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# Supremum estimates for parabolic stochastic partial differential equations

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## Abstract

We generalize the approach à la De Giorgi developed in [HWW17] to get some estimates on the moments of the sup-norm of the solutions to parabolic stochastic partial differential equations. We also provide an alternative method, based on duality and estimates  $L^1 - L^r$  à la Boccardo-Gallouët, [BG89], for a backward stochastic partial differential equation.

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## 1 Introduction

Let  $U$  be a smooth open bounded subset of  $\mathbb{R}^d$ , let  $T > 0$  and let  $Q_T = U \times (0, T)$ ,  $T \in (0, +\infty)$ . Let

$$A = (a^{ij})_{1 \leq i, j \leq d}: U \rightarrow \mathcal{S}_d(\mathbb{R}) \quad (1.1)$$

be measurable and satisfy the ellipticity condition and bound

$$\forall \xi \in \mathbb{R}^d, \text{ for a.e. } x \in U, \lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \lambda^{-1} |\xi|^2, \quad (1.2)$$

where  $\lambda$  is a positive constant. In (1.1),  $\mathcal{S}_d(\mathbb{R})$  denotes the set of symmetric,  $d \times d$  real matrices. In (1.2), and throughout the paper, we will use the convention of summation over repeated indexes. Our first concern is to get supremum estimate for the solution  $u$  to the stochastic parabolic partial differential equation

$$du - \partial_i(a^{ij} \partial_j u) dt = f dt + g_k dB^k(t) \text{ in } Q_T \quad (1.3)$$

$$u = 0 \text{ on } \partial U \times (0, T) \quad (1.4)$$

$$u = u_0 \text{ on } U \times \{0\}. \quad (1.5)$$

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Such estimates have already been derived and by different methods, in particular:

- De Giorgi's method by truncature has been used in [DG17, HWW17, Qiu20],
- Moser's approach by iterative estimates on the  $L^p$ -norms has been adapted to the stochastic setting in [DMS05, Wan18, DG19],
- in [DDMH15], the Da Prato - Debussche trick is used, and estimates separately given on the stochastic convolution and deterministic parabolic equations (the latter exploiting in particular the theory in [LSU68], where truncations as in De Giorgi's approach are used at some point).

Note that in some of the references just mentioned, the framework may be much more general than in (1.3). In particular, [DDMH15, DG19] apply to quasilinear equations, and degeneracy of the superparabolic condition is addressed in [Qiu20]. In the present paper, we provide two results:

1. a generalization of the result of [HWW17], based on De Giorgi's approach via truncation of the natural energy, see Theorem 2.1. Our interest is mainly to get a more explicit dependence of the supremum estimate on the norms in  $L^\mu(Q_T)$  and  $L^{2\mu}(Q_t; \ell^2)$  of  $f$  and  $(g_k)$  respectively, where  $\mu > 1 + d/2$ . Indeed, in [HWW17], the exponent is  $\mu = +\infty$  and the bound on the data is assumed to be deterministic. We will also consider a slightly more general situation where the data depend on the unknown in a (nice) sub-linear way.
2. a different estimate, which is obtained by a duality method, based on  $L^1 - L^r$  estimates for a backward stochastic parabolic partial differential equation with terminal datum in  $L^1(U)$ , see Theorem 4.1. These  $L^1 - L^r$  estimates are obtained by adapting, in the framework of backward stochastic parabolic partial differential equations, the estimates by Boccardo and Gallouët as they can be found in [BG89]. Note that a stochastic duality method based on the theory of backward SPDE was already used in [DRV21].

The results of Theorem 2.1 and Theorem 4.1 are slightly different, with minimal differences however. The proof by duality is, in our opinion, more elementary than the proof by De Giorgi's truncation. Nevertheless, this is this latter proof by truncation, and not the former proof, by duality, that we manage to adapt to derive supremum estimates in the paper [LV23].

We will consider weak solutions to (1.3)-(1.4)-(1.5) defined as follows.

**Definition 1.1** (Weak solution). A function  $u \in L^2(\Omega \times (0, T), \mathcal{P}; H_0^1(U))$  (where  $\mathcal{P}$  is the predictable  $\sigma$ -algebra) is said to be a weak solution to (1.3)-(1.4)-(1.5) if

$$u \in C([0, T]; L^2(U)), \quad \mathbb{P} - a.s., \quad (1.6)$$

and for all  $\varphi \in C_c^\infty(U)$ , for a.e.  $(\omega, t) \in \Omega \times [0, T]$ ,

$$\int_U u(t)\varphi dx + \int_0^t \int_U a^{ij} \partial_i \varphi \partial_j u(s) dx ds = \int_U u_0 \varphi dx + \int_0^t \int_U g_k(s) \varphi dx dB^k(s). \quad (1.7)$$

## 2 Supremum estimate by De Giorgi's method

**Theorem 2.1** (Supremum estimate). *Let  $u$  be a solution<sup>1</sup> to (1.3)-(1.4)-(1.5). Let  $\mu > 1 + \frac{d}{2}$ . Assume that  $f$  and  $g_k$  are some random functions of the variables  $(t, x, u)$  satisfying*

$$|f(t, x, u)| \leq \bar{f}(t, x) + a(t, x)|u|, \quad \sum_k |g_k(t, x, u)|^2 \leq \mathbf{G}^2(t, x) + (b(t, x)u)^2, \quad (2.1)$$

---

<sup>1</sup>specify

where  $\bar{f}$ ,  $\mathbf{G}$ ,  $a$ ,  $b$  are some random predictable functions of  $(t, x)$  such that the deterministic bound

$$R := \|a\|_{L^\mu(Q_T)} + \|b\|_{L^{2\mu}(Q_T)} < +\infty, \quad \mathbb{P} - a.s., \quad (2.2)$$

is satisfied, while, for all  $p \geq 1$ , the following moments are finite:

$$\mathbb{E} \left[ \|\bar{f}\|_{L^\mu(Q_T)}^p + \|\mathbf{G}\|_{L^{2\mu}(Q_T)}^p \right] < +\infty. \quad (2.3)$$

Let  $\alpha, \beta \in [1, +\infty)$  satisfy the condition

$$\alpha < \beta. \quad (2.4)$$

We have then

$$\left\{ \mathbb{E} \left[ \|u\|_{L^\infty(Q_T)}^\alpha \right] \right\}^{\frac{1}{\alpha}} \leq C(\alpha, \beta) \left\{ \mathbb{E} \left[ \|u_0\|_{L^\infty(U)}^\beta \right] + \mathbb{E} \left[ \|\bar{f}\|_{L^\mu(Q_T)}^\beta + \|\mathbf{G}\|_{L^{2\mu}(Q_T)}^\beta \right] \right\}^{\frac{1}{\beta}}, \quad (2.5)$$

where  $C(\alpha, \beta)$  is a constant depending on  $\alpha, \beta, U, T, d, \mu, R$  and on the ellipticity constant  $\lambda$  in (1.2) only.

*Proof of Theorem 2.1.* We will assume that  $u_0$  is deterministic, but all that follows is valid when  $u_0$  is  $\mathcal{F}_0$ -measurable, we simply have to work with  $\mathbb{P}(\cdot|\mathcal{F}_0)$  instead of  $\mathbb{P}$ . We will also prove Theorem 2.1 under the restriction<sup>2</sup> that  $T$  is sufficiently small,  $T \leq T_1$ , where  $T_1 > 0$  depends on  $U, T, d, \mu, R$  and on the ellipticity constant  $\lambda$  in (1.2) only. The estimate on the global interval  $[0, T]$  results from the iteration of this preliminary partial estimate.

For  $u \in \mathbb{R}, \xi \geq 0$ , let us introduce the quantities

$$H(u; \xi) = (u - \xi)_+ + (u + \xi)_-, \quad \eta(u; \xi) = \frac{1}{2}H(u; \xi)^2 = \frac{1}{2}(u - \xi)_+^2 + \frac{1}{2}(u + \xi)_-^2, \quad (2.6)$$

and the truncated energy

$$U(\xi) = \sup_{0 \leq t \leq T} \int_U \eta(u; \xi) dx dt + \iint_{Q_T} |\nabla H(u; \xi)|^2 dx dt. \quad (2.7)$$

In (2.7),  $\xi \in \mathbb{R}_+$  is a given truncation level. Let  $\Lambda$  be a deterministic quantity, homogeneous to  $u, U^{\frac{1}{2}}$ , etc., that will play the role of a given threshold. We assume

$$\Lambda \geq \|u_0\|_{L^\infty(U)}. \quad (2.8)$$

Our aim will be to estimate the probability  $p_\varepsilon$  to have

$$\|u\|_{L^\infty(Q_T)} > \frac{\Lambda}{\varepsilon}, \quad (2.9)$$

so will study the probability that  $U(\xi) = 0$ , with

$$\xi = \frac{\Lambda}{\varepsilon}. \quad (2.10)$$

**Notation.** We will use the notation  $A \lesssim B$  when  $A \leq CB$  with a constant  $C$  depending on  $U, T, d, \mu, R$  and on the ellipticity constant  $\lambda$  only. When necessary, we will let the letter  $C$  be explicit, although it may vary from line to line.

