Explicit parameters of exponential stability and contractivity of globally Lipschitz semi-linear systems

Mario Spirito, Daniele Astolfi

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Mario Spirito[1] and Daniele Astolfi[1]

***Abstract*— In this paper, we propose a change of coordinates that brings the state matrix of an autonomous linear system into a modified Jordan Block form. This change of coordinates and the obtained modified Jordan Block form, allows us to obtain exact values of the scaling factor and the convergence rate of the exponential stability bound for linear systems. The analysis is then applied to obtain sufficient conditions for the stability of semilinear systems with Lipshitz nonlinearity. Numerical examples are also provided to illustrate the effectiveness of the proposed approach.**

I. Introduction

Since the origin of Lyapunov's direct method [1], assessing the stability of an unforced system is done by considering a nonnegative scalar function of the state vector and checking that its value is monotonically decreasing in time. The approach was also described and extended in [2], [3], [4], [5], [6], see also [7] to review on the development of the definition of stability during last century and [8] for a historical perspective on stability concept.

It is well-known and worldwide accepted that, for linear systems, time-(in)dependent quadratic forms are used to study the stability property of systems. Thus, to prove the stability of an autonomous linear system $\dot{x} = Ax$, with $x(0) = x_0 \in R^n$, one has to look for a symmetric positive definite matrix P such that $PA + A^\top P < -2\alpha P$, for some real $\alpha > 0$. However, rather than directly specifying such a matrix P, a standard approach [9][Ch.7] involves an additional positive definite matrix Q, and one has to look for a P solution of $PA + A^\top P = -Q$, where Q is a degree of freedom, for example, it can be taken as $Q = 2qI$, for some real $q > 0$. It is also well-known, that such a solution P exists and is also unique if and only if A has all negative real parts, and such a P is given by

$$P = \int_0^\infty \exp(A^\top s) Q \exp(As) ds, \qquad Q = Q^\top > 0.$$

It is also possible to numerical compute P by exploiting the vectorization of matrices as introduced in [10]. Other approaches to compute the Lyapunov function for nonlinear systems can be found in [11], or [12], [13], see [14] for a recent review on the topic. Such a pair P, Q can be exploited to determine the positive real constants κ and α, respectively, the *scaling factor* and the *convergence rate*, in the standard definition of uniform exponential stability, [9], [11], i.e.,

$$|x(t)| \leq \kappa \exp(-\alpha t)|x(0)|.$$

In particular, by considering $\sigma_{\min}(Q)$ to be the minimum singular value of Q is easy to show, by comparison lemma [11], that we have

$$|x(t)| \leq \frac{\sigma_{\max}(P)}{\sigma_{\min}(P)} \exp\left(-\frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)} t\right) |x(0)|, \quad \forall t \geq 0, \quad (1)$$

where κ is given by the condition number of P and the maximum eigenvalue of P is involved in the convergence rate α, along with the minimum eigenvalue of Q. Although very general, the obtained constants for the norm upper bound, thus highly depend on the choice of the matrix Q. It is not an easy task to select Q in order to obtain the optimal values of the exponential convergence, i.e., to estimate κ and α such that the upper bound is as close as possible to the real norm evolution. Usually, taking Q with a large $\sigma_{\min}(Q)$ (aiming at obtaining a large convergence rate α), makes P very ill-conditioned, thus increasing the condition number κ in the upper bound.

To the best of the authors' knowledge, there is no other general approach available in the literature to explicitly obtain the scaling factor κ and the convergence rate α for linear systems. The closer results we were able to find are [15] in which the authors test the exponential convergence rate using integral quadratic constraints, and in [16] the author obtains the rate of decay of solutions of a class of convolution Volterra difference equations. However, none of the available results in the literature are suitable to explicitly obtain the parameters of exponential convergence κ and α in closed form, neither, with respect to the eigenvalues and generalized eigenvectors properties in the case of linear time-invariant systems.

The paper is structured as follows. We first collect some preliminary results in Section II. Then in Section III we propose a modification of the standard Jordan block normal form of the state matrix, through which we can obtain the explicit exponential stability parameters (the values of κ and α) in terms of the algebraic properties of the matrix eigenvalues and the (generalized) eigenvectors. In Section IV, we employ the obtained explicit parameters to study the stability of semilinear systems with (regional) Lipschitz nonlinearity. Finally, in Section V, we show the effectiveness of the proposed bounds on an autonomous system, with large geometric multiplicity, and an application of the stability of semilinear systems. Some conclusions and perspectives are then given in Section VI.

