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# Explicit convergence rate parameters for linear autonomous systems

Mario Spirito, Daniele Astolfi \*

\* *Université Lyon 1– CNRS, UMR 5007, LAGEPP, Villeurbanne, France (e-mail: name.surname@univ-lyon1.fr)*

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## Abstract:

In this paper, we propose a change of coordinates that brings the state matrix of an autonomous linear system into a modified Jordan Block form. Such a change of coordinates allows us to obtain exact values of the scaling factor and the convergence rate of the exponential stability bound for linear systems. The analysis is then applied to a control design with requirements on the evolution of the state norm. Numerical examples are also provided to illustrate the effectiveness of the proposed approach.

*Keywords:* Exponential convergence parameters, Lyapunov equation, Lyapunov inequality, modified Jordan form

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## 1. INTRODUCTION AND MOTIVATIONS

Since the origin of Lyapunov's direct method (Lyapunov, 1992), assessing the stability of an unforced system is done by considering a nonnegative scalar function of the state vector and checking that its value is monotonically decreasing in time. The approach was also described and extended in (Bertram and Kalman, 1960; Massera, 1956; Barbashin and Krasovskii, 1961; Hahn et al., 1967), see also (Martynuk, 2002) for a review on the development of the definition of stability during last century and (Leine, 2010) for a historical perspective on stability concept.

For autonomous linear system  $\dot{x} = Ax$ , with  $x(0) = x_0 \in \mathbb{R}^n$ , stability analysis is usually done by looking for a time-(in)dependent quadratic Lyapunov function, i.e., one has to look for a symmetric positive definite matrix  $P$  solution of a particular Linear Matrix Inequality (LMI), i.e., Lyapunov inequality

$$PA + A^T P \prec 0. \quad (1)$$

Such a solution  $P$  can be computed via numerical tools (i.e. optimization solvers), but, to the best of the authors' knowledge, it has not been solved in any direct nor closed form. As a consequence, the standard approach to obtain  $P$  is reverted, (Rugh, 1996)[Ch.7, pg.124], i.e., rather than directly specifying the matrix  $P$ , one has to involve an additional positive definite matrix  $Q = Q^T > 0$ , and look for a  $P$  solution of so-called Lyapunov Equation

$$PA + A^T P = -Q, \quad (2)$$

where  $Q$  now is a degree of freedom. For example, it can be taken as  $Q = 2qI$ , for some real  $q > 0$ .

It is also well-known, that such a solution  $P$  to (2), and thus to (2), exists and is also unique if and only if  $A$  has all negative real parts, and such a  $P$  is given by

$$P = \int_0^\infty \exp(A^T s) Q \exp(As) ds.$$

It is also possible to numerically compute  $P$  by exploiting the vectorization of matrices as introduced in (Bellman, 1957). Other approaches to compute the Lyapunov function for nonlinear systems can be found in (Khalil, 2002), or (Davison and Kurak, 1971), (Blanchini, 1995), see (Giesel and Hafstein, 2015) for a recent review on the topic.

Such a pair  $P, Q$  can be exploited to determine the positive real constants  $\kappa$  and  $\alpha$ , respectively, the *scaling factor* and the *convergence rate*, in the standard definition of uniform exponential stability, (Rugh, 1996; Khalil, 2002), i.e.,

$$|x(t)| \leq \kappa \exp(-\alpha t) |x(0)|. \quad (3)$$

In particular, by considering  $\sigma_{\min}(Q)$  to be the minimum singular value of  $Q$  is easy to show, by comparison lemma (Khalil, 2002), that we have

$$|x(t)| \leq \frac{\sigma_{\max}(P)}{\sigma_{\min}(P)} \exp\left(-\frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)} t\right) |x(0)|, \quad \forall t \geq 0, \quad (4)$$

where  $\kappa$  is given by the condition number of  $P$  and the maximum eigenvalue of  $P$  is involved in the convergence rate  $\alpha$ , along with the minimum eigenvalue of  $Q$ . Although very general, the obtained constants for the norm upper bound, thus highly depend on the choice of the matrix  $Q$ . To the best of the authors' knowledge, it does not exist any (constructive) approach to select  $Q$  in order to obtain the (in some sense) optimal values of the exponential convergence, i.e., to estimate  $\kappa$  and  $\alpha$  such that the upper bound is as close as possible to the real norm evolution. Usually, taking  $Q$  with a large  $\sigma_{\min}(Q)$  (aiming at obtaining a large convergence rate  $\alpha$ ), makes  $P$  very ill-conditioned, thus increasing the condition number  $\kappa$  in the upper bound. Although this can be accepted by common sense, or simply obtained by intuition, we were not able to find any result in the literature properly describing these facts.

