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Turbulence modeling for remeshed vortex methods

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Abstract : This work uses a semi-Lagrangian approach combining both Lagrangian and Eulerian methods to solve the incompressible Navier-Stokes equations in their velocity-vorticity formulation. Different closure models for the filtered equations are explored in the context of large eddy simulations for turbulent flows. Their performance are evaluated on the Taylor Green vortex at $Re = 5000$, showing good performances compared to pure Eulerian approaches.

Mots clefs : incompressible flows, particle methods, semi-Lagrangian vortex methods, turbulence, Large Eddy Simulation

1 Introduction

Vortex methods are a type of Lagrangian methods used to solve incompressible Navier-Stokes equations, where the vorticity is discretized on numerical particles following the fluid dynamics. In this work we use a semi-Lagrangian approach, called the remeshed vortex method, in which particles are relocated on a fixed mesh after being transported in a Lagrangian way. This “hybridization” through remeshing allows for the use of Eulerian methods in an initially Lagrangian algorithm. Their Lagrangian aspect gives them many strong advantages: they are closer to the physics of the fluid, they are low-dispersive and low-diffusive (see [1] for a comparison with a LBM method) and do not require a CFL condition constraining the advection time step to the grid size.

This method has proved to be efficient for a number of laminar and highly transitional flows [2], illustrating the flexibility provided by the optimal coupling between Lagrangian and Eulerian schemes. However, this method has been mainly used as a DNS technique until now. In this context, the treatment of turbulent flows with such approach is unaffordable and therefore needs the design of turbulent models, adapted to the vortex-method framework. According to the strengths cited above (especially the low diffusivity property), the present approach represents a legitimate candidate to perform large eddy simulations with artificial viscosity models.

Following the pioneer, but very few, works on turbulent models for vortex methods [3, 4] and semi-Lagrangian vortex methods [5] in the context of large eddy simulations (LES), the present study explores different LES models for the velocity-vorticity formulation of the filtered Navier-Stokes equations and numerically compares them through the present semi-Lagrangian solver.

The paper is structured this way: first we will describe the remeshed vortex method, then we will expose the different subgrid scale models tested, finally we will present the performance of such models on the Taylor Green Vortex at $Re = 5000$ and discuss the most suitable modelling of the subgrid-scale vorticity stress.

2 Remeshed Vortex methods

We present in this section a brief overview of the numerical methods used in this study to discretize the incompressible Navier-Stokes equations. For a more complete introduction, see [6, 2].

Vortex methods are Lagrangian methods. They are based on the vorticity($\boldsymbol{\omega}$)-velocity(\mathbf{u}) formulation of the incompressible Navier-Stokes equations:

$$\partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} = \frac{1}{Re} \Delta \boldsymbol{\omega} \quad \Delta \mathbf{u} = -\nabla \times \boldsymbol{\omega} \quad (2.1)$$

where $\boldsymbol{\omega} := \nabla \times \mathbf{u}$ and where $(\mathbf{u} \cdot \nabla) \boldsymbol{\omega}$ and $(\boldsymbol{\omega} \cdot \nabla) \mathbf{u}$ denote respectively the advection and stretching terms. The vorticity field is discretized on a set of numerical particles with position \mathbf{x}_p and volume \mathbf{v}_p . At a given point \mathbf{x} of the domain and a given time t , the vorticity field is discretized as:

$$\boldsymbol{\omega}(\mathbf{x}, t) = \sum_p \alpha_p \zeta_\varepsilon(\mathbf{x} - \mathbf{x}_p(t)). \quad (2.2)$$

where ζ_ε is a smooth radial basis function, of radius ε , that converges to the Dirac distribution δ as ε goes to 0. The position \mathbf{x}_p of the numerical particles and the local circulation $\alpha_p = \int_{\mathbf{v}_p} \boldsymbol{\omega} d\mathbf{x} = \boldsymbol{\omega}_p \mathbf{v}_p$ are updated in a Lagrangian way by solving the system of ODE's:

$$\frac{d\mathbf{x}_p}{dt} = \mathbf{u}(\mathbf{x}_p(t), t) \quad \frac{d\alpha_p}{dt} = \alpha_p \cdot \nabla \mathbf{u}(\mathbf{x}_p(t), t) \quad (2.3)$$

In order to avoid a distortion of the vorticity field and to preserve the convergence of the vortex method, the distance between two particles needs to be controlled at all times. One way of doing this is to remesh the particles: the particles are regularly projected on an underlying mesh. The vorticity at a node i of the mesh is then obtained with:

$$\boldsymbol{\omega}_i^{n+1}(\mathbf{x}) = \sum_p \boldsymbol{\omega}_p^n(\mathbf{x}) \Lambda \left(\frac{\mathbf{x}_p^{n+1} - \mathbf{x}_i}{\Delta \mathbf{x}} \right) \quad (2.4)$$

where Λ is a piecewise polynomial with compact support, called the remeshing kernel.

