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# INFINITARY NOETHERIAN CONSTRUCTIONS II. TRANSFINITE WORDS AND THE REGULAR SUBWORD TOPOLOGY

JEAN GOUBAULT-LARRECQ, SIMON HALFON, AND ALIAUME LOPEZ

ABSTRACT. We show that the spaces of transfinite words, namely ordinal-indexed words, over a Noetherian space, is also Noetherian, under a natural topology which we call the regular subword topology. We characterize its sobrification and its specialization ordering, and we give an upper bound on its sobrification rank and on its stature.

## 1. INTRODUCTION

Given a well-quasi-order  $X$  (wqo, for short), the space of infinite words  $X^\omega$  (of length  $\omega$ ) need not be wqo in the subword preordering. One way of correcting this anomaly is to turn to the stronger notion of better quasi-orderings [11]. Another one is to turn to the weaker notion of Noetherian space. Noetherian spaces are a natural, topological generalization of wqos with many similar properties [5, Section 9.7]. For example, there are Noetherian analogues of Higman's theorem and of Kruskal's theorem. Noetherianity is also preserved by some infinitary constructions such as powerset.

In part I of this work [6] we have shown that, given a Noetherian space  $X$ ,  $X^\omega$  is again Noetherian, with a natural topology, the subword topology. The same works for the set of finite-or-infinite words  $X^{\leq\omega}$ . The purpose of the present paper is to extend this to spaces  $X^{<\alpha}$  of transfinite words, namely words indexed by ordinals strictly smaller than a fixed ordinal bound  $\alpha$ .

The topology we choose is a natural generalization of that of part I. The bulk of the work consists in showing that if  $X$  is Noetherian, then so is  $X^{<\alpha}$ . *Outline.* Section 2 recapitulates a few basic notions, and is also where we state our basic tool for showing that  $X^{<\alpha}$  is Noetherian, as Proposition 2.1. We define the regular subword topology on  $X^{<\alpha}$  in Section 3, based on so-called transfinite products. After a few basic results on transfinite words in Section 4, we show that the  $\alpha$ -products, a specific kind of transfinite products, form an irredundant subbase of closed subsets of  $X^{<\alpha}$ , where  $\alpha$  is a

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special kind of ordinal which we call a bound; this is one of the conditions of Proposition 2.1. We characterize inclusion of transfinite products in Section 6, and we show that every transfinite product has a canonical form in Section 7. This allows us to show that the collection of  $\alpha$ -products is well-founded in Section 8; this is another condition of Proposition 2.1. The final condition requires us to express finite intersections of  $\alpha$ -products as finite unions of  $\alpha$ -products, which we do in Section 9. In the process, we obtain an upper bound on the stature and reduced sobrification rank of  $X^{<\alpha}$ . Finally, we describe the specialization preordering of  $X^{<\alpha}$  in Section 10.

## 2. PRELIMINARIES

**2.1. Topology.** Most of the following can be found in [5].

Every topological space has a *specialization* preordering  $\leq$ , defined by  $x \leq y$  if and only if every open neighborhood of  $x$  contains  $y$ . The closure of  $\{x\}$  is the *principal ideal*  $\downarrow x \stackrel{\text{def}}{=} \{y \in X \mid y \leq x\}$ . We silently consider any topological space as a preordered set under  $\leq$ .

The *Alexandroff topology* of a preordering  $\leq$  is its family of upwards-closed sets. Among the topologies with a given specialization preordering  $\leq$ , it is the finest. The coarsest is the *upper topology*, whose closed sets are intersections of sets of the form  $\downarrow E$ ,  $E$  finite; we write  $\downarrow E$  for  $\bigcup_{x \in E} \downarrow x$ .

A *Noetherian* space is a topological space in which every open set is compact. We do not require compactness to imply separation.

A strict partial ordering  $<$  is *well-founded* if and only if there is no infinite strictly descending chain. By a slight abuse of language, we say that a preordering  $\leq$  is well-founded if and only if its strict part ( $x < y$  if and only if  $x \leq y$  and  $y \not\leq x$ ) is well-founded. A space is Noetherian if and only if its set of closed subsets is well-founded under inclusion.

A closed subset  $C$  is *irreducible* if and only if  $C \neq \emptyset$ , and for all closed sets  $F_1, F_2$  such that  $C \subseteq F_1 \cup F_2$ ,  $C$  is included in  $F_1$  or in  $F_2$ . The closure  $\downarrow x$  of every point  $x$  is irreducible closed. The product  $C_1 \times C_2$  of two irreducible closed subsets is irreducible in the product topology. A space is *sober* if and only if every irreducible closed subset is the closure  $\downarrow x$  of a unique point  $x$ .

An important property is that, in a Noetherian space  $X$ , every closed subset is a *finite* union of irreducible closed subsets.

The *sobrification*  $\mathcal{S}X$  of a topological space  $X$  is its set of irreducible closed subsets, with the sets  $\diamond U \stackrel{\text{def}}{=} \{C \in \mathcal{S}X \mid C \cap U \neq \emptyset\}$ ,  $U$  open in  $X$ , as open sets. The specialization ordering of  $\mathcal{S}X$  is inclusion.  $\mathcal{S}X$  is sober,

and its lattice of open subsets is order-isomorphic to that of  $X$ , through  $U \mapsto \diamond U$ . In particular,  $X$  is Noetherian if and only if  $\mathcal{S}X$  is.

$\mathcal{S}$  defines a functor: for every continuous map  $f: X \rightarrow Y$ ,  $\mathcal{S}f$  maps every  $C \in \mathcal{S}X$  to the closure  $cl(f[C])$  in  $Y$  of  $f[C]$ . (We write  $f[C]$  for the direct image  $\{f(x) \mid x \in C\}$ .) In particular,  $cl(f[C])$  is irreducible closed.

The sober Noetherian spaces can be characterized order-theoretically: they are exactly the sets  $X$  with a well-founded preordering  $\leq$  such that every finite intersection of principal ideals can be expressed as a finite union of principal ideals; the topology of  $X$  is the upper topology of  $\leq$ . Then the closed subsets are exactly the sets  $\downarrow E$  with  $E$  finite.

The following proposition refines this, and will be the core of our constructions. A family of sets  $\mathcal{P}$  is *irredundant* if and only if every element of  $\mathcal{P}$  is irreducible in  $\mathcal{P}$ , namely: no element of  $\mathcal{P}$  is empty, and for all  $P, P_1, P_2 \in \mathcal{P}$ , if  $P \subseteq P_1 \cup P_2$  then  $P \subseteq P_1$  or  $P \subseteq P_2$ .

**Proposition 2.1.** *Let  $\mathcal{P}$  be a family of subsets of a set  $X$ , such that:*

- (1)  $\mathcal{P}$  is well-founded under inclusion;
- (2)  $X$  can be written as a finite union of elements of  $\mathcal{P}$ ;
- (3) for all  $P, Q \in \mathcal{P}$ ,  $P \cap Q$  is a finite union of elements of  $\mathcal{P}$ .

*Then  $X$ , with the coarsest topology that makes every element of  $\mathcal{P}$  closed, is Noetherian. If  $\mathcal{P}$  is irredundant, its irreducible closed subsets are exactly the elements of  $\mathcal{P}$ , and  $\mathcal{S}X$  equals  $\mathcal{P}$  with the upper topology of inclusion.*

*Proof.* By assumption,  $\mathcal{P}$  is sober Noetherian in the upper topology of  $\subseteq$ .

For every  $x \in X$ , since  $\mathcal{P}$  is well-founded (property (1)), there is a minimal element  $P$  of  $\mathcal{P}$  that contains  $x$ . For every  $Q \in \mathcal{P}$  that contains  $x$ , we can write  $P \cap Q$  as  $\bigcup_{i=1}^n P_i$  where each  $P_i$  is in  $\mathcal{P}$ , by (3). Then  $x$  is in one  $P_i$ , and by minimality of  $P$ ,  $P = P_i$ . It follows that  $P = P_i \subseteq P \cap Q$ , so  $P \subseteq Q$ . Hence  $P$  is the smallest element of  $\mathcal{P}$  that contains  $x$ . Let us write that element as  $\eta(x)$ ; for every  $Q \in \mathcal{P}$ ,  $\eta(x) \subseteq Q$  if and only if  $x \in Q$ .

This defines a map  $\eta: X \rightarrow \mathcal{P}$ . For every finite subset  $E \stackrel{\text{def}}{=} \{P_1, \dots, P_n\}$  of  $\mathcal{P}$ ,  $\eta^{-1}(\downarrow E) = \{x \in X \mid \exists i, \eta(x) \subseteq P_i\} = \bigcup_{i=1}^n P_i$  is closed in  $X$ . Hence  $\eta$  is continuous. Taking  $n \stackrel{\text{def}}{=} 1$ , we obtain that every element  $P$  of  $\mathcal{P}$  can be written as  $\eta^{-1}(\downarrow \{P\})$ . Since  $\eta^{-1}$  commutes with all intersections and (finite) unions,  $\eta$  is a *full* map, viz. every closed subset of  $X$  is the inverse image of some closed subset of  $\mathcal{P}$ —this was called an *initial* map in [6]. Lemma 8 of that paper states that any space from which there is a full map to a Noetherian space is itself Noetherian. Therefore  $X$  is Noetherian.

By Lemma 9 of the same paper, every irreducible closed subset  $C$  of  $X$  is of the form  $\eta^{-1}(\mathcal{C})$  for some irreducible closed subset  $\mathcal{C}$  of  $\mathcal{P}$ . Since  $\mathcal{P}$  is sober,  $\mathcal{C} = \downarrow P$  for some unique  $P \in \mathcal{P}$ , hence  $C = P$ .

Conversely, and assuming  $\mathcal{P}$  irredundant, we claim that every element  $P$  of  $\mathcal{P}$  is irreducible in  $X$ . By assumption,  $P$  is non-empty. Since  $\eta$  is full, the closed subsets of  $X$  are exactly the inverse images of closed subsets of  $\mathcal{P}$ , namely the finite unions of elements of  $\mathcal{P}$ . Hence it suffices to show that if  $P$  is included in a finite union of elements of  $\mathcal{P}$ , then it is included in one of them. This follows directly from the fact that  $\mathcal{P}$  is irredundant.  $\square$

**2.2. Ordinals.** An *indecomposable* ordinal is one of the form  $\omega^\beta$ , where  $\beta$  is an ordinal; equivalently, an ordinal that cannot be written as the sum of two strictly smaller ordinals. The other ordinals are *decomposable*. For all ordinals  $\alpha, \beta, \gamma$ , we have: (1) if  $\gamma$  is indecomposable and  $\alpha, \beta < \gamma$ , then  $\alpha + \beta < \gamma$ ; (2) if  $\gamma$  is indecomposable,  $\alpha < \gamma$ , and  $\beta \leq \gamma$ , then  $\alpha + \beta \leq \gamma$ ; (3) every ordinal  $\alpha$  can be written in a unique way as a finite sum of indecomposable ordinals  $\gamma_1 + \dots + \gamma_m$  (its *Cantor normal form*), where  $\alpha \geq \gamma_1 \geq \dots \geq \gamma_m$ ; (4) the ordering on Cantor normal forms is lexicographic:  $\gamma_1 + \dots + \gamma_m < \gamma'_1 + \dots + \gamma'_n$  (where both sides are in Cantor normal form) if and only if for some  $i \geq 1$ ,  $\gamma_1 = \gamma'_1, \dots, \gamma_{i-1} = \gamma'_{i-1}$  and either  $i = m + 1 \leq n$ , or  $i \leq m, n$  and  $\gamma_i < \gamma'_i$ .

### 3. TRANSFINITE WORDS

We call *transfinite word* on a set  $X$  any map from  $\alpha$  to  $X$ , where  $\alpha$  is any ordinal. Such a word  $w$  has *length*  $|w| \stackrel{\text{def}}{=} \alpha$ . Seen as the set of ordinals  $\beta$  strictly less than  $\alpha$ ,  $|w|$  is also the *domain* of  $w$ , and we write  $w(\beta)$  for the letter at position  $\beta$  in  $w$ , for every  $\beta \in |w|$  (equivalently,  $\beta < |w|$ ).

When  $X$  is preordered by  $\leq$ , the *subword preordering*  $\leq$  is defined on transfinite words by  $w \leq_* w'$  if and only if there is a strictly increasing map  $f: |w| \rightarrow |w'|$  such that  $w(\beta) \leq w'(f(\beta))$  for every  $\beta, 0 \leq \beta < |w|$ . We also say that  $f$  *exhibits*  $w$  as a subword of  $w'$ .

We write  $X^{<\alpha}$  for the set of transfinite words  $w$  of length  $|w| < \alpha$ .  $X^{\leq\alpha}$  denotes  $X^{<\alpha+1}$ . For example,  $X^{<\omega}$  is the set  $X^*$  of finite words on  $X$ , and  $X^{<\omega+1} = X^{\leq\omega}$  is the set of finite-or-infinite words studied in Part I [6].

The *concatenation*  $ww'$  of  $w$  and  $w'$  is the transfinite word of length  $|w| + |w'|$  such that  $(ww')(\beta) = w(\beta)$  for every  $\beta < |w|$ , and  $(ww')(|w| + \beta) = w'(\beta)$  for every  $\beta < |w'|$ . We write  $AB$  for the set  $\{ww' \mid w \in A, w' \in B\}$ .

We are interested in the following topology. We cannot work on the (proper) class of all transfinite words over  $X$ , for foundational reasons. Instead we work on *sets*  $Y$  of transfinite words; this leads us to take intersections with  $Y$  here and there. Usually,  $Y$  will be a set of the form  $X^{<\alpha}$ .

**Definition 3.1** (Regular subword topology). The *regular subword topology* on any set of transfinite words  $Y$  on a space  $X$  is the coarsest topology that makes the sets  $(F_1^{<\alpha_1} F_2^{<\alpha_2} \dots F_n^{<\alpha_n}) \cap Y$  closed, where  $n \in \mathbb{N}$ ,  $F_1, F_2, \dots, F_n$  are closed subsets of  $X$ , and  $\alpha_1, \alpha_2, \dots, \alpha_n$  are ordinals.

The following class of ordinals will be ubiquitous.

