $T^{-1}Z^2_T(X)$ is irrelevant for the large deviations. Therefore (5.4) expresses the Gallavotti-Cohen observable as a function of the empirical process. However, as $\widetilde{W}_{[0,1]}$ involves the Itô integral, this function is not continuous. By a truncation procedure that it is not detailed, see [26, Lemma 6.2] for a similar argument, we can however construct a continuous, exponentially good approximation of $\widetilde{W}_{[0,1]}$ and deduce the large deviation principle for $\tilde{W}_{[0,T]}$ by contraction principle from Theorem 2.2.

The convexity of the rate function s readily follows from its definition while the fluctuation theorem is a corollary of Proposition 5.1. □

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