The present work is based on a semi-Lagrangian or remeshed vortex methods (RVM) which is characterized by the use of both Lagrangian vortex methods and Eulerian methods to solve the Navier-Stokes equations (2.1), thus allowing to keep the strengths from both approaches. This resolution relies on a fractional step algorithm, which consists at each time step to successively solve the different equations described below:

	Fractional steps	Time discretization	Space discretization
1) Solenoidal reprojection	$\Delta \boldsymbol{\omega} = \Delta \boldsymbol{\omega}^* - \nabla(\nabla \cdot \boldsymbol{\omega}^*)$	-	spectral method (grid)
2) Poisson equation	$\Delta \mathbf{u} = -\nabla \times \boldsymbol{\omega}$	-	spectral method (grid)
3) Stretching	$\partial_t \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u}$	RK3	4 th order centered FD (grid)
4) Diffusion	$\partial_t \boldsymbol{\omega} = \frac{1}{Re} \Delta \boldsymbol{\omega}$	implicit Euler	spectral method (grid)
5) Advection	$\begin{cases} \partial_t \mathbf{x}_p = \mathbf{u}(\mathbf{x}_p(t), t) \\ \partial_t \boldsymbol{\omega} = 0 \end{cases}$	RK2	Lagrangian (particles)
6) Remeshing	$\boldsymbol{\omega}(\mathbf{x}) = \sum_p \boldsymbol{\omega}_p(\mathbf{x}) \Lambda \left(\frac{\mathbf{x}_p - \mathbf{x}}{\Delta \mathbf{x}} \right)$	-	$\Lambda_{4,2}$ remeshing kernel
7) Adaptive time step	$\Delta t_{\text{adapt}} = \text{LCFL} / \ \nabla \mathbf{u}\ _\infty$	-	4 th order centered FD (grid)

3 Turbulence modeling

In order to perform large eddy simulations, we write the filtered Navier-Stokes equations in their velocity-vorticity formulation:

$$\frac{\partial \bar{\boldsymbol{\omega}}}{\partial t} + \nabla \cdot (\bar{\boldsymbol{\omega}} \otimes \bar{\mathbf{u}} - \bar{\mathbf{u}} \otimes \bar{\boldsymbol{\omega}}) = \nu \Delta \bar{\mathbf{u}} - \nabla \cdot \mathbf{R} \quad (3.1)$$

where the subgrid scale vorticity stress to be modelled is expressed as:

$$\mathbf{R} = \overline{\boldsymbol{\omega} \otimes \mathbf{u}} - \overline{\boldsymbol{\omega}} \otimes \overline{\mathbf{u}} - \overline{\mathbf{u}} \otimes \overline{\boldsymbol{\omega}} + \overline{\mathbf{u}} \otimes \overline{\boldsymbol{\omega}} \quad (3.2)$$

Subgrid scale models In a velocity-pressure formulation, artificial viscosity models are usually of the form

$$\boldsymbol{\tau} \approx \boldsymbol{\tau}_{SGS} = \nu_{SGS}(\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^T) \quad (3.3)$$

where $\boldsymbol{\tau}$ is the subgrid scale stress tensor to model and where ν_{SGS} is the kinematic eddy viscosity. A first approach to model \mathbf{R} in the vorticity-velocity formulation is to take the curl of the divergence of (3.3):

$$\nabla \times \nabla \cdot \boldsymbol{\tau}_{SGS} = \nabla \times (\nu_{SGS} \nabla \cdot (\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^T)) + \nabla \times (\nabla \nu_{SGS} \cdot (\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^T)) \quad (3.4)$$