**Definition 3.2** (Bound). A *bound* is an ordinal of the form  $\omega^\beta$  or  $\omega^\beta + 1$ ,  $\beta \geq 0$ . The *trivial bound* is  $\omega^0$  ( $= 1$ ), all others are *non-trivial*. A *proper bound* is one of the form  $\omega^\beta$  or  $\omega^\beta + 1$  with  $\beta \geq 1$ .

**Definition 3.3** (Preatom, atom, product). Let  $X$  be a topological space. A *preatom* is an expression of the form  $F^{<\gamma}$ , where  $F$  is a closed subset of  $X$  and  $\gamma$  is a bound. An *atom* is a preatom  $F^{<\gamma}$  such that  $\gamma$  is non-trivial, and if  $\gamma = \omega^0 + 1$  then  $F$  is irreducible closed in  $X$ .

A *transfinite product*  $P$  is any set of the form  $A_1 A_2 \dots A_n$ , where  $n \in \mathbb{N}$  and each  $A_i$  is an atom. We write  $\varepsilon$  when  $n = 0$ , namely  $\varepsilon \stackrel{\text{def}}{=} \{\epsilon\}$ .

When  $\beta = 0$ ,  $F^{<\omega^0} = \varepsilon = \{\epsilon\}$ , and  $F^{<\omega^0+1}$  is sometimes written as  $F^?$ : that is the set of words of length at most 1, whose only letter if any is in  $F$ .

**Lemma 3.4.** *For every set  $F$ , for every ordinal  $\alpha$ , one can write  $F^{<\alpha}$  as  $F^{<\gamma_1} F^{<\gamma_2} \dots F^{<\gamma_m}$ , where  $m \in \mathbb{N}$  and each  $\gamma_i$  is a non-trivial bound.*

*Proof.* If  $\alpha = 0$ , we take  $m \stackrel{\text{def}}{=} 0$ . Otherwise, let us write  $\alpha$  in Cantor normal form, as a finite sum  $\gamma_1 + \dots + \gamma_m$  of indecomposable ordinals, where  $m \neq 0$  and  $\alpha \geq \gamma_1 \geq \dots \geq \gamma_m$ . We claim that  $F^{<\alpha}$  is equal to  $F^{\leq \gamma_1} F^{\leq \gamma_2} \dots F^{\leq \gamma_{m-1}} F^{<\gamma_m}$ . The result will follow, since the latter is equal to  $F^{<\gamma_1+1} F^{<\gamma_2+1} \dots F^{<\gamma_{m-1}+1} F^{<\gamma_m}$ : when  $\gamma_m \neq 1$ , all the superscripts are non-trivial bounds; when  $\gamma_m = 1$ , this simplifies to  $F^{<\gamma_1+1} F^{<\gamma_2+1} \dots F^{<\gamma_{m-1}+1}$ , where all superscripts are non trivial bounds, too.

$F^{\leq \gamma_1} \dots F^{\leq \gamma_{m-1}} F^{<\gamma_m}$  is included in  $F^{<\alpha}$ . Conversely, let  $w \in F^{<\alpha}$ , and let us write  $|w|$  in Cantor normal form as  $\gamma'_1 + \dots + \gamma'_n$ . We can write  $w$  as a concatenation  $w_1 \dots w_n$  where  $|w_1| = \gamma'_1, \dots, |w_n| = \gamma'_n$ . Since  $|w| < \alpha$ , there is a number  $i \geq 1$  such that  $\gamma'_1 = \gamma_1, \dots, \gamma'_{i-1} = \gamma_{i-1}$  and either  $i = n+1 \leq m$ , or  $i \leq m, n$  and  $\gamma'_i < \gamma_i$ . In the first case,  $w \in F^{\leq \gamma_1} \dots F^{\leq \gamma_m} \subseteq F^{\leq \gamma_1} \dots F^{\leq \gamma_{m-1}} F^{<\gamma_m}$ . In the second case, since  $\gamma_i > \gamma'_i \geq \gamma'_{i+1} \geq \dots \geq \gamma'_n$

and  $\gamma_i$  is indecomposable,  $\gamma'_i + \dots + \gamma'_n < \gamma_i$ . Then  $w_1 \in F^{\leq \gamma_1}, \dots, w_{i-1} \in F^{\leq \gamma_{i-1}}$ , and  $w_i w_{i+1} \dots w_n$  is in  $F^{< \gamma_i}$ , hence in  $F^{\leq \gamma_i} \dots F^{\leq \gamma_{m-1}} F^{< \gamma_m}$ .  $\square$

**Proposition 3.5.** *The regular subword topology on any set of transfinite words  $Y$  on a Noetherian space  $X$  is the coarsest topology that has the intersections of  $Y$  with transfinite products as closed sets.*

*Proof.* Let us consider a set of the form  $F_1^{< \alpha_1} F_2^{< \alpha_2} \dots F_n^{< \alpha_n}$ , where  $F_1, \dots, F_n$  are closed subsets of  $X$ , and  $\alpha_1, \alpha_2, \dots, \alpha_n$  are ordinals. We claim that we can rewrite it as a finite union of transfinite products.

If some  $\alpha_i$  equals 0, then  $F_1^{< \alpha_1} F_2^{< \alpha_2} \dots F_n^{< \alpha_n}$  is empty. Otherwise, using Lemma 3.4, we may assume that every  $\alpha_i$  is a non-trivial bound. We may also remove the preatoms  $F_i^{< \alpha_i}$  such that  $F_i = \emptyset$ , since in that case  $F_i^{\alpha_i} = \varepsilon$ . Let  $I$  be the subset of those indices  $i$ ,  $1 \leq i \leq n$ , such that  $\alpha_i = \omega^0 + 1$ . For each  $i \in I$ , we can write  $F_i$  as a finite union of irreducible closed subsets  $C_{i1}, \dots, C_{ik_i}$  (and  $k_i \neq 0$  since  $F_i \neq \emptyset$ ), since  $X$  is Noetherian. For every function  $f$  mapping each  $i \in I$  to an element of  $\{1, \dots, k_i\}$ , let  $P_f$  be the transfinite product obtained from  $F_1^{< \alpha_1} F_2^{< \alpha_2} \dots F_n^{< \alpha_n}$  by replacing each preatom  $F_i^{< \alpha_i}$ ,  $i \in I$ , by  $C_{if(i)}$ . Then  $F_1^{< \alpha_1} F_2^{< \alpha_2} \dots F_n^{< \alpha_n}$  is the finite union of the transfinite products  $P_f$ , when  $f$  varies over the finitely many possible functions.  $\square$

#### 4. ELEMENTARY COMBINATORICS ON TRANSFINITE WORDS

**Lemma 4.1.** *Let  $\gamma, \gamma'$  be two bounds, and  $u$  and  $v$  be transfinite words.*

- (1) *If  $\gamma$  is indecomposable and  $|u|, |v| < \gamma$ , then  $|uv| < \gamma$ .*
- (2) *If  $\gamma < \gamma'$  and  $|u| < \gamma, |v| < \gamma'$ , then  $|uv| < \gamma'$ .*
- (3) *If  $\gamma$  is proper and  $|u| < \gamma$ , then for every  $x \in X$ ,  $|xw| < \gamma$ .*

*Proof.* (1) By definition of indecomposability, using  $|uv| = |u| + |v|$ .

(2) follows from (1) if  $\gamma'$  is indecomposable. Otherwise, let us write  $\gamma'$  as  $\omega^\beta + 1$ . Then  $\gamma < \gamma'$  means  $\gamma \leq \omega^\beta$ . Since  $|u| < \gamma, |u| < \omega^\beta$  and since  $|v| < \gamma', |v| \leq \omega^\beta$ . Hence  $|uv| = |u| + |v| \leq \omega^\beta < \gamma'$ , because  $\omega^\beta$  is indecomposable.

(3) Since  $\gamma$  is proper, we have  $|x| = \omega^0 < \gamma$ ; then (3) follows from (2).  $\square$

**Lemma 4.2.** *Every closed set in the regular subword topology is downwards-closed with respect to  $\leq_*$ .*

*Proof.* Given any two downwards-closed subsets  $A$  and  $B$  with respect to  $\leq_*$ , their product  $AB$  is, too: if  $w \leq_* w'$  and  $w' \in AB$ , then we can write  $w'$  as  $u'v'$  where  $u' \in A$  and  $v' \in B$ , and it is then easy to show that  $w = uv$  for some  $u \leq_* u'$  and  $v \leq_* v'$ . It is clear that every atom is downwards-closed, as well as  $\varepsilon$ , so every transfinite product is downwards-closed, hence also every intersection of finite unions of transfinite products.  $\square$

For every indecomposable ordinal  $\gamma$ , there is a so-called *Hessenberg pairing* map  $H: \gamma \times \gamma \rightarrow \gamma$ , which is injective; for all  $\alpha < \beta < \gamma$  and  $\delta < \gamma$ ,  $H(\alpha, \delta) < H(\beta, \delta)$  and  $H(\delta, \alpha) < H(\delta, \beta)$  [10, Exercise 2.23 (ii)], and it is easy to see that  $H(\alpha, 0) \geq \alpha$  for every  $\alpha < \gamma$ .

**Lemma 4.3.** *Let  $F$  be a non-empty subset of a set  $X$ . For every transfinite word  $w$  on  $X$  such that  $|w|$  is an indecomposable ordinal  $\gamma$ , and whose letters are in  $F$ , there is a word  $w'$  of length  $\gamma$ , whose letters are in  $F$  again, such that for every way of writing  $w'$  as a concatenation  $uv$  with  $|u| < \gamma$ ,  $w \leq_* v$ .*

*Proof.* Let us pick  $x \in F$ . We build  $w'$  as the following word of length  $\gamma$ :  $w'(H(\alpha, \beta)) \stackrel{\text{def}}{=} w(\beta)$  for all  $\alpha, \beta < \gamma$ , and  $w'(\delta) \stackrel{\text{def}}{=} x$  for every position  $\delta < \gamma$  that is not in the range of  $H$ . Now let us write  $w'$  as  $uv$  with  $|u| < \gamma$ . Then  $H(|u|, -)$  exhibits  $w$  as a subword of  $v$ , using the fact that  $H(|u|, 0) \geq |u|$ .  $\square$

## 5. CONTINUITY AND IRREDUNDANCY

**Lemma 5.1.** *Let  $X$  be a Noetherian space.*

- (1) *For any set  $Y$  of transfinite words on  $X$  containing  $X^{\leq 1}$ , the function  $i: X \rightarrow Y$  mapping  $x \in X$  to the one-letter word  $x$  is continuous.*
- (2) *For every ordinal  $\beta$ , the concatenation map  $\text{cat}: X^{< \omega^\beta} \times X^{\leq \omega^\beta} \rightarrow X^{\leq \omega^\beta}$  (resp.,  $\text{cat}: X^{< \omega^\beta} \times X^{< \omega^\beta} \rightarrow X^{< \omega^\beta}$ ) is continuous.*

*Proof.* (1) For every preatom  $F^{< \gamma}$  with  $\gamma$  non-trivial,  $i^{-1}(F^{< \gamma}) = F$  is closed. Then, the inverse image of any transfinite product  $A_1 A_2 \cdots A_n \cap Y$  is  $i^{-1}(A_1) \cup \cdots \cup i^{-1}(A_n)$ , which is closed.

(2) First,  $\text{cat}$  is well-defined by Lemma 4.1 (1). As far as continuity is concerned, let  $P \stackrel{\text{def}}{=} A_1 A_2 \cdots A_n$  be any transfinite product. We show that  $\text{cat}^{-1}(P)$  (or rather,  $\text{cat}^{-1}(P \cap X^{\leq \omega^\beta})$ ) is (the intersection of  $X^{< \omega^\beta} \times X^{\leq \omega^\beta}$  with) a finite union  $\bigcup_{i=1}^m P_i \times Q_i$  of products of pairs of transfinite word products  $P_i, Q_i$ , by induction on  $n$ . If  $n = 0$ , then  $\text{cat}^{-1}(P)$  only contains  $(\epsilon, \epsilon)$ , hence is equal to  $\epsilon \times \epsilon$ . Otherwise, let  $P' \stackrel{\text{def}}{=} A_2 \cdots A_n$ , and let us write  $A_1$  as  $F^{< \gamma}$ , where  $\gamma$  is a non-trivial bound, and  $F$  is closed. By induction hypothesis,  $\text{cat}^{-1}(P')$  is a finite union  $\bigcup_{i=1}^m P_i \times Q_i$ , where  $P_i$  and  $Q_i$  are transfinite word products. The pairs of words  $u, v$  whose concatenation are in  $P$  are those such that  $u$  is of the form  $u_1 u_2$  with  $u_1 \in A_1$  and  $(u_2, v) \in \text{cat}^{-1}(P')$  (namely, the elements of  $\bigcup_{i=1}^m A_1 P_i \times Q_i$ , since, as one sees easily, concatenation distributes over union) or such that  $v$  is of the form  $v_1 v_2$  with  $u v_1 \in A_1$  and  $v_2 \in P'$ . In order to conclude, it therefore suffices to show that the set  $A$  of pairs  $(u, v)$  with  $v$  of the form  $v_1 v_2$ ,  $u v_1 \in A_1$  and  $v_2 \in P'$ , is a finite union of products of pairs of transfinite products.



If  $\gamma$  is of the form  $\omega^\beta$ , then  $uv_1 \in A_1$  if and only if  $u \in F^{<\gamma}$  and  $v_1 \in F^{<\gamma}$ . The only if direction is clear, and the if direction is by Lemma 4.1 (1). Hence  $A = F^{<\gamma} \times F^{<\gamma}P'$  in this case.

If  $\gamma$  is of the form  $\omega^\beta + 1$ , then  $uv_1 \in A_1$  if and only if  $u \in F^{<\omega^\beta}$  and  $v_1 \in F^{\leq\omega^\beta}$ , or  $u \in F^{\leq\omega^\beta}$  and  $v_1 = \epsilon$ . In the only if direction, we reason by cases, depending whether  $|u| = \omega^\beta$  or not. In the if direction, the case  $v_1 = \epsilon$  is obvious, while  $u \in F^{<\omega^\beta}$  and  $v_1 \in F^{\leq\omega^\beta}$  imply  $uv_1 \in A_1 = F^{<\gamma}$  by Lemma 4.1 (2). Hence  $A = (F^{<\omega^\beta} \times F^{<\gamma}P') \cup (F^{<\gamma} \times P')$  in this case.  $\square$

On spaces of the form  $X^{<\alpha}$ , the following refinement of the notion of transfinite product will be the family  $\mathcal{P}$  we will use Proposition 2.1 on.

