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► **To cite this version:**

Sylvie Monniaux. MAXIMAL REGULARITY AS A TOOL FOR PARTIAL DIFFERENTIAL EQUATIONS. 2023. hal-04283279

**HAL Id: hal-04283279**

**<https://hal.science/hal-04283279>**

Preprint submitted on 13 Nov 2023

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# MAXIMAL REGULARITY AS A TOOL FOR PARTIAL DIFFERENTIAL EQUATIONS

SYLVIE MONNIAUX

ABSTRACT. In the last decades, a lot of progress has been made on the subject of maximal regularity. The property of maximal  $L^p$  regularity is an a priori estimate and reads as follows:

For  $A$  the negative generator of an analytic semigroup on a Banach space  $X$ , for  $1 < p < \infty$ , for  $0 < T < \infty$ , does there exist  $C_p > 0$  a constant such that for all  $f \in L^p(0, T; X)$ , there exists a unique solution  $u \in L^p(0, T; D(A)) \cap W^{1,p}(0, T; X)$  of the equation  $\partial_t(u) + Au = f$ ,  $u(0) = 0$  with the estimate

$$\|\partial_t(u)\|_{L^p(0,T;X)} + \|Au\|_{L^p(0,T;X)} \leq C_p \|f\|_{L^p(0,T;X)}?$$

It started with a paper by De Simon in 1964 in which the author proved maximal  $L^p$  regularity for negative generators of analytic semigroups in Hilbert spaces. The next big step has been made by Dore and Venni in their 1987 paper on operators admitting bounded imaginary powers. The final result on maximal regularity was done by Weis in his 2001 paper where he gave a characterisation of negative generators of analytic semigroups in Banach spaces which have the maximal regularity property. These results are all about linear theory of unbounded operators in Banach spaces.

I will then show how to use this property to find solutions or to prove uniqueness of solutions of semi linear partial differential equations of parabolic type such as the non linear heat equation and the Navier-Stokes system.

## 1. INTRODUCTION

The property of maximal  $L^p$  regularity ( $1 < p < \infty$ ) can be formulated as follows:

let  $A$  be an (unbounded) operator with dense domain  $D(A)$  defined on a Banach space  $X$ , let  $0 < T \leq \infty$ , for  $f \in L^p(0, T; X)$ , does there exist  $u \in L^p(0, T; D(A))$  such that  $\partial_t u + Au = f$  and  $u(0) = 0$ ? And in that case, the closed graph theorem implies that there is a constant  $C > 0$  such that

$$\|u\|_{L^p(0,T;X)} + \|\partial_t u\|_{L^p(0,T;X)} + \|Au\|_{L^p(0,T;X)} \leq C \|f\|_{L^p(0,T;X)}. \quad (1.1)$$

It is clear that if  $A$  has the maximal  $L^p$  regularity property for  $T = \infty$ , then this is also the case for  $T < \infty$ .

In a first part, we will study this property of maximal  $L^p$  regularity: what does this imply on the operator  $A$ ? What are the operators with this property? The non zero initial condition will also be investigated.

In a second part, we will focus on how to apply this property to study non linear equations, and more precisely semi linear problems. Examples will range from

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2000 *Mathematics Subject Classification.* 47D06, 34G10, 35B45, 35Q30.

*Key words and phrases.* Maximal regularity, partial differential equations, evolution equations, Navier-Stokes equations, existence of solutions, uniqueness of solutions.

the non linear heat equation to the Navier-Stokes system: existence of solutions, uniqueness of mild solutions...

## 2. MAXIMAL REGULARITY

**2.1. Analytic semigroup.** The maximal  $L^p$  regularity property for operators defined on a Hilbert space has been studied by De Simon [15] in 1964. Let us start with easy properties that an operator with the maximal  $L^p$  property enjoys.

**Proposition 2.1.** *Let  $p \in (1, \infty)$  and  $T = \infty$ . Assume that  $A$  has the maximal  $L^p$  regularity property. Then  $-A$  generates an analytic semigroup.*

*Proof.* Let  $z \in \mathbb{C}$  with  $\Re(z) > 0$  and define the function  $f_z \in L^p(0, \infty)$  by

$$f_z(t) = e^{zt} \text{ if } 0 \leq t \leq \frac{1}{\Re(z)} \text{ and } f_z(t) = 0 \text{ if } t > \frac{1}{\Re(z)}.$$

Let  $x \in X$ : denote by  $u_z$  the solution of  $\partial_t u + Au = f_z x$ ,  $u_z(0) = 0$  and let

$$R_z x := \Re(z) \int_0^\infty e^{-zt} u_z(t) dt.$$

One can prove that, using (1.1),  $\|R_z x\|_X \leq C_p \|x\|_X$ . An easy integration by parts gives also that

$$R_z x = \frac{\Re(z)}{z} \int_0^\infty e^{-zt} \partial_t u_z(t) dt$$

and that  $\|R_z x\|_X \leq C'_p \frac{1}{|z|} \|x\|_X$ . Putting these two estimates together gives

$$\|R_z x\|_X \leq M_p \frac{1}{1 + |z|} \|x\|_X,$$

where  $M_p$  is a constant depending only on  $p$  and on the constant  $C$  in (1.1).