<sup>2</sup>we get directly the result with  $T_1 = T$  if  $R = 0$  in (2.2)

**Energy estimate.** Note that

$$\partial_u \eta(u; \xi) = (u - \xi)_+ - (u + \xi)_-, \quad |\partial_u \eta(u; \xi)| \leq H(u; \xi), \quad \partial_u^2 \eta(u; \xi) = \mathbf{1}_{|u| \geq \xi}. \quad (2.11)$$

By (1.3), (2.11) and the Itô formula (the Itô formula can be justified as in [DHV16, Proposition A.1]), we have

$$U(\xi) \lesssim \iint_{Q_T} \left[ |f| H(u; \xi) + \frac{1}{2} \sum_k g_k^2 \mathbf{1}_{|u| \geq \xi} \right] dx dt + M_T(\xi)^*, \quad (2.12)$$

where

$$M_t(\xi) = \int_0^t \int_U g_k \partial_u \eta(u; \xi) dx dB^k(s), \quad M_T(\xi)^* = \sup_{0 \leq t \leq T} |M_t(\xi)|. \quad (2.13)$$

From the structure hypothesis (2.1), and from the inequalities

$$|u| \leq H(u; \xi) + \xi, \quad u^2 \leq 2\eta(u; \xi) + 2\xi^2, \quad (2.14)$$

we deduce from (2.12) that

$$\begin{aligned} U(\xi) &\lesssim \iint_{Q_T} \{ \bar{f} H(u; \xi) + a(\eta(u; \xi) + \xi H(u; \xi)) \} dx dt \\ &\quad + \iint_{Q_T} \{ \mathbf{G}^2 \mathbf{1}_{|u| \geq \xi} + b^2 (\eta(u; \xi) + \xi^2 \mathbf{1}_{|u| \geq \xi}) \} dx dt + M_T(\xi)^*. \end{aligned} \quad (2.15)$$

We apply the Hölder inequality to various terms in (2.15) to obtain

$$\begin{aligned} U(\xi) &\lesssim \left\{ \Theta_T \|H(u; \xi)\|_{L^{\mu'}(Q_T)} + \|\xi H(u; \xi)\|_{L^{\mu'}(Q_T)} + \|\eta(u; \xi)\|_{L^{\mu'}(Q_T)} \right\} \\ &\quad + \left\{ \Theta_T^2 \|\mathbf{1}_{|u| \geq \xi}\|_{L^{\mu'}(Q_T)} + \|\xi^2 \mathbf{1}_{|u| \geq \xi}\|_{L^{\mu'}(Q_T)} \right\} + M_T(\xi)^*, \end{aligned} \quad (2.16)$$

where

$$\Theta_T = \|\bar{f}\|_{L^\mu(Q_T)} + \|\mathbf{G}\|_{L^{2\mu}(Q_T)}. \quad (2.17)$$

**Non-linear estimate.** The energy  $U(\xi)$  is dominating both  $\|H\|_{L_t^\infty L_x^2}^2$  and  $\|\nabla H\|_{L_{x,t}^2}^2$ , where  $H := H(u; \xi)$ . By interpolation and Sobolev's inequality, the bound

$$\|H(u; \xi)\|_{L^\gamma(Q_T)}^2 \lesssim U(\xi), \quad \gamma := 2 \frac{d+2}{2} = 2p_F, \quad (2.18)$$

is satisfied (see (3.30) below for the derivation of a similar bound). The hypothesis  $\mu > 1 + d/2$ , equivalent to  $\mu' < p_F$ , gives

$$\rho := \frac{\gamma}{2\mu'} = \frac{p_F}{\mu'} > 1. \quad (2.19)$$

Let  $\zeta \in \mathbb{R}_+$  be a multiple  $\zeta = (1 - \theta)\xi$ , so that  $\xi - \zeta = \theta\xi$ . We will use the bounds from above

$$\mathbf{1}_{|u| \geq \xi} \leq \left( \frac{H(u; \zeta)}{(\xi - \zeta)} \right)^p, \quad H(u; \xi) \leq H(u; \zeta), \quad (2.20)$$

for different values of  $p$ . For instance, with  $n = 0$  or  $1$ ,

$$\|\xi^n H(u; \xi)\|_{L^{\mu'}(Q_T)} \leq \frac{\xi^n}{(\theta\xi)^{2\rho-1}} \|H(u; \zeta)\|_{L^\gamma(Q_T)}^{2\rho}, \quad (2.21)$$

and

$$\|\xi^{2n} \mathbf{1}_{|u| \geq \xi}\|_{L^{\mu'}(Q_t)} \leq \frac{\xi^{2n}}{(\theta\xi)^{2\rho}} \|H(u; \zeta)\|_{L^\gamma(Q_t)}^{2\rho}, \quad (2.22)$$

and

$$\|\eta(u; \xi)\|_{L^{\mu'}(Q_T)} \leq \frac{1}{(\theta\xi)^{2(\rho-1)}} \|H(u; \zeta)\|_{L^\gamma(Q_T)}^{2\rho}. \quad (2.23)$$

Reporting (2.21)-(2.22)-(2.23) in (2.16) and using (2.18) leads to the non-linear estimate

$$U(\xi) \lesssim \frac{\xi^2 + \Theta_T^2}{(\theta\xi)^{2\rho}} U(\zeta)^\rho + M_T(\xi)^*. \quad (2.24)$$

**Recursion in the deterministic case.** To deal with the martingale term  $M_T(\xi)^*$  in (2.24), we use a procedure which is better understood if we recall first the approach in the deterministic case. Indeed, assuming that  $M_T(\xi)^* \equiv 0$  in (2.24), and that  $\varepsilon = 1$  in (2.10), we apply (2.24) with  $\xi = \xi_{k+1}$ ,  $\zeta = \xi_k := (1 - 2^{-k-1})\bar{\xi}$ . This yields the recursive inequality

$$U_{k+1} \leq U_b^{1-\rho} A^k U_k^\rho, \quad (2.25)$$

where  $U_k = U(\xi_k)$ ,  $A$  is a constant depending on  $U$ ,  $T$ ,  $d$ ,  $\mu$ ,  $R$  and  $\lambda$  only, and

$$U_b := \left[ \frac{1}{\bar{\xi}^{2(\rho-1)}} + \frac{\Theta_T^2}{\bar{\xi}^{2\rho}} \right]^{-\frac{1}{\rho-1}}. \quad (2.26)$$

Applying (2.25) recursively leads to

$$U_{k+1} \leq U_b^{1-\rho^{k+1}} A^{S\rho^{k+1}} U_0^{\rho^{k+1}}, \quad S := \sum_{j \geq 0} j\rho^{-j-1} < +\infty, \quad (2.27)$$

and then, assuming  $\Theta_T \leq \Lambda$  and  $U_0 \leq \delta\Lambda^2$ , where  $\delta < 1$ , finally gives a doubly exponential decay  $U_k \leq \delta\rho^k \Lambda^2$ .

**Recursion in the stochastic case.** Let  $\underline{\rho} \in \mathbb{R}$  satisfy

$$1 < \underline{\rho} < \frac{1 + \rho}{2}. \quad (2.28)$$

Let  $\xi$  be defined by (2.10) and set

$$\xi_k = (1 - 2^{-k-1})\xi, \quad U_k = U(\xi_k). \quad (2.29)$$

We examine the occurrence of the bound

$$U_k \leq \delta \rho^k \Lambda^2. \quad (2.30)$$

Assume first that (2.30) is satisfied for a given  $k \geq 0$ . We apply (2.24) with  $(\xi, \zeta) = (\xi_{k+1}, \xi_k)$ . Then  $\xi_{k+1} = (1 - \theta)\xi_k$ , with

$$\theta = \frac{2^{-k-2}}{1 - 2^{-k-1}} \geq 2^{-k-2}, \quad (2.31)$$

so there is a constant  $A = 4^\rho$  such that

$$U_{k+1} \leq \left[ \frac{1}{\xi^{2(\rho-1)}} + \frac{\Theta_T^2}{\xi^{2\rho}} \right] A^k U_k^\rho + M_T(\xi_{k+1})^*. \quad (2.32)$$

Using (2.30), we find out that

$$U_{k+1} \leq \delta^{\rho^{k+1}} \Lambda^2 \left[ (\xi^{-1} \Lambda)^{2(\rho-1)} + (\Theta_T \Lambda^{-1})^2 (\xi^{-1} \Lambda)^{2\rho} \right] \varepsilon^{2(\rho-1)} A^k \delta^{\rho^k (\rho-\rho)} + M_T(\xi_{k+1})^*. \quad (2.33)$$

The control (2.30) will therefore be satisfied at the next rank  $k+1$  if

$$\left[ (\xi^{-1} \Lambda)^{2(\rho-1)} + (\Theta_T \Lambda^{-1})^2 (\xi^{-1} \Lambda)^{2\rho} \right] A^k \delta^{\rho^k (\rho-\rho)} \leq \frac{1}{2}, \quad (2.34)$$

and

$$M_T(\xi_{k+1})^* \leq \frac{1}{2} \delta^{\rho^{k+1}} \Lambda^2. \quad (2.35)$$

For  $k \geq 1$ , let  $E_{k+1}$  denote the event  $E_{k+1} = \{(2.34) \& (2.35)\}$  and let  $H_k$  denote the event (2.30). Our aim is to evaluate the probability  $\mathbb{P}(\mathbf{H})$  of the event

$$\mathbf{H} = \bigcap_{k \geq 0} H_k = \bigcap_{k \geq 0} \mathbf{H}_k, \quad \mathbf{H}_k := \bigcap_{j=0}^k H_j. \quad (2.36)$$

We will first estimate  $\sum_{k \geq 0} p_k - p_{k+1}$ , where  $p_k = \mathbb{P}(\mathbf{H}_k)$ , and then, in the next step, evaluate  $p_0$ . Since  $\mathbf{H}_k \cap E_{k+1} \subset \mathbf{H}_{k+1}$ , we have

$$p_k - p_{k+1} \leq \mathbb{P}(\mathbf{H}_k) - \mathbb{P}(\mathbf{H}_k \cap E_{k+1}) = \mathbb{P}(\mathbf{H}_k \cap E_{k+1}^c) \quad (2.37)$$