**Definition 5.2** ( $\alpha$ -product). For a topological space  $X$  and a bound  $\alpha$ , the  $\alpha$ -products are the products of the form  $F_1^{<\gamma_1}F_2^{<\gamma_2}\dots F_n^{<\gamma_n}$  where  $n \in \mathbb{N}$ ,

- $\gamma_i \leq \alpha$  and  $F_i$  is non-empty for each  $i$ ,  $1 \leq i \leq n$ ,
- and if  $\alpha$  is decomposable, then  $\gamma_i < \alpha$  for every  $i$ ,  $1 \leq i < n$ ; namely, the only  $\gamma_i$  that is equal to  $\alpha$ , if any, is obtained with  $i = n$ .

**Proposition 5.3.** *For every topological space  $X$  and every bound  $\alpha$ , the  $\alpha$ -products are closed in  $X^{<\alpha}$ , and their complements form a subbase of the regular subword topology.*

*Proof.* Let  $P \stackrel{\text{def}}{=} F_1^{<\gamma_1}F_2^{<\gamma_2}\dots F_n^{<\gamma_n}$  be an  $\alpha$ -product. Let us consider any transfinite word  $w$  in  $P$ , and let us write it as  $u_1u_2\dots u_n$ , where  $u_i \in F_i^{<\gamma_i}$  for each  $i$ . If  $\alpha$  is indecomposable, then  $|w| < \alpha$  because  $|u_i| < \alpha$  for every  $i$ , and using Lemma 4.1 (1). Otherwise, let us write  $\alpha$  as  $\omega^\beta + 1$ . If  $\gamma_i \leq \omega^\beta$  for every  $i$ , by the same argument  $|w| < \omega^\beta$ , hence  $|w| < \alpha$ . By the second item in the definition of  $\alpha$ -products, the only remaining possibility is that  $\gamma_i \leq \omega^\beta$  for every  $i$ ,  $1 \leq i < n$ , that  $n \geq 1$  and that  $\gamma_n = \omega^\beta + 1 = \alpha$ . Then  $|u_n| < \alpha$ ; since  $|u_{n-1}| < \gamma_{n-1} < \alpha$ , we obtain  $|u_{n-1}u_n| < \alpha$  by Lemma 4.1 (2); then  $|u_{n-2}u_{n-1}u_n| < \alpha$ ,  $\dots$ , and eventually  $|w| < \alpha$ .

To show the second part of the proposition, we claim that the intersection of every transfinite product  $P \stackrel{\text{def}}{=} F_1^{\gamma_1}F_2^{<\gamma_2}\dots F_n^{<\gamma_n}$  with  $X^{<\alpha}$  is a finite union of  $\alpha$ -products. If  $n = 0$ ,  $P$  is already an  $\alpha$ -product, so let us assume  $n \neq 0$ . Given any  $w \in P \cap X^{<\alpha}$ , we can write  $w$  as  $u_1u_2\dots u_n$ , where  $u_i \in F_i^{<\gamma_i}$  for each  $i$ . For each  $i$ , not only  $|u_i| < \gamma_i$  but also  $|u_i| \leq |w| < \alpha$ , so we can assume without loss of generality that  $\gamma_i \leq \alpha$  for every  $i$ ,  $1 \leq i \leq n$ .

If  $\alpha$  is indecomposable, then this makes  $P$  an  $\alpha$ -product. Henceforth let us assume that  $\alpha$  is decomposable, say  $\alpha \stackrel{\text{def}}{=} \omega^\beta + 1$ . Then we can rewrite every transfinite product of the form  $F^{<\alpha}Q$  included in  $X^{<\alpha}$  as  $F^{<\omega^\beta}Q \cup F^{<\alpha}$ . Indeed, every word  $w \stackrel{\text{def}}{=} uv$  with  $u \in F^{<\alpha}$  and  $v \in Q$  is either such

that  $|u| < \omega^\beta$  (then  $w \in F^{<\omega^\beta}Q$ ), or  $|u| = \omega^\beta$ . If  $|u| = \omega^\beta$ , then  $v = \epsilon$ , otherwise  $|w| > \omega^\beta$ , which is impossible since  $w \in X^{<\alpha}$ ; so  $w \in F^{<\alpha}$ .

We can therefore rewrite  $P$  as follows. Let  $i_1 < \dots < i_k$  be the list of indices  $i$  between 1 and  $n - 1$  such that  $\gamma_i = \alpha$ . Let  $\gamma'_i \stackrel{\text{def}}{=} \gamma_i$  for every  $i$  different from  $i_1, \dots, i_k$ , and  $\gamma'_i \stackrel{\text{def}}{=} \omega^\beta$  otherwise. (I.e., we replace the exponents equal to  $\alpha = \omega^\beta + 1$  by  $\omega^\beta$ .) Then  $P$  is the union of the  $\alpha$ -products  $F_1^{<\gamma'_1} F_2^{<\gamma'_2} \dots F_{i_j-1}^{<\gamma'_{i_j-1}} F_{i_j}^{<\alpha}$ ,  $1 \leq j \leq k$ , and  $F_1^{<\gamma'_1} F_2^{<\gamma'_2} \dots F_n^{<\gamma'_n}$ .  $\square$

**Proposition 5.4.** *Let  $X$  be a Noetherian space. For every bound  $\alpha$ , every  $\alpha$ -product is irreducible in  $X^{<\alpha}$ . Hence the family of  $\alpha$ -products is irredundant.*

*Proof.* We first show that every atom  $F^{<\gamma}$  is irreducible in  $X^{<\alpha}$ .

If  $\gamma = \omega^0 + 1$  and  $F$  is irreducible, then  $i$  is continuous (Lemma 5.1 (1)), so  $\mathcal{S}i(F) = cl(i[F])$  is irreducible closed in  $X^{<\alpha}$ . Then  $cl(i[F])$  is non-empty, and downwards-closed under  $\leq_*$  by Lemma 4.2, so it contains  $\epsilon$ . Clearly  $i[F] \subseteq cl(i[F])$ , so  $F^{<\gamma} = F^\gamma = \{\epsilon\} \cup i[F] \subseteq cl(i[F])$ . Also,  $i[F] \subseteq F^{<\gamma}$ , and  $F^{<\gamma}$  is closed. Therefore  $F^{<\gamma} = cl(i[F])$ , so  $F^{<\gamma}$  is irreducible.

Let now  $\gamma$  be a proper bound. We show that  $F^{<\gamma}$  is directed in  $\leq_*$ , namely that  $F^{<\gamma} \neq \emptyset$  and that any two elements  $u, v$  of  $F^{<\gamma}$  have an upper bound  $w$  in  $F^{<\gamma}$ . If  $\gamma$  is indecomposable, then  $w \stackrel{\text{def}}{=} uv$  fits, using Lemma 4.1 (1). Otherwise, let  $\gamma \stackrel{\text{def}}{=} \omega^\beta + 1$ . If  $|u| < \omega^\beta$ , then  $uv$  fits again, by Lemma 4.1 (2). If  $|v| < \omega^\beta$ , then we pick  $vu$  instead. Finally, if  $|u| = |v| = \omega^\beta$ , we define  $w$  as the one-for-one interleaving of  $u$  and  $v$ , namely as the word of length  $\omega^\beta$  such that, for every ordinal  $\lambda + n < \omega^\beta$  (where  $\lambda$  is 0 or a limit ordinal, and  $n \in \mathbb{N}$ ),  $w(\lambda + 2n) = u(\lambda + n)$  and  $w(\lambda + 2n + 1) = v(\lambda + n)$ .

Since  $F^{<\gamma}$  is directed,  $F^{<\gamma}$  is irreducible. Indeed, if  $F^{<\gamma} \subseteq \mathcal{C}_1 \cup \mathcal{C}_2$  where  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are closed in  $X^{<\alpha}$ , but  $F^{<\gamma} \not\subseteq \mathcal{C}_1, \mathcal{C}_2$ , then we can pick  $u \in F^{<\gamma} \setminus \mathcal{C}_1$  and  $v \in F^{<\gamma} \setminus \mathcal{C}_2$ ; let  $w$  be an upper bound of  $u$  and  $v$  in  $F^{<\gamma}$ , then  $w$  is neither in  $\mathcal{C}_1$  nor in  $\mathcal{C}_2$ , since those sets are downwards-closed under  $\leq_*$  (Lemma 4.2), and therefore  $w \in F^{<\gamma} \setminus (\mathcal{C}_1 \cup \mathcal{C}_2)$ , which is impossible.

We now prove that every  $\alpha$ -product  $P$  is irreducible in  $X^{<\alpha}$ , by induction on the number  $n$  of atoms in  $P$ . If  $n = 0$ , then  $P = \epsilon$ , and this is clear. If  $n \geq 1$ , we have just seen that  $P = F^{<\gamma}$  is irreducible. If  $n \geq 2$ , we write  $P$  as  $F^{<\gamma}Q$  where  $Q$  is shorter, hence irreducible by induction hypothesis.

If  $\alpha$  is indecomposable, then  $F^{<\gamma}$  is irreducible in  $X^{<\alpha}$ , so  $F^{<\gamma} \times Q$  is irreducible in  $X^{<\alpha} \times X^{<\alpha}$ . By Lemma 5.1 (2),  $cat$  is continuous from the latter to  $X^{<\alpha}$ ; so  $\mathcal{S}cat(F^{<\gamma} \times Q) = cl(cat[F^{<\gamma} \times Q])$  is irreducible in  $X^{<\alpha}$ . Now  $cat[F^{<\gamma} \times Q] = F^{<\gamma}Q$  is closed, hence equal to its own closure.

If  $\alpha$  is decomposable, then let  $\alpha \stackrel{\text{def}}{=} \omega^\beta + 1$ . By the second item in the definition of  $\alpha$ -products, we must have  $\gamma < \alpha$ , hence  $\gamma \leq \omega^\beta$ . Then  $F^{<\gamma}$  is

irreducible in  $X^{<\omega^\beta}$ , while  $Q$  is irreducible in  $X^{\leq\omega^\beta}$ . We use Lemma 5.1 (2):  $\text{cat}$  is continuous from  $X^{<\omega^\beta} \times X^{\leq\omega^\beta}$  to  $X^{\leq\omega^\beta}$ . Then we conclude as above that  $F^{<\gamma}Q = \text{cat}[F^{<\gamma} \times Q] = \text{cl}(\text{cat}[F^{<\gamma} \times Q])$  is irreducible.  $\square$

## 6. INCLUSION OF TRANSFINITE PRODUCTS

We start with necessary conditions for inclusion of transfinite products. We abbreviate “ $\gamma = \gamma'$  and  $\gamma$  is decomposable” as “ $\gamma = \gamma'$  is decomposable”.

**Lemma 6.1.** *Let  $\gamma, \gamma'$  be two non-trivial bounds, and let  $F^{<\gamma}P, F'^{<\gamma'}P'$  be two transfinite products. If  $F^{<\gamma}P \subseteq F'^{<\gamma'}P'$ , then:*

- (1)  $P \subseteq F'^{<\gamma'}P'$ .
- (2) If  $F \not\subseteq F'$  then  $F^{<\gamma}P \subseteq P'$ .
- (3) If  $F$  is non-empty and if  $\gamma > \gamma'$ , then  $F^{<\gamma}P \subseteq P'$ .
- (4) If  $F \subseteq F'$  and  $F \neq \emptyset$ , and if  $\gamma = \gamma'$  is decomposable, then  $P \subseteq P'$ .

*Proof.* Let us assume that  $F^{<\gamma}P \subseteq F'^{<\gamma'}P'$ .

(1)  $P \subseteq F^{<\gamma}P$ , because  $\epsilon \in F^{<\gamma}$ , and  $F^{<\gamma}P \subseteq F'^{<\gamma'}P'$  by assumption.

(2) Let us assume  $F \not\subseteq F'$ . Then there is a letter  $x$  in  $F \setminus F'$ . Let  $w \in F^{<\gamma}P$  be arbitrary, and let us write  $w$  as  $uv$  where  $u \in F^{<\gamma}$  and  $v \in P$ .

If  $\gamma$  is proper, then  $xu$  is in  $F^{<\gamma}$  again, by Lemma 4.1 (3), so  $xw = xuv$  is in  $F^{<\gamma}P$ , hence in  $F'^{<\gamma'}P'$ . Since  $x \notin F'$ ,  $xuv$  is in  $P'$ , so  $w = uv$  is in  $P'$ , by Lemma 4.2 and since  $w \leq_* xuv$ . Since  $w$  is arbitrary,  $F^{<\gamma}P \subseteq P'$ .

It remains to deal with the case  $\gamma = \omega^0 + 1$ ,  $F$  irreducible. Let  $A \stackrel{\text{def}}{=} \{y \in X \mid \forall v \in P, yv \in P'\}$ . For every  $y \in F \setminus F'$ ,  $y$  is in  $A$ : indeed, for every  $v \in P$ ,  $yv$  is in  $F^?P = F^{<\gamma}P$  hence in  $F'^{<\gamma'}P'$ , and since  $y \notin F'$ ,  $yv$  must be in  $P'$ . This means that  $F \subseteq F' \cup A$ .  $A$  is also equal to  $\bigcap_{v \in P} \text{cat}(i(\cdot), v)^{-1}(P')$ , where  $\text{cat}(i(\cdot), v): y \mapsto \text{cat}(i(y), v) = yv$  is continuous by Lemma 5.1. Hence  $A$  is closed. Since  $F$  is irreducible, and since  $F \not\subseteq F'$ ,  $F$  must be included in  $A$ ; equivalently,  $FP \subseteq P'$ . Since  $P'$  is downwards-closed with respect to  $\leq_*$  (Lemma 4.2), and since every element  $u$  of  $P$  is a subword of  $xu \in FP$ ,  $F^?P = P \cup FP$  is also included in  $P'$ —namely,  $F^{<\gamma}P \subseteq P'$ .

(3) Let us assume  $F^{<\gamma}P \subseteq F'^{<\gamma'}P'$ , with  $F \neq \emptyset$  and  $\gamma > \gamma'$ . We pick  $x \in F$ . For every  $\alpha$ , let  $x^\alpha$  be the word of length  $\alpha$  whose sole letter is  $x$ .