Now, for  $x \in D(A)$ , it is easy to see that  $R_z(zx + Ax) = x$ , so that since  $D(A)$  is dense in  $X$   $R_z$  is the resolvent of  $-A$  in  $z$ .  $\square$

**Remark 2.2.** Since  $-A$  generates an analytic semigroup  $(e^{-tA})_{t \geq 0}$ , the solution  $u$  of  $\partial_t u + Au = f$ ,  $u(0) = 0$  is given by

$$u(t) = \int_0^t e^{-(t-s)A} f(s) ds, \quad t \geq 0. \quad (2.1)$$

The next result concerns the independence of the maximal  $L^p$  regularity property with respect to  $p \in (1, \infty)$ .

**Proposition 2.3.** *Let  $p_0 \in (1, \infty)$ . Assume that  $A$  is the negative generator of an analytic semigroup with the maximal  $L^{p_0}$  regularity property. Then  $A$  has the maximal  $L^p$  regularity property for all  $p \in (1, \infty)$ .*

The proof of this proposition relies on a more general result by A. Benedek, A.P. Calderón, R. Panzone [1, Theorem 2].

**Theorem 2.4.** *Let  $X$  be a Banach space. Let  $p \in (1, \infty)$  and denote by  $k : \mathbb{R} \rightarrow \mathcal{L}(X)$  a measurable function  $k \in L^1_{\text{loc}}(\mathbb{R} \setminus \{0\}, \mathcal{L}(X))$ . Let  $S \in \mathcal{L}(L^p(\mathbb{R}, X))$  be the convolution operator with  $k$ , i.e.,*

$$Sf(t) = \int_{\mathbb{R}} k(t-s)f(s) ds, \quad \forall f \in L^\infty_c(\mathbb{R}, X), t \notin \text{supp}(f).$$

Assume that there is a constant  $c > 0$  such that

$$\int_{|t|>2|s|} \|k(t-s) - k(t)\|_{\mathcal{L}(X)} dt \leq c, \quad \forall s \in \mathbb{R}.$$

Then  $S \in \mathcal{L}(L^q(\mathbb{R}, X))$  for all  $q \in (1, \infty)$ .

*Proof.* The proof relies on a Calderón-Zygmund decomposition of  $f$  to prove that  $S$  is of weak type  $(1,1)$ . Then by interpolation (Marcinkiewicz theorem), one deduces boundedness of  $S$  on  $L^q(\mathbb{R}, X)$  for all  $q \in (1, p_0]$ . To prove boundedness for  $q > p_0$ , we argue by duality.  $\square$

*Proof of Proposition 2.3.* Relying on (2.1), we apply previous theorem with

$$k(t) = Ae^{-tA} \text{ for } t > 0 \text{ and } k(t) = 0 \text{ if } t \leq 0.$$

It suffices to show that this kernel  $k$  satisfies the assumptions of the theorem.  $\square$

We finish this first insight into maximal  $L^p$  regularity by giving a reverse statement of Proposition 2.1 in the case  $X$  is a Hilbert space. This result is due to De Simon [15].

**Theorem 2.5.** *Let  $A$  be the negative generator of a bounded analytic semigroup on a Hilbert space  $X$ . Then  $A$  has the maximal  $L^p$  regularity.*

*Proof.* Using (2.1) to represent the solution of  $\partial_t u + Au = f$ ,  $u(0) = 0$  with  $f \in L^2(0, \infty; X)$  and taking the Fourier-in-time transform  $\mathcal{F}$  of  $Au$  shows that  $\mathcal{F}(Au) \in L^2(0, \infty; X)$  (since  $-A$  generates a bounded analytic semigroup) and then  $\|Au\|_{L^2(0, \infty; X)} \lesssim \|f\|_{L^2(0, \infty; X)}$ . Note that

$$\mathcal{F}(Au)(\tau) = A(i\tau + A)^{-1} \mathcal{F}(f)(\tau), \quad \tau \in \mathbb{R}.$$

We conclude by applying Proposition 2.3.  $\square$

It has been a long open problem whether this last theorem was true in general Banach spaces or not. A result by T. Coulhon and D. Lamberton [3] shows that it is a necessary condition that the Banach space  $X$  has the  $UMD$  property, i.e., the Hilbert transform

$$\mathcal{H} : L^2(\mathbb{R}; X) \rightarrow L^2(\mathbb{R}; X); \quad \mathcal{H}(f)(t) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(s)}{t-s} ds, \quad t \in \mathbb{R}$$

is a bounded operator. Examples of  $UMD$  spaces are: Hilbert spaces,  $L^p(\Omega, X)$  ( $\Omega \subset \mathbb{R}^d$  for  $p \in (1, \infty)$ ) if  $X$  is a  $UMD$  space. A  $UMD$  Banach space is necessarily reflexive.