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Nomenclature

The L_2-norm of matrix M is denoted by $\|M\|$, and we denote the condition number of M as

$$\mu(M) := \sqrt{\frac{\sigma_{\max}(M^\top M)}{\sigma_{\min}(M^\top M)}} = \mu(M^{-1}) = \sqrt{\frac{\sigma_{\max}(M^{-\top} M^{-1})}{\sigma_{\min}(M^{-\top} M^{-1})}}.$$

We consider an orthogonally diagonalizable matrix M if it has an orthonormal set of eigenvectors. Given a complex number $\mu \in \mathbb{C}$ and a natural number $n \in \mathbb{N}$, we denote

$$D_n(\mu) := \begin{cases} \operatorname{diag}(1, \mu, \ldots, \mu^{n-1}), & \text{if } \mu \neq 0, \\ I_n, & \text{if } \mu = 0. \end{cases} \tag{2}$$

Then, the following identity

$$N_n D_n(\mu) = \mu D_n(\mu) N_n \tag{3}$$

holds for any $\mu \in \mathbb{C}$ and $n \in \mathbb{N}$, where $N_n \in \mathbb{R}^{n \times n}$ is a the matrix of zero elements, with only 1 on the first super diagonal.

II. Preliminaries

A. Norm of the Matrix exponential

We recall some properties of the matrix exponential norm. We first consider the fact that the exponential of a skewsymmetric (or symplectic) matrix is always orthonormal (and hence has unitary norm), i.e., for $S = -S^\top$, $\|\exp(S)\|_2 = \sqrt{\|\exp(S^\top)\exp(S)\|_2} = \sqrt{\|\exp(0)\|_2} = 1$.
The general formula of the products of the exponential of two non-commuting matrices A and B, is given by

$$\exp(A)\exp(B) = \exp(A+B)\exp\left(\frac{1}{2}(AB - BA)\right).$$

In particular, when we consider $A = B^\top$, we have

$$\exp(B^\top)\exp(B) = \exp(B^\top + B)\exp\left(\operatorname{skew}(B^\top B)\right),$$

where $\operatorname{sk}(B^\top B)$ is the skew symmetric part of $B^\top B$. Hence, we can always write

$$\begin{aligned} \|\exp(B^\top)\exp(B)\| &\leq \|\exp(B^\top + B)\| \|\exp\left(\operatorname{sk}(B^\top B)\right)\| \\ &\leq \|\exp(B^\top + B)\|. \end{aligned}$$

We hence proved that

Lemma 1. *Given a matrix M and its matrix exponential* $\exp(M)$*, then*

$$\|\exp(M^\top)\exp(M)\| \leq \|\exp(M^\top + M)\|.$$

B. Some explicit eigenvalues

From [17, Ex. 7.2.5], we have an explicit formula for the eigenvalues of a matrix $A \in \mathbb{R}^{n \times n}$, with $a \neq 0$ and $c \neq 0$,

$$A = \begin{bmatrix} b & a & & & \\ c & b & a & & \\ & \ddots & \ddots & \ddots & \\ & & c & b & a \\ & & & c & b \end{bmatrix} = bI + aN + cN^\top \tag{4}$$

are explicitly given by

$$\lambda_i = b + 2a\sqrt{\frac{c}{a}}\cos\left(\pi \frac{i}{n+1}\right), \quad i \in \{1, \ldots, n\}. \tag{5}$$

For our purposes we only deal with the case $c = a$, as also exploited in [18], leading to the explicit eigenvalues form given by

$$\lambda_i = b + 2a\cos\left(\pi \frac{i}{n+1}\right), \quad i \in \{1, \ldots, n\}. \tag{6}$$

C. Contractive systems with constant metric

Consider the system

$$\dot{x} = f(x) \tag{7}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Let $\phi(x_0, t)$ denote the trajectory of (7) at time t originated at x_0 at time $t = 0$. We then have the following definition.