Whether  $\gamma'$  is equal to  $\omega^{\beta'}$  or to  $\omega^{\beta'+1}$ , we define  $\gamma'^-$  as  $\omega^{\beta'}$ . For every transfinite word  $v$ , if  $x^{\gamma'^-}v \in F'^{<\gamma'}P'$ , then  $v \in P'$ . Indeed, assuming  $x^{\gamma'^-}v \in F'^{<\gamma'}P'$ , we can write  $x^{\gamma'^-}v$  as  $v_1v_2$  where  $v_1 \in F'^{<\gamma'}$  and  $v_2 \in P'$ . If  $\gamma' = \omega^{\beta'}$ , then  $|x^{\gamma'^-}| = \omega^{\beta'} > |v_1|$ ; if  $\gamma' = \omega^{\beta'+1}$ , then  $|x^{\gamma'^-}| = \omega^{\beta'} \geq |v_1|$ . In any case,  $|v_1| \leq |x^{\gamma'^-}|$ , so  $v_1$  is a prefix of  $x^{\gamma'^-}$ , and  $v_2$  is of the form  $x^\alpha v$  for some ordinal  $\alpha$ . Since  $v_2 \in P'$ , and  $v \leq_* x^\alpha v = v_2$ ,  $v \in P'$  by Lemma 4.2.

Let  $w \in F^{<\gamma}P$  be arbitrary, and let us write  $w$  as  $uv$  where  $u \in F^{<\gamma}$  and  $v \in P$ . With the aim of showing that  $w \in P'$ , we form the transfinite word  $x^{\gamma^-}u$ . Its letters are in  $F$ , and its length is  $\gamma^- + |u|$ .

If  $\gamma^- + |u| < \gamma$ , then  $x^{\gamma^-}u$  is in  $F^{<\gamma}$ , so  $x^{\gamma^-}w = x^{\gamma^-}uv$  is in  $F^{<\gamma}P$ , hence in  $F'^{<\gamma'}P'$ . We have seen that this implies  $w \in P'$ .

Henceforth, we assume that  $\gamma^- + |u| \geq \gamma$ . We recall that  $\gamma^- \leq \gamma' < \gamma$  and that  $|u| < \gamma$ . If  $\gamma$  were indecomposable, then we would have  $\gamma^- + |u| < \gamma$ , contradicting our assumption. Hence  $\gamma = \omega^\beta + 1$  for some ordinal  $\beta$ . Then  $\gamma' \leq \omega^\beta$  and  $|u| \leq \omega^\beta$ . If  $\gamma' < \omega^\beta$ , then  $\gamma^- + |u| \leq \omega^\beta < \gamma$ , which is impossible again. Therefore  $\gamma' = \omega^\beta$ .

To sum up,  $|u| < \gamma = \omega^\beta + 1$ , and  $\gamma' = \omega^\beta$ . Let  $W$  be  $u$  itself if  $|u| = \omega^\beta$ , else  $ux^{\omega^\beta}$ . In each case,  $W$  is a word of length  $\omega^\beta$  whose letters are all in  $F$ , so  $W \in F^{<\gamma}$ . We use Lemma 4.3: let  $W'$  be a word of length  $\omega^\beta$ , whose letters are all in  $F$ , and such that for every way of writing  $W'$  as  $UV$  with  $|U| < \omega^\beta$ ,  $W \leq_* V$ . Then  $W'v$  is in  $F^{<\gamma}P$ , hence in  $F'^{<\gamma'}P'$ . Let us write  $W'v$  as  $Uv'$  where  $|U| < \gamma'$  and  $v' \in P'$ . Since  $|U| < \gamma' = \omega^\beta$ ,  $U$  is a prefix of  $W'$ , and we can therefore write  $W'$  as  $UV$  for some transfinite word  $V$ , and  $v'$  as  $Vv$ . By construction,  $W \leq_* V$ . Therefore  $u \leq_* W \leq_* V$ , and hence  $w = uv \leq_* Vv = v'$ . Since  $v'$  is in  $P'$ , so is  $w$ , by Lemma 4.2.

(4) Since  $F \neq \emptyset$ , let us pick  $x \in F$ . We write  $\gamma = \gamma'$  as  $\omega^\beta + 1$ . For every  $w \in P$ ,  $x^{\omega^\beta}w$  is in  $F^{<\gamma}P$ , hence in  $F'^{<\gamma'}P'$ . Hence we can write  $x^{\omega^\beta}w$  as  $uv$  where  $|u| < \gamma'$  (namely,  $|u| \leq \omega^\beta$ ) and  $v \in P'$ . Since  $|u| \leq \omega^\beta = |x^{\omega^\beta}|$ , we can write  $v$  as  $x^\alpha w$  for some ordinal  $\alpha$ . In particular,  $w \leq_* v$ , and since  $v \in P'$ ,  $w$  is in  $P'$ , by Lemma 4.2.  $\square$

We turn to sufficient conditions. There are three cases, depending on the relative positions and indecomposability statuses of  $\gamma$  and  $\gamma'$ .

**Lemma 6.2.** *Let  $\gamma, \gamma'$  be two non-trivial bounds, and let  $F^{<\gamma}P, F'^{<\gamma'}P'$  be two transfinite products. Assuming that  $\gamma < \gamma'$ , or that  $\gamma = \gamma'$  is indecomposable,  $F^{<\gamma}P \subseteq F'^{<\gamma'}P'$  if and only if:*

- (1)  $F \subseteq F'$  and  $P \subseteq F'^{<\gamma'}P'$ ,
- (2) or  $F \not\subseteq F'$  and  $F^{<\gamma}P \subseteq P'$ .

*Proof.* The ‘only if’ direction is by Lemma 6.1 (1) and (2). We deal with the ‘if’ direction. Note that  $\gamma \leq \gamma'$ ; also, if  $\gamma'$  is decomposable, then  $\gamma < \gamma'$ .

(1) For every  $w \in F^{<\gamma}P$ , let us write  $w$  as  $uv$  with  $u \in F^{<\gamma}$  and  $v \in P$ . Since  $P \subseteq F'^{<\gamma'}P'$ ,  $v$  is in  $F'^{<\gamma'}P'$ . Let us write  $v$  as  $v_1v_2$  with  $v_1 \in F'^{<\gamma'}$  and  $v_2 \in P'$ . Then the letters of  $uv_1$  are all in  $F'$ , and  $|uv_1| < \gamma'$  by Lemma 4.1 (2). It follows that  $uv_1$  is in  $F'^{<\gamma'}$ , so  $w = uv_1v_2$  is in  $F'^{<\gamma'}P'$ .

(2) If  $F^{<\gamma}P \subseteq P'$ , then  $F^{<\gamma}P \subseteq P' \subseteq F'^{<\gamma'}P'$ , where the last inequality is because every  $w \in P'$  can be written as  $\epsilon w \in F'^{<\gamma'}P'$ .  $\square$

**Lemma 6.3.** *Let  $\gamma, \gamma'$  be two non-trivial bounds, and  $F^{<\gamma}P, F'^{<\gamma'}P'$  be two transfinite products. Assuming that  $\gamma > \gamma'$ ,  $F^{<\gamma}P \subseteq F'^{<\gamma'}P'$  if and only if:*

- (1)  $F$  is empty and  $P \subseteq F'^{<\gamma'}P'$ ,
- (2) or  $F$  is non-empty and  $F^{<\gamma}P \subseteq P'$ .

*Proof.* (1) If  $F = \emptyset$  then  $F^{<\gamma}P = P$ , and the equivalence is clear.

(2) If  $F^{<\gamma}P \subseteq P'$ , then  $F^{<\gamma}P \subseteq F'^{<\gamma'}P'$  since  $P'$  is trivially included in  $F'^{<\gamma'}P'$ . The ‘only if’ direction is by Lemma 6.1 (3).  $\square$

**Lemma 6.4.** *Let  $\gamma, \gamma'$  be two non-trivial bounds, and let  $F^{<\gamma}P, F'^{<\gamma'}P'$  be two transfinite products. Assuming that  $\gamma = \gamma'$  is decomposable,  $F^{<\gamma}P \subseteq F'^{<\gamma'}P'$  if and only if:*

- (1)  $F$  is empty and  $P \subseteq F'^{<\gamma'}P'$ ,
- (2) or  $F$  is non-empty,  $F \subseteq F'$ , and  $P \subseteq P'$ ,
- (3) or  $F \not\subseteq F'$  and  $F^{<\gamma}P \subseteq P'$ .

*Proof.* If  $F$  is empty then  $F^{<\gamma}P = P$ , so  $F^{<\gamma}P \subseteq F'^{<\gamma'}P'$  is equivalent to  $P \subseteq F'^{<\gamma'}P'$ . Henceforth, we assume  $F$  non-empty.

‘If’ direction. If  $F \subseteq F'$  and  $P \subseteq P'$ , then  $F^{<\gamma}P \subseteq F'^{<\gamma'}P'$  is obvious (recall that  $\gamma = \gamma'$ ). If  $F^{<\gamma}P \subseteq P'$ , then  $F^{<\gamma}P \subseteq P' \subseteq F'^{<\gamma'}P'$ .

‘Only if’. Let us assume  $F^{<\gamma}P \subseteq F'^{<\gamma'}P'$ . If  $F \subseteq F'$ , then  $P \subseteq P'$  by Lemma 6.1 (4). Otherwise,  $F^{<\gamma}P \subseteq P'$  by Lemma 6.1 (2).  $\square$

## 7. REDUCED PRODUCTS

We can write transfinite products in many equivalent ways. For example,  $\emptyset^{<\gamma} = \varepsilon$  for every non-trivial bound  $\gamma$ . Here are a few other cases.

**Lemma 7.1.** *Let  $F, F'$  be non-empty closed subsets of a topological space  $X$ , and  $\gamma, \gamma'$  be non-trivial bounds. If  $F \subseteq F'$ , then:*

- (1) If  $\gamma < \gamma'$  or if  $\gamma = \gamma'$  is indecomposable, then  $F^{<\gamma}F'^{<\gamma'} = F'^{<\gamma'}$ .
- (2) If  $\gamma'$  is indecomposable, and  $\gamma \leq \gamma'$ , then  $F'^{<\gamma'}F^{<\gamma} = F'^{<\gamma'}$ .

*Proof.* The right-hand sides are always included in the left-hand sides.

(1) By Lemma 6.2, and remembering that  $F \subseteq F'$ ,  $F^{<\gamma}F'^{<\gamma'} \subseteq F'^{<\gamma'}$  if and only if  $F'^{<\gamma'} \subseteq F'^{<\gamma'}$ , which is simply true.

(2) Since  $\gamma'$  is indecomposable, by the same lemma,  $F'^{<\gamma'}F^{<\gamma} \subseteq F'^{<\gamma'}$  if and only if  $F^{<\gamma} \subseteq F'^{<\gamma'}$ , which holds since  $F \subseteq F'$  and  $\gamma \leq \gamma'$ .  $\square$

We will see that this leads to canonical forms for transfinite products. As in [7, Theorem 4.22], and to reduce excessive pedantry related to the difference between syntax and semantics, we write  $\mathbf{A}, \mathbf{B}$  (resp.,  $\mathbf{P}, \mathbf{Q}$ ) to denote atoms, resp. sequences of atoms (syntax), and  $A, B, P, Q$  for their respective semantics. Hence if  $\mathbf{P} = \mathbf{A}_1 \mathbf{A}_2 \cdots \mathbf{A}_n$  (as a sequence), then  $P = A_1 A_2 \cdots A_n$  (as a product). The (syntactic) atoms  $\mathbf{A}$  are pairs  $(F, \gamma)$  of a closed set  $F$  and a non-trivial bound  $\gamma$  (with  $F$  irreducible if  $\gamma = \omega^0 + 1$ ), and then  $A = F^{<\gamma}$ . Note that  $\mathbf{P} = \mathbf{Q}$  implies  $P = Q$ , but the converse may fail.

**Definition 7.2** (Reduced). A sequence of atoms  $\mathbf{P} \stackrel{\text{def}}{=} \mathbf{A}_1, \mathbf{A}_2 \cdots \mathbf{A}_n$  on  $X$ , where  $\mathbf{A}_i \stackrel{\text{def}}{=} (F_i, \gamma_i)$  for each  $i$ , is *reduced* if and only if:

- (1)  $F_i$  is a non-empty closed subset of  $X$  ( $1 \leq i \leq n$ );
- (2) for every  $i$ ,  $1 \leq i < n$ , such that  $\gamma_i < \gamma_{i+1}$ ,  $F_i$  is not included in  $F_{i+1}$ ;
- (3) for every  $i$ ,  $1 \leq i < n$ , such that  $\gamma_i = \gamma_{i+1}$  is indecomposable,  $F_i$  and  $F_{i+1}$  are incomparable;
- (4) and for every  $i$ ,  $1 \leq i < n$ , such that  $\gamma_i$  is indecomposable and  $\gamma_i > \gamma_{i+1}$ ,  $F_i$  does not contain  $F_{i+1}$ .

**Lemma 7.3.** For all non-empty closed subsets  $F$  and  $F'$  of a space  $X$ , for all non-trivial bounds  $\gamma, \gamma'$ ,

- (1)  $F^{<\gamma} \neq \varepsilon$ ;
- (2)  $F^{<\gamma} \subseteq F'^{<\gamma'}$  if and only if  $F \subseteq F'$  and  $\gamma \leq \gamma'$ .
- (3)  $F^{<\gamma} = F'^{<\gamma'}$  if and only if  $F = F'$  and  $\gamma = \gamma'$ .

*Proof.* (1) Since  $F$  is non-empty, let us pick  $x$  in  $F$ . Since  $\gamma$  is non-trivial, the one-letter word  $x$  is in  $F^{<\gamma}$ , whence the conclusion.