## 2.2. Maximal regularity in Banach spaces - weighted maximal regularity.

We focus now on the maximal  $L^p$  regularity property for negative generators of analytic semigroups in  $UMD$  Banach space.

D. Lamberton [10] proved that the negative generator of an analytic semigroup of contractions on  $X = L^2(\Omega, \mu)$  (where  $(\Omega, \mu)$  is a measure space) satisfying  $\|e^{-tA} f\|_q \lesssim \|f\|_q$  for all  $f \in L^2(\Omega, \mu) \cap L^q(\Omega, \mu)$  for all  $q \in (1, \infty)$  has the maximal  $L^p$  regularity property.

**Example 2.6.** The negative Laplacian with Dirichlet, Neumann or Robin boundary conditions (on a domain for which the divergence theorem for  $L^1$  functions applies; this is the case for instance if it is a domain with a Lipschitz boundary) has the maximal  $L^p$  regularity property.

A positive result was given by G. Dore and A. Venni the same year in [4] where they proved that if  $A$  has bounded imaginary powers with angle strictly less than  $\frac{\pi}{2}$ , then it has the maximal  $L^p$  regularity property.

Later, M. Hieber and J. Prß [7] and T. Coulhon and X.T. Duong [2] proved that if the semigroup generated by  $-A$  has a kernel with gaussian estimates, then  $A$  has the maximal  $L^p$  regularity property. This result was generalised by P.C. Kunstmann [9] for negative generators of semigroups with integrated gaussian estimates.

A characterisation of operators with the maximal  $L^p$  regularity property was given by L. Weis [16] using the concept of  $\mathcal{R}$  boundedness.

**Definition 2.7.** A family  $\tau$  of bounded linear operators from  $X$  to  $Y$  is called  $\mathcal{R}$  bounded if there exists a constant  $C > 0$  such that for all  $n \geq 1$ , for all  $T_1, \dots, T_n \in \tau$  and all  $x_1, \dots, x_n \in X$ ,

$$\int_0^1 \left\| \sum_{j=1}^n r_j(s) T_j x_j \right\|_Y ds \leq C \int_0^1 \left\| \sum_{j=1}^n r_j(s) x_j \right\|_X ds$$

where  $(R_j)_{j=1, \dots, n}$  is a sequence of independent  $\{-1, 1\}$  valued random variables on  $[0, 1]$ ; for example, the Rademacher functions  $r_j(t) = \text{sgn}(\sin(2^j \pi t))$ . The  $\mathcal{R}$  bound of  $\tau$  denoted by  $\mathcal{R}(\tau)$  is then the smallest  $C$  for which the above estimate holds.

**Remark 2.8.** If  $X$  and  $Y$  are Hilbert spaces, then a family  $\tau$  is  $\mathcal{R}$  bounded if, and only if, it is bounded.

We are now in the position to state the result which characterises operators with the maximal  $L^p$  regularity property due to L. Weis [16].

**Theorem 2.9.** *Let  $A$  be the negative generator of an analytic semigroup on a UMD Banach space. Then  $A$  has the maximal  $L^p$  regularity property if, and only if, the family of resolvents  $\{i\sigma(i\sigma I + A)^{-1}, \sigma \in \mathbb{R}\}$  is  $\mathcal{R}$  bounded.*

We finish this section with the notion of weighted maximal  $L^p$  regularity. The following result is due to J. Prß and G. Simonett [14, Theorem 2.4].

**Theorem 2.10.** *Assume that  $A$  is an operator having the maximal  $L^p$  regularity property on a Banach space  $X$  with the UMD property. Then  $A$  has the weighted maximal  $L^p$  regularity property, i.e., for all  $\mu \in (\frac{1}{p}, 1]$ , for all  $f \in L^p_\mu(0, \infty; X)$ , there exists a unique solution  $u \in L^p_\mu(0, \infty; \mathcal{D}(A))$  with  $\partial_t u \in L^p_\mu(0, \infty; X)$  of  $\partial_t u + Au = f$  with  $u(0) = 0$ , where*

$$L^p_\mu(0, \infty; X) = \{u : (0, \infty) \rightarrow X; t \mapsto t^{1-\mu} u(t) \in L^p(0, \infty; X)\}$$

*equipped with the norm  $\|u\|_{L^p_\mu(0, \infty; X)} = \|t \mapsto t^{1-\mu} u(t)\|_{L^p(0, \infty; X)}$ .*

**Remark 2.11.** The weighted maximal  $L^p$  regularity property for  $\mu = 1$  is the usual maximal  $L^p$  regularity property.