One particular form of the Lyapunov inequality (1), that can be used to directly obtain the parameters of the exponential stability in (3) is the following

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$$PA + A^\top P \preceq -2\alpha P. \quad (5)$$

where  $\alpha$  is precisely the convergence rate of the definition, while  $\kappa$  is the  $P$  conditioning number. However, solving (5) has never been done analytically, and this problem is usually addressed via numerical optimization algorithms. Moreover, the value of  $\alpha$  cannot be chosen arbitrarily. When  $A$  is diagonalizable, one can select  $\alpha = |\Re \lambda_{\min}(A)|$ , but for a non trivial geometric multiplicity of  $\lambda_{\min}(A)$  one can find a positive definite solution  $P$  to (5) only with  $\alpha < |\Re \lambda_{\min}(A)|$ .

To the best of the authors' knowledge, there is no other general approach available in the literature to **explicitly** relate the scaling factor  $\kappa$  and the convergence rate  $\alpha$  for linear systems to the eigenstructure of the state matrix  $A$ . Although this might be intuitive and well-understood by the community, to the best of our knowledge there is no available result that properly formalizes this connection. The closer results we were able to find are (Hu and Seiler, 2016) in which the authors test the exponential convergence rate using integral quadratic constraints, and (Appelby et al., 2006) the author obtains the rate of decay of solutions of a class of convolution Volterra difference equations. However, none of the available results in the literature are suitable to explicitly obtain the parameters of exponential convergence  $\kappa$  and  $\alpha$  in closed form, neither, with respect to the eigenvalues and generalized eigenvectors properties in the case of linear time-invariant systems.

It is well known that the exponential rate of convergence for a linear autonomous system can be deduced from the eigenvalues of the corresponding system's state matrix  $A$ . However, in this paper, we want to explicitly characterize the dependence between the eigenvalues (more properly, the eigenstructure) of the state matrix  $A$  and the convergence rate  $\alpha$ , and provide an explicit link of such an eigenstructure of  $A$  with the scaling factor  $\kappa$ , in the definition of exponential stability (3). The reader can easily check that this type of characterization is easy to get in two particular cases, i.e., the symmetric part of the state matrix  $A$  is negative definite (Rugh, 1996)[Ch.7, pg 114], or when the matrix  $A$  is diagonalizable. Indeed, when  $A$  is not diagonalizable, due to the structure of its Jordan blocks, the exponential behavior due to the system eigenvalues is "rescaled" by a polynomial function of time that is not immediate to upper-bound, see discussion in Section 3.

Thus the aim of this paper is twofold:

- (1) Provide a review of the fundamental concepts related to the stability of linear autonomous systems;
- (2) Provide a characterization of the solution of the Lyapunov inequality (5), or equivalently (1), also for non-diagonalizable matrices  $A$ ;

The rest of this article is structured as follows. We first collect some preliminary results in Section 2, by recalling some definitions from the Jordan block form and propose a modification of this normal form. Based on such a transformation, we define next a Lyapunov function which allows to obtain the explicit exponential stability parameters (the values of  $\kappa$  and  $\alpha$ ) in terms of the algebraic properties of the matrix eigenvalues and the (generalized) eigenvectors, see as shown in Section 3. In the same section, we also dis-

cuss some properties of the proposed solution compared to a numerical solution. In Section 4, we show an application of the analysis in the design of closed loop system performances. Finally, in Section 5, we show the effectiveness of the proposed bounds on an autonomous system, with large geometric multiplicity. Some conclusions and perspectives are given in Section 6.

*Notation.*  $\mathbb{R}$ , resp.  $\mathbb{N}$ , resp.  $\mathbb{C}$  denotes the real, resp. natural, resp. complex, numbers.  $|\cdot|$  denote the standard Euclidean norm. Given a matrix  $A \in \mathbb{R}^{n \times n}$ , we denote the spectrum of  $A$  as  $\sigma(A)$ , where  $\lambda_i(A) \in \sigma(A)$  is the  $i$ -th eigenvalue of  $A$ , and  $\sigma_i(A)$  is its  $i$ -th singular value. Furthermore,  $\sigma_{\max}(A)$  and  $\sigma_{\min}(A)$  are respectively the maximum and minimum singular value of  $A$ . The  $L_2$ -norm of matrix  $M \in \mathbb{R}^{n \times n}$  is denoted by  $\|M\|$ , and we denote the condition number of  $M$  as

$$\mu(M) := \sqrt{\frac{\lambda_{\max}(M^\top M)}{\lambda_{\min}(M^\top M)}}$$

and  $\mu(M) = \mu(M^{-1})$ . For two square matrices  $A, B \in \mathbb{R}^{n \times n}$  we have

$$\begin{aligned} \mu(AB) &= \|AB\| \|(AB)^{-1}\| \\ &\leq \|A\| \|B\| \|B^{-1}\| \|A^{-1}\| = \mu(A)\mu(B). \end{aligned}$$