(2) If  $\gamma < \gamma'$  or if  $\gamma = \gamma'$  is indecomposable, then by Lemma 6.2,  $F^{<\gamma} \subseteq F'^{<\gamma'}$  if and only if  $F \subseteq F'$  and  $\varepsilon \subseteq F'^{<\gamma'}$  (true), or  $F \not\subseteq F'$  and  $F^{<\gamma} \subseteq \varepsilon$  (false, by (1)). Hence  $F^{<\gamma} \subseteq F'^{<\gamma'}$  if and only if  $F \subseteq F'$  in this case. If  $\gamma > \gamma'$ , by Lemma 6.3  $F^{<\gamma} \subseteq F'^{<\gamma'}$  reduces to  $F^{<\gamma} \subseteq \varepsilon$ , which is false by (1). If  $\gamma = \gamma'$  is decomposable, then by Lemma 6.4,  $F^{<\gamma} \subseteq F'^{<\gamma'}$  if and only if  $F \subseteq F'$  and  $\varepsilon \subseteq \varepsilon$ , or  $F \not\subseteq F'$  and  $F^{<\gamma} \subseteq \varepsilon$ ; equivalently, if  $F \subseteq F'$ .

(3) follows immediately from (2).  $\square$

**Lemma 7.4.** The only reduced sequence of atoms with semantics  $\{\varepsilon\}$  is  $\varepsilon$ .

*Proof.* Let  $\mathbf{P}$  be a reduced sequence of atoms of length  $n \geq 1$ , say  $\mathbf{A}_1 \cdots \mathbf{A}_n$ . By Lemma 7.3 (1), each  $A_i$  contains a non-empty word, and their concatenation is a non-empty word in  $P$ .  $\square$

**Lemma 7.5.** For all atoms  $A \stackrel{\text{def}}{=} F^{<\gamma}$ ,  $B \stackrel{\text{def}}{=} F'^{<\gamma'}$ , with  $F, F' \neq \emptyset$ , and all transfinite products  $P, Q$ , if  $AP \subseteq BQ$  and  $A \not\subseteq B$ , then  $AP \subseteq Q$ .

*Proof.* Since  $A \not\subseteq B$ , we have  $F \not\subseteq F'$  or  $\gamma > \gamma'$ , by Lemma 7.3 (2).

If  $\gamma > \gamma'$ , then Lemma 6.3 and  $F^\gamma P \subseteq F'^{<\gamma'} Q$  imply  $F^{<\gamma} P \subseteq Q$ . Let us therefore assume  $\gamma \leq \gamma'$ , and  $F \not\subseteq F'$ . If  $\gamma < \gamma'$  or  $\gamma = \gamma'$  is indecomposable, then by Lemma 6.2,  $F^{<\gamma} P \subseteq Q$  again. If  $\gamma = \gamma'$  is decomposable, then we reach the same conclusion by using Lemma 6.4.  $\square$

**Lemma 7.6.** *For any reduced sequence of atoms  $AP$ ,  $AP = A$  implies  $P = \varepsilon$ .*

*Proof.* By contradiction, let us assume that  $P = A'Q$ , and let us write  $A$  as  $F^{<\gamma}$  and  $A'$  as  $F'^{<\gamma'}$ . Since  $AP \subseteq A$ , either  $\gamma$  is indecomposable and  $P \subseteq A$  by Lemma 6.2, or  $\gamma$  is decomposable and  $P \subseteq \varepsilon$  by Lemma 6.4. The latter is impossible by Lemma 7.4. Hence  $\gamma$  is indecomposable, and  $P = A'Q \subseteq A$ .

If  $\gamma' \leq \gamma$ , then by Lemma 6.2  $F' \subseteq F$  and  $Q \subseteq A$ , or  $F' \not\subseteq F$  and  $P \subseteq \varepsilon$ . The latter is impossible, as above. The former is impossible, too, because  $AA'Q$  is reduced, using Definition 7.2 (3), (4). If  $\gamma' > \gamma$ , then  $P = A'Q \subseteq \varepsilon$  by Lemma 6.3, which is impossible by Lemma 7.4.  $\square$

**Lemma 7.7.** *For every reduced sequence of atoms  $AP$  and for every atom  $B \stackrel{\text{def}}{=} F'^{<\gamma'}$  with  $F' \neq \emptyset$ , if  $AP = B$  then  $P = \varepsilon$ .*

*Proof.* By induction on the number  $n \geq 1$  of atoms in  $AP$ . If  $n = 1$ , this is vacuous, so let  $n \geq 2$ . If  $B \subseteq A$ , then  $B \subseteq A \subseteq AP = B$ , so  $A = B$ . Then  $AP = A$ , so  $P = \varepsilon$  by Lemma 7.6.

We now assume  $B \not\subseteq A$ , and we will show that this is impossible. By Lemma 7.5 applied to  $B \subseteq AP$ , we have  $B \subseteq P$ . Since  $P \subseteq AP = B$ ,  $B = P$ . By induction hypothesis, if we write  $P$  as  $A'Q$  where  $A'$  is an atom, then  $Q = \varepsilon$ , so  $P = A'$ . Then  $P = B$  entails  $P = A' = B$  by Lemma 7.3 (3). Let us write  $A$  as  $(F, \gamma)$  and  $B$  as  $(F', \gamma')$ . Since  $A \subseteq AP = B$ , Lemma 7.3 (2) entails that  $F \subseteq F'$  and  $\gamma \leq \gamma'$ . But  $AP = (F, \gamma)(F', \gamma')$  is reduced, and this contradicts Definition 7.2 (2) if  $\gamma < \gamma'$ , (3) if  $\gamma = \gamma'$  is indecomposable. Hence  $\gamma = \gamma'$  is decomposable. Then  $AP = F^{<\gamma} F'^{<\gamma} \subseteq B = F'^{<\gamma}$  implies  $F'^{<\gamma} \subseteq \varepsilon$  by Lemma 6.4, and that is impossible by Lemma 7.3 (1).  $\square$

**Proposition 7.8.** *For all reduced sequences of atoms  $P$  and  $Q$ ,  $P = Q$  if and only if  $P = Q$ .*

*Proof.* The ‘if’ direction is trivial. We show that  $P = Q$  implies  $P = Q$  by induction on the sum  $|P| + |Q|$  of the sizes of  $P$  and  $Q$ , where by size we mean number of atoms. Henceforth we assume  $P = Q$ .

If  $|P| = 0$ , then  $P = Q = \{\epsilon\}$ . By Lemma 7.4,  $Q = \varepsilon = P$ . The situation is symmetric if  $|Q| = 0$ . Let us assume  $|P|, |Q| \geq 1$ . If  $|Q| = 1$ ,  $Q$  is an atom  $B$ . By Lemma 7.7, we must have  $|P| = 1$ . Then  $P = Q$  implies  $P = Q$  by

Lemma 7.3 (3). Similarly if  $|P| = 1$ . The interesting case is the remaining one:  $|P|, |Q| \geq 2$ . Let us write  $P$  as  $A_1A_2P'$  and  $Q$  as  $B_1B_2Q'$ .

We first claim that if  $A_1 \not\subseteq B_1$ , then  $P = B_2Q'$ . Indeed, under that assumption, and since  $P = A_1A_2P' \subseteq Q = B_1B_2Q'$ , we obtain  $P \subseteq B_2Q'$  by Lemma 7.5. In turn,  $B_2Q' \subseteq B_1B_2Q' = Q = P$ , so  $P = B_2Q'$ . The induction hypothesis then yields  $P = B_2Q'$ . Similarly, if  $B_1 \not\subseteq A_1$  then  $Q = A_2P'$ .

It follows that we cannot have  $A_1 \not\subseteq B_1$  and  $B_1 \not\subseteq A_1$ . Otherwise,  $Q = B_1B_2Q' = B_1P = B_1A_1A_2P' = B_1A_1Q$ , which is impossible since  $|B_1A_1Q| \neq |Q|$ .

Hence  $A_1$  is included in  $B_1$ , or conversely. Without loss of generality, let us assume  $B_1 \subseteq A_1$ . We claim that, in fact,  $A_1 = B_1$ . We reason by contradiction, and we assume  $A_1 \not\subseteq B_1$ . Then we have seen that  $P = B_2Q'$ . By syntactic matching,  $A_1 = B_2$  (and  $Q' = A_2P'$ ). Let us write  $B_1$  as  $(F, \gamma)$  and  $B_2$  as  $(F', \gamma')$ . Since  $B_1 \subseteq A_1 = B_2$ ,  $F \subseteq F'$  and  $\gamma \leq \gamma'$  by Lemma 7.3 (2). By Definition 7.2 (2) and (3) applied to  $B_1B_2Q'$ , it is impossible that  $\gamma < \gamma'$ , or that  $\gamma = \gamma'$  is indecomposable. Hence  $\gamma = \gamma'$  is decomposable. Now  $Q = B_1B_2Q' = F^{<\gamma}B_2Q'$  is included in  $P = A_1A_2P' = B_2A_2P' = F'^{<\gamma}A_2P'$  (since  $A_1 = B_2$  and  $\gamma = \gamma'$ ), so  $B_2Q' \subseteq A_2P'$  by Lemma 6.4 (2). Since  $A_2P' \subseteq P = B_2Q'$  (because  $P = B_2Q'$ ),  $A_2P' = B_2Q'$ . By induction hypothesis,  $A_2P' = B_2Q'$ , so  $A_2P' = P$ . This is impossible since  $P = A_1A_2P'$ . Having reached a contradiction, we conclude that  $A_1 = B_1$ , so  $A_1 = B_1$  by Lemma 7.3 (3).

We now claim that  $A_2P' \subseteq B_2Q'$ . We know that  $P = A_1A_2P'$  is included in  $Q = B_1B_2Q' = A_1B_2Q'$  (since  $A_1 = B_1$ ). Let us write  $A_1$  as  $(F, \gamma)$  and  $A_2$  as  $(F', \gamma')$ . If  $\gamma$  is decomposable, then  $A_2P' \subseteq B_2Q'$  by Lemma 6.4 (2). Let therefore  $\gamma$  be indecomposable. Since  $A_1A_2P'$  is reduced, we cannot have  $F \supseteq F'$  and  $\gamma \geq \gamma'$ , by Definition 7.2 (3) and (4), so  $A_2 \not\subseteq A_1$ , by Lemma 7.3 (2). Then  $A_2P' \subseteq A_1A_2P' \subseteq A_1B_2Q'$ , so  $A_2P' \subseteq B_2Q'$  by Lemma 7.5.

Symmetrically,  $B_2Q'$  is included in  $A_2P'$ , so  $A_2P' = B_2Q'$ . By the induction hypothesis  $A_2P' = B_2Q'$ . We remember that  $A_1 = B_1$ , so  $P = Q$ .  $\square$

We can always rewrite any transfinite product into a reduced product with the same semantics, using Lemma 7.1, whence the following.

**Corollary 7.9.** *Every transfinite product is equal to  $P$  for some unique reduced sequence of atoms  $P$ .*

Corollary 7.9 allows us to conflate the notions of transfinite product and of reduced sequence of atoms. By abuse of language, we will call *reduced product* any transfinite product  $P$  written in such a way that  $P$  is reduced. A *reduced  $\alpha$ -product* is an  $\alpha$ -product that is reduced in this sense.



## 8. WELL-FOUNDEDNESS

The *rank* (or *height*) of an element  $x$  in a well-founded poset  $P$  is defined by well-founded induction as the least ordinal strictly larger than the ranks of all elements  $y < x$ . We write  $\|F\|$  for the rank of  $F$  in the lattice of closed subsets of a Noetherian space  $X$ . For any Noetherian space  $F$ ,  $\|F\|$  is the *stature* of  $F$  [8], generalizing the notion of the same name on wqos [1].

Given any two ordinals  $\alpha \stackrel{\text{def}}{=} \omega^{\alpha_1} + \dots + \omega^{\alpha_m}$  and  $\beta \stackrel{\text{def}}{=} \omega^{\beta_1} + \dots + \omega^{\beta_n}$  in Cantor normal form, their *natural sum*  $\alpha \oplus \beta$  is defined as  $\omega^{\gamma_1} + \dots + \omega^{\gamma_{m+n}}$ , where  $\gamma_1 \geq \dots \geq \gamma_{m+n}$  is the list obtained by sorting the list  $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$  in decreasing order. This operation is associative and commutative, and strictly monotonic in both arguments.

An ordinal  $\delta$  is *critical* if and only if  $\omega^\delta = \delta$ . For every ordinal  $\alpha$ , let  $\alpha^\circ$  be  $\alpha+1$  if  $\alpha = \delta+n$  for some critical ordinal  $\delta$  and some  $n \in \mathbb{N}$ , and  $\alpha$  otherwise. Then  $\alpha < \omega^{\alpha^\circ}$ , and  $\alpha \mapsto \alpha^\circ$  is strictly monotonic [8, Lemmata 12.3 and 12.4]. For every proper bound  $\gamma$ , we define  $[\gamma]$  as the rank of  $\gamma$  in the poset of all proper bounds less than or equal to  $\gamma$ . Explicitly, and writing  $n$  for a natural number and  $\lambda$  for a limit ordinal,  $[\omega^n] = 2n - 2$  and  $[\omega^n + 1] = 2n - 1$  if  $n \geq 1$ ,  $[\omega^{\lambda+n}] = \lambda + 2n$ , and  $[\omega^{\lambda+n} + 1] = \lambda + 2n + 1$ .

**Definition 8.1.** Let  $X$  be a Noetherian space. For every atom  $F^{<\gamma}$ , let  $\varphi(F^{<\gamma})$  be  $\|F\|$  if  $\gamma = \omega^0 + 1$ , and  $\omega^{(\|F\| \oplus [\gamma])^\circ}$  otherwise. For every reduced product  $P \stackrel{\text{def}}{=} A_1 \cdots A_n$ , let  $\varphi(P) \stackrel{\text{def}}{=} \bigoplus_{i=1}^n \varphi(A_i)$ .

**Proposition 8.2.** *Let  $X$  be a Noetherian space. For all reduced products  $P$  and  $P'$ ,  $P \subseteq P'$  implies  $\varphi(P) \leq \varphi(P')$ , and  $P \subsetneq P'$  implies  $\varphi(P) < \varphi(P')$ .*

*Proof.* We first claim that  $\varphi$  is strictly monotonic on reduced atoms: for all atoms  $F^{<\gamma}$  and  $F'^{<\gamma'}$  with  $F, F' \neq \emptyset$  and  $F^{<\gamma} \subsetneq F'^{<\gamma'}$ ,  $\varphi(F^{<\gamma}) < \varphi(F'^{<\gamma'})$ . By Lemma 7.3,  $F \subseteq F'$  and  $\gamma \leq \gamma'$ , and not both are equalities. If  $\gamma = \gamma' = \omega^0 + 1$  and  $F \subsetneq F'$ , then  $\varphi(F^{<\gamma}) = \|F\| < \|F'\| = \varphi(F'^{<\gamma'})$ . If  $\gamma = \omega^0 + 1 < \gamma'$ , then  $\varphi(F^{<\gamma}) = \|F\| \leq \|F\| \oplus [\gamma] < \|F'\| \oplus [\gamma'] < \omega^{(\|F'\| \oplus [\gamma'])^\circ} = \varphi(F'^{<\gamma'})$ . If both  $\gamma$  and  $\gamma'$  are proper, then  $\varphi(F^{<\gamma}) = \omega^{(\|F\| \oplus [\gamma])^\circ} < \omega^{(\|F'\| \oplus [\gamma'])^\circ} = \varphi(F'^{<\gamma'})$  since  $\oplus$ ,  $[\cdot]$ ,  $^\circ$  and  $\|\cdot\|$  are strictly monotonic.