### 3. SEMI-LINEAR EQUATIONS IN CRITICAL SPACES

In this section, we describe two semi-linear pdes of the form

$$\partial_t u + Au = f(u), \quad u(0) = u_0.$$

We will give existence and uniqueness results using the maximal  $L^p$  regularity property.

**3.1. Mild solutions.** First, let us define what we mean with “critical space” for a pde of the previous form. Assume that  $A$  is an operator on a space depending of a space variable  $x$  homogeneous in the sense that  $Au_\lambda(x) = \lambda^\alpha Au(x)$  for all  $\lambda > 0$ ,  $x \in \mathbb{R}^n$  where  $u_\lambda(x) = u(\lambda x)$ . Assume moreover that the function  $f$  is also homogeneous in the sense that  $f(\lambda w) = \lambda^\gamma f(w)$  and  $f(v_\lambda(t, x)) = \lambda^\beta f(v(\lambda^\alpha t, \lambda x))$  where  $v_\lambda(t, x) = v(\lambda^\alpha t, \lambda x)$  for  $\lambda > 0$ ,  $t > 0$  and  $x \in \Omega \subset \mathbb{R}^n$ . We assume that if  $\alpha \neq \beta$ , then  $\gamma \neq 1$ . In that case, if  $u$  is a solution of  $\partial_t u + Au = f(u)$ , then  $(t, x) \mapsto \lambda^{\frac{\alpha-\beta}{\gamma-1}} u(\lambda^\alpha t, \lambda x)$  is also a solution of the same equation for all  $\lambda > 0$ .

**Definition 3.1.** In this situation, we call  $Y$  a critical space for  $\partial_t u + Au = f(u)$  a time-space variable space for which

$$\|(t, x) \mapsto \lambda^{\frac{\alpha-\beta}{\gamma-1}} u(\lambda^\alpha t, \lambda x)\|_Y = \|u\|_Y.$$

**Example 3.2.** If  $Y$  is of the form  $Y = L_t^p(L_x^q)$  for  $1 < p, q < \infty$ ,  $Y$  is a critical space if, and only if,  $\frac{\alpha-\beta}{\gamma-1} = \frac{n}{q} + \frac{\alpha}{p}$ .

**Definition 3.3.** Assume that  $A$  is the negative generator of an analytic semigroup on a Banach space  $X$ . A mild solution on  $[0, T]$  ( $0 < T \leq +\infty$ ) of the equation  $\partial_t u + Au = f(u)$  with initial value  $u_0 \in X$  is a continuous function in  $t \in [0, T]$  with values in  $X$  satisfying the Duhamel formula:

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(u(s)) ds, \quad 0 \leq t < T.$$

In other words,  $u$  is a fixed point of the map  $v \mapsto a + F(v)$  where  $a(t) = e^{-tA}u_0$ ,  $t \geq 0$  and  $F(v)(t) = \int_0^t e^{-(t-s)A} f(v(s)) ds$ ,  $0 \leq t < T$ . If  $T = +\infty$ ,  $u$  is called a global mild solution.

The method to prove the existence of global mild solutions for small initial data or local mild solutions if the initial condition has no size restriction relies on the following fixed point theorem.

**Theorem 3.4** (Fixed point theorem). *Let  $Y$  be a Banach space,  $a \in Y$  and  $F : Y \rightarrow Y$  with the property that  $F(0) = 0$  and there exists  $\epsilon > 0$  and  $M > 0$  such that*

$$\|F(u) - F(v)\|_Y \leq M\|u - v\|_Y (\|u\|_Y^\epsilon + \|v\|_Y^\epsilon), \quad \forall u, v \in Y. \quad (3.1)$$

*Then for  $\delta < \frac{1}{2(2M)^\epsilon}$ , for all  $a \in \overline{B}_Y(0, \delta)$ , the ball in  $Y$  of center 0 and radius  $\delta$ , the map  $\Phi : u \mapsto a + F(u)$  admits a fixed point, unique in  $\overline{B}_Y(0, 2\delta)$ .*

*Proof.* The proof of the theorem follows the lines of [11, Theorem 15.1], Picard contraction principle.

The choice of  $\delta$  implies that  $(2\delta)^\epsilon < \frac{1}{2M}$ .