Definition 1. *We say that system* (7) *defines a contraction if there exist real positive $\alpha, \kappa > 0$ such that*

$$\|\phi(x_1, t) - \phi(x_2, t)\| \leq \kappa \exp(-\alpha t)\|x_1 - x_2\|$$

for all initial conditions $x_1, x_2 \in X \subseteq \mathbb{R}^n$ and for all $t \geq 0$.

A sufficient condition to determine whether system (7) defines a contraction is the existence of a constant metric P for which the distance with this metric between any two system trajectories is monotonically decreasing in time. We thus have the following well-known result, also known as Demidovic condition.

Lemma 2. *System* (7) *is a contraction on X if there exists a constant $P \in \mathbb{R}^{n \times n}$ symmetric and positive definite, such that*

$$P\frac{df(x)}{dx} + \frac{df(x)}{dx}^\top P < 0 \tag{8}$$

for all $x \in X \subset \mathbb{R}^n$.

This result has been proved in [19], [20], [21], [22], [23] and extended in [24] and [25] in the more general case of nonlinear Riemannian metrics. See also [26] for the case of non-Euclidean L_1 and L_∞ metrics.

III. Explicit parameters for Continous time LTI systems

Consider an asymptotically stable system

$$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n, \tag{9}$$

i.e. A is Hurwitz with appropriate dimension. We denote with $\{\lambda_1, \ldots, \lambda_n\}$ the eigenvalues of A, which are ordered, without loss of generality, in decreasing order with respect to the real part, namely

$$\Re\{\lambda_1\} \geq \cdots \geq \Re\{\lambda_n\}.$$

Furthermore, we denote with g_i the geometric multiplicity[1] of λ_i, for $i = 1, \ldots, m$, where m is the number of distinct Jordan blocks. By construction, $\sum_{i=1}^m g_i = n$. We define the diagonal matrix of the ordered eigenvalues of A, i.e., $\Lambda =$

[1] i.e., the dimension of the eigenspace associated with λ_i.

diag $(\lambda_1, \lambda_2, \dots, \lambda_n)$, where for some i and j, we might have $\lambda_i = \lambda_j$, according to the relative algebraic and geometric multiplicity of each eigenvalue.

A. The case of real eigenvalues

We first suppose that all the eigenvalues of A are real. The case of complex eigenvalues will be detailed in the next section.

We can define the Jordan normal form J as the block diagonal matrix composed of the Jordan blocks of the eigenvalues of A. It is possible to find such a Jordan normal form, by means of a change of coordinates, obtained by defining T as the matrix composed of the associated normalized (generalized) T_i, $i = 1, \dots, n$ i.e.

$$T = \begin{bmatrix} T_1 \vdots \cdots \vdots T_n \end{bmatrix}$$

where for each i-th eigenvalue, λ_i, with $i = 1, \dots, m$, we have

$$AT_i = T_i\lambda_i, \text{ or}$$
$$(A - \lambda_i I)\, T_{i+j} = T_{i+j-1}, \quad j = 1, \dots, g_i - 1$$

if the associated eigenvalue λ_i has geometric multiplicity $g_i > 1$. Through this matrix T we can construct a change of coordinates and obtain the *standard* Jordan block form of A, i.e. $T^{-1}AT = J$, defined as

$$J = \text{blckdiag}(J_1, \dots, J_m), \qquad J_i = \lambda_i I_{g_i} + N_{g_i}.$$

We now define, the modified Jordan normal form, $\mathbb{J}$, by defining the diagonal matrix

$$\mathbb{D} = \text{blckdiag}\left(D_{g_1}(\lambda_1), D_{g_2}(\lambda_2) \dots, D_{g_m}(\lambda_m)\right), \tag{10}$$

and the change of coordinates $\mathbb{T} = T\mathbb{D}$ where the columns of $\mathbb{T}$, $\mathbb{T}_i$, $i = 1, \dots, n$ are the (generalized) eigenvectors in scaled version solution of

$$A\mathbb{T}_i = \mathbb{T}_i\lambda_i, \text{ or}$$
$$(A - \lambda_i I)\, \mathbb{T}_{i+j} = \lambda_i \mathbb{T}_{i+j-1}, \quad j = 1, \dots, g_i - 1$$