We prove the proposition by induction on the sum of the lengths of  $P$  and  $P'$ . Let  $P \subseteq P'$ . If  $P = \varepsilon$ , then  $\varphi(P) = 0 \leq \varphi(P')$ . If additionally  $P' \neq \varepsilon$ , then  $\varphi(P')$  is the natural sum of at least one term, and all those terms are non-zero: they are of the form  $\varphi(F^{<\gamma})$ , and when  $\gamma = \omega^0 + 1$ ,  $\varphi(F^{<\gamma}) = \|F\| \neq 0$  since  $F \neq \emptyset$ , otherwise  $\varphi(F^{<\gamma})$  is a power of  $\omega$ .

We now assume that  $P \neq \varepsilon$ . We write  $P$  as  $A_1 \cdots A_n$ , where  $A_1, \dots, A_n$  are atoms and  $n \geq 1$ . Let also  $Q \stackrel{\text{def}}{=} A_2 \cdots A_n$ , so  $P = A_1 Q$ . Since  $P \subseteq P'$ ,

$P' \neq \varepsilon$ , so  $P' = A'_1 Q'$  for some atom  $A'_1$  and some (reduced) product  $Q'$ . We write  $A'_1$  as  $F'^{<\gamma'}$ , and  $A_i$  as  $F_i^{<\gamma_i}$  for each  $i$ ,  $1 \leq i \leq n$ .

If  $A_1 \not\subseteq A'_1$ , then Lemma 7.5 entails that  $P = A_1 Q \subseteq Q'$ . By induction hypothesis,  $\varphi(P) \leq \varphi(Q')$ . Now  $\varphi(P') = \varphi(A'_1) \oplus \varphi(Q') > \varphi(Q')$ , so  $\varphi(P) < \varphi(P')$ . We turn to the other cases: from now on,  $A_1 \subseteq A'_1$ .

If  $\gamma'$  is indecomposable. We say that  $A_i$  is *small* if  $A_i \subseteq A'_1$ , equivalently  $F_i \subseteq F'$  and  $\gamma_i \leq \gamma'$ , by Lemma 7.3 (2).  $A_1$  is small. Let  $k$  be largest such that  $A_1, \dots, A_k$  are small, and  $R \stackrel{\text{def}}{=} A_{k+1} \cdots A_n$ . Then  $R \subseteq Q'$ . This is clear if  $R = \varepsilon$ ; otherwise  $k < n$ ,  $A_{k+1} \not\subseteq A'_1$ , then  $R \subseteq P \subseteq P' = A'_1 Q'$  implies  $R \subseteq Q'$  by Lemma 7.5. By the induction hypothesis,  $\varphi(R) \leq \varphi(Q')$ .

If some  $A_i$  with  $1 \leq i \leq k$  is equal to  $A'_1 = F'^{<\gamma'}$ , then  $\gamma_i = \gamma'$  is indecomposable, and  $F_i = F'$ . We cannot have  $i \geq 2$ , since that would contradict Definition 7.2 (2) or (3) at positions  $i-1$  and  $i$ ; similarly,  $i \leq k-1$  would contradict Definition 7.2 (4) or (3) at positions  $i$  and  $i+1$ . Hence  $k = 1$ . Then  $\varphi(P) = \varphi(A_1) \oplus \varphi(R) \leq \varphi(A'_1) \oplus \varphi(Q') = \varphi(P')$ . Additionally, if  $P \neq P'$ , then (since  $k = 1$  and  $A_1 = A'_1$ )  $R$  is strictly included in  $Q'$ , so  $\varphi(R) < \varphi(Q')$  by induction hypothesis, from which  $\varphi(P) < \varphi(P')$  follows.

Otherwise,  $k \geq 1$  and  $A_1, \dots, A_k$  are all strictly included in  $A'_1$ . Then  $\varphi(A_i) < \varphi(A'_1)$  for every  $i$  with  $1 \leq i \leq k$ . Since  $\gamma'$  is indecomposable,  $\gamma' \neq \omega^0 + 1$ , so  $\varphi(A'_1) = \omega^{(\|F'\| \oplus \|\gamma'\)}^\circ$ . The latter is indecomposable, so  $\varphi(A_1) \oplus \cdots \oplus \varphi(A_k) < \varphi(A'_1)$ . Together with  $\varphi(R) \leq \varphi(Q')$ , this implies that  $\varphi(P) = \varphi(A_1) \oplus \cdots \oplus \varphi(A_k) \oplus \varphi(R) < \varphi(A'_1) \oplus \varphi(Q') = \varphi(P')$ .

If  $\gamma'$  is decomposable. In that case, we say that  $A_i$  is *small* if and only if  $F_i \subseteq F'$  and  $\gamma_i < \gamma'$  (not  $\leq$ ). Let  $k$  be largest such that  $A_1, \dots, A_k$  are small, and  $R \stackrel{\text{def}}{=} A_{k+1} \cdots A_n$ .  $A_1, \dots, A_k$  are all strictly included in  $A'_1$ , so  $\varphi(A_i) < \varphi(A'_1)$  for every  $i$  with  $1 \leq i \leq k$ . If  $k \geq 1$ , then  $\gamma_1 < \gamma'$ , so  $\gamma' \neq \omega^0 + 1$ , and therefore  $\varphi(A'_1)$  is indecomposable. Hence  $\varphi(A_1) \oplus \cdots \oplus \varphi(A_k) < \varphi(A'_1)$ . If  $k = 0$ , the same inequality holds, vacuously.

If  $R = \varepsilon$ , then  $\varphi(P) = \varphi(A_1) \oplus \cdots \oplus \varphi(A_k) < \varphi(A'_1) \leq \varphi(P')$ . We now assume  $R \neq \varepsilon$ . Then,  $k < n$ , and  $A_{k+1}$  is not small.

If  $A_{k+1} \not\subseteq A'_1$ , then  $R \subseteq P \subseteq P' = A'_1 Q'$  implies  $R \subseteq Q'$  by Lemma 7.5, hence  $\varphi(R) \leq \varphi(Q')$  by induction hypothesis. Then  $\varphi(P) = \varphi(A_1) \oplus \cdots \oplus \varphi(A_k) \oplus \varphi(R) < \varphi(A'_1) \oplus \varphi(Q') = \varphi(P')$ .

There remains one case, where  $A_{k+1} \subseteq A'_1$  and  $A_{k+1}$  is not small. Then  $F_{k+1} \subseteq F'$  and  $\gamma_{k+1} = \gamma'$ , which is decomposable. Let  $R' \stackrel{\text{def}}{=} A_{k+2} \cdots A_n$ .  $R = A_{k+1} R' \subseteq P \subseteq P' = A'_1 Q'$  implies  $R' \subseteq Q'$  by Lemma 6.4. By induction hypothesis,  $\varphi(R') \leq \varphi(Q')$ . If  $k = 0$ , then  $P = A_1 R'$ ,  $P' = A'_1 Q'$ ,  $A_1 \subseteq A'_1$ , and  $R' \subseteq Q'$ , so  $\varphi(P) = \varphi(A_1) \oplus \varphi(R') \leq \varphi(A'_1) \oplus \varphi(Q') = \varphi(P')$ ;

additionally, if  $P \subsetneq P'$  then  $A_1 \subsetneq A'_1$  or  $R' \subsetneq Q'$ , which implies  $\varphi(P) < \varphi(P')$ . Let us now assume  $k \geq 1$ . We claim that the inclusion  $A_{k+1} \subseteq A'_1$  is strict: if  $A_{k+1} = A'_1$ , then  $A_k \subseteq A'_1 = A_{k+1}$  and  $\gamma_k < \gamma' = \gamma_{k+1}$ , contradicting Definition 7.2 (2). Hence  $A_1, \dots, A_k$ , and also  $A_{k+1}$ , are strictly included in  $A'_1$ . We recall that, since  $k \geq 1$ ,  $\varphi(A'_1)$  is indecomposable, so  $\varphi(P) = \varphi(A_1) \oplus \dots \oplus \varphi(A_k) \oplus \varphi(A_{k+1}) \oplus \varphi(R') < \varphi(A'_1) \oplus \varphi(Q') = \varphi(P')$ .  $\square$

**Corollary 8.3.** *Let  $X$  be a Noetherian space. The inclusion ordering on transfinite products on  $X$  is well-founded. Additionally, the ordinal rank of  $X^{<\alpha}$  in the poset of all  $\alpha$ -products is at most  $\omega^{(|X| \oplus [\alpha])^\circ}$ , for every bound  $\alpha$ .*

## 9. INTERSECTIONS OF TRANSFINITE PRODUCTS

**Lemma 9.1.** *Let  $X$  be a topological space. The intersection of two transfinite products satisfies the following properties:*

- (1)  $\varepsilon \cap P' = P \cap \varepsilon = \varepsilon$ .
- (2) If  $\gamma < \gamma'$  or if  $\gamma = \gamma'$  is indecomposable, then  $F^{<\gamma} P \cap F'^{<\gamma'} P'$  is equal to the union of  $(F \cap F')^{<\gamma} (P \cap F'^{<\gamma'} P')$  and of  $(F \cap F')^{<\gamma} (F^{<\gamma} P \cap P')$ .
- (3) If  $\gamma = \gamma'$  is decomposable, say  $\gamma = \gamma' = \omega^\beta + 1$ , then  $F^{<\gamma} P \cap F'^{<\gamma'} P'$  is the union of  $(F \cap F')^{<\gamma} (P \cap P')$ , of  $(F \cap F')^{<\omega^\beta} (P \cap F'^{<\gamma'} P')$ , and of  $(F \cap F')^{<\omega^\beta} (F^{<\gamma} P \cap P')$ .

*Proof.* (1) is clear. Let us deal with the left to right inclusions for the other cases, as a first step. For every  $w \in F^{<\gamma} P \cap F'^{<\gamma'} P'$ , let us write  $w$  as  $uv$  where  $u \in F^{<\gamma}$ ,  $v \in P$  and also as  $u'v'$  where  $u' \in F'^{<\gamma'}$ ,  $v' \in P'$ .

(2) If  $|u| \leq |u'|$ ,  $u$  is a prefix of  $u'$ , so its letters are not just in  $F$ , but also in  $F'$ . Therefore  $u \in (F \cap F')^{<\gamma}$ . Also,  $v$  is in  $P$ , and is a suffix of  $w$ . In particular,  $v \leq_* w$ , so  $v \in F'^{<\gamma'} P'$  by Lemma 4.2. If  $|u'| \leq |u|$ , then symmetrically  $u'$  is in  $(F \cap F')^{<\gamma}$  and  $v'$  is in  $P'$  and in  $F^{<\gamma} P$ .

(3) If  $|u| = |u'| = \omega^\beta$ , then  $u = u'$  is in  $(F \cap F')^{<\gamma}$ , and  $v = v'$  is in  $P \cap P'$ , so  $w \in (F \cap F')^{<\gamma} (P \cap P')$ . Otherwise,  $w$  is in  $F^{<\omega^\beta} P$  and in  $F'^{<\gamma'} P'$ , or in  $F^{<\gamma} P$  and in  $F'^{<\omega^\beta} P'$ . In the first case, by (2) with  $\omega^\beta$  in lieu of  $\gamma$ ,  $w$  is in  $(F \cap F')^{<\omega^\beta} (P \cap F'^{<\gamma'} P') \cup (F \cap F')^{<\omega^\beta} (F^{<\omega^\beta} P \cap P')$ , hence in  $(F \cap F')^{<\omega^\beta} (P \cap F'^{<\gamma'} P') \cup (F \cap F')^{<\omega^\beta} (F^{<\gamma} P \cap P')$ . In the second case, a similar argument leads to the same result.

We now deal with the right to left inclusions.

(2)  $(F \cap F')^{<\gamma} (P \cap F'^{<\gamma'} P')$  is included both in  $F^{<\gamma} P$  (because  $F \cap F' \subseteq F$ ) and in  $(F \cap F')^{<\gamma} F'^{<\gamma'} P' = F'^{<\gamma'} P'$ , by Lemma 7.1 (1). Similarly for  $(F \cap F')^{<\gamma} (F^{<\gamma} P \cap P')$ .

(3)  $(F \cap F')^{<\gamma} (P \cap P')$  is included both in  $F^{<\gamma} P$  and in  $F'^{<\gamma} P' = F'^{<\gamma'} P'$ .  $(F \cap F')^{<\omega^\beta} (P \cap F'^{<\gamma'} P')$  is included both in  $(F \cap F')^{<\omega^\beta} P \subseteq F^{<\omega^\beta} P \subseteq$

$F^{<\gamma}P$  and in  $(F \cap F')^{<\omega^\beta} F'^{\gamma'} P' = F'^{\gamma'} P'$  (by Lemma 7.1 (1)). Finally,  $(F \cap F')^{<\omega^\beta} (F^{<\gamma}P \cap P')$  is included both in  $(F \cap F')^{<\omega^\beta} F^{<\gamma}P = F^{<\gamma}P$  (by Lemma 7.1 (1)) and in  $(F \cap F')^{<\omega^\beta} P' \subseteq F'^{<\omega^\beta} P' \subseteq F'^{<\gamma'} P'$ .  $\square$

**Corollary 9.2.** *Let  $X$  be a Noetherian space. The intersection of any two transfinite products is a finite union of transfinite products.*

*Proof.* By induction on the sum of their sizes, using Lemma 9.1. The case where one of them is  $\varepsilon$  is obvious, so we deal with the intersection of two transfinite products  $F^{<\gamma}P$  and  $F'^{<\gamma'}P'$ . Without loss of generality,  $\gamma \leq \gamma'$ .