First, we prove that if  $a \in \overline{B}_Y(0, \delta)$ , then  $\overline{B}_Y(0, 2\delta)$  is stable under the map  $\Phi$ . The estimate on  $F$  and the fact that  $F(0) = 0$  give for  $u \in \overline{B}_Y(0, 2\delta)$  the estimate  $\|F(u)\|_Y \leq M\|u\|_Y^{1+\epsilon} \leq M(2\delta)^{1+\epsilon} \leq \delta$ . With  $\|a\|_Y \leq \delta$ , we obtain  $\|\Phi(u)\|_Y \leq 2\delta$ .

Next,  $\Phi : \overline{B}_Y(0, 2\delta) \rightarrow \overline{B}_Y(0, 2\delta)$  is a contraction. We have that

$$\|\Phi(u) - \Phi(v)\|_Y \leq M(\|u\|_Y^\epsilon + \|v\|_Y^\epsilon)\|u - v\|_Y \leq 2M(2\delta)^\epsilon\|u - v\|_Y$$

with  $2M(2\delta)^\epsilon < 1$  as established in the previous step.

We conclude by Picard contraction principle that  $\Phi$  has a unique fixed point in  $\overline{B}_Y(0, 2\delta)$ .  $\square$

### 3.2. The non linear heat equation: existence and uniqueness of solutions.

The non linear heat equation in  $\mathbb{R}^n$  has the form

$$\begin{cases} \partial_t u - \Delta u &= |u|^{\nu-1} u & \text{in } (0, \infty) \times \mathbb{R}^n \\ u(0) &= 0 & \text{in } \mathbb{R}^n \end{cases} \quad (\text{NLHE1})$$

or

$$\begin{cases} \partial_t u - \Delta u &= |u|^\nu & \text{in } (0, \infty) \times \mathbb{R}^n \\ u(0) &= 0 & \text{in } \mathbb{R}^n \end{cases} \quad (\text{NLHE2})$$

for  $1 < \nu < \infty$ . We propose here to give a proof of existence and uniqueness of solutions of (NLHE1) or (NLHE2) as in [17, Theorem 1] using the property of maximal regularity (see also [18]. With the notations used to define a critical space for these equations, we have in our case  $\alpha = 2$ ,  $\beta = 0$  and  $\gamma = \nu$  so that a critical space for (NLHE1) or (NLHE2) is of the form  $L_t^p(L_x^q)$  with  $\frac{n}{q} + \frac{2}{p} = \frac{2}{\nu-1}$ . In the special case  $\nu = 2$  (the quadratic non linear heat equation), this condition becomes  $\frac{n}{q} + \frac{2}{p} = 2$ .

**Theorem 3.5** (Existence). *Let  $\nu \in (1, \infty)$ . Assume that  $\nu < p, q < \infty$  satisfy  $\frac{n}{q} + \frac{2}{p} = \frac{2}{\nu-1}$ . There exists  $\eta > 0$  such that for all  $u_0 \in \dot{B}_{p,q}^{-2/p}$  with  $\|u_0\|_{\dot{B}_{p,q}^{-2/p}} \leq \eta$ , the equations (NLHE1) and (NLHE2) have a mild solution in  $L_t^p(L_x^q)$ .*

*Proof.* The treatments of (NLHE1) and (NLHE2) are the same, so we will focus only on solutions of (NLHE1). We start by rewriting (NLHE1) in the integral form as in Definition 3.3:  $u \in Y = L_t^p(L_x^q)$  is a mild solution of (NLHE1) if

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} |u(s)|^{\nu-1} u(s) ds, \quad t \in (0, \infty), \quad (3.2)$$

and define  $a : t \mapsto e^{t\Delta} u_0$  and

$$F(u)(t) = \int_0^t e^{(t-s)\Delta} |u(s)|^{\nu-1} u(s) ds, \quad t \in (0, \infty). \quad (3.3)$$

In that case,  $u$  is a mild solution in  $Y$  if  $u = a + F(u)$ . *Claim 1:  $a$  belongs to  $Y$ .*

By definition of the Besov space  $\dot{B}_{p,q}^{-2/p}$ ,  $a \in Y$  and  $\|a\|_Y \leq c\eta$  where  $c$  is a constant.

*Claim 2:  $F$  maps  $Y$  to  $Y$ .*

This comes from the maximal regularity property satisfied by the heat semigroup and the mixed derivative result:

$$W_t^{1,p}(L_x^q) \cap L_t^p(W_x^{2,q}) \hookrightarrow W_t^{s,p}(W_x^{2(1-s),q}), \quad s \in (0, 1). \quad (3.4)$$

For  $u \in L_t^p(L_x^q)$ , we have that  $|u|^{\nu-1} u \in L_t^{p/\nu}(L_x^{q/\nu})$  and the mixed derivative result for  $p/\nu$  instead of  $p$ ,  $q/\nu$  instead of  $q$  and  $s = \frac{\nu-1}{\nu}$ , using the condition linking  $p$  and  $q$  and the Sobolev embeddings

$$W_t^{\frac{\nu-1}{p}, \frac{p}{\nu}} \hookrightarrow L_t^p \text{ in dimension 1} \quad \text{and} \quad W_x^{\frac{n(\nu-1)}{q}, \frac{q}{\nu}} \hookrightarrow L_x^q \text{ in dimension } n \quad (3.5)$$

yield the claim.