and we obtain, as a change of coordinates

$$\mathbb{T}^{-1}A\mathbb{T} = \mathbb{J} = \Lambda(I_n + N_n).$$

We hence take $z = \mathbb{T}^{-1}x$, with $z(0) = \mathbb{T}^{-1}x(0)$, whose dynamics is given by

$$\dot{z} = \mathbb{T}^{-1}A\mathbb{T}z = \mathbb{J}z$$

and its solution is simply $z(t) = \exp(\mathbb{J}t)\, z(0)$. Hence, we obtain the solution in the original coordinates

$$x(t) = \mathbb{T}z(t) = \mathbb{T}\exp(\mathbb{J}t)\mathbb{T}^{-1}x(0),$$

where $\mathbb{T} = T\mathbb{D}$ and $\mathbb{T}^{-1} = \mathbb{D}^{-1}T^{-1}$. Then we can write the evolution of $x(t)$ passing for $x(0)$ as

$$x(t) = T\mathbb{D}\exp(\mathbb{J}t)\mathbb{D}^{-1}T^{-1}x(0). \tag{11}$$

Hence, we have the following.

Theorem 1. *The norm of the state evolution associated with the state matrix A can be explicitly bound as*

$$\mu(\mathbb{T})\exp(\underline{\lambda}t)|x(0)| \le |x(t)| \le \mu(\mathbb{T})\exp(\overline{\lambda}t)|x(0)| \tag{12}$$

where $\overline{\lambda} = \Re\{\lambda_{\max}(\text{sym}(\mathbb{J}))\}$ *and* $\underline{\lambda} = \Re\{\lambda_{\min}(\text{sym}(\mathbb{J}))\}$.

Proof: The proof comes as an application of the norm in the $z(t)$ coordinates and from Lemma 1. In particular, from the norm of $z(t)$ we have

$$\begin{aligned} |z| &= \sqrt{z^\top z} = \sqrt{z^\top(0)\exp(\mathbb{J}^\top t)\exp(\mathbb{J}t)z(0)} \\ &\le |z(0)|\sqrt{\|\exp((\mathbb{J}^\top + \mathbb{J})t)\|} \le \|\exp(\text{sym}(\mathbb{J})t)\|\,|z(0)|. \end{aligned}$$

Now, from the definition of z we have

$$\begin{aligned} |\mathbb{T}^{-1}x| &\le \|\exp(\text{sym}(\mathbb{J})t)\|\,|\mathbb{T}^{-1}x(0)| \\ |x| &\le \|\mathbb{T}\|\|\mathbb{T}\|^{-1}\|\exp(\text{sym}(\mathbb{J})t)\|\,|x(0)| \end{aligned}$$

where by the properties of the matrix 2-norm $\|\mathbb{T}\|\|\mathbb{T}\|^{-1} = \mu(\mathbb{T})$. Now, since sym$(\mathbb{J})$ is symmetric and negative definite, there exists an orthonormal change of coordinates that brings it into diagonal form, and thus we can write

$$\|\exp(\text{sym}(\mathbb{J}))\| \le \exp(\Re\{\lambda_{\max}(\text{sym}(\mathbb{J}))\}).$$

Hence, we have the upper bound for $|x|$. The analogous analysis can be carried out to obtain the lower bound. □

Remark. *The eigenvalues of* $\mathbb{J}^\top + \mathbb{J}$ *can be explicitly written as a function of the eigenvalues of A and of the related geometric multiplicity since each block of* $\mathbb{J}^\top + \mathbb{J}$ *has the form of* (6).

More explicitly, thanks to the structure of $\mathbb{J}$, we have

$$\begin{aligned} \underline{\lambda} &= \min_{i\in\{1,\dots,m\}} \left\{ \Re\{\lambda_i\} \left[1 - \cos\left(\frac{\pi g_i}{g_i + 1} \right) \right] \right\}, \\ \overline{\lambda} &= \max_{i\in\{1,\dots,m\}} \left\{ \Re\{\lambda_i\} \left[1 - \cos\left(\frac{\pi}{g_i + 1} \right) \right] \right\}. \end{aligned} \tag{13}$$

As a consequence, we can explicitly write the system convergence rate α and the scaling factor κ as

$$\alpha = -\overline{\lambda}, \quad \kappa = \mu(\mathbb{T}). \tag{14}$$

We want to highlight that these values are explicitly given as properties of the eigenvalues and the eigenvectors of the matrix A.