By induction hypothesis,  $P \cap F'^{\gamma'}P'$ ,  $F^{<\gamma}P \cap P'$  and  $P \cap P'$  are finite unions of transfinite products, say  $\bigcup_i P_i$ ,  $\bigcup_j Q_j$ , and  $\bigcup_k R_k$  respectively.

The intersection  $F^{<\gamma}P \cap F'^{<\gamma'}P'$  can then be expressed as  $\bigcup_i (F \cap F')^{<\gamma} P_i \cup \bigcup_j (F \cap F')^{<\gamma} Q_j$  when  $\gamma < \gamma'$  or if  $\gamma = \gamma'$  is indecomposable. If  $\gamma = \gamma'$  is of the form  $\omega^\beta + 1$ , then it can be expressed as the union  $\bigcup_k (F \cap F')^{<\gamma} R_k \cup \bigcup_i (F \cap F')^{<\omega^\beta} P_i \cup \bigcup_j (F \cap F')^{<\omega^\beta} Q_j$ .

This is a finite union of transfinite products, except when  $\gamma = \omega^0 + 1$ . In that case, we need to refine the expressions above. If  $F \cap F'$  is empty, then  $(F \cap F')^{<\gamma} = \varepsilon$ , so  $F^{<\gamma}P \cap F'^{<\gamma'}P' = \bigcup_i P_i \cup \bigcup_j Q_j$  when  $\omega^0 + 1 < \gamma'$ , and  $F^{<\gamma}P \cap F'^{<\gamma'}P' = \bigcup_k R_k \cup \bigcup_i P_i \cup \bigcup_j Q_j$  otherwise.

We now assume that  $F \cap F' \neq \emptyset$ .  $F$  and  $F'$  are irreducible closed, and since  $X$  is Noetherian,  $F \cap F'$  is a finite union  $\bigcup_{\ell=1}^n C_\ell$  of irreducible closed subsets. Since  $F \cap F' \neq \emptyset$ ,  $n$  is non-zero. Then  $(F \cap F')^{<\gamma} = (F \cap F')^? = \bigcup_{\ell=1}^n C_\ell^?$  (an equality that would fail if  $n$  were zero), and  $(F \cap F')^{<\omega^0} = \varepsilon$ . Therefore  $F^{<\gamma}P \cap F'^{<\gamma'}P'$  is equal to  $\bigcup_{i,\ell} C_\ell^? P_i \cup \bigcup_{j,\ell} C_\ell^? Q_j$  when  $\omega^0 + 1 < \gamma'$ , or to  $\bigcup_{k,\ell} C_\ell^? R_k \cup \bigcup_i P_i \cup \bigcup_j Q_j$  if  $\gamma' = \omega^0 + 1$ .  $\square$

We can now use Proposition 2.1. If  $X$  is Noetherian, the class  $\mathcal{P}$  of  $\alpha$ -products is well-founded under inclusion by Corollary 8.3,  $X^{<\alpha}$  is a finite union of elements of  $\mathcal{P}$  since it is, in fact, an  $\alpha$ -product. The intersection of any two elements of  $\mathcal{P}$  is a finite union of elements of  $\mathcal{P}$  by Corollary 9.2, and  $\mathcal{P}$  is irredundant by Proposition 5.4. Moreover, the regular subword topology on  $X^{<\alpha}$  is the coarsest that makes every element of  $\mathcal{P}$  a closed set, by Proposition 5.3. Since every set of transfinite words on  $X$  is included in  $X^{<\alpha}$  for some bound  $\alpha$ , we obtain the following.

**Theorem 9.3.** *For every Noetherian space  $X$ , every space  $Y$  of transfinite words on  $X$  is Noetherian in the regular subword topology. For every bound  $\alpha$ , the irreducible closed subsets of  $X^{<\alpha}$  are the  $\alpha$ -products.*

Given any non-empty Noetherian space  $Z$ ,  $Z$  has finitely many maximal irreducible closed subsets  $C_1, \dots, C_n \in \mathcal{SZ}$  ( $n \geq 1$ ). The *reduced sobri-fication rank*  $\text{rsob } Z$  is the maximum of the ranks of  $C_1, \dots, C_n$  in  $\mathcal{SZ}$  [8, Lemma 4.2]; then  $\|Z\| \leq \omega^{\text{rsob } Z} \otimes n$  [8, Proposition 4.5]. For  $Z \stackrel{\text{def}}{=} X^{<\alpha}$ ,  $n = 1$  since  $X^{<\alpha}$  is the unique largest element of  $\mathcal{S}(X^{<\alpha})$ , so the  $\otimes n$  part vanishes. Corollary 8.3 then gives us the following quantitative information.

**Proposition 9.4.** *For every Noetherian space  $X$ , for every bound  $\alpha$ , we have  $\text{rsob } X^{<\alpha} \leq \omega^{(\|X\| \oplus [\alpha])^\circ}$  and  $\|X^{<\alpha}\| \leq \omega^{\omega^{(\|X\| \oplus [\alpha])^\circ}}$ .*

It is not our purpose to give an exact formula for  $\text{rsob } X^{<\alpha}$  and  $\|X^{<\alpha}\|$  here. When  $\alpha = \omega^1$ ,  $X^{<\alpha} = X^*$ ,  $[\alpha] = 0$ , and Proposition 9.4 implies  $\text{rsob } X^* \leq \omega^{\|X\|^\circ}$  and  $\|X^*\| \leq \omega^{\omega^{\|X\|^\circ}}$ . The former is essentially optimal, since if  $X \neq \emptyset$ , then  $\text{rsob } X^* = \omega^{\|X\|^\circ}$  [8, Theorem 12.13], while the latter is close to optimal: if  $\|X\|$  is infinite,  $\|X^*\| = \omega^{\omega^{\|X\|^\circ}}$  [8, Theorem 12.23], generalizing Schmidt's formula on well-partial-orders [13, Theorem 9]. But when  $\|X\|$  is finite and non-empty,  $\|X^*\| = \omega^{\omega^{\|X\|^\circ - 1}} < \omega^{\omega^{\|X\|^\circ}}$ .

## 10. THE SPECIALIZATION ORDERING

A non-empty transfinite word  $w$  is indecomposable if and only if for every way of writing  $w$  as  $uv$  where  $v \neq \epsilon$ , we have  $w \leq_* v$ . Pouzet [12], confirming a conjecture of Jullien [9], shows that  $\leq$  is a better-quasi-ordering if and only if every transfinite word (of countable length) is a concatenation of finitely many indecomposable words.

Given a topological space  $X$  and an ordinal  $\alpha$ , let us call  $w \in X^{<\alpha}$  *topologically indecomposable* if and only  $w \neq \epsilon$  and, for every way of writing  $w$  as  $uv$  where  $v \neq \epsilon$ ,  $w$  is in the closure  $\bar{v}$  of  $v$  in  $X^{<\alpha}$ . (We write  $\bar{v}$  instead of the more cumbersome notation  $\text{cl}(\{v\})$ .) When  $w = uv$ , we have  $v \leq_* w$ , so  $v \in \bar{w}$  by Lemma 4.2. Hence  $w$  is topologically indecomposable if and only if for every way of writing  $w$  as  $uv$  where  $v \neq \epsilon$ ,  $\bar{v} = \bar{w}$ .

**Lemma 10.1.** *Let  $X$  be a topological space. Every indecomposable transfinite word  $w$  on  $X$  is topologically indecomposable.*

*Proof.* Whenever  $w = uv$  with  $v \neq \epsilon$ ,  $w \leq_* v$ ; so  $w \in \bar{v}$  by Lemma 4.2.  $\square$

**Lemma 10.2.** *Every transfinite word  $w \in X^{<\alpha}$ , where  $X$  is Noetherian and  $\alpha$  is a bound, can be written as a finite concatenation of topologically indecomposable transfinite words.*

*Proof.* By induction on  $|w|$ . The claim is clear if  $w = \epsilon$ . Otherwise, for every  $\beta < |w|$ , we write  $w$  as  $u_\beta v_\beta$  where  $u_\beta$  is its prefix of length  $\beta$ , and  $v_\beta \neq \epsilon$ .

For all ordinals  $\beta < \gamma < |w|$ ,  $v_\gamma \leq_* v_\beta$ , so  $v_\gamma \in \overline{v_\beta}$  by Lemma 4.2, and therefore  $\overline{v_\gamma} \subseteq \overline{v_\beta}$ . Hence the closed sets  $\overline{v_\beta}$  with  $\beta < |w|$  form a chain. Since  $X^{<\alpha}$  is Noetherian (Theorem 9.3), inclusion is well-founded on its closed subsets, so there is a non-empty suffix  $v_\beta$  of  $w$  whose closure  $\overline{v_\beta}$  is smallest.

We claim that  $v_\beta$  is topologically indecomposable. Let us write  $v_\beta$  as  $uv$  with  $v \neq \epsilon$ . Then  $v = v_\gamma$ , where  $\gamma \stackrel{\text{def}}{=} \beta + |u|$ , and  $u_\gamma = u_\beta u$ . Since  $\overline{v_\beta}$  is least and  $\overline{v_\gamma} \subseteq \overline{v_\beta}$ , owing to the fact that  $\gamma \geq \beta$ , we deduce that  $\overline{v} = \overline{v_\gamma} = \overline{v_\beta}$ .

By induction hypothesis (indeed,  $|u_\beta| = \beta < |w|$ ),  $u_\beta$  is a finite concatenation of topologically indecomposable words; hence so is  $w = u_\beta v_\beta$ .  $\square$

**Lemma 10.3.** *The length of a topologically indecomposable transfinite word is indecomposable.*

*Proof.* Let  $w$  be topologically indecomposable in  $X^{<\alpha}$ , and let us assume that  $|w|$  is decomposable. We write  $|w|$  as  $\beta + \gamma$ , where  $0 < \beta, \gamma < |w|$ , and then  $w$  as  $uv$  where  $|u| = \beta$ . We have  $|v| = \gamma \neq 0$ , so  $v \neq \epsilon$ . Since  $w$  is topologically indecomposable,  $w$  is in  $\overline{v}$ . Now  $v$  is in  $X^{<\gamma+1}$ , which is closed, so  $\overline{v} \subseteq X^{<\gamma+1}$ . Thus  $w$  is in  $X^{<\gamma+1}$ , which is impossible since  $\gamma < |w|$ .  $\square$

**Lemma 10.4.** *Let  $X$  be a Noetherian space and  $\alpha$  be a bound. For every non-empty transfinite word  $w \in X^{<\alpha}$ ,  $w$  is topologically indecomposable if and only if for every closed subset of  $X^{<\alpha}$  that is the concatenation  $\mathcal{C}_1\mathcal{C}_2$  of two closed subsets of  $X^\alpha$ , if  $w \in \mathcal{C}_1\mathcal{C}_2$  then  $w$  is in  $\mathcal{C}_1$  or in  $\mathcal{C}_2$ .*

*Proof.* Let  $w$  be topologically indecomposable. Since  $w \in \mathcal{C}_1\mathcal{C}_2$ , we can write  $w$  as  $uv$  where  $u \in \mathcal{C}_1$  and  $v \in \mathcal{C}_2$ . If  $v = \epsilon$ , then  $w = u$  is in  $\mathcal{C}_1$ . Otherwise, since  $w$  is topologically indecomposable,  $\overline{v} = \overline{w}$ . But  $\overline{v} \subseteq \mathcal{C}_2$ , so  $w \in \overline{v} \subseteq \mathcal{C}_2$ .

Conversely, let us assume that  $w \in \mathcal{C}_1\mathcal{C}_2$  implies  $w \in \mathcal{C}_1$  or  $w \in \mathcal{C}_2$  for every concatenation of two closed sets  $\mathcal{C}_1\mathcal{C}_2$ . For any way of writing  $w$  as  $uv$  with  $v \neq \emptyset$ , we let  $\mathcal{C}_1 \stackrel{\text{def}}{=} \overline{u}$  and  $\mathcal{C}_2 \stackrel{\text{def}}{=} \overline{v}$ . Then  $w \in \overline{u}$  or  $w \in \overline{v}$ . Let  $\beta \stackrel{\text{def}}{=} |u|$ . We note that  $\beta < |w|$ , since  $|w| = \beta + |v|$  and  $v \neq \epsilon$ . Clearly,  $u$  is in the closed set  $X^{<\beta+1}$ , so  $\overline{u} \subseteq X^{<\beta+1}$ . If  $w \in \overline{u}$ , then  $w$  is in  $X^{<\beta+1}$ , so  $|w| \leq \beta < |w|$ , which is impossible. Therefore  $w$  is in  $\overline{v}$ .  $\square$

Let us write  $\text{Im } w$  for the set of letters in a transfinite word  $w$ . We call *support*  $\text{supp } w$  of  $w$  the closure  $cl(\text{Im } w)$  of  $\text{Im } w$ .

**Lemma 10.5.** *Let  $X$  be Noetherian, and  $\alpha$  be a bound. For every topologically indecomposable word  $w \in X^{<\alpha}$ ,  $\overline{w} = (\text{supp } w)^{<\gamma+1}$ , where  $\gamma \stackrel{\text{def}}{=} |w|$ .*

*Proof.* Let  $F \stackrel{\text{def}}{=} \text{supp } w$ . Clearly,  $w$  is in  $F^{<\gamma+1}$ , so  $\overline{w} \subseteq F^{<\gamma+1}$ . Conversely,  $\overline{w}$  is irreducible closed, hence is an  $\alpha$ -product  $P \stackrel{\text{def}}{=} A_1 \cdots A_n$ , by Theorem 9.3.