Denote by  $G_k$  the event

$$G_k = \{\Theta_T \leq 2^k \Lambda\}. \quad (2.38)$$

We have

$$\mathbb{P}(\mathbf{H}_k \cap E_{k+1}^c) \leq \mathbb{P}(\mathbf{H}_k \cap E_{k+1}^c \cap G_k) + \mathbb{P}(G_k^c). \quad (2.39)$$

We use the exponential martingale inequality

$$\mathbb{P}(M_\infty^* \geq \alpha + \beta \langle M, M \rangle_\infty) \leq e^{-2\alpha\beta}, \quad (2.40)$$

with  $M_t = M_{t \wedge T}(\xi_{k+1})$  and some deterministic numbers  $\alpha = V_k$ ,  $\beta = \hat{V}_k^{-1}$  to get

$$\mathbb{P}(\mathbf{H}_k \cap E_{k+1}^c \cap G_k) \leq e^{-2V_k \hat{V}_k^{-1}} + \mathbb{P}(B_k \cap G_k) + \mathbb{P}(L_k \cap G_k), \quad (2.41)$$

where  $B_k$  is the event

$$B_k = \mathbf{H}_k \cap \left\{ \frac{1}{2} \delta^{\rho^{k+1}} \Lambda^2 \leq V_k + \hat{V}_k^{-1} \langle M(\xi_{k+1}), M(\xi_{k+1}) \rangle_T \right\}, \quad (2.42)$$

while  $L_k$  denote the event

$$L_k = \left\{ \left[ (\xi^{-1} \Lambda)^{2(\rho-1)} + (\Theta_T \Lambda^{-1})^2 (\xi^{-1} \Lambda)^{2\rho} \right] A^k \delta^{\rho^k (\rho-\rho)} > \frac{1}{2} \right\}. \quad (2.43)$$

The quadratic variation of  $M_t(\xi)$  at time  $T$  is

$$\langle M(\xi), M(\xi) \rangle_T = \int_0^T \sum_{k \geq 0} \left| \int_U g_k \partial_u \eta(u; \xi) dx \right|^2 dt, \quad (2.44)$$

which can be bounded as follows:

$$\langle M(\xi), M(\xi) \rangle_T \leq \int_0^T \sum_{k \geq 0} \left[ \int_U |g_k|^2 \mathbf{1}_{|u| \geq \xi} dx \right] \left[ \int_U \eta(u; \xi) dx \right] dt. \quad (2.45)$$

Proceeding as in (2.15)-(2.16) and using the bound

$$\int_U \eta(u; \xi)(t) dx \leq 2U(\xi), \quad t \in [0, T], \quad (2.46)$$

we obtain

$$\langle M(\xi), M(\xi) \rangle_T \leq \left\{ \Theta_T^2 \|\mathbf{1}_{|u| \geq \xi}\|_{L^{\mu'}(Q_T)} + R \|\xi^2 \mathbf{1}_{|u| \geq \xi}\|_{L^{\mu'}(Q_T)} + R \|\eta(u; \xi)\|_{L^{\mu'}(Q_T)} \right\} U(\xi). \quad (2.47)$$

By (2.21) (or a similar bound), we have

$$\int_U H(u; \xi) dx \leq \frac{1}{\theta \xi} \int_U \eta(u; \zeta) dx \leq \frac{2}{\theta \xi} U(\zeta). \quad (2.48)$$

Using also (2.22), (2.23), we deduce from (2.47) the non-linear estimate

$$\langle M(\xi), M(\xi) \rangle_T \lesssim \left[ \frac{1}{(\theta \xi)^{2(\rho-1)}} + \frac{\Theta_T^2}{(\theta \xi)^{2\rho}} \right] U(\zeta)^{1+\rho}. \quad (2.49)$$

We take  $\xi = \xi_{k+1}$  and  $\zeta = \xi_k$  in (2.49) to obtain

$$\langle M(\xi_{k+1}), M(\xi_{k+1}) \rangle_T \lesssim \left[ \frac{1}{\xi^{2(\rho-1)}} + \frac{\Theta_T^2}{\xi^{2\rho}} \right] A^k U_k^{1+\rho}. \quad (2.50)$$

If  $U_k$  additionally satisfies (2.30), then we infer from (2.50) the bound

$$\langle M(\xi_{k+1}), M(\xi_{k+1}) \rangle_T \leq C_1 \left[ (\xi^{-1} \Lambda)^{2(\rho-1)} + (\Theta_T \Lambda^{-1})^2 (\xi^{-1} \Lambda)^{2\rho} \right] \Lambda^4 A^k \delta \rho^k (1+\rho). \quad (2.51)$$

If  $G_k$  is realized, then  $\Theta_T \Lambda^{-1} \leq 4^k$ , and since  $\xi = \varepsilon^{-1} \Lambda$  (cf. (2.10)), (2.51) yields

$$\langle M(\xi_{k+1}), M(\xi_{k+1}) \rangle_T \leq C_1 \varepsilon^{2(\rho-1)} \Lambda^4 \tilde{A}^k \delta \rho^k (1+\rho), \quad (2.52)$$

where  $\tilde{A} = 16A$ . Let  $V_k$  and  $\hat{V}_k$  be defined by

$$V_k = \frac{1}{8} \delta \rho^{k+1} \Lambda^2, \quad \hat{V}_k^{-1} \cdot C_1 \varepsilon^{2(\rho-1)} \Lambda^4 \tilde{A}^k \delta \rho^k (1+\rho) = V_k. \quad (2.53)$$

The estimate (2.52) then ensures that  $\mathbb{P}(B_k \cap G_k) = 0$ . Note that

$$V_k(\Lambda) \hat{V}_k^{-1}(\Lambda) = C_2 \varepsilon^{-2(\rho-1)} \tilde{A}^{-k} \delta^{-\rho^k (1+\rho-2\rho)}. \quad (2.54)$$

By similar computations, we also obtain

$$L_k \cap G_k \subset \left\{ \tilde{A}^k \delta \rho^k (\rho-\rho) > \frac{1}{2} \varepsilon^{-2(\rho-1)} \right\}. \quad (2.55)$$

We assume  $\varepsilon \leq 1$ . There exists a constant  $C_*$  depending on  $U, T, d, \mu, R$  and  $\lambda$  only such that

$$C_* \delta \leq 1 \implies \forall k \geq 0, \tilde{A}^k \delta \rho^k (\rho-\rho) \leq \frac{1}{2}. \quad (2.56)$$



Assuming the condition

$$C_*\delta \leq 1, \quad (2.57)$$

we have  $\mathbb{P}(L_k \cap G_{\Lambda_k}) = 0$  for all  $k$ . Using the Markov inequality to estimate  $\mathbb{P}(G_k^c)$ , we deduce from (2.37)-(2.41)-(2.54) the bound from above

$$p_k - p_{k+1} \leq \exp\left(-C_2\varepsilon^{-2(\rho-1)}\tilde{A}^{-k}\delta^{-\rho^k(1+\rho-2\rho)}\right) + \frac{3}{2^{km}}\frac{\mathbb{E}[\Theta_T^m]}{\Lambda^m}, \quad (2.58)$$

where  $m$  is an arbitrary exponent in  $[1, +\infty)$ . By (2.28), the quantity  $\alpha = 1 + \rho - 2\rho$  is positive. We use the domination

$$\sum_{k \geq 0} p_k - p_{k+1} \leq \int_0^\infty \Phi(t) dt + 3\frac{\mathbb{E}[\Theta_T^m]}{\Lambda^m}, \quad (2.59)$$

where

$$\Phi(t) := \exp\left(-\varepsilon^{-2(\rho-1)}\exp(\gamma\rho^t - pt)\right), \quad \gamma = \alpha|\ln(\delta)|, \quad p = \ln(\tilde{A}). \quad (2.60)$$

To justify the comparison (2.59), we assume that  $C_*$  in (2.57) is big enough to ensure that  $\gamma \ln(\rho) \geq 2p$ . Then  $t \mapsto \gamma\rho^t - pt$  is non-decreasing and it is also easy to prove that

$$|\Phi'(t)| = -\Phi'(t) \geq \frac{1}{2}\gamma\varepsilon^{-2(\rho-1)}\ln(\rho)\Phi(t), \quad (2.61)$$

and thus (using (2.57) which implies  $1 \lesssim \gamma$ )

$$\int_0^\infty \Phi(t) dt \leq \frac{\exp(-\varepsilon^{-2(\rho-1)}e^\gamma)}{\gamma\varepsilon^{-2(\rho-1)}\ln(\rho)} \leq C_3\varepsilon^{2(\rho-1)}\exp(-C_4\varepsilon^{-2(\rho-1)}). \quad (2.62)$$

Finally, we deduce from (2.59) and (2.62) the estimate

$$\mathbb{P}(\mathbf{H}) \geq p_0 - C_3\varepsilon^{2(\rho-1)}\exp(-C_4\varepsilon^{-2(\rho-1)}) - 3\frac{\mathbb{E}[\Theta_T^m]}{\Lambda^m}. \quad (2.63)$$

**Initial smallness condition.** Let us now estimate the probability  $p_0$  that the smallness condition

$$U_0 = U(\xi/2) \leq \delta\Lambda^2 \quad (2.64)$$

should be satisfied. We apply (2.24) with  $\zeta = \xi/4$  to obtain

$$U(\xi/2) \leq C_5\left[\frac{1}{\xi^{2(\rho-1)}} + \frac{\Theta_T^2}{\xi^{2\rho}}\right]U(\xi/4)^\rho + C_5M_T^*(\xi/2). \quad (2.65)$$

Let  $V, \hat{V}$  be some given constant (deterministic) numbers, and let

$$D = \left\{M_T^*(\xi/2) \leq V + \hat{V}^{-1}\langle M(\xi/2), M(\xi/2)_T \rangle\right\}. \quad (2.66)$$