*Claim 3:  $F$  satisfies (3.1) with  $\epsilon = \nu - 1$ .*

It is immediate that  $F(0) = 0$ . Next, for all  $x, y \in \mathbb{R}$ , we have

$$||x|^{\nu-1} x - |y|^{\nu-1} y| \leq \nu(|x|^{\nu-1} + |y|^{\nu-1})|x - y|. \quad (3.6)$$

Therefore using the same tools as in Claim 1, we obtain

$$\|F(u) - F(v)\|_Y \leq M \|u - v\|_Y (\|u\|_Y^{\nu-1} + \|v\|_Y^{\nu-1})$$

with  $M = \nu C_{p/\nu, q/\nu} M_{\frac{\nu-1}{p}, p/\nu, q/\nu} \alpha_{\nu, p} \beta_{\nu, q}$ , where  $C_{p, q}$  is the constant appearing in (1.1) for  $T = \infty$ ,  $X = L_x^q$  and  $p$ ,  $M_{s, p, q}$  is the embedding constant in the mixed derivative result (3.4),  $\alpha_{\nu, p}$  and  $\beta_{\nu, q}$  are the constants in the Sobolev embeddings (3.5).

To end the proof, it suffices to apply Theorem 3.4 for  $\delta = c\eta$  ( $\eta$  small enough so that the condition on  $\delta$  is satisfied).  $\square$

The next result is a uniqueness result for solutions of (NLHE1) or (NLHE2) in  $\mathcal{C}([0, T]; L_x^q)$ . The limiting case  $p = \infty$  in the previous result gives  $q = \frac{n(\nu-1)}{2}$ : the condition  $\frac{q}{\nu} > 1$  implies then that  $n > \frac{2\nu}{\nu-1}$ . Remark that in the case of the quadratic non linear heat equation ( $\nu = 2$ ), this imposes that we work in dimension  $n \geq 5$ .

**Theorem 3.6** (Uniqueness). *Let  $\nu > 1$ , let  $n > \frac{2\nu}{\nu-1}$ , let  $q = \frac{n(\nu-1)}{2}$ . Let  $T > 0$ . Then for all  $u_0 \in L_x^q$  there exists at most one mild solution in  $\mathcal{C}([0, T]; L_x^q)$  of (NLHE1) or (NLHE2).*

*Proof.* The strategy of the proof is to assume that there are two mild solutions  $u$  and  $v$  in  $\mathcal{C}([0, T]; L_x^q)$  for the same initial value  $u_0$  so that  $u - v = F(u) - F(v)$  for  $F$  defined in (3.3). Then we show that  $u$  and  $v$  necessarily coincide in  $L^p(0, \tau; L_x^q)$  for  $\tau > 0$  and one  $p > 1$ :  $\|u - v\|_{L^p(0, \tau; L_x^q)} = 0$ . Iterating this procedure ultimately shows that  $u = v$  on  $[0, T]$ .

**Step 1:** Let  $p > 1$ . Using (3.6), we have the following estimate for every  $0 < \tau \leq T$

$$\|F(u) - F(v)\|_{L^p(0, \tau; L_x^q)} \leq M \|u - v\|_{L^p(0, \tau; L_x^q)} (\|u\|_{L^\infty(0, \tau; L_x^q)}^{\nu-1} + \|v\|_{L^\infty(0, \tau; L_x^q)}^{\nu-1})$$

with  $M$  independent of  $\tau$ . This follows from the same arguments as in the proof of Claim 3 of Theorem 3.5.

**Step 2:** Let  $\epsilon > 0$ . We can approximate  $u_0 \in L_x^q$  by a  $u_{0, \epsilon} \in \mathcal{C}_c^\infty(\mathbb{R}^n)$  such that

$$\|u_{0, \epsilon} - u_0\|_{L_x^q} \leq \epsilon.$$

We denote by  $a_\epsilon$  the function defined on  $(0, \infty)$  by  $a_\epsilon(t) = e^{t\Delta} u_{0, \epsilon}$  and we have for all  $\epsilon > 0$

$$\begin{aligned} \|u\|_{L^\infty(0, \tau; L_x^q)}^{\nu-1} - \|a_\epsilon\|_{L^\infty(0, \tau; L_x^q)}^{\nu-1} &\xrightarrow{\tau \rightarrow 0} \|u_0\|_{L_x^q}^{\nu-1} - \|u_{0, \epsilon}\|_{L_x^q}^{\nu-1} \\ \text{and } \|v\|_{L^\infty(0, \tau; L_x^q)}^{\nu-1} - \|a_\epsilon\|_{L^\infty(0, \tau; L_x^q)}^{\nu-1} &\xrightarrow{\tau \rightarrow 0} \|u_0\|_{L_x^q}^{\nu-1} - \|u_{0, \epsilon}\|_{L_x^q}^{\nu-1}. \end{aligned} \quad (3.7)$$