We define  $M \in \mathbb{R}^{n \times n}$  to be orthogonally diagonalizable if it has an orthonormal set of eigenvectors. Given a matrix  $M \in \mathbb{R}^{n \times n}$ ,  $\text{sym}(M)$  refers to the symmetric part of such a matrix, i.e.,  $\text{sym}(M) = (M + M^\top)/2$ . Given a complex number  $\mu \in \mathbb{C}$  and a explicit number  $n \in \mathbb{N}$ , we denote

$$D_n(\lambda) := \begin{cases} \text{diag}(1, \lambda, \dots, \lambda^{n-1}), & \text{if } \lambda \neq 0, \\ I_n, & \text{if } \lambda = 0, \end{cases} \quad (6)$$

where  $I_n$  is the identity matrix of dimension  $n$ . Then, the following identity

$$N_n D_n(\lambda) = \lambda D_n(\lambda) N_n \quad (7)$$

holds for any  $\lambda \in \mathbb{C}$  and  $n \in \mathbb{N}$ , where  $N_n \in \mathbb{R}^{n \times n}$  is a the matrix of zero elements, with only 1 on the first super diagonal. Similarly, for a matrix  $A \in \mathbb{R}^{r \times r}$  we can define

$$D_n(\Lambda) := \begin{cases} \text{blkdiag}(I_r, \Lambda, \dots, \Lambda^{n-1}), & \text{if } \Lambda \neq 0, \\ I_{rn}, & \text{if } \Lambda = 0, \end{cases} \quad (8)$$

and in this case, the following identity can be verified

$$D_n(\Lambda)^{-1} (N_n \otimes I_r) D_n(\Lambda) = N_n \otimes \Lambda. \quad (9)$$

Finally, given any  $n \in \mathbb{N}$ , we define the following matrix

$$\mathbb{J}_n := I_n + N_n. \quad (10)$$

## 2. PRELIMINARIES

### 2.1 Some explicit eigenvalues

From (Meyer, 2000, Ex. 7.2.5), we have an explicit formula for the eigenvalues of a matrix  $M \in \mathbb{R}^{n \times n}$ , with  $a \neq 0$  and  $c \neq 0$ ,

$$M = \begin{bmatrix} b & a & & & \\ c & b & a & & \\ & \ddots & \ddots & \ddots & \\ & & c & b & a \\ & & & c & b \end{bmatrix} = bI + aN + cN^\top \quad (11)$$

are explicitly given by

$$\lambda_i(M) = b + 2a\sqrt{\frac{c}{a}} \cos\left(\pi \frac{i}{n+1}\right), \quad i \in \{1, \dots, n\}. \quad (12)$$

For our purposes we only deal with the case  $c = a$ , as also exploited in (Baggio and Zampieri, 2022), leading to the explicit eigenvalues form given by

$$\lambda_i(M) = b + 2a \cos\left(\pi \frac{i}{n+1}\right), \quad i \in \{1, \dots, n\}. \quad (13)$$

## 2.2 The Jordan normal form

In the next subsection, we introduce a modification of the Jordan normal form that will be instrumental in showing the result of the paper. We first recall and define some quantities from the standard Jordan form, in this subsection.

Given a matrix  $A$  of dimensions  $n \times n$  we suppose, without loss of generality, that its eigenvalues are ordered in decreasing order with respect to the real part, namely

$$\Re\{\lambda_1\} \geq \Re\{\lambda_2\} \geq \dots \geq \Re\{\lambda_n\}.$$

Let  $m \leq n$  be the total number of linearly independent (non-generalized) eigenvectors  $T_i^1 \neq 0$  associated with an eigenvalue  $\bar{\lambda}_i \in \sigma(A)$ ,  $i = 1, \dots, m$ , such that

$$AT_i^1 = \bar{\lambda}_i T_i^1 \quad \forall i = 1, \dots, m.$$

**Definition 1.** (Jordan blocks dimension). *For each  $i \in \{1, \dots, m\}$ , we define the values  $g_i \geq 1$  satisfying  $\sum_{i=1}^m g_i = n$ , such that there exist  $g_i - 1$  linearly independent generalized eigenvectors  $T_i^k \neq 0$ , for  $k = 2, \dots, g_i$ , associated to the corresponding eigenvalue  $\bar{\lambda}_i$  and satisfying*

$$(A - \bar{\lambda}_i I)T_i^k = T_i^{k-1} \quad \forall k = 2, \dots, g_i.$$

The introduced notation allows us to determine in advance the number of distinct Jordan blocks  $m$  and their relative dimensions  $g_i$ , when the matrix  $A$  is transformed into its Jordan form  $J$ . Moreover, to all these  $m$  distinct Jordan blocks we associated, differently from the standard notation,  $m$  distinct eigenvalues  $\bar{\lambda}_i \in \sigma(A)$ ,  $i = 1, \dots, m$ .