Conditioning to the occurrence of  $G_0 \cap D$ , we obtain by (2.40),

$$\mathbb{P}(U(\xi/2) > \delta\Lambda^2) \leq \mathbb{P}(\{U(\xi/2) > \delta\Lambda^2\} \cap G_0 \cap D) + \frac{\mathbb{E}[\Theta_T^m]}{\Lambda^m} + e^{-2V\hat{V}^{-1}}. \quad (2.67)$$

If we set  $V = C_5^{-1}\frac{1}{2}\delta\Lambda^2$ ,  $\hat{V}^{-1} = C_5\Lambda^{-2}\Upsilon$ , then we can insert the bound given in (2.66) in (2.65) to get

$$\{U(\xi/2) > \delta\Lambda^2\} \cap D \subset K_U \cup K_M,$$

where

$$K_U = \left\{ \frac{1}{4} \delta \Lambda^2 < C_5 \left[ \frac{1}{\xi^{2(\rho-1)}} + \frac{\Theta_T^2}{\xi^{2\rho}} \right] U(\xi/4)^\rho \right\},$$

and

$$K_M = \left\{ \frac{1}{4} \delta \Lambda^2 < C_5 \hat{V}^{-1} \langle M(\xi/2), M(\xi)/2 \rangle_T \right\}.$$

It results then from (2.67) and the Markov inequality that

$$\begin{aligned} \mathbb{P}(U(\xi/2) > \delta \Lambda^2) &\leq \left( \frac{4C_5}{\delta \Lambda^2} \right)^q \left[ \frac{1}{\xi^{2(\rho-1)}} + \frac{\Lambda^2}{\xi^{2\rho}} \right]^q \mathbb{E} [U(\xi/4)^{q\rho}] \\ &\quad + \left( \frac{4C_5}{\delta \Lambda^2} \hat{V}^{-1} \right)^p \mathbb{E} [\langle M(\xi/2), M(\xi)/2 \rangle_T^p \mathbf{1}_{G_0}] + \frac{\mathbb{E}[\Theta_T^m]}{\Lambda^m} + e^{-\delta\Upsilon}, \end{aligned} \quad (2.68)$$

where  $p, q$  are some arbitrary positive exponent. We use the estimate (2.49) on the quadratic variation of  $M(\xi)$  with  $\xi = \varepsilon^{-1}\Lambda$ , to deduce from (2.68) that

$$\begin{aligned} \mathbb{P}(U(\xi/2) > \delta \Lambda^2) &\leq C(p, q) \left\{ \left( \frac{\varepsilon^{2(\rho-1)}}{\delta} \right)^q \mathbb{E} \left[ \frac{U(\xi/4)^{q\rho}}{\Lambda^{2q\rho}} \right] + \left( \frac{\Upsilon \varepsilon^{2(\rho-1)}}{\delta} \right)^p \mathbb{E} \left[ \frac{U(\xi/4)^{p(1+\rho)}}{\Lambda^{2p(1+\rho)}} \right] \right\} \\ &\quad + \frac{\mathbb{E}[\Theta_T^m]}{\Lambda^m} + e^{-\delta\Upsilon}. \end{aligned} \quad (2.69)$$

Admit for the moment the bound

$$\mathbb{E} [U(\xi/4)^p] \leq C(p) \mathbb{E} [\mathbf{F}_{2p}], \quad \mathbf{F}_p := \Theta_T^p + \|u_0\|_{L^\infty(U)}^p, \quad p > 0. \quad (2.70)$$

We will take  $p = \frac{\rho}{1+\rho}q$  in (2.69), and also assume  $\varepsilon^{2(\rho-1)} \leq \delta$  to get

$$\mathbb{P}(U(\xi/2) > \delta \Lambda^2) \leq C(q) \left( \frac{\varepsilon^{2(\rho-1)}}{\delta} \right)^{\frac{\rho}{1+\rho}q} \left( 1 + \left( \frac{\Upsilon}{\delta} \right)^p \right) \frac{\mathbb{E} [\mathbf{F}_{2q\rho}]}{\Lambda^{2q\rho}} + \frac{\mathbb{E} [\mathbf{F}_m]}{\Lambda^m} + e^{-\delta\Upsilon}. \quad (2.71)$$

Combining (2.71) with (2.63), we obtain finally

$$\begin{aligned} \mathbb{P}(\mathbf{H}^c) &\leq C_3 \varepsilon^{2(\rho-1)} \exp \left( -C_4 \varepsilon^{-2(\rho-1)} \right) + C(q) \left( \frac{\varepsilon^{2(\rho-1)}}{\delta} \right)^p \left( 1 + \left( \frac{\Upsilon}{\delta} \right)^p \right) \frac{\mathbb{E} [\mathbf{F}_{2q\rho}]}{\Lambda^{2q\rho}} \\ &\quad + 4 \frac{\mathbb{E} [\mathbf{F}_m]}{\Lambda^m} + e^{-\delta\Upsilon}, \end{aligned} \quad (2.72)$$

with  $p, q, m \geq 1$ ,  $p = \frac{\rho}{1+\rho}q$ .

**Conclusion.** We make the following choice of parameters in (2.72): we take  $\Upsilon = a |\ln(\varepsilon)|$ ,  $\delta > 0$  fixed satisfying the smallness condition (2.57). If the constant  $a$  is large enough, then (2.72) gives

$$\mathbb{P} \left[ \|u\|_{L^\infty(Q_T)} > \frac{\Lambda}{\varepsilon} \right] \leq C_5(N, q, r) \left[ \varepsilon^N + \varepsilon^r \frac{\mathbb{E} [\mathbf{F}_{2q\rho}]}{\Lambda^{2q\rho}} + \frac{\mathbb{E} [\mathbf{F}_m]}{\Lambda^m} \right], \quad (2.73)$$

where  $N := a\delta$  can be arbitrary large and  $r \in (1, 2(\rho-1)p)$ . We choose then, for a given  $\kappa > 0$ ,

$$\Lambda = \frac{\bar{\Lambda}}{\varepsilon^\kappa} \quad (2.74)$$

and obtain the tail estimate

$$\mathbb{P} \left[ \|u\|_{L^\infty(Q_T)} > \frac{\bar{\Lambda}}{\varepsilon^{1+\kappa}} \right] \leq C_5(N, q, r) \left[ \varepsilon^N + \varepsilon^{r+2q\rho\kappa} \frac{\mathbb{E}[\mathbf{F}_{2q\rho}]}{\bar{\Lambda}^{2q\rho}} + \varepsilon^{m\kappa} \frac{\mathbb{E}[\mathbf{F}_m]}{\bar{\Lambda}^m} \right], \quad (2.75)$$

valid for  $\varepsilon \leq C_6^{-1}$ . We have then, for  $\alpha > 0$ ,

$$\mathbb{E} \left[ \|u\|_{L^\infty(Q_T)}^\alpha \right] = \int_0^\infty \alpha t^{\alpha-1} \mathbb{P} [\|u\|_{L^\infty(Q_T)} > t] dt, \quad (2.76)$$

and after a simple change of variable,

$$\mathbb{E} \left[ \|u\|_{L^\infty(Q_T)}^\alpha \right] \lesssim \bar{\Lambda}^\alpha + \bar{\Lambda}^\alpha \int_0^{C_6^{-1}} \frac{\alpha}{\varepsilon^{\alpha(1+\kappa)+1}} \mathbb{P} \left[ \|u\|_{L^\infty(Q_T)} > \frac{\bar{\Lambda}}{\varepsilon^{1+\kappa}} \right] d\varepsilon. \quad (2.77)$$

If  $N, q, m$  are large enough, so that

$$\alpha(1+\kappa) < \min(N, m\kappa, r+2q\rho\kappa), \quad (2.78)$$

then, we deduce from (2.75) and (2.77) the bound

$$\mathbb{E} \left[ \|u\|_{L^\infty(Q_T)}^\alpha \right] \leq C_7(N, q, r, \alpha) \bar{\Lambda}^\alpha \left( 1 + \frac{\mathbb{E}[\mathbf{F}_{2q\rho}]}{\bar{\Lambda}^{2q\rho}} + \frac{\mathbb{E}[\mathbf{F}_m]}{\bar{\Lambda}^m} \right). \quad (2.79)$$

Clearly, (2.78) will be satisfied if

$$\alpha < m, \quad \alpha < 2q\rho, \quad (2.80)$$

by a choice of  $\kappa$  sufficiently large. If  $\alpha$  and  $\beta$  satisfy the condition (2.4) now, then we select  $q, m$  satisfying (2.80) and  $m, 2q\rho \leq \beta$ . We set  $\bar{\Lambda} = \{\mathbb{E}[\mathbf{F}_\beta]\}^{\frac{1}{\beta}}$ . Since

$$\mathbb{E}[\mathbf{F}_\gamma] \leq \bar{\Lambda}^\gamma, \quad \gamma \leq \beta, \quad (2.81)$$

by the Hölder inequality, (2.79) gives

$$\mathbb{E} \left[ \|u\|_{L^\infty(Q_T)}^\alpha \right] \leq C(\alpha, \beta) \{\mathbb{E}[\mathbf{F}_\beta]\}^{\frac{\alpha}{\beta}}, \quad (2.82)$$

which is the desired estimate (2.5).