A first model, used in [3], consists in taking only the first term of this expression, the second one being negligible in practice. We obtain the following expression for $\mathbf{g} = \nabla \cdot \mathbf{R}_{SGS} \approx \nabla \cdot \mathbf{R}$:

$$\mathbf{g} = \nabla \times (\nu_{SGS} \nabla \cdot (\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^T)) = \nabla \times (\nu_{SGS} \Delta \overline{\mathbf{u}}) = -\nabla \times (\nu_{SGS} \nabla \times \overline{\boldsymbol{\omega}}) \quad (3.5)$$

it is equivalent to

$$\mathbf{g} = \nabla \cdot (\nu_{SGS}(\nabla \overline{\boldsymbol{\omega}} - \nabla \overline{\boldsymbol{\omega}}^T)) \quad (3.6)$$

thus,

$$\mathbf{R}_{SGS} = \nu_{SGS}(\nabla \overline{\boldsymbol{\omega}} - \nabla \overline{\boldsymbol{\omega}}^T). \quad (3.7)$$

We note that the tensor \mathbf{R}_{SGS} is anti-symmetric, as is the tensor \mathbf{R} to be modelled. Another model used in [5] is

$$\mathbf{R}_{SGS} = \nu_{SGS}(\nabla \overline{\boldsymbol{\omega}} + \nabla \overline{\boldsymbol{\omega}}^T) \quad (3.8)$$

where \mathbf{R}_{SGS} is here symmetric.

Finally, following [7] and [5], we explore a third model. Let \overline{f} be some resolved field, we define, in Fourier space, the largest of the resolved scales of \overline{f} by

$$\widehat{\overline{f}}(k) = \widehat{G}(k)\overline{f}(k) \quad (3.9)$$

where \widehat{G} is some test filter, and the small resolved scales f_S by

$$\overline{f}_S = \overline{f} - \widehat{\overline{f}} \quad (3.10)$$

One therefore defines the third model by taking the smallest of the resolved scales of the vorticity field such that

$$\mathbf{R}_{SGS} = \nu_{SGS}(\nabla \overline{\boldsymbol{\omega}}_S + (\nabla \overline{\boldsymbol{\omega}}_S)^T) \quad (3.11)$$

We will refer to model (3.7) as the "anti-symmetric" model, model (3.8) as the "symmetric" model and model (3.11) as the "small" model.

Artificial viscosity We chose the classical Smagorinsky model for artificial viscosity ν_{SGS}

$$\nu_{SGS} = (C\Delta)^2 |\overline{\mathbf{S}}| \quad (3.12)$$

where C is a coefficient to be chosen, Δ taken to be equal to the grid size, $\overline{\mathbf{S}} = \frac{1}{2}(\nabla \overline{\mathbf{u}} + (\nabla \overline{\mathbf{u}})^T)$ and $|\overline{\mathbf{S}}| = \sqrt{2\overline{\mathbf{S}}_{ij}\overline{\mathbf{S}}_{ij}}$ is the magnitude of \mathbf{S} .

Scale separation Let Δ be the grid size on which the field to be filtered is resolved. In the following we use two different Fourier filters : a Gaussian filter as a test filter in *a priori* tests to reproduce the implicit filtering from the grid at a coarser scale $\bar{\Delta} = \alpha\Delta$

$$\widehat{G}(k) = \exp\left(\frac{-\bar{\Delta}^2 k^2}{4\gamma}\right), \quad \text{with } \gamma = 6 \quad (3.13)$$

and a compact second order filter to select the smallest scales

$$\widehat{G}(k) = (1 - \sin^2(\Delta k_x/2))(1 - \sin^2(\Delta k_y/2))(1 - \sin^2(\Delta k_z/2)) \quad (3.14)$$

4 Numerical experiments

In this section we present results on the Taylor Green Vortex (TGV) test case at $Re = 5000$, which consists of a good benchmark to study the evolution of turbulence and the apparition of small scales. The simulations are performed in a periodic cubic box of side length $L = 2\pi$.

A posteriori tests In a first approach, we apply our LES models to the TGV simulation at a filtered resolution of 96^3 (the grid filtering is applied implicitly through the grid discretization). In figure 4.1, the time evolution of kinetic energy dissipation is represented. First of all, we note that all the proposed models greatly improve the solution compared to a no-model simulation which corresponds to a 96^3 resolution DNS, too coarse to represent accurately the energy in the smallest scales. We further note that the anti-symmetric model (3.7) is less dissipative than the symmetric one (3.8) although we obtain similar performance with the symmetric "small" model (3.11) by filtering the smallest scales of the resolved vorticity field. We also compare our results to the solution obtained in [8] through a Smagorinsky model and based on a high order finite-differences solver. We obtain similar trend and find our approach to be less dissipative for the three considered models.