**Lemma 1.** *Let  $A$  be a  $n \times n$  matrix, let  $m$  be the number of distinct eigenvalues  $\bar{\lambda}_i$  with associated Jordan block dimension  $g_i$ , so that  $\sum_{i=1}^m g_i = n$ . Then, there exists a  $T \in \mathbb{R}^{n \times n}$  such that*

$$T^{-1}AT = J, \quad \begin{cases} J := \text{blkdiag}(J_{\bar{\lambda}_1}, \dots, J_{\bar{\lambda}_m}), \\ J_{\bar{\lambda}_i} := \bar{\lambda}_i I_{g_i} + N_{g_i}, \quad i = 1, \dots, m. \end{cases} \quad (14)$$

For complex eigenvalues, one can always consider a real Jordan form in which a Jordan block associated with a complex pair of eigenvalues  $\bar{\lambda}_{i,1/2} = a_i \pm ib_i$ , with  $g_i - 1$  pairs of associated generalized eigenvectors, is given by

$$J_{\bar{\lambda}_i} := I_{g_i} \otimes \bar{\Lambda}_i + N_{g_i} \otimes I_2 = \begin{bmatrix} \bar{\Lambda}_i & I_2 & 0 & \dots & 0 \\ 0 & \bar{\Lambda}_i & I_2 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & \bar{\Lambda}_i & I_2 & \\ 0 & \dots & 0 & \bar{\Lambda}_i & \end{bmatrix}_{2g_i \times 2g_i}$$

where  $\bar{\Lambda}_i$  is defined as

$$\bar{\Lambda}_i = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}$$

and one can verify that  $\sigma(\bar{\Lambda}_i) = \{a + ib, a - ib\}$ . Note also that

$$\bar{\Lambda}_i + \bar{\Lambda}_i^\top = 2a_i I_2 = 2\Re\{\bar{\lambda}_{i,1/2}\}I_2.$$

## 2.3 The modified Jordan Form

We now introduce the modification of the Jordan normal form. In particular, for any non-zero distinct eigenvalue  $\bar{\lambda} \neq 0$  with associate Jordan block dimension  $g$ , we can use the matrix  $D_g(\bar{\lambda})$  defined in (6) to obtain

$$D_g^{-1}(\bar{\lambda})J_{\bar{\lambda}}D_g(\bar{\lambda}) = \bar{\lambda}J_g \quad (15)$$

with the matrix  $J_g$  defined according to (10). Hence, by defining the matrix

$$\mathbb{D} := \text{blkdiag}(D_{g_1}(\bar{\lambda}_1), \dots, D_{g_m}(\bar{\lambda}_m)) \quad (16)$$

we introduce the *modified Jordan form*  $\mathbb{J}$  defined as

$$\mathbb{J} := \mathbb{T}^{-1}A\mathbb{T}, \quad \mathbb{T} := T\mathbb{D}, \quad (17)$$

in which  $T$  satisfies (14). One can verify, using the property (15) that  $\mathbb{J}$  as the following form

$$\mathbb{J} = \text{blkdiag}(\bar{\lambda}_1 J_{g_1}, \dots, \bar{\lambda}_m J_{g_m}) = \Lambda(I + N), \quad (18)$$

with the matrix  $J_{g_i}$  defined as in (10), for each  $i = 1, \dots, m$ , where

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$$

and

$$N = \text{blkdiag}(N_{g_1}, \dots, N_{g_m}).$$

As we shall see in the next section, the matrix  $\mathbb{J}$  is now in a more convenient form to study Lyapunov matrix inequalities. Note that in the case of complex eigenvalues, one has to define a slightly different change of coordinates in order to deal with the generalized Jordan block (2.2). In particular, by using the definition (8) one can verify the following identity

$$J_{g_i} \otimes \bar{\Lambda}_i = (I_{g_i} + N_{g_i}) \otimes \bar{\Lambda}_i = D_{g_i}^{-1}(\bar{\Lambda}_i)J_{\bar{\Lambda}_i}D_{g_i}(\bar{\Lambda}_i)$$

which gives

$$J_{g_i} \otimes \bar{\Lambda}_i = \begin{bmatrix} \bar{\Lambda}_i & \bar{\Lambda}_i & 0 & \dots & 0 \\ 0 & \bar{\Lambda}_i & \bar{\Lambda}_i & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & \bar{\Lambda}_i & \bar{\Lambda}_i & \\ 0 & \dots & 0 & \bar{\Lambda}_i & \end{bmatrix}. \quad (19)$$

In the following, we give two relevant properties of the matrix  $\mathbb{J}_n$ .

**Lemma 2.** *For any  $n \in \mathbb{N}$ , the matrix  $\mathbb{J}_n$  satisfies the following properties.*

(1) *The eigenvalues of  $\mathbb{J}_n + \mathbb{J}_n^\top$  satisfies*

$$\sigma_i(\mathbb{J}_n + \mathbb{J}_n^\top) = 2 \left[ 1 + \cos\left(\pi \frac{i}{n+1}\right) \right], \quad i = \{1, \dots, n\}.$$

(2) *The matrix  $\mathbb{J}_n + \mathbb{J}_n^\top$  satisfies*

$$2 \left[ 1 + \cos\left(\pi \frac{n}{n+1}\right) \right] I_n \preceq \mathbb{J}_n + \mathbb{J}_n^\top \preceq 2 \left[ 1 + \cos\left(\pi \frac{1}{n+1}\right) \right] I_n.$$

*Proof.* The explicit formula of the eigenvalues of  $\mathbb{J} + \mathbb{J}^\top$  follows from (13), showing items 1 and 2.  $\square$

**Remark 1.** From the first item of the Lemma right above, one can easily check that  $\mathbb{J}_n + \mathbb{J}_n^\top$  is a positive definite matrix for any  $n \in \mathbb{N}$ .