B. Dealing with complex conjugate eigenvalues pairs

For the sake of completeness and clarity of exposition, we want to separately treat how to obtain a real transformation matrix $\mathbb{T}$ in the case of complex conjugate eigenvalues.
In particular, note that the analysis above leads to complex transformation T and Jordan form J matrices in case A have some complex conjugate eigenvalues pair. Because the (generalized) eigenvectors associated with such complex conjugate eigenvalues pair are complex conjugate as well, one can replace the two columns in the transformation matrix T, say columns T_i and $T_{i+1} = T_i^\star$, with the real and the imaginary part of T_i respectively, i.e. $T_j = \Re\{T_i\}$ and $T_{j+1} =$

$\Im\{T_i\}$, with $j=i$. This new transformation T will provide a matrix of dimension 2 replacing the complex diagonal matrix with the complex conjugate pair on the main diagonal. When the geometric multiplicity of the pair is larger than one, their associated standard Jordan block form will not have a set of 1 on the super diagonal but rather have a set of identity matrices of dimension 2 on the super diagonal.

Since we are interested in the real realization of the transformation and in the modified Jordan normal form, we claim that once the real transformation T for the standard Jordan normal form is found, we can easily get the real transformation $\mathbb{T}$ leading to the real modified Jordan normal form.

Theorem 2. *Assume $A \in \mathbb{R}^{2n\times 2n}$ have a pair of complex conjugate eigenvalues $\lambda_{1/2} = \mathrm{a} \pm \mathrm{ib}$ (not in the origin, i.e. $\lambda_{1/2} \neq 0$), with geometric multiplicity $g = n$, and consider T to be the (real) matrix transformation to obtain the real Jordan normal form J, with*

$$D_\lambda = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ -\mathrm{b} & \mathrm{a} \end{bmatrix}$$

on the main diagonal. Then, by defining

$$\mathbb{D} = D_{g-1}(D_\lambda) = \mathrm{diag}(I, D_\lambda^1, D_\lambda^2, \ldots, D_\lambda^{g-1})$$

the (real) transformation that puts A into the real modified Jordan normal form, is given by $\mathbb{T} = T\mathbb{D}$.

Proof: The objective is prove that the transformation $\mathbb{T} = T\mathbb{D}$ provides the (real) modified Jordan block associated with D_λ, i.e. $\mathbb{J} = \mathbb{T}^{-1}A\mathbb{T}$ looks like

$$\mathbb{J} = (I+N)\otimes D_\lambda = \begin{bmatrix} D_\lambda & D_\lambda & 0 & \ldots & 0 \\ 0 & D_\lambda & D_\lambda & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ & & & D_\lambda & D_\lambda \\ 0 & \ldots & & 0 & D_\lambda \end{bmatrix}.$$

Indeed, this is the case because $A = TJT^{-1}$, where

$$J = I\otimes D_\lambda + N\otimes I_2 = \begin{bmatrix} D_\lambda & I & 0 & \ldots & 0 \\ 0 & D_\lambda & I & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ & & & D_\lambda & I \\ 0 & \ldots & & 0 & D_\lambda \end{bmatrix}.$$

So, $\mathbb{J} = \mathbb{T}^{-1}A\mathbb{T} = \mathbb{T}^{-1}TJT^{-1}\mathbb{T}$ where $\mathbb{T} = T\mathbb{D}$ and $\mathbb{T}^{-1} = \mathbb{D}^{-1}T^{-1}$, hence by very simple manipulations we have that $\mathbb{J} = \mathbb{D}^{-1}J\mathbb{D}$. □

We provided the result only for a single pair of complex conjugate eigenvalues $\lambda = \mathrm{a}+\mathrm{ib}$ and $\lambda^\star = \mathrm{a}-\mathrm{ib}$, the extension to the general case is just an application of the proposed procedure to each involved complex conjugate pair.
To obtain a real modified Jordan matrix when A has complex conjugate eigenvalues with geometric multiplicity larger than 1, we can simply rely on a different definition of the diagonal matrix $\mathbb{D}$, that now becomes block diagonal with all real entries, and on the real transformation matrix T used to obtain a real Jordan block matrix.