**Energy estimate.** There remains to prove (2.70), which we do here. Without loss of generality, we assume  $4\varepsilon \leq 1$ , with the consequence (since  $\Lambda \geq \|u_0\|_{L^\infty(U)}$  by (2.8)) that  $\xi \geq 4\|u_0\|_{L^\infty(U)}$ . The energy  $U(\xi)$  defined in (2.7) has also the expression

$$U(\xi) = \sup_{0 \leq t \leq T} \int_U \eta(u; \xi) dx dt + \iint_{Q_T} \mathbf{1}_{|u| > \xi} |\nabla u|^2 dx dt, \quad (2.83)$$

on which it is apparent that  $\xi \mapsto U(\xi)$  is non-increasing. Consequently, with  $\xi$  given by (2.10), we have  $U(\xi/4) \leq U(\xi_0)$ ,  $\xi_0 := \|u_0\|_{L^\infty(U)}$ . We use (2.16) to get the bound

$$U(\xi_0) \lesssim \left\{ (\Theta_T + \xi_0) \|H(u; \xi_0)\|_{L^{\mu'}(Q_T)} + \|\eta(u; \xi_0)\|_{L^{\mu'}(Q_T)} \right\} + (\Theta_T^2 + \xi_0^2) + M_T(\xi_0)^*. \quad (2.84)$$

By the Hölder inequality and (2.18),

$$\|H(u; \xi_0)\|_{L^{\mu'}(Q_T)} \lesssim U(\xi_0)^{\frac{1}{2}}, \quad \|\eta(u; \xi_0)\|_{L^{\mu'}(Q_T)} \lesssim T^{\frac{1}{\rho'}} U(\xi_0). \quad (2.85)$$

If  $T$  satisfies the smallness condition<sup>3</sup>

$$CT \leq 1, \quad (2.86)$$

then (2.84)-(2.85) imply

$$U(\xi_0) \lesssim (\Theta_T^2 + \xi_0^2) + M_T(\xi_0)^*. \quad (2.87)$$

Let  $T_1$  be such that  $CT_1 = \frac{1}{2}$ , where  $C$  is the constant in (2.86). Then for  $T \leq T_1$ , we have

$$\mathbb{E}[U(\xi_0)^p] \leq C(p)(\Theta_T^{2p} + \xi_0^{2p}) + C(p)\mathbb{E}[|M_T(\xi_0)^*|^p]. \quad (2.88)$$

By the Burkholder-Davis-Gundy inequality, we get

$$\mathbb{E}[U(\xi_0)^p] \leq C(p)(\Theta^{2p} + \xi_0^{2p}) + C(p)\mathbb{E}\left[\langle M(\xi_0), M(\xi_0) \rangle_T^{\frac{p}{2}}\right]. \quad (2.89)$$

The estimate (2.47) on the quadratic variation of  $M$  and (2.85) give the bound from above

$$\mathbb{E}\left[\langle M(\xi_0), M(\xi_0) \rangle_T^{\frac{p}{2}}\right] \leq C(p)(\Theta^{2p} + \xi_0^{2p})U(\xi_0)^{\frac{p}{2}} + C(p)T^{p/2\rho'}U(\xi_0)^p, \quad (2.90)$$

so, under the size constraint (2.86), the estimate (2.70) follows from (2.89) and (2.90).  $\square$

### 3 Backward stochastic parabolic partial differential equation

We consider the setting given in the introductory part, Section 1. Suppose additionally that  $U$  is of class  $C^2$ . Let  $\phi$  be an  $L^1(U)$ -valued,  $\mathcal{F}_T$ -measurable random variable. By [DT12] (see Remark 3.1 and Lemma 3.1) we can consider  $(\psi, Z)$ , the solution to the BSPDE

$$d\psi + \partial_i(a^{ij}\partial_j\psi)dt = Z_k dB^k(t) \text{ in } Q_T \quad (3.1)$$

$$\psi = 0 \text{ on } \partial U \times (0, T) \quad (3.2)$$

$$\psi = \phi \text{ on } U \times \{T\}. \quad (3.3)$$

Solutions are understood in the following sense, [DT12, Definition 2.1-ii]):

$$\psi \in L^2(\Omega \times (0, T), \mathcal{P}; H_0^1(U)), \quad (Z_k) \in L^2(\Omega \times (0, T), \mathcal{P}; \ell^2(\mathbb{N}; L^2(U))), \quad (3.4)$$

where  $\mathcal{P}$  is the predictable  $\sigma$ -algebra,

$$\psi \in C([0, T]; L^2(U)), \quad \mathbb{P} - a.s., \quad (3.5)$$

and for all  $\varphi \in C_c^\infty(U)$ , for a.e.  $(\omega, t) \in \Omega \times [0, T]$ ,

$$\int_U \phi \varphi dx = \int_U \psi(T) \varphi dx + \int_t^T \int_U a^{ij} \partial_i \varphi \partial_j \psi(s) dx ds + \int_t^T \int_U Z_k(s) \varphi dx dB^k(s). \quad (3.6)$$

**Theorem 3.1** (Boccardo-Gallouët estimate for BSPDE). *Let  $r, p \in \mathbb{R}$  satisfy*

$$1 \leq r < p_F := \frac{d+2}{d}, \quad 1 \leq p < \frac{d+2}{d+1}. \quad (3.7)$$

---

<sup>3</sup>the condition (2.86) has the more precise version  $CR\rho'T \leq 1$ , where  $C$  depends on  $U$ ,  $d$ ,  $\mu$  and  $\lambda$ , and also on a bounded way  $T$ , but not on  $R$ , this explains why we need the bound by  $R$  in (2.2) to be deterministic. It also justifies that there is no constraint on the length of the time interval when  $R = 0$ .

Then there exists a constant  $C \geq 0$  depending on the dimension  $d$ , on the domain  $U$ , on the time  $T$ , on the ellipticity constant  $\lambda$  and on  $r, p$  only, such that

$$\left\{ \mathbb{E} \left[ \|\psi\|_{L^r(Q_T)}^r \right] \right\}^{1/r} + \left\{ \mathbb{E} \left[ \|\mathbf{Z}\|_{L^p(Q_T)}^p \right] \right\}^{1/p} \leq C \mathbb{E} [\|\phi\|_{L^1(U)}], \quad (3.8)$$

where  $\mathbf{Z} = \|(Z_k)\|_{\ell^2(\mathbb{N})}$ .

*Proof of Theorem 3.1.* We will denote by  $C_1, C_2, \dots$  some constant depending on  $d$ , on the domain  $U$ , on the time  $T$ , on the ellipticity constant  $\lambda$  and on  $r, p$  only. The proof breaks into several steps.

**Step 1. Itô Formula.** Let  $J: \mathbb{R} \rightarrow \mathbb{R}$  be a convex non-negative function of class  $C^2$ , such that  $J'$  and  $J''$  are bounded and  $J(0) = 0$  (this last condition ensures in particular that  $J(\psi(t)) \in H_0^1(U)$  for all  $t \in [0, T]$ ). We have then, by Itô's formula (which can be justified as in [DHV16, Proposition A.1] for instance), and for all non-negative  $\varphi \in C_c^\infty(U)$ ,

$$\begin{aligned} \int_U J(\psi(T))\varphi dx &= \int_U J(\psi(t))\varphi dx + \int_t^T \int_U J''(\psi) a^{ij} \partial_i \psi \partial_j \psi \varphi dx ds \\ &+ \int_t^T \int_U a^{ij} \partial_i \varphi \partial_j J(\psi) \varphi dx ds + \frac{1}{2} \int_t^T \int_U J''(\psi) \mathbf{Z}^2 \varphi dx ds + \int_t^T \int_U J'(\psi) Z_k \varphi dx dB^k(s), \end{aligned} \quad (3.9)$$

and, as a consequence of (1.2),

$$\begin{aligned} \int_U J(\psi(T))\varphi dx &\geq \int_U J(\psi(t))\varphi dx + \lambda \int_t^T \int_U J''(\psi) |\nabla \psi|^2 \varphi dx ds + \int_t^T \int_U a^{ij} \partial_i \varphi \partial_j J(\psi) \varphi dx ds \\ &+ \frac{1}{2} \int_t^T \int_U J''(\psi) \mathbf{Z}^2 \varphi dx ds + \int_t^T \int_U J'(\psi) Z_k \varphi dx dB^k(s), \end{aligned} \quad (3.10)$$

where we recall that

$$\mathbf{Z}^2 := \|\mathbf{Z}\|_{\ell^2}^2 = \sum_{k \geq 0} |Z_k|^2. \quad (3.11)$$

Since  $x \mapsto J(\psi(x, t))$  is non-negative and vanishes on  $\partial U$ , we can then take  $\varphi = 1$  in (3.10), while keeping the same sign in the inequality. To justify this, we proceed as in [MPT02] for instance. For  $\delta > 0$ , let  $\zeta_\delta$  denote the solution to the problem

$$-\delta^2 \partial_i (a^{ij} \partial_j \zeta_\delta) + \zeta_\delta = 1 \text{ in } U, \quad \zeta_\delta = 0 \text{ on } \partial U. \quad (3.12)$$

By the Appendix in [MPT02], we have  $0 \leq \zeta_\delta \leq 1$  and  $\zeta_\delta \uparrow 1$ . Setting  $\varphi = \zeta_\delta$  in (3.10), we can take the limit  $\delta \rightarrow 0$  of all the terms but the ‘‘boundary layer’’ term

$$\int_t^T \int_U a^{ij} \partial_i \zeta_\delta \partial_j J(\psi) \varphi dx ds = \int_t^T \int_U J(\psi) \frac{1 - \zeta_\delta}{\delta^2} dx ds. \quad (3.13)$$

Since  $J(\psi) \geq 0$  and  $\zeta_\delta \leq 1$ , this term is however non-negative, which yields

$$\begin{aligned} \int_U J(\psi(T)) dx &\geq \int_U J(\psi(t)) dx \\ &+ \lambda \int_t^T \int_U J''(\psi) |\nabla \psi|^2 dx ds + \frac{1}{2} \int_t^T \int_U J''(\psi) \mathbf{Z}^2 dx ds + \int_t^T \int_U J'(\psi) Z_k dx dB^k(s). \end{aligned} \quad (3.14)$$

**Step 2. Reduction to a non-negative  $\psi$ .** According to the decomposition  $\phi = \phi^+ - \phi^-$  as sum of the positive and (minus) negative part, we have, by uniqueness of the solution to (3.1)-(3.2)-(3.3), a decomposition

$$(\psi, Z) = (\psi^{(+)}, Z^{(+)}) - (\psi^{(-)}, Z^{(-)}), \quad \psi^{(\pm)}(T) = \phi^{\pm}. \quad (3.15)$$

It is sufficient to establish (3.8) for only one of the component in (3.15), for instance the (+) one, so we can assume  $\phi \geq 0$  a.e., a.s. Let us show that  $\psi \geq 0$  a.e., a.s. then: we apply (3.14) with a non-negative function  $J$  such that  $J(s) = 0$  is equivalent to  $s \geq 0$  (typically, a suitable regularization of  $s \mapsto s^-$ ). We take expectation and use the non-negative sign of the terms involving  $J''$  to obtain

$$\mathbb{E} \left[ \int_U J(\psi(t)) dx \right] \leq \mathbb{E} \left[ \int_U J(\phi) dx \right] = 0,$$

hence  $\psi \geq 0$  a.e., a.s.