Hence, as a consequence, by defining

$$\mathbb{J} = \text{blkdiag}(J_{g_1}, \dots, J_{g_m}) = I + N$$

we can also say that  $\text{sym}(\mathbb{J})$  is a positive definite matrix.

## 2.4 A high-gain interpretation

In some sense, we construct the modified Jordan block form from the standard one by defining the  $\mathbb{D}$  change of coordinates. This sort of change of coordinates is also used in the high gain (observer/stabilizer) framework, see, e.g. (Bernard et al., 2022, Section 6) or (Isidori, 1995, Chapter 4.7). In this case, for  $i = 1, \dots, m$ , by exploiting for each distinct eigenvalue  $\bar{\lambda}_i$  the transformation matrix  $D_{g_i}(\bar{\lambda}_i)$ , we obtain on the super diagonal the value of the interested eigenvalue. More in particular, for each Jordan block  $J_i$ ,  $i = 1, \dots, m$

$$D_{g_i}(\lambda_i)^{-1}(\lambda_i I + N_i)D_{g_i}(\lambda_i) = \lambda_i I + D_{g_i}(\lambda_i)^{-1}N_i D_{g_i}(\lambda_i) = \lambda_i I + \lambda_i N_i = \lambda_i(I + N_i).$$

In the general case, where we have  $m$  distinct Jordan blocks for the matrix  $A$ ,

$$\mathbb{D} = \text{diag}(D_{g_1}(\lambda_1), D_{g_2}(\lambda_2), \dots, D_{g_m}(\lambda_m))$$

then we can write the following identity

$$\begin{aligned} \mathbb{J} &= \mathbb{D}^{-1}\mathbb{J}\mathbb{D} = \mathbb{D}^{-1}T^{-1}AT\mathbb{D} = \mathbb{T}^{-1}A\mathbb{T} \\ &= \mathbb{D}^{-1}(\Lambda + N)\mathbb{D} = \Lambda + \mathbb{D}^{-1}N\mathbb{D} = \Lambda + \Lambda N = \Lambda\mathbb{J}. \end{aligned}$$

### 3. EXPONENTIAL PARAMETERS FROM AN EXPLICIT LYAPUNOV FUNCTION

Given a Hurwitz matrix  $A$ , via the Jordan blocks normal form we can explicitly write the solution to the system dynamics

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n, \quad (20)$$

We denote with  $\{\lambda_1, \dots, \lambda_n\}$  the eigenvalues of  $A$ , which are as

$$x(t) = T \exp(\Lambda t) \sum_{k=0}^{g_{\max}} \frac{N^k t^k}{k!} T^{-1} x(0)$$

where  $g_{\max} = \max_{i \in \{1, \dots, m\}} \{g_i\}$ . The solution proposed in this paper allows us to upper bound the norm of  $x(t)$  via an exponential convergence function

$$\begin{aligned} |x(t)| &= |T \exp(\Lambda t) \sum_{k=0}^{g_{\max}} \frac{N^k t^k}{k!} T^{-1} x(0)| \\ &\leq \kappa \exp(-\alpha t) |x(0)| \end{aligned}$$

such that  $\alpha$  dependence explicitly on the eigenstructure, and not only on the eigenvalues, of the system. This upper bound cannot be obtained by exploiting the normal form coordinates  $z = T^{-1}x$  and for this coordinates taking an identity Lyapunov matrix, i.e.,  $P = I_n$ , as solution to the Lyapunov matrix inequality  $J + J^\top \preceq -\alpha I$  for some  $\alpha > 0$ . In particular, by computing the eigenvalues of  $J + J^\top$  for each Jordan block form  $J_{\bar{\lambda}_i}$  applying (13), we have

$$\sigma_{\max}(J_{\bar{\lambda}_i} + J_{\bar{\lambda}_i}^\top) = 2\Re\{\bar{\lambda}_i\} + 2 \cos\left(\frac{\pi}{g_i + 1}\right)$$

which is negative only for  $\Re\{\bar{\lambda}_i\} \leq -\cos(\pi/(g_i + 1)) \leq -1$ , and it does not general.