C. A high-gain interpretation

In some sense, we construct the modified Jordan block form from the standard one by defining the $\mathbb{D}$ change of coordinates. This sort of change of coordinates is also used in the high gain (observer/stabilizer) framework, see, e.g. [27, Section 6] or [28, Chapter 4.7]. In this case, for $i = 1,\ldots,m$, by exploiting for each eigenvalue λ_i the transformation matrix $D_{g_i}(\lambda_i)$, we are able to obtain on the super diagonal the value of the interested eigenvalue. More in particular, for each Jordan block J_i, $i = 1,\ldots,m$

$$D_{g_i}(\lambda_i)^{-1}(\lambda_i I + N_i)D_{g_i}(\lambda_i) = \lambda_i I + D_{g_i}(\lambda_i)^{-1}ND_{g_i}(\lambda_i) = \lambda_i I + \lambda_i N_i = \lambda_i(I+N_i).$$

In the general case, where we have m distinct Jordan blocks for the matrix A,

$$\mathbb{D} = \mathrm{diag}\,(D_{g_1}(\lambda_1), D_{g_2}(\lambda_2), \ldots, D_{g_m}(\lambda_m))$$

and, with $\Lambda = \mathrm{diag}\,(\lambda_1,\ldots,\lambda_n)$,

$$\mathbb{J} = \mathbb{D}^{-1}J\mathbb{D} = \mathbb{D}^{-1}T^{-1}AT\mathbb{D} = \mathbb{T}^{-1}A\mathbb{T} = \mathbb{D}^{-1}(\Lambda+N)\mathbb{D} = \Lambda + \mathbb{D}^{-1}N\mathbb{D} = \Lambda + \Lambda N = \Lambda(I+N).$$

IV. Application: Contractivity of Lipschitz semi-linear systems

Consider the general semi-linear system

$$\dot{x} = Ax + \varphi(x) \tag{15}$$

where φ is Lipschitz with Lipschitz constant γ, i.e., for any x and x' in $X \subset \mathbb{R}^n$

$$|\varphi(x') - \varphi(x)| \leq \gamma |x' - x|$$

or equivalently if $|d\varphi(x)/dx| \leq \gamma$ for all $x \in X$. φ is globally Lipschitz if $X \equiv \mathbb{R}^n$.

Theorem 3. *Consider the semi-linear system (15), with φ being Lipschitz in X with Lipschitz constant γ. If $\overline{\lambda} + 2\gamma < 0$, with $\overline{\lambda}$ being the convergence rate of A as defined in (13), then the system is exponentially contracting on $X \subseteq \mathbb{R}^n$, with constant metric $P = \mathbb{T}^{-\top}\mathbb{T}^{-1}$.*

Proof: Consider two solutions x_1, x_2 to system (15) and set $\tilde{x} = x_1 - x_2$. The $\tilde{x}$-dynamics are given by

$$\dot{\tilde{x}} = A\tilde{x} + \tilde{\varphi}(\tilde{x}, x_2), \qquad \tilde{\varphi}(\tilde{x}, x_2) := \varphi(\tilde{x} + x_2) - \varphi(x_2).$$

Consider the Lyapunov function $V = \tilde{x}^\top \mathbb{T}^{-\top}\mathbb{T}^{-1}\tilde{x}$, we then have

$$\dot{V} = 2\tilde{x}^\top \mathbb{T}^{-\top}\mathbb{T}^{-1}\left(A\tilde{x} + \tilde{\varphi}(\tilde{x}, x_2)\right)$$

By using the mean value theorem we further obtain

$$\begin{aligned} \dot{V} &= 2\tilde{x}^\top \mathbb{T}^{-\top}\mathbb{T}^{-1}\left(A + \int_0^1 \frac{d\varphi}{dx}(\tilde{x} + s x_2)ds\right)\mathbb{T}\mathbb{T}^{-1}\tilde{x} \\ &= \tilde{x}^\top \mathbb{T}^{-\top}\left(\mathbb{J} + 2\mathbb{T}^{-1}\int_0^1 \frac{d\varphi}{dx}(\tilde{x} + s x_2)ds\,\mathbb{T}\right)\mathbb{T}^{-1}\tilde{x} \\ &\leq \tilde{x}^\top \mathbb{T}^{-\top}\left(\overline{\lambda} + 2\gamma\right)\mathbb{T}^{-1}\tilde{x} = -\varepsilon V \end{aligned}$$

for $\varepsilon = -(\overline{\lambda} + 2\gamma) > 0$. Thus, we proved the theorem. □