A priori tests So far we have used a single subgrid scale model to model the whole tensor \mathbf{R} . In the perspective of a model specifically derived for the present method, we are interested to know the relative importance played by the convection and the stretching respectively in the development of small scales in turbulence; to that end, we decompose \mathbf{R} in two parts:

$$\mathbf{R} = \mathbf{R}_C - \mathbf{R}_S, \quad \mathbf{R}_C = \overline{\boldsymbol{\omega} \otimes \mathbf{u}} - \overline{\boldsymbol{\omega}} \otimes \overline{\mathbf{u}} \quad \mathbf{R}_S = \overline{\mathbf{u} \otimes \boldsymbol{\omega}} + \overline{\mathbf{u}} \otimes \overline{\boldsymbol{\omega}} \quad (4.1)$$

where \mathbf{R}_C reflects the vorticity transport by subfilter scale (SFS) velocity fluctuations and \mathbf{R}_S represents the SFS vortex stretching due to the unresolved motion. We make use of test filters such that \mathbf{R} is computed as $\mathbf{R} = (\overline{\boldsymbol{\omega} \otimes \mathbf{u}} - \widehat{\boldsymbol{\omega}} \otimes \widehat{\mathbf{u}}) - (\overline{\mathbf{u} \otimes \boldsymbol{\omega}} + \widehat{\mathbf{u}} \otimes \widehat{\boldsymbol{\omega}})$ where $\overline{\mathbf{u}}$ represents the resolved velocity and $\widehat{\mathbf{u}}$ the test-filtered velocity. Here we compare $\mathbf{r} = \nabla \cdot \mathbf{R}$, $\mathbf{r}_C = \nabla \cdot \mathbf{R}_C = \overline{\mathbf{u} \cdot \nabla \boldsymbol{\omega}} - \widehat{\mathbf{u}} \cdot \nabla \widehat{\boldsymbol{\omega}}$ and $\mathbf{r}_S = \nabla \cdot \mathbf{R}_S = \overline{\boldsymbol{\omega} \cdot \nabla \mathbf{u}} - \widehat{\boldsymbol{\omega}} \cdot \nabla \widehat{\mathbf{u}}$ on a 256^3 simulation with fields filtered to a 64^3 resolution. In figure 4.2 (top) we represent the norm of \mathbf{r} , \mathbf{r}_C and \mathbf{r}_S on a slice of the domain at time $t = 8.5$ of the TGV simulation at $Re = 5000$. We note that \mathbf{r}_C and \mathbf{r}_S have different structures that both play a significant role in \mathbf{r} . Furthermore, according to figure 4.2 (bottom), the relative importance of \mathbf{r}_S with respect to \mathbf{r}_C increases with the Reynolds number. This suggests that decoupling the turbulence modelling to account for the specificity of each subgrid scale tensor would be more accurate.

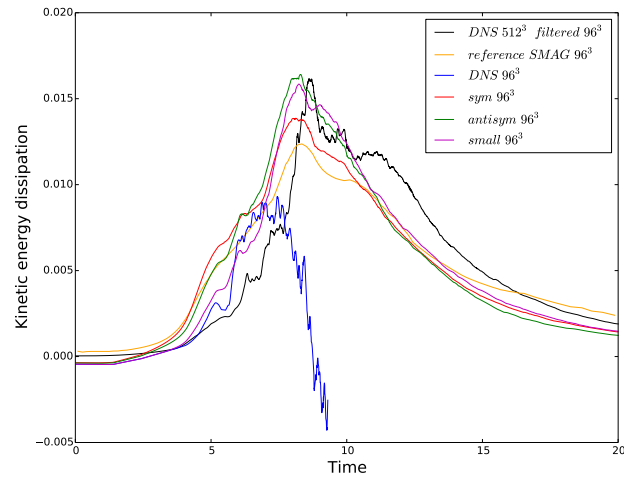


Figure 4.1: TGV at $Re = 5000$, time history of kinetic energy dissipation. Yellow curve reports the solution obtained with a Smagorinsky model in [8] from a finite differences solver.