This problem does not arise if we take into account the modified Jordan form coordinates by defining  $z = \mathbb{T}x$ , and we thus define a Lyapunov candidate

$$V = x^\top \mathbb{T}^{-\top} \mathbb{T}^{-1} x, \quad (21)$$

that we call the *explicit* Lyapunov functions<sup>2</sup>. One can easily notice that  $\mathbb{T}^{-\top} \mathbb{T}^{-1}$  is a positive definite matrix,

<sup>2</sup> We denote as Explicit Lyapunov function any Lyapunov function that can be written directly in terms of the algebraic property of a

because  $\mathbb{T}$  has all linearly independent columns, and so it is full rank, hence  $V(x) > 0$  for  $x \neq 0$ , with  $V(0) = 0$ , and it is radially unbounded

$$\sigma_{\min}(\mathbb{T}^{-1})|x|^2 \leq V \leq \sigma_{\max}(\mathbb{T}^{-1})|x|^2.$$

**Theorem 1.** Consider an asymptotically stable system (20), and the transformation matrix  $\mathbb{T}$  defined in (17). Then, the norm of the system state is upper bounded by (3), where

$$\kappa = \mu(\mathbb{T}), \quad \alpha = -\Re\{\lambda_{\max}(\text{sym}(\mathbb{J}))\}. \quad (22)$$

*Proof.* By taking the time derivative of  $V$  in 21, we have

$$\begin{aligned} \dot{V} &= 2x^\top \mathbb{T}^{-\top} \mathbb{T}^{-1} \dot{x} = 2x^\top (\mathbb{T}^{-\top} \mathbb{T}^{-1} A) x \\ &= 2x^\top \mathbb{T}^{-\top} (\mathbb{T}^{-1} A \mathbb{T}) \mathbb{T}^{-1} x = x^\top \mathbb{T}^{-\top} (\mathbb{J}^\top + \mathbb{J}) \mathbb{T}^{-1} x \\ &\leq (\Re\{\lambda_{\max}(\mathbb{J}^\top + \mathbb{J})\}) V. \end{aligned}$$

Hence,  $\dot{V}$  is strictly less than zero, and, in particular,  $V$  is exponentially decreasing. Indeed, by the comparison Lemma (Khalil, 2002)[Lemma 3.4], we have

$$V(t) \leq \exp(\Re\{\lambda_{\max}(\mathbb{J}^\top + \mathbb{J})t\}) V(0)$$

from which

$$|x|^2 \leq \frac{\sigma_{\max}(\mathbb{T}^{-1})}{\sigma_{\min}(\mathbb{T}^{-1})} \exp(\Re\{\lambda_{\max}(\mathbb{J}^\top + \mathbb{J})t\}) |x(0)|^2$$

and taking the square root of both terms of the inequality proves the theorem.  $\square$

The eigenvalues of  $\mathbb{J}^\top + \mathbb{J}$  can be explicitly written as a function of the eigenvalues of  $A$  and of the related geometric multiplicity since each block of  $\mathbb{J}^\top + \mathbb{J}$  has the form of (13). We thus provide an explicit expression for  $\alpha$ , as the following

$$\alpha = - \max_{i \in \{1, \dots, m\}} \left\{ \Re\{\bar{\lambda}_i\} \left[ 1 - \cos\left(\frac{\pi}{g_i + 1}\right) \right] \right\}. \quad (23)$$

**Remark 2.** The values of  $\alpha, \kappa$  given in Theorem 1 are explicitly given as properties of the eigenvalues and the (generalized) eigenvectors of the matrix  $A$ .

**Remark 3.** Due to the presence of the generalized eigenvectors, i.e., given by the geometric multiplicity of the associated eigenvalue  $\bar{\lambda}_i$ , the columns of  $\mathbb{T}$ , i.e.,  $\mathbb{T}_i$ ,  $i = 1, \dots, n$ , are not all orthogonal one each other. Thus, they yield a bad condition number of matrix  $\mathbb{T}$ . Only when  $A$  is *orthogonally diagonalizable*, we have  $\mathbb{D} = I$  and we can always normalize the columns of  $T$  so to get an orthonormal matrix, i.e.  $T^\top T = I$ . In this case, the  $x$  state evolution norm upper bound simplifies to the one one with standard *dominant pole* convergence rate, i.e.  $\alpha = |\lambda_1|$ ,

$$|x| \leq \exp(-|\lambda_1|t) |x(0)|.$$

#### 3.1 Comparison with a standard solution of 5

When we solve for 5, with a fixed convergence rate (23), the obtained solution  $P$  can be always rewritten as  $P = \mathbb{T}^{-\top} P' \mathbb{T}^{-1}$ , for some positive definite  $P'$ . We can thus write, from the dynamics of  $V' = x^\top P x = x^\top \mathbb{T}^{-\top} P' \mathbb{T}^{-1} x$ , that

$$|x| \leq \mu(\mathbb{T}^{-\top} P' \mathbb{T}^{-1}) \exp(-\alpha t) |x(0)|.$$

However, we cannot determine a priori if  $\mu(\mathbb{T}^{-\top} \mathbb{T}^{-1}) \leq \mu(\mathbb{T}^{-\top} P' \mathbb{T}^{-1})$ , we can only provide a conservative upper

Hurwitz  $A$  without considering the solution of a Lyapunov equation (2), or inequality (1), and thus without the choice of positive definite matrices  $P$  and  $Q$ .

bound of the form  $\mu(\mathbb{T}^{-\top} P' \mathbb{T}^{-1}) \leq \mu(\mathbb{T}^{-\top} \mathbb{T}^{-1}) \mu(P')$ . Thus, the condition number of  $P$  could be in principle better than the one provided by the choice  $\mathbb{T}^{-\top} \mathbb{T}^{-1}$ . However, we cannot mathematically prove that the solution of the LMI (5) performs better or worse than the particular choice  $\mathbb{T}^{-\top} \mathbb{T}^{-1}$ , and as discussed above, we can only get some conservative upper bounds. We thus report here the results from a set of numerical tests to show that most of the time  $\mathbb{T}^{-\top} \mathbb{T}^{-1}$  is better conditioned than the numerical solution  $P$  of the LMI, and on average performs better, see Sec. 5.2.