Remark. *We can moreover obtain an upper bound for the evolution of $\tilde{x}$, i.e., transforming from V back to $\tilde{x}$ we have*

$$|\tilde{x}| \leq \mu(\mathbb{T}) \exp\left((\overline{\lambda} + 2\gamma)t\right) |\tilde{x}(0)|.$$

V. Examples

A. Autonomous system example

We consider the evolution of the norm of $z(t)$ for an eigenvalue, $\lambda = -1$, with associated geometric multiplicity $g = n = 10$, i.e., we consider an autonomous system dynamics with state matrix $A = -1(I+N) \in \mathbb{R}^{10\times 10}$.
The evolution of the components of the $z(t)$ vector are shown in Fig.1, starting from an initial condition

$$\begin{aligned} z_0 = [&0.01001979,\ 0.02185996,\ -0.01413963 \\ &0.08315555,\ -0.14693675,\ 0.31947075 \\ &-0.4683713,\ 0.59627956,\ -0.4827674 \\ &0.24631203]^\top. \end{aligned} \tag{16}$$

Note here that the presence of the overshoot, associated with the standard Jordan block form, is not present. We also

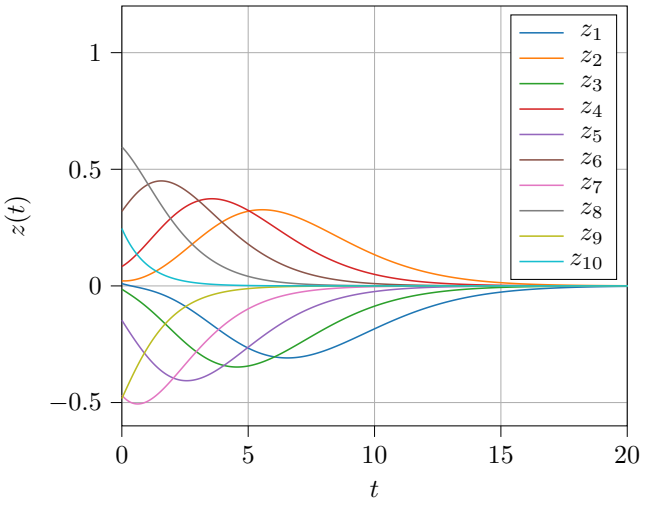

Fig. 1. Evolution of the components of the $z(t)$ state, whose dynamics is associated to the eigenvalue $\lambda = -1$ with geometric multiplicity $g = 10$, initialized at $z(0) = z_0$.

show, in Fig.2, the associated norm evolution and the natural obtained bound with convergence rate $-\overline{\lambda} \approx 0.0405$, while its evolution is lower bounded by the exponential function with convergence rate $-\underline{\lambda} \approx 1.9595$, even if the eigenvalue of the dynamics is -1.

B. An example of semilinear stability

We consider the case in which the system is already in the z coordinates, that we have a single negative eigenvalue $\lambda \in \mathbb{R}$ with geometric multiplicity $g = n$, and that the nonlinearity part is given by $\varphi(z)$ with $\varphi_i(z) = z^\top z$, $i = 1,\dots,n$, i.e.