**Step 3.  $L^1$ -norm of  $\psi$ .** We use the fact that  $\psi \geq 0$  and apply (3.9) with  $J(s) = s$  to obtain

$$\|\phi\|_{L^1(U)} = \|\psi(T)\|_{L^1(U)} \geq \|\psi(t)\|_{L^1(U)} + (N_T - N_t), \quad N_t := \int_0^t \int_U Z_k dx dB^k(s). \quad (3.16)$$

Let  $(K_t)$  be the martingale defined by

$$K_t = \mathbb{E} [\|\phi\|_{L^1(U)} | \mathcal{F}_t]. \quad (3.17)$$

Since  $(K_t)$  is adapted, taking conditional expectation in (3.16) gives us

$$\|\psi(t)\|_{L^1(U)} \leq K_t. \quad (3.18)$$

In particular, we have

$$\mathbb{E} \|\psi(t)\|_{L^1(U)} \leq \mathbb{E} \|\phi\|_{L^1(U)}, \quad (3.19)$$

for all  $t \in [0, T]$ .

**Step 4. Elementary blocks.** Let  $\Gamma$  be a given constant that will be fixed later (see (3.46)). For  $n \in \mathbb{Z}$ , let  $J_n$  be defined by

$$J_n(s) = \int_0^s \min [1, (\Gamma 2^n)^{-1}(\sigma - \Gamma 2^n)^+] d\sigma = \begin{cases} 0 & s \leq \Gamma 2^n, \\ (\Gamma 2^{n+1})^{-1}(s - \Gamma 2^n)^2 & \Gamma 2^n \leq s \leq \Gamma 2^{n+1}, \\ s - \frac{3}{2}\Gamma 2^n & \Gamma 2^{n+1} \leq s. \end{cases} \quad (3.20)$$

Then  $J_n \in C^2(\mathbb{R} \setminus \{\Gamma 2^n, \Gamma 2^{n+1}\})$  with

$$J_n'' = \frac{1}{\Gamma 2^n} \mathbf{1}_{[\Gamma 2^n, \Gamma 2^{n+1})} \text{ on } \mathbb{R} \setminus \{\Gamma 2^n, \Gamma 2^{n+1}\}.$$

Although  $J_n$  is not strictly of class  $C^2$ , we can use (3.14) and an approximation procedure to justify the following inequality:

$$\begin{aligned} \int_U J_n(\psi(T)) dx &\geq \int_U J_n(\psi(t)) dx \\ &+ \frac{\lambda}{\Gamma 2^n} \iint_{B_n \cap Q_{t,T}} |\nabla \psi|^2 dx ds + \frac{1}{\Gamma 2^{n+1}} \iint_{B_n \cap Q_{t,T}} \mathbf{Z}^2 dx ds + (M_n(T) - M_n(t)), \end{aligned} \quad (3.21)$$

where  $Q_{t,T} = U \times (t, T)$ ,

$$M_n(t) = \int_0^t \int_U J'_n(\psi) Z_k dx dB^k(s), \quad (3.22)$$

and

$$B_n = \{(x, t) \in Q_T; \Gamma 2^n \leq \psi(x, t) < \Gamma 2^{n+1}\}.$$

Whether the end-point sets  $\{\psi = \Gamma 2^v\}$ ,  $v = n, n+1$ , are taken into account in the integral

$$\iint_{B_n} |\nabla \psi|^2 dx dt$$

has no importance since  $\nabla \psi = 0$  a.e. on a set  $\{\psi = \text{cst}\}$ . We may have to take care to the term involving  $\mathbf{Z}^2$  in (3.21), but (3.21) is simply obtained by considering the regularization of  $J''_n$  given by

$$s \mapsto \frac{1}{\Gamma 2^n} \min [\mathbf{1}_{[\Gamma 2^n, \Gamma 2^{n+1})}(s), \varepsilon^{-1}(s - \Gamma 2^n + \varepsilon)^+, \varepsilon^{-1}(s - \Gamma 2^{n+1} - \varepsilon)^-] \geq \frac{1}{\Gamma 2^n} \mathbf{1}_{[\Gamma 2^n, \Gamma 2^{n+1})}(s).$$

Averaging (3.21) with respect to  $\mathbb{P}$ , we get

$$\begin{aligned} \mathbb{E} \int_U J_n(\psi(t)) dx + \frac{\lambda}{\Gamma 2^n} \mathbb{E} \iint_{B_n \cap Q_{t,T}} |\nabla \psi|^2 dx ds + \frac{1}{\Gamma 2^{n+1}} \mathbb{E} \iint_{B_n \cap Q_{t,T}} \mathbf{Z}^2 dx ds \\ \leq \mathbb{E} \int_U J_n(\psi(T)) dx \leq \mathbb{E} \int_U \psi(T) dx \leq \mathbb{E} [\|\phi\|_{L^1(U)}]. \end{aligned} \quad (3.23)$$

**Step 5. Estimate on  $\psi$ .** Set

$$D_m = \bigcup_{n \geq m} B_m = \{(x, t) \in Q_T; \Gamma 2^m \leq \psi(x, t)\}, \quad (3.24)$$

and note that

$$J_m(\psi) \leq J_{m+1}(\psi) + \frac{3}{2} \Gamma 2^m. \quad (3.25)$$

We will estimate the  $L^r_{\omega, t, x}$ -norm of  $J_m(\psi)$  as follows. Let  $q = \frac{d}{d+1} r$  and let  $v \in L^q(\Omega \times (0, T); W_0^{1,q}(U))$ . For a.e.  $(\omega, t)$ , we have

$$\int_U |v(t, x)|^q dx \leq C_1 \left( \int_U |\nabla v(t)|^q \right)^{q^*/q}, \quad \frac{1}{q^*} = \frac{1}{q} - \frac{1}{d}, \quad (3.26)$$

by Sobolev's inequality. Note that

$$\frac{1}{q^*} = \frac{1}{r} \frac{d+1}{d} - \frac{1}{d} \iff \frac{1}{r} = \frac{1}{d+1} + \frac{d}{d+1} \frac{1}{q^*}, \quad (3.27)$$

so  $1 \leq r \leq q^*$ , and by Hölder's inequality,

$$\left( \mathbb{E} \int_U |v(t)|^r dx \right)^{\frac{1}{r}} \leq \left( \mathbb{E} \int_U |v(t)| dx \right)^{\frac{1}{d+1}} \left( \mathbb{E} \int_U |v(t)|^{q^*} dx \right)^{\frac{d}{d+1} \frac{1}{q^*}}. \quad (3.28)$$

We deduce then from (3.26) that

$$\mathbb{E} \int_U |v(t)|^r dx \leq \left( \mathbb{E} \int_U |v(t)| dx \right)^{\frac{q}{d}} \mathbb{E} \int_U |\nabla v(t)|^q dx, \quad (3.29)$$

and thus after integration in time, using Fubini's theorem,

$$\mathbb{E} \iint_{Q_T} |v|^r dxdt \leq C_1 \sup_{0 \leq t \leq T} (\mathbb{E} \|v(t)\|_{L^1(U)})^{q/d} \mathbb{E} \iint_{Q_T} |\nabla v|^q dxdt. \quad (3.30)$$

Let  $m \in \mathbb{Z}$  be an arbitrary integer (we may take  $m = 0$  typically). We apply (3.30) to  $v = J_{m+1}(\psi)$ . Since

$$\mathbb{E} \|J_{m+1}(\psi)(t)\|_{L^1(U)} \leq \mathbb{E} \|\psi(t)\|_{L^1(U)} = \mathbb{E} \|\psi(T)\|_{L^1(U)} \quad (3.31)$$

by (3.19), we obtain

$$\mathbb{E} \iint_{Q_T} J_{m+1}(\psi)^r dxdt \leq C_1 (\mathbb{E} \|\psi(T)\|_{L^1(U)})^{q/d} \mathbb{E} \iint_{Q_T} |J'_{m+1}(\psi) \nabla \psi|^q dxdt, \quad (3.32)$$

and so

$$\mathbb{E} \iint_{Q_T} J_{m+1}(\psi)^r dxdt \leq C_1 (\mathbb{E} \|\psi(T)\|_{L^1(U)})^{q/d} \iint_{D_{m+1}} |\nabla \psi|^q dxdt, \quad (3.33)$$

which gives

$$\mathbb{E} \iint_{Q_T} J_{m+1}(\psi)^r dxdt \leq C_1 (\mathbb{E} \|\psi(T)\|_{L^1(U)})^{q/d} \sum_{n \geq m+1} \mathbb{E} \iint_{B_n} |\nabla \psi|^q dxdt. \quad (3.34)$$