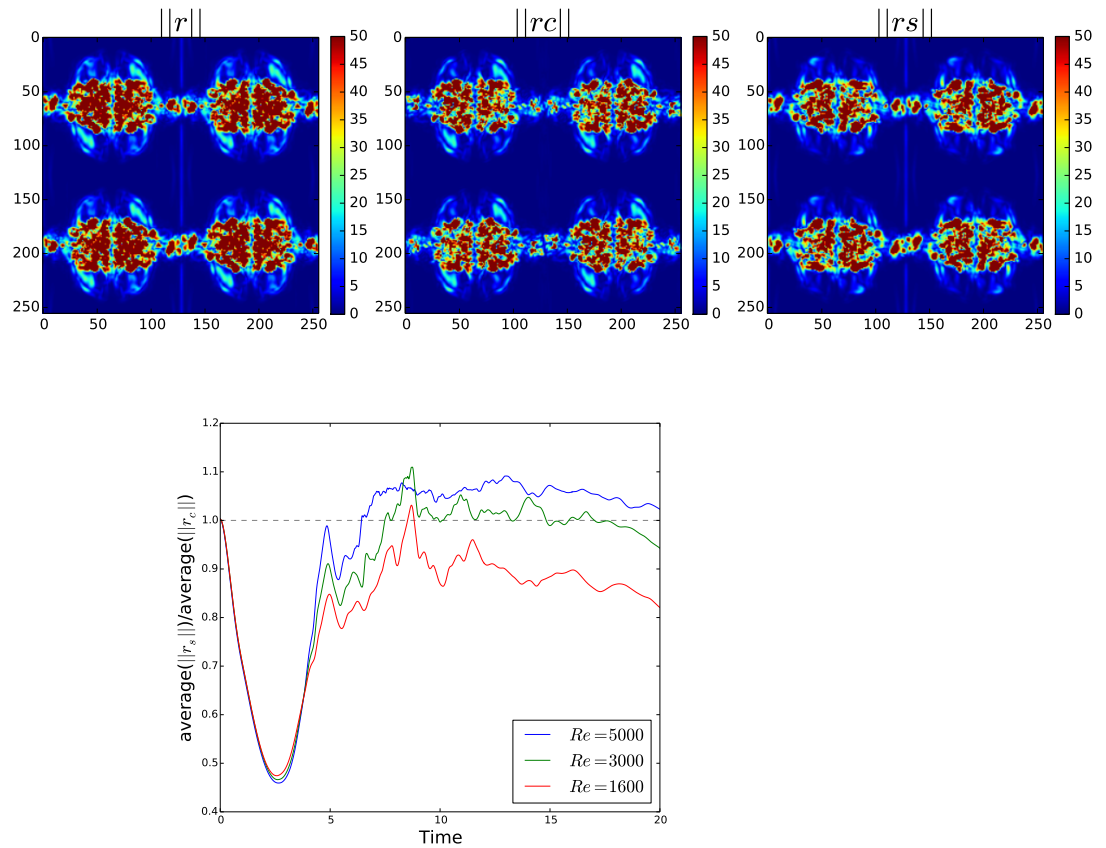


Figure 4.2: (top) Norm of \mathbf{r} compared to the norm of \mathbf{r}_C and \mathbf{r}_S (the different parts of the subgrid scale tensor) at time $t = 8.5$ of a TGV simulation at $Re = 5000$. (bottom) Time evolution of the ratio between averaged norm of \mathbf{r}_S and \mathbf{r}_C for TGV simulations at different Reynolds numbers.

5 Conclusion and perspectives

In this work we presented and tested various subgrid scale models for the vortex method based on the velocity-vorticity formulation of the Navier-Stokes equations. Tests on the Taylor Green Vortex benchmark give satisfying results even with a simple Smagorinsky model, confirming the potential of semi-Lagrangian vortex method in LES. Comparison between different models seems to indicate that an anti-symmetric model is the less dissipative one that still represents correctly the physics of a turbulent flow. The study also shows the interest of focusing on the smallest scales of the resolved vorticity field to reduce the over-dissipation of a classical Smagorinsky model. Furthermore, an *a priori* study shows that the vorticity-velocity subgrid scale tensor is composed of two sub-tensors of comparable importance. Future studies could exploit this feature to offer a decoupled model more adapted to the present method.

Following [5], we first chose $C^2 = 0.027$ for the symmetric and anti-symmetric model and $C^2 = 1.39 \times 0.027$ for the "small" model. However it might not be the most appropriate choice. A Germano-type dynamical approach [9] or a selective model in which the model is multiplied by a turbulence-sensor function [10, 11] would allow to select dynamically in time and space the most adapted coefficient.

While the Taylor-Green test case