$$\dot{z} = \lambda(I+N)z + \varphi(z).$$

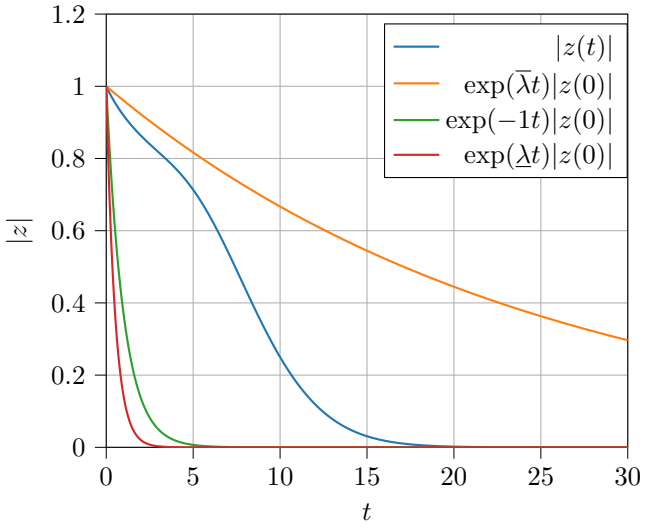

Fig. 2. Evolution of the norm of $z(t)$, and its upper bound evolution through the explicit value of the convergence rate $-\overline{\lambda}$, orange curve. While its evolution is lower bounded by the exponential function with convergence rate $-\underline{\lambda}$, in red. Moreover, see that the exponential function with convergence rate given by the associated eigenvalue $\lambda = -1$ does not give any particular information.

We analyze the gradient of $\varphi(z) = z^\top z$ to obtain its local Lipschitz constant γ. In particular, we have for $i = 1,\dots,n$

$$\frac{\partial \varphi_i(z)}{\partial z} = 2z^\top \implies \left\| \frac{\partial \varphi(z)}{\partial z} \right\| = 2n|z|.$$

Because we are considering the case in which z is contracting in a certain domain we have that $|z| \leq |z(0)|$, we can write $2n|z| < 2n|z(0)| = \gamma$, thus defining the Lipschitz constant on the hypersphere encompassing the set of initial conditions, $|z(0)| \leq \rho_z$, for some ρ_z defining the radius of this hypersphere.

From Theorem 3, we have as sufficient condition for the exponential contraction of z, $\overline{\lambda} + 2\gamma < 0$, or more explicitly

$$\lambda(1 - \cos(\frac{\pi}{n+1})) + 4n\|z(0)\| < 0 \implies$$

$$\lambda < -\frac{4n\|z(0)\|}{1 - \cos(\frac{\pi}{n+1})}$$

Consider the case $n = 3$ and z initialized on the unitary sphere, i.e., $|z(0)| = 1$, we have as sufficient condition for exponential stability of z

$$\lambda < -\frac{4n\|z(0)\|}{1 - \cos(\frac{\pi}{n+1})} = -\frac{24}{2 - \sqrt{2}} = -40.9706.$$

Hence, setting $\lambda = -41$ we have an exponential decrease as shown by the simulation results reported in Fig. 3.

It is clear from the simulation results shown in Fig. 3, that the sufficient condition on the stability of semilinear systems is very conservative although it provides explicit values for guaranteeing exponential convergence of the system states. The result is conservative because our stability property only relies upon the linear part of the dynamics, whereas the

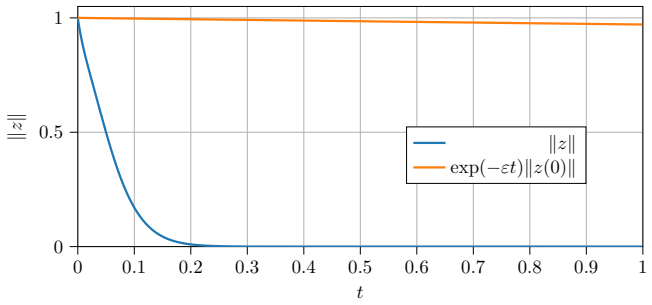

Fig. 3. Norm behavior of $|z(t)|$ of semilinear system, with initial condition $z(0) = (0.26726124, -0.53452248, 0.80178373)$.

nonlinear part could contribute to the stability of the system, possibly increasing its convergence rate. However, the analysis in this case would become much more complicated and usually rely upon a not-easy-to-find metric P which possibly depends on the system's states. In our case, instead, the metric is not only given but it is also constant, with all the positive consequences that a constant metric provides.

VI. Conclusions

In this work, we present a modification of the Jordan Block normal form of the state matrix of an autonomous linear system. We use this analysis to obtain, in closed form, the constant values for the scaling factor κ and the converge rate α for the uniform exponential upper bound of stable linear autonomous systems. This approach allows us to obtain a constructive characterization of the convergence property of a linear system. Moreover, we show some sufficient conditions to describe the exponential stability of a class of semilinear systems with Lipschitz nonlinearity. Numerical examples are correlated to show the effectiveness of the analysis.

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