Next, by Hölder's inequality, we have

$$\mathbb{E} \iint_{B_n} |\nabla \psi|^q dxdt \leq \left[ \mathbb{E} \iint_{B_n} |\nabla \psi|^2 dxdt \right]^{q/2} [\mathbb{E} |B_n|]^{1-q/2}. \quad (3.35)$$

By the Markov inequality, we can estimate  $|B_n|$  from above as follows:

$$|B_n| \leq \iint_{B_n} \frac{\psi^r}{(\Gamma 2^n)^r} \quad (3.36)$$

If  $\Gamma 2^n \leq \psi$ , and  $n \geq m + 1$ , then

$$\psi \leq J_m(\psi) + \frac{3}{2} \Gamma 2^m \leq J_m(\psi) + \frac{3}{4} \psi, \quad (3.37)$$

so  $\psi \leq 4J_m(\psi)$  and (3.36) gives

$$|B_n| \leq 4^r \iint_{B_n} \frac{J_m(\psi)^r}{(\Gamma 2^n)^r}. \quad (3.38)$$

Inserting (3.38) in (3.33)-(3.35) gives us

$$\begin{aligned} \mathbb{E} \iint_{Q_T} J_{m+1}(\psi)^r dxdt &\leq C_2 (\mathbb{E} \|\psi(T)\|_{L^1(U)})^{q/d} \\ &\times \sum_{n \geq m+1} (\Gamma 2^n)^{-\beta q/2} \left[ \frac{1}{\Gamma 2^n} \mathbb{E} \iint_{B_n} |\nabla \psi|^2 dxdt \right]^{q/2} \left[ \mathbb{E} \iint_{B_n} J_m(\psi)^r \right]^{(1-q/2)}, \end{aligned} \quad (3.39)$$

where

$$\beta = r(2/q - 1) - 1 = p_F - r > 0. \quad (3.40)$$



By Young's inequality with a parameter  $\varepsilon > 0$ , we obtain

$$\begin{aligned} \mathbb{E} \iint_{Q_T} J_{m+1}(\psi)^r dxdt &\leq C_3(\varepsilon) (\mathbb{E} \|\psi(T)\|_{L^1(U)})^{2/d} \sum_{n \geq m+1} (\Gamma 2^n)^{-\beta} \left[ \frac{1}{\Gamma 2^n} \mathbb{E} \iint_{B_n} |\nabla \psi|^2 dxdt \right] \\ &\quad + \varepsilon \sum_{n \geq m+1} \mathbb{E} \iint_{B_n} J_m(\psi)^r, \end{aligned} \quad (3.41)$$

and so

$$\begin{aligned} \mathbb{E} \iint_{Q_T} J_{m+1}(\psi)^r dxdt \\ \leq C_3(\varepsilon) (\mathbb{E} \|\psi(T)\|_{L^1(U)})^{2/d} \sum_{n \geq m+1} (\Gamma 2^n)^{-\beta} \left[ \frac{1}{\Gamma 2^n} \mathbb{E} \iint_{B_n} |\nabla \psi|^2 dxdt \right] + \varepsilon \mathbb{E} \iint_{Q_T} J_m(\psi)^r. \end{aligned} \quad (3.42)$$

By (3.25), we also have

$$J_m(\psi)^r \leq 2^r J_{m+1}(\psi)^r + C_3(\Gamma 2^m)^r, \quad (3.43)$$

and so, for  $\varepsilon$  small enough,

$$\begin{aligned} \mathbb{E} \iint_{Q_T} J_m(\psi)^r dxdt \\ \leq C_5 [\Gamma 2^m]^r + C_5 (\mathbb{E} \|\psi(T)\|_{L^1(U)})^{2/d} \sum_{n \geq m} (\Gamma 2^n)^{-\beta} \left[ \frac{1}{\Gamma 2^n} \mathbb{E} \iint_{B_n} |\nabla \psi|^2 dxdt \right]. \end{aligned} \quad (3.44)$$

We use (3.23) to obtain

$$\mathbb{E} \iint_{Q_T} J_m(\psi)^r dxdt \leq C_5 [\Gamma 2^m]^r + C_5 (\mathbb{E} \|\psi(T)\|_{L^1(U)})^{2/d} (\Gamma 2^m)^{-\beta} \mathbb{E} [\|\phi\|_{L^1(U)}]. \quad (3.45)$$

We take  $\Gamma = \|\psi(0)\|_{L^1(U)}$ . Then  $(K_t)$  being defined by (3.17)

$$\Gamma = \mathbb{E} [K_0] = \mathbb{E} [K_T] = \mathbb{E} [\|\psi(T)\|_{L^1(U)}] = \mathbb{E} [\|\phi\|_{L^1(U)}], \quad (3.46)$$

so

$$\mathbb{E} \iint_{Q_T} J_m(\psi)^r dxdt \leq C_6(m) (\mathbb{E} [\|\phi\|_{L^1(U)}])^r. \quad (3.47)$$

Since  $0 \leq \psi \leq J_m(\psi) + \Gamma 2^{m+1}$ , (3.46) and (3.47) imply the first estimate in (3.8).

**Step 6. Estimate on  $\mathbf{Z}$ .** Let  $1 < p < \frac{2r}{r+1}$ . By Hölder's inequality, we have the bound

$$\mathbb{E} \left[ \|\mathbf{Z}\|_{L^p(B_n)}^p \right] \leq \left[ \mathbb{E} \iint_{B_n} \mathbf{Z}^2 dxdt \right]^{p/2} (\mathbb{E} |B_n|)^{1-p/2}, \quad (3.48)$$

and then, using (3.38),

$$\mathbb{E} \left[ \|\mathbf{Z}\|_{L^p(D_{m+1})}^p \right] \leq C_8 \sum_{n > m} (\Gamma 2^n)^{-\frac{p}{2}\gamma} \left[ \frac{1}{\Gamma 2^n} \mathbb{E} \iint_{B_n} \mathbf{Z}^2 dxdt \right]^{p/2} \left[ \mathbb{E} \iint_{B_n} J_m(\psi)^r dxdt \right]^{1-p/2}, \quad (3.49)$$

where

$$\gamma = r(2/p - 1) - 1 > 0. \quad (3.50)$$

By Young's inequality, we obtain

$$\mathbb{E} \left[ \|\mathbf{Z}\|_{L^p(D_{m+1})}^p \right] \leq C_9 \mu^{2/p} \sum_{n \geq m} (\Gamma 2^n)^{-\gamma} \left[ \frac{1}{\Gamma 2^n} \iint_{B_n} \mathbf{Z}^2 dx dt \right] + \mu^{-(1-p/2)^{-1}} \mathbb{E} \iint_{Q_T} J_m(\psi)^r dx dt, \quad (3.51)$$

where  $\mu$  is a positive constant. We take  $\mu = \Gamma^{(r-p)(1-p/2)}$ . Note that  $\frac{2r}{r+1} < r$  since  $1 < r$ , so  $p < r$  and  $\mu$  is a positive power of  $\Gamma$ . Note also that  $\mu^{2/p} = \Gamma^{p+\gamma-1}$ . By (3.46)-(3.47), we obtain

$$\mathbb{E} \left[ \|\mathbf{Z}\|_{L^p(D_{m+1})}^p \right] \leq C_9 \Gamma^{p+\gamma-1} \sum_{n \geq m} (\Gamma 2^n)^{-\gamma} \left[ \frac{1}{\Gamma 2^n} \iint_{B_n} \mathbf{Z}^2 dx dt \right] + C_6(m) (\mathbb{E} [\|\phi\|_{L^1(U)}])^p. \quad (3.52)$$

Finally, using the estimate (3.23) yields the bound

$$\mathbb{E} \left[ \|\mathbf{Z}\|_{L^p(D_{m+1})}^p \right] \leq C_{10}(m) (\mathbb{E} [\|\phi\|_{L^1(U)}])^p. \quad (3.53)$$

When  $n \leq m$ , we use the crude bound  $|B_n| \leq |Q_T|$  in (3.48) to obtain

$$\mathbb{E} \left[ \|\mathbf{Z}\|_{L^p(B_n)}^p \right] \leq C_1 1 \left[ \frac{1}{\Gamma 2^n} \mathbb{E} \iint_{B_n} \mathbf{Z}^2 dx dt \right]^{p/2} (\Gamma 2^n)^{p/2}. \quad (3.54)$$

We use (3.23), (3.46) and sum (3.54) over  $n \leq m$  to get

$$\mathbb{E} \left[ \|\mathbf{Z}\|_{L^p(Q_T \setminus D_{m+1})}^p \right] \leq C_{12}(m) (\mathbb{E} [\|\phi\|_{L^1(U)}])^p, \quad (3.55)$$

which, combined with (3.53), yields the second half of (3.8).  $\square$

## 4 Supremum estimate

**Theorem 4.1.** *Let  $u$  be a solution to (1.3)-(1.4)-(1.5) as in Definition 1.1. Assume that  $f$  and  $g_k$  are some random functions of the variables  $(t, x, u)$  satisfying (2.1), where  $\bar{f}$ ,  $\mathbf{G}$ ,  $a$ ,  $b$  are some random predictable functions of  $(t, x)$  such that*

$$\mathfrak{B} := \|a\|_{L^\mu(\Omega \times Q_T)} + \|b\|_{L^{2\mu}(\Omega \times Q_T)} < +\infty, \quad (4.1)$$

and

$$\|\bar{f}\|_{L^\mu(\Omega \times Q_T)} + \|\mathbf{G}\|_{L^{2\mu}(\Omega \times Q_T)} < +\infty. \quad (4.2)$$