Since  $w \in P$  is topologically indecomposable hence non-empty,  $n \geq 1$ . Using Lemma 10.4,  $w$  is in some  $A_i$ . Let us write  $A_i$  as  $F'^{<\gamma'}$ . Since  $|w| = \gamma$ ,  $\gamma < \gamma'$ . Every letter of  $w$  is in  $F'$ , so  $\text{Im } w \subseteq F'$ , and taking closures,  $F \subseteq F'$ . Then  $F^{<\gamma+1} \subseteq F'^{<\gamma'}$ , so  $F^{<\gamma+1} \subseteq P = \bar{w}$ .  $\square$

**Lemma 10.6.** *Let  $X$  be a Noetherian space,  $\beta$  be an ordinal, and  $\alpha$  be  $\omega^\beta$  or  $\omega^\beta + 1$ . The map  $\text{cat}$  is closed and continuous from  $X^{<\omega^\beta} \times X^{<\alpha}$  to  $X^{<\alpha}$ .*

*Proof.* It is well-defined and continuous by Lemma 5.1. By Theorem 9.3, the closed subsets of  $X^{<\omega^\beta}$  (resp.,  $X^{<\alpha}$ ) are the finite unions of  $\omega^\beta$ -products (resp.,  $\alpha$ -products). Given any two such closed sets expressed as finite unions of such products  $\mathcal{C}_1 \stackrel{\text{def}}{=} \bigcup_{i=1}^m P_i$  and  $\mathcal{C}_2 \stackrel{\text{def}}{=} \bigcup_{j=1}^n Q_j$ , the image  $\mathcal{C}_1\mathcal{C}_2$  of  $\mathcal{C}_1 \times \mathcal{C}_2$  by  $\text{cat}$  is the (finite) union over all  $i$  and  $j$  of the products  $P_iQ_j$ .  $\square$

**Lemma 10.7.** *Let  $X$  be a Noetherian space, and  $\alpha$  be a bound. For all sets of transfinite words  $A$  and  $B$  such that  $AB \subseteq X^{<\alpha}$ ,  $\text{cl}(AB) = \text{cl}(A)\text{cl}(B)$ .*

*Proof.* We use Lemma 10.6. If  $\alpha$  is indecomposable,  $\text{cat}$  is continuous from  $X^{<\alpha} \times X^{<\alpha}$  to  $X^{<\alpha}$ , so  $\text{cl}(A)\text{cl}(B) \subseteq \text{cl}(AB)$ ;  $\text{cat}$  is closed, so  $\text{cl}(A)\text{cl}(B)$  is closed, and contains  $AB$ , so it contains  $\text{cl}(AB)$ .

If  $\alpha = \omega^\beta + 1$ , then either  $B \subseteq \{\epsilon\}$ , in which case  $\text{cl}(AB) = \text{cl}(A) = \text{cl}(A)\text{cl}(B)$ ; or  $A \subseteq X^{<\omega^\beta}$  (else we could pick  $u \in A$  of length  $\omega^\beta$ ,  $v \neq \epsilon$  in  $B$ , and then  $uv \in AB$  would not be in  $X^{<\alpha}$ ), then we reason as above, using the fact that  $\text{cat}$  is closed and continuous from  $X^{<\omega^\beta} \times X^{<\alpha}$  to  $X^{<\alpha}$ .  $\square$

**Proposition 10.8.** *For every Noetherian space  $X$  and every bound  $\alpha$ , the regular subword topology on  $X^{<\alpha}$  is the coarsest one such that  $F^{<\gamma}$  is closed for every closed subset  $F$  of  $X$  and every ordinal  $\gamma \leq \alpha$ , and such that  $\mathcal{C}_1\mathcal{C}_2$  is closed for all closed subsets  $\mathcal{C}_1$  and  $\mathcal{C}_2$  such that  $\mathcal{C}_1\mathcal{C}_2 \subseteq X^{<\alpha}$ .*

*Proof.* Let us call admissible any topology  $\tau$  containing the sets  $F^{<\gamma}$  as closed sets, and closed under concatenations  $\mathcal{C}_1\mathcal{C}_2$ , as described above. The regular subword topology is admissible, since  $\mathcal{C}_1\mathcal{C}_2 = \text{cl}(\mathcal{C}_1)\text{cl}(\mathcal{C}_2) = \text{cl}(\mathcal{C}_1\mathcal{C}_2)$ , by Lemma 10.7. Given any admissible topology  $\tau$ , we see that the  $\alpha$ -products are closed in  $\tau$ , so  $\tau$  is finer than the regular subword topology.  $\square$

There is a full, continuous map  $\eta_Y : y \mapsto \downarrow y$  from any space  $Y$  to  $\mathcal{S}Y$ . The specialization preordering  $\leq$  on  $Y$  is characterized by  $y \leq y'$  if and only if  $y \in \eta_Y(y')$ . We note that  $\eta_{X^{<\alpha}}(w)$  is simply  $\bar{w}$ .

**Theorem 10.9.** *Let  $X$  be a Noetherian space,  $\alpha$  be a bound. Let  $\leq_*^{\text{top}}$  be the specialization preordering of  $X^{<\alpha}$ .*

- (1) For every  $w \in X^{<\alpha}$ , one can write  $w$  as a finite concatenation of topologically indecomposable words  $w_1 \cdots w_n$ , and then  $\eta_{X^{<\alpha}}(w) = \bar{w} = \bar{w}_1 \cdots \bar{w}_n$ , and  $\bar{w}_i = (\text{supp } w_i)^{<|w_i|+1}$  for every  $i$ ,  $1 \leq i \leq n$ .
- (2) For all transfinite words  $w \stackrel{\text{def}}{=} w_1 \cdots w_m$  and  $w' \stackrel{\text{def}}{=} w'_1 \cdots w'_n$  written as finite concatenations of topologically indecomposable words,  $w \leq_*^{\text{top}} w'$  if and only if there are indices  $0 = i_0 \leq i_1 \leq \cdots \leq i_{n-1} \leq i_n = m$  such that for every  $j$  with  $1 \leq j \leq n$ ,  $\bigcup_{i=i_{j-1}+1}^{i_j} \text{supp } w_i \subseteq \text{supp } w'_j$  and  $|w_{i_{j-1}+1}| + \cdots + |w_{i_j}| \leq |w'_j|$ .

*Proof.* (1) We write  $w$  as  $w_1 \cdots w_n$  where each  $w_i$  is topologically indecomposable by Lemma 10.2. By Lemma 10.7 (and since  $\bar{\varepsilon} = \varepsilon$  in the base case  $n = 0$ ),  $\bar{w} = \bar{w}_1 \cdots \bar{w}_n$ . Finally,  $\bar{w}_i = (\text{supp } w_i)^{<|w_i|+1}$  by Lemma 10.5.

(2) Lemma 10.4 has the following consequence. Given any closed set of the form  $\mathcal{C}_1\mathcal{C}_2$  with  $\mathcal{C}_1$  and  $\mathcal{C}_2$  closed, for every transfinite word  $w \stackrel{\text{def}}{=} w_1 \cdots w_m$  in  $\mathcal{C}_1\mathcal{C}_2$ , written as a finite concatenation of topologically indecomposable words, there is an index  $i$  with  $0 \leq i \leq m$  such that  $w_1 \cdots w_i \in \mathcal{C}_1$  and  $w_{i+1} \cdots w_m \in \mathcal{C}_2$ . Indeed, since  $w \in \mathcal{C}_1\mathcal{C}_2$ , there is an index  $j$  with  $1 \leq j \leq m$  such that one can write  $w_j$  as  $uv$ , and  $w_1 \cdots w_{j-1}u \in \mathcal{C}_1$ ,  $vw_j \cdots w_m \in \mathcal{C}_2$ . Since  $\text{cat}$  is continuous (Lemma 5.1 (2)),  $\mathcal{C}'_1 \stackrel{\text{def}}{=} \text{cat}(w_1 \cdots w_{j-1}, -)^{-1}(\mathcal{C}_1)$  and  $\mathcal{C}'_2 \stackrel{\text{def}}{=} \text{cat}(-, w_j \cdots w_m)^{-1}(\mathcal{C}_2)$  are closed. Now  $w_j = uv \in \mathcal{C}'_1\mathcal{C}'_2$ , so  $w_j$  is in  $\mathcal{C}'_1$  or in  $\mathcal{C}'_2$ ; we define  $i$  as  $j$  in the first case, as  $j - 1$  in the second case.

By induction on  $n$ , if  $w$  belongs to a finite product  $\mathcal{C}_1 \cdots \mathcal{C}_n$  of closed subsets of  $X^{<\alpha}$  included in  $X^{<\alpha}$ , we can find indices  $0 = i_0 \leq i_1 \leq \cdots \leq i_{n-1} \leq i_n = m$  such that  $w_{i_{j-1}+1} \cdots w_{i_j} \in \mathcal{C}_j$  for every  $j$  with  $1 \leq j \leq n$ .

Let us assume that  $w \leq_*^{\text{top}} w'$ , namely  $\bar{w} \subseteq \bar{w}'$ . By (1),  $\bar{w}_1 \cdots \bar{w}_m \subseteq \bar{w}'_1 \cdots \bar{w}'_n$ . Letting  $\mathcal{C}_j \stackrel{\text{def}}{=} \bar{w}'_j$ , we obtain indices  $0 = i_0 \leq i_1 \leq \cdots \leq i_{n-1} \leq i_n = m$  such that  $w_{i_{j-1}+1} \cdots w_{i_j} \in \bar{w}'_j$  for each  $j$ . Since  $w'_j \in X^{<|w'_j|+1}$ ,  $\bar{w}'_j \subseteq X^{<|w'_j|+1}$ , so  $|w_{i_{j-1}+1}| + \cdots + |w_{i_j}| = |w_{i_{j-1}+1} \cdots w_{i_j}| \leq |w'_j|$ . Also, for every  $i$  with  $i_{j-1} + 1 \leq i \leq i_j$ ,  $w_i \leq_* w_{i_{j-1}+1} \cdots w_{i_j}$ , so  $w_i$  is in  $\bar{w}'_j$  by Lemma 4.2; hence  $\text{supp } w_i \subseteq \text{supp } w'_j$ , using Lemma 10.5 and Lemma 7.3 (2).

Conversely, if  $\bigcup_{i=i_{j-1}+1}^{i_j} \text{supp } w_i \subseteq \text{supp } w'_j$  and  $|w_{i_{j-1}+1}| + \cdots + |w_{i_j}| \leq |w'_j|$  for every  $j$ , then  $w_{i_{j-1}+1} \cdots w_{i_j}$  is in  $(\text{supp } w'_j)^{<|w'_j|+1} = \bar{w}'_j$  (by Lemma 10.5); so  $w = w_1 \cdots w_m$  is in  $\bar{w}'_1 \cdots \bar{w}'_n = \bar{w}'$ , by (1).  $\square$

How does this compare to the subword ordering  $\leq_*$ ? By Lemma 4.2,  $w \leq_* w'$  implies  $w \leq_*^{\text{top}} w'$ . The converse holds on  $X^* = X^{<\omega}$  [5, Exercise 9.7.29], and on  $X^{<\omega+1}$  if  $X$  is a wqo in its Alexandroff topology [6, Proposition 5.16]. Otherwise, the result may fail, as the following demonstrates.



**Example 10.10.** Let  $X$  be  $\mathbb{N}$  with the cofinite topology. Since its non-trivial closed subsets are finite,  $X$  is Noetherian. In order to see that  $\leq_*^{\text{top}}$  and  $\leq_*$  differ, let  $w \stackrel{\text{def}}{=} 0\ 1\ 2\ \dots$ . For every non-empty suffix  $v$  of  $w$ ,  $\text{Im } v$  is infinite, so  $\text{supp } v = \mathbb{N}$ . Hence  $w$  is topologically indecomposable. By Theorem 10.9 (1),  $\bar{w} = X^{<\omega+1}$ , so  $w' \leq_*^{\text{top}} w$  for every  $w' \in X^{<\omega+1}$ . Hence, for example,  $0^\omega \leq_*^{\text{top}} w$ , but  $0^\omega \not\leq_* w$ .

For spaces  $X^{<\alpha}$  with  $\alpha \geq \omega^2$ , we have the following.

**Example 10.11.** Let  $X$  be any Noetherian space with two elements  $a$  and  $b$  such that  $a \not\leq b$  (say,  $\{a, b\}$  with the discrete topology). We claim that  $\leq_*^{\text{top}}$  and  $\leq_*$  differ on  $X^{<\alpha}$ , for any bound  $\alpha \geq \omega^2$ . Let  $w' \stackrel{\text{def}}{=} (ab^\omega)^\omega$  (explicitly,  $w': \omega^2 \rightarrow X$ ,  $w'(\omega \cdot m + n) \stackrel{\text{def}}{=} a$  if  $n = 0$ ,  $b$  otherwise);  $w'$  is indecomposable, hence topologically indecomposable by Lemma 10.1, and  $\text{supp } w' = \downarrow\{a, b\}$ . Let  $w \stackrel{\text{def}}{=} a^{\omega^2}$ . This is also indecomposable, and  $\text{supp } w = \downarrow a$ . Hence, by Theorem 10.9,  $w \leq_*^{\text{top}} w'$ . However,  $w \not\leq_* w'$ , since there is no strictly increasing map from  $\omega^2$  into  $\omega$ , the subset of positions of  $w'$  where  $a$  occurs.

## 11. CONCLUSION AND OPEN PROBLEMS

We have described a Noetherian topology on spaces of transfinite words over a Noetherian space  $X$ , in particular on spaces  $X^{<\alpha}$ , where  $\alpha$  is a bound. In the latter situation, we have characterized its irreducible closed subsets, and given upper bounds on the stature and reduced sobrification rank of  $X^{<\alpha}$ . We have also characterized the specialization preordering  $\leq_*^{\text{top}}$  of  $X^{<\alpha}$ .

Although we have not stressed it, the syntax of  $\alpha$ -products naturally yields an S-representation of  $X^{<\alpha}$ , in the sense of [3, 6], provided we restrict our bounds to lie in some class of ordinals with a computable representation, decidable ordering, and decidable equality. S-representations are important in forward analysis procedures for well-structured transition systems [2].

We finish with two questions. First, it is frustrating that  $\leq_*^{\text{top}}$  and  $\leq_*$  differ in general. Is there a natural, finer Noetherian topology on  $X^{<\alpha}$  that would have  $\leq_*$  as specialization preordering? The specialization preordering of a Noetherian space is necessarily well-founded. Hence, if the desired topology exists,  $X^{<\alpha}$  is well-founded under  $\leq_*$ . By an argument similar to Lemma 10.2, every non-empty word  $w \in X^{<\alpha}$  would have an indecomposable suffix, in other words  $X$  would have to be  $\beta$ -better-quasi-ordered in the sense of [12, Definition IV-2] or of [4, Chapter 8, 5.1], for every  $\beta < \alpha$ .

Second, what is the exact stature of  $X^{<\alpha}$ ? its reduced sobrification rank?

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