**Step 3:** Since  $s \mapsto |a_\epsilon(s)|^{\nu-1} (u(s) - v(s))$  belongs to  $L^p(0, \tau; L_x^{\frac{nq}{n+q}})$  whenever  $u, v \in L^p(0, \tau; L_x^q)$  and  $a_\epsilon \in L^\infty(0, \tau; L_x^{\frac{n}{\nu-1}})$ , we have

$$\begin{aligned} &\left\| t \mapsto \int_0^t e^{(t-s)\Delta} |a_\epsilon(s)|^{\nu-1} (u(s) - v(s)) \, ds \right\|_{L^p(0, \tau; L_x^q)} \\ &\lesssim \sqrt{\tau} \|u - v\|_{L^p(0, \tau; L_x^q)} \|a_\epsilon\|_{L^\infty(0, \tau; L_x^{\frac{n}{\nu-1}})}^{\nu-1} \end{aligned}$$

using the fact that

$$\|e^{r\Delta} f\|_{L_x^q} = \|(-\Delta)^{\frac{1}{2}} e^{r\Delta} (-\Delta)^{-\frac{1}{2}} f\|_{L_x^q} \lesssim \frac{1}{\sqrt{r}} \|(-\Delta)^{-\frac{1}{2}} f\|_{L_x^q} \lesssim \frac{1}{\sqrt{r}} \|f\|_{L_x^{\frac{nq}{n+q}}}$$

for all  $r > 0$  and all  $f \in L_x^{\frac{nq}{n+q}}$ , by the Sobolev embedding  $W_x^{1, \frac{nq}{n+q}} \hookrightarrow L_x^q$  in dimension  $n$ .



**Step 4:** For every  $\epsilon > 0$ , we estimate  $F(u) - F(v)$  as

$$\begin{aligned} |F(u)(t) - F(v)(t)| &\leq \int_0^t e^{(t-s)\Delta} \left( (|u(s)|^{\nu-1} - |a_\epsilon(s)|^{\nu-1}) \right. \\ &\quad \left. + (|v(s)|^{\nu-1} - |a_\epsilon(s)|^{\nu-1}) + 2|a_\epsilon(s)|^{\nu-1} (u(s) - v(s)) \right) ds. \end{aligned}$$

Therefore, applying Step 2 and Step 3, there exists a constant  $C > 0$  such that

$$\begin{aligned} \|F(u) - F(v)\|_{L^p(0,\tau;L_x^q)} &\leq C \|u - v\|_{L^p(0,\tau;L_x^q)} \left( \|u\|_{L^\infty(0,\tau;L_x^q)}^{\nu-1} - \|a_\epsilon\|_{L^\infty(0,\tau;L_x^q)}^{\nu-1} \right) \\ &\quad + \|v\|_{L^\infty(0,\tau;L_x^q)}^{\nu-1} - \|a_\epsilon\|_{L^\infty(0,\tau;L_x^q)}^{\nu-1} \\ &\quad + \sqrt{\tau} \|a_\epsilon\|_{L^\infty(0,\tau;L_x^{\frac{n}{\nu-1}})}^{\nu-1} \end{aligned}$$

Since  $u - v = F(u) - F(v)$  for  $u$  and  $v$  mild solutions of (NLHE1) We conclude by choosing  $\epsilon > 0$  small enough such that  $\|u_0\|_{L_x^q}^{\nu-1} - \|u_{0,\epsilon}\|_{L_x^q}^{\nu-1} \leq \frac{1}{8C}$  and then  $\tau > 0$  small enough such that

$$\begin{aligned} \|u\|_{L^\infty(0,\tau;L_x^q)}^{\nu-1} - \|a_\epsilon\|_{L^\infty(0,\tau;L_x^q)}^{\nu-1} &\leq \frac{1}{4C}, \\ \|v\|_{L^\infty(0,\tau;L_x^q)}^{\nu-1} - \|a_\epsilon\|_{L^\infty(0,\tau;L_x^q)}^{\nu-1} &\leq \frac{1}{4C}, \end{aligned}$$

and

$$\sqrt{\tau} \|a_\epsilon\|_{L^\infty(0,\tau;L_x^{\frac{n}{\nu-1}})}^{\nu-1} \leq \frac{1}{4C}.$$

We then obtain that  $\|u - v\|_{L^p(0,\tau;L_x^q)} \leq \frac{3}{4} \|u - v\|_{L^p(0,\tau;L_x^q)}$  which shows that  $u = v$  on  $[0, \tau]$ .