Then, for all  $\alpha \geq 1$ , there is a constant  $C \geq 0$  depending on the dimension  $d$ , on the domain  $U$ , on the time  $T$ , on the ellipticity constant  $\lambda$  on  $\mu$  and on the quantity  $\mathfrak{B}$  in (4.1) only, such that

$$\mathbb{E} \left[ \|u\|_{L^\infty(Q_T)}^\alpha \right] \leq C (\|u_0\|_{L^\infty(U)} + \|\bar{f}\|_{L^\mu(\Omega \times Q_T)} + \|\mathbf{G}\|_{L^{2\mu}(\Omega \times Q_T)})^\alpha. \quad (4.3)$$

*Proof of Theorem 4.1.* For simplicity, we will assume  $u_0 \equiv 0$ , since the case  $u_0 \not\equiv 0$  can be treated by superposition with the solution to (1.3)-(1.4)-(1.5) with  $f \equiv 0$ ,  $g_k \equiv 0$ . By iteration of the result if necessary, it is sufficient to prove (4.3) under the condition that  $T$  is small enough, say  $T \leq T_1$ , where  $T_1 > 0$  depends on the dimension  $d$ , on the domain  $U$ , on the time  $T$ , on the ellipticity constant  $\lambda$ , on  $\mu$  and on the quantity  $\mathfrak{B}$  in (4.1) only. We will also need to assume that

$\mathbb{E} \left[ \|u\|_{L^\infty(Q_T)}^\alpha \right] < +\infty$ , an admissible hypothesis, provided we approximate  $u$  by the solutions to some regularized problems. Finally, we note that it is sufficient to establish (4.3) for  $\alpha \geq \alpha_1$ , for a given  $\alpha_1 \geq 1$ .

We denote by  $C_1, C_2, \dots$  some constants depending on the dimension  $d$ , on the domain  $U$ , on the time  $T$ , on the ellipticity constant  $\lambda$ , on  $\mu$  and on the quantity  $\mathfrak{B}$  only. We also set

$$\Upsilon = \|\bar{f}\|_{L^\mu(\Omega \times Q_T)} + \|\mathbf{G}\|_{L^{2\mu}(\Omega \times Q_T)}. \quad (4.4)$$

Let  $(\psi, Z)$  be the solution to (3.1)-(3.2)-(3.3) with given terminal condition  $\phi$ . By Itô's formula (again this can be justified as in [DHV16, Proposition A.1]), we have

$$\mathbb{E} \left[ \int_U u(T) \phi dx \right] = \mathbb{E} \left[ \iint_{Q_T} f \psi dx dt + \iint_{Q_T} Z_k g_k dx dt \right]. \quad (4.5)$$

Note that the condition  $\mu > 1 + \frac{d}{2}$  is equivalent to  $r := \mu' < p_F = 1 + \frac{2}{d}$  or  $p := (2\mu)' < \frac{d+2}{d+1}$ , where  $q'$  denote the conjugate exponent to  $q$ . We fix an exponent  $\mu_1$  such that  $\mu > \mu_1 > 1 + \frac{d}{2}$ . As a consequence of (2.1), we have

$$\begin{aligned} \mathbb{E} \left[ \int_U u(T) \phi dx \right] &\leq C_1 \left[ \|\bar{f}\|_{L^{\mu_1}(\Omega \times Q_T)} + \|au\|_{L^{\mu_1}(\Omega \times Q_T)} \right] \|\psi\|_{L^{\mu_1'}(\Omega \times Q_T)} \\ &\quad + C_1 \left[ \|\mathbf{G}\|_{L^{2\mu_1}(\Omega \times Q_T)} + \|bu\|_{L^{2\mu_1}(\Omega \times Q_T)} \right] \|\mathbf{Z}\|_{L^{(2\mu_1)'}(\Omega \times Q_T)}. \end{aligned} \quad (4.6)$$

From the estimate (3.8) for the backward equation, we deduce the estimate

$$\mathbb{E} \left[ \int_U u(T) \phi dx \right] \leq C_1 \left[ \Upsilon + \|au\|_{L^{\mu_1}(\Omega \times Q_T)} + \|bu\|_{L^{2\mu_1}(\Omega \times Q_T)} \right] \|\phi\|_{L^1(\Omega \times Q_T)}. \quad (4.7)$$

By the Hölder inequality, we have

$$\|au\|_{L^{\mu_1}(\Omega \times Q_T)}^{\mu_1} \leq \mathbb{E} \left[ \|a\|_{L^{\mu_1}(Q_T)}^{\mu_1} \|u\|_{L^\infty(Q_T)}^{\mu_1} \right] \leq \left\{ \mathbb{E} \left[ \|a\|_{L^{\mu_1}(Q_T)}^{\mu_1} \right] \right\}^{\mu_1/\mu} \left\{ \mathbb{E} \left[ \|u\|_{L^\infty(Q_T)}^{\alpha_1} \right] \right\}^{\mu_1/\alpha_1}, \quad (4.8)$$

where  $\alpha_1 = \mu\mu_1/(\mu - \mu_1)$ , and similarly

$$\|bu\|_{L^{2\mu_1}(\Omega \times Q_T)}^{2\mu_1} \leq \left\{ \mathbb{E} \left[ \|b\|_{L^{2\mu_1}(Q_T)}^{2\mu_1} \right] \right\}^{\mu_1/\mu} \left\{ \mathbb{E} \left[ \|u\|_{L^\infty(Q_T)}^{\alpha_2} \right] \right\}^{2\mu_1/\alpha_2}, \quad (4.9)$$

where  $\alpha_2 = 2\alpha_1$ . We use then Hölder's inequality again to obtain

$$\|a\|_{L^{\mu_1}(Q_T)} \leq \|a\|_{L^\mu(Q_T)} |Q_T|^{1/\alpha_1}, \quad \|b\|_{L^{2\mu_1}(Q_T)} \leq \|b\|_{L^{2\mu}(Q_T)} |Q_T|^{1/\alpha_2}. \quad (4.10)$$

Inserting (4.8), (4.9), (4.10) in (4.7) gives us

$$\mathbb{E} \left[ \int_U u(T) \phi dx \right] \leq C_2 \Upsilon' \|\phi\|_{L^1(\Omega \times Q_T)}, \quad \Upsilon' := \Upsilon + T^\delta \|u\|_{L^{\alpha_2}(\Omega; L^\infty(Q_T))}, \quad (4.11)$$

where  $\delta := 2\mu_1/\alpha_2 > 0$ . Let  $p \in 2\mathbb{N}$ ,  $p \geq 2 \vee \alpha$ . We apply (4.7) with

$$\phi = (\|u(T)\|_{L^p(U)} + \varepsilon)^{\alpha-p} u(T)^{p-1}, \quad (4.12)$$

where  $\varepsilon > 0$ , to obtain

$$\mathbb{E} \left[ (\|u(T)\|_{L^p(U)} + \varepsilon)^{\alpha-p} \|u(T)\|_{L^p(U)}^p \right] \leq C_1 \Upsilon' \mathbb{E} \left[ (\|u(T)\|_{L^p(U)} + \varepsilon)^{\alpha-p} \|u(T)\|_{L^{p-1}(U)}^{p-1} \right]. \quad (4.13)$$

We use the bound  $\|u(T)\|_{L^{p-1}(U)}^{p-1} \leq \|u(T)\|_{L^p(U)}^{p-1} |U|^{1/p}$  to obtain

$$\mathbb{E} \left[ (\|u(T)\|_{L^p(U)} + \varepsilon)^{\alpha-p} \|u(T)\|_{L^p(U)}^p \right] \leq C_3 |U|^{1/p} \Upsilon' \mathbb{E} \left[ (\|u(T)\|_{L^p(U)} + \varepsilon)^{\alpha-p} \|u(T)\|_{L^p(U)}^{p-1} \right]. \quad (4.14)$$

We can let  $\varepsilon \rightarrow 0$  in (4.14) to get

$$\mathbb{E} \left[ \|u(T)\|_{L^p(U)}^\alpha \right] \leq C_3 |U|^{1/p} \Upsilon' \mathbb{E} \left[ \|u(T)\|_{L^p(U)}^{\alpha-1} \right], \quad (4.15)$$

and thus

$$\mathbb{E} \left[ \|u(T)\|_{L^p(U)}^\alpha \right] \leq C_3 |U|^{1/p} \Upsilon' \left\{ \mathbb{E} \left[ \|u(T)\|_{L^p(U)}^\alpha \right] \right\}^{\frac{\alpha-1}{\alpha}}, \quad (4.16)$$

by Hölder's inequality. Then (4.16) gives

$$\mathbb{E} \left[ \|u(T)\|_{L^p(U)}^\alpha \right] \leq \left( C_3 |U|^{1/p} \Upsilon' \right)^\alpha. \quad (4.17)$$

Note well that the constant  $C_3$  in (4.17) is independent on  $p$ , so that we can let  $p \rightarrow +\infty$  in (4.17) to get

$$\|u(T)\|_{L^\alpha(\Omega; L^\infty(U))} \leq C_3 \Upsilon' = C_3 (\Upsilon + T^\delta \|u\|_{L^{\alpha_2}(\Omega; L^\infty(Q_T))}). \quad (4.18)$$

We choose  $\alpha \geq \alpha_2$  and  $T$  arbitrary in the interval  $[0, T_1]$  with  $C_3 T_1^\delta = \frac{1}{2}$ , to deduce (4.3) from our last estimate (4.18).  $\square$

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