#### 4. APPLICATION IN CONTROLLER DESIGN

Consider the controllable linear system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \quad (24)$$

with  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ , for which to design of a stabilizing state feedback gain  $K$  such that  $A - BK$  is Hurwitz, and the state norm has some desired performance to be guaranteed for *all initial conditions*  $x_0$  in a ball of radius  $\rho_0$ , i.e.,  $x_0 \in \mathcal{B}_{\rho_0}$ . These performances on the state norm  $|x(t)|$  can be related to

- limiting the ‘overshoot’ with respect to the initial condition, i.e., the evolution of the state norm must be constrained to remain inside a certain ball of fixed radius  $\bar{\rho} \geq \rho_0$ ;
- finding a ‘good’ approximation of the time  $t^*$  after which the state norm reaches a certain ball of radius  $\underline{\rho}$ , i.e., find  $t^* \in \mathbb{R}$  such that  $|x(t)| \leq \underline{\rho}$  for all  $t \geq t^*$ , when the system is initialized in  $\mathcal{B}_{\rho_0}$ ;
- forcing the state norm to reach a certain ball  $|x(t)| \leq \underline{\rho}$  within a prescribed fixed time  $t^*$ .

In order to illustrate the potential of the application on an example, let us consider, for the sake of exposition, the case in which

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and take as static feedback gain  $K = [\alpha_1 \alpha_2, \alpha_1 + \alpha_2]$ , with  $\alpha_1 = \gamma$  and  $\alpha_2 = 2\gamma$ , as in the standard high-gain formalism, where  $\gamma > 0$  plays the role of the high gain and imposes the closed loop eigenvalues to be  $\lambda_1 = -\gamma$  and  $\lambda_2 = -2\gamma$ .

Since the closed loop matrix is in companion form, it can be diagonalized (due to the presence of simple eigenvalues) via the Vandermonde matrix

$$T = \begin{bmatrix} 1 & 1 \\ -\alpha_1 & -\alpha_2 \end{bmatrix}, \quad T^{-1} = \frac{1}{\alpha_1 - \alpha_2} \begin{bmatrix} -\alpha_2 & -1 \\ \alpha_1 & 1 \end{bmatrix},$$

whose conditioning number  $\mu(T)$ , for  $\alpha_1 = \gamma$  and  $\alpha_2 = 2\gamma$ , reads as

$$\mu(T) = \sqrt{\frac{5\gamma^2 + 2 + \sqrt{25\gamma^4 + 16\gamma^2 + 4}}{5\gamma^2 + 2 - \sqrt{25\gamma^4 + 16\gamma^2 + 4}}}$$

and thus have  $|x(t)| \leq \mu(T) \exp(-\gamma t) |x(0)|$ . For this choice of the closed loop characteristic polynomial, we have a minimum of  $\mu(T)$  at  $\gamma = \sqrt{\frac{2}{5}} \approx 0.6325$ , to which corresponds a condition number  $\mu(t) \approx 6.1623$ .

However, we cannot prove that this solution is the best for this set of parameters, although, to the best of the authors’ knowledge, this is the only constructive approach available

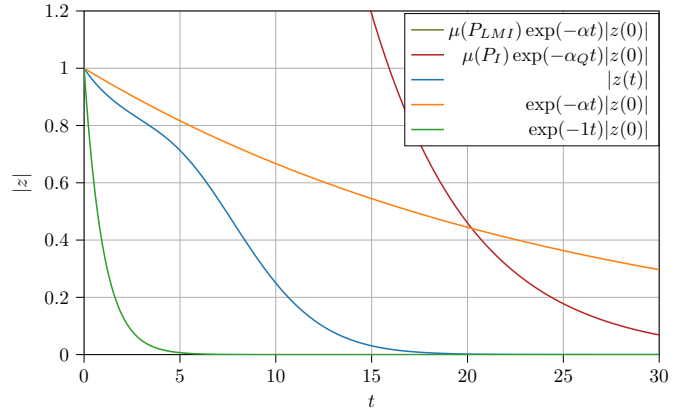


Fig. 1. Evolution of the norm of  $z(t)$ , compared to (3) with different parameters, i.e., the one obtained from (5), from (2), and the one in (22).

in the literature to analytically achieve this kind of design and that is independent of testing all initial conditions in a compact set  $\mathcal{B}_{\rho_0}$ .