**Step 5:** By continuity in time of  $u$  and  $v$ , we have equality on the closed segment  $[0, \tau]$ . If  $\tau < T$ , we can repeat this argument starting from  $\tau$  instead of 0 until we reach  $T$ .  $\square$

**3.3. Navier-Stokes equations.** The same arguments can be applied to Navier-Stokes equations:

$$\begin{cases} \partial_t u - \Delta u + \nabla \pi + \nabla \cdot (u \otimes u) &= 0 & \text{in } (0, \infty) \times \mathbb{R}^n \\ \operatorname{div} u &= 0 & \text{in } (0, \infty) \times \mathbb{R}^n \\ u(0) &= u_0 & \text{in } \mathbb{R}^n, \end{cases} \quad (\text{NS})$$

where  $u \otimes v$  denotes the matrix  $(u_i v_j)_{1 \leq i, j \leq n}$  and  $\nabla \cdot M$ , the divergence of the matrix  $M = (m_{i,j})_{1 \leq i, j \leq n}$ , is the vector obtained by taking the divergence of the lines:  $\nabla \cdot M = (\sum_{i=1}^n \partial_i m_{i,j})_{j=1, \dots, n}$ . It is easy to see that the invariance for this system is given by  $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$  and  $\pi_\lambda(t, x) = \lambda^2 \pi(\lambda^2 t, \lambda x)$  so that a critical space of the form  $L_t^p(L_x^q)$  ( $1 < p, q < \infty$ ) for (NS) must satisfy the relation  $\frac{2}{p} + nq = 1$ . A mild solution of (NS) is given by the formula

$$u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}(\nabla \cdot (u(s) \otimes u(s))) ds, \quad t \geq 0, \quad (3.8)$$

where  $\mathbb{P}$  is the Helmholtz projection onto divergence-free vector fields (its Fourier symbol is given by  $(\delta_{i,j} - \frac{\xi_i \xi_j}{|\xi|^2})_{1 \leq i, j \leq n}$ ). Existence results of mild solutions of (NS) go back to Fujita and Kato [5] and Kato [8]. The same methods as in the case of the non linear heat equation apply. We obtain the following existence and uniqueness results.

**Theorem 3.7** (Existence). *Assume that  $1 < p, q < \infty$  satisfy  $\frac{n}{q} + \frac{2}{p} = 1$ . There exists  $\eta > 0$  such that for all  $u_0 \in \dot{B}_{p,q}^{-2/p}$  with  $\operatorname{div} u_0 = 0$  and  $\|u_0\|_{\dot{B}_{p,q}^{-2/p}} \leq \eta$ , the system (NS) has a mild solution in  $L_t^p(L_x^q)$ .*

*Proof.* The proof follows the lines of the existence result for (NLHE1) using Theorem 3.4, having in mind that  $\mathbb{P}$  is a bounded operator on every  $W_x^{s,q}$  if  $s \in \mathbb{R}$  and  $1 < q < \infty$ .  $\square$

In the limit  $p = \infty$ , the relation  $\frac{2}{p} + \frac{n}{q} = 1$  imposes  $q = n$ . We state next a uniqueness result for mild solutions in  $\mathcal{C}([0, T]; L_x^n)$ . This result was originally proved by Furioli, Lemarié-Rieusset and Terraneo in [6] by different methods and in [12] using maximal regularity (see also the book by Lemarié-Rieusset [11] where he gave at least 3 proofs in Chapter 27). For the case of Navier-Stokes equations in bounded Lipschitz domains, see for instance [13].

**Theorem 3.8** (Uniqueness). *Let  $T > 0$ . For all  $u_0 \in L_x^n$  there exists at most one mild solution in  $\mathcal{C}([0, T]; L_x^n)$  of (NS).*

*Proof.* Again, the strategy of the proof and the tools used are the same as in the case of (NLHE1). We assume that there are two mild solutions  $u$  and  $v$  of (NS), i.e.,  $u$  and  $v$  satisfy both (3.8). We prove that for some (all)  $p \in (1, \infty)$ , there exists  $\tau \in (0, T]$  such that  $\|u - v\|_{L^p(0,\tau;L_x^n)} \leq c\|u - v\|_{L^p(0,\tau;L_x^n)}$  with  $0 < c < 1$ , so that  $u = v$  on  $[0, \tau]$  (by continuity) and then iterate the procedure.  $\square$

#### ACKNOWLEDGEMENTS

The research is financially supported by a grant ANR-18-CE40-0012 “Real Analysis and GEometry” (french research agency).

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