Other design scenarios, such as the design of observer dynamics with prescribed convergence performances and their application in output dynamical feedback controller design (motivating this work at an early stage) are possible and postponed to the journal version of this paper, due to lack of space. See for example Spirito et al. (2024) to see an application to define the Hamiltonian structural matrices of a stable linear system.

#### 5. SOME NUMERICAL RESULTS

##### 5.1 An autonomous system example

We consider the evolution of the norm of  $z(t)$  for an eigenvalue,  $\lambda = -1$ , with associated geometric multiplicity  $g = n = 10$ , i.e., we consider an autonomous system dynamics with state matrix  $A = -1(I + N) \in \mathbb{R}^{10 \times 10}$ , whose corresponding convergence rate given (23), is  $\alpha \approx 0.0405$ . The evolution of the state norm  $|z(t)|$ , from initial conditions

$$z_0 = [0.01001979, 0.02185996, -0.01413963, 0.08315555, -0.14693675, 0.31947075, -0.4683713, 0.59627956, -0.4827674, 0.24631203]^\top. \quad (25)$$

is depicted in Fig.1. When compared to the numerical solution of the LMI (5), by fixing  $\alpha$  as in (23), we get a matrix  $P_{LMI}$  with condition number of  $\mu(P_{LMI}) = 5.0957$ , which is worse than the unitary condition number of  $\mathbb{T} = I$ . Whereas, solving (2), with  $Q = I$  or with  $Q = \alpha$ , where  $\alpha$  in (23), gives solutions  $P_I$  and  $P_\alpha$  whose maximum eigenvalues are respectively  $\lambda_{\max}(P_I) \approx 5.2605$  and  $\lambda_{\max}(P_\alpha) \approx 0.2131$ . Furthermore, they share the same condition number is  $\mu(P_I) \approx 20.5959$ , and provide the same convergence rate  $\alpha_Q = \lambda_{\min}(Q)/\lambda_{\max}(P_I) \approx 0.1901$  which is approximately 5 times larger than the closed form solution.

In Fig.1, we compare the proposed parameters (23) with the two set of parameters obtain from the solutions of (5) and (2). In particular, we can notice that during a first transient, the proposed approach provides a better approx-

imation of norm evolution, while the one obtained from (2) performs better on a larger time interval. Due to the fact the solution of (5) can only be obtained numerically, it provides by far the most conservative result<sup>3</sup>.

## 5.2 Numerical tests

We consider as numerical test a matrix  $A$  with dimension 9 with 3 distinct eigenvalues  $\bar{\lambda}$ , i.e.,  $\{-0.5, -2, -4\}$  with associated Jordan block dimensions  $(g_1, g_2, g_3) = (3, 5, 1)$ . That is, according to the value convergence rate formula (23) we have

$$\alpha = 0.5 \left( 1 + \cos \left( \frac{3}{4}\pi \right) \right) \approx 0.1464$$

with  $P = \mathbb{T}^{-\top} \mathbb{T}^{-1}$ , thus the condition number of  $P$  is equivalent to the condition number of  $\mathbb{T}$ . By comparing the solution  $P_{LMI}$  of LMI solver (getlmis()) implemented in the Robust Control Toolbox™ of Matlab version R2022a with the given  $\alpha$ , we get approximatively, over 5000 comparisons, 99% of the times a better condition number of  $\mathbb{T}$  than that of  $P_{LMI}$ . The numerical testing details are summarized in Table 1. Other testing results are also

Table 1. Numerical results

# tests	success %	average $\mu(\mathbb{T}^{\top} \mathbb{T})$	average $\mu(P_{LMI})$
5000	99%	1.4839e + 03	1.0417e + 06

available in (Spirito and Astolfi, 2024), in which we show that with a free convergence rate  $\alpha$  in (5), the solution  $\mathbb{T}^{-\top} \mathbb{T}^{-1}$  performance better than the numerical solution obtain by the Matlab solver.

## 6. CONCLUSIONS

In this work, we present a modification of the Jordan block normal form of the state matrix of an autonomous linear system. We use this analysis to obtain, in closed form, the constant values for the scaling factor  $\kappa$  and the converge rate  $\alpha$  for the uniform exponential upper bound of stable linear autonomous systems. This approach allows us to obtain a constructive characterization of the convergence property of a linear system. Moreover, we apply such a characterization to enhance the closed loop performances of a controllable system. Numerical examples are correlated to show the effectiveness of the analysis.

The presented result is a preliminary work that has been extended, in the submitted paper (Spirito and Astolfi, 2024), to the case of non-Hurwitz matrices typical of  $k$ -contraction scenarios, see e.g. (Zoboli et al., 2023).

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<sup>3</sup> Note that solving (5) by fixing  $\alpha = \lambda_{\min}(I)/\lambda_{\max}(P_I) \approx 0.1901$ , provides a numerical solution with conditioning number  $\mu(P) \approx 729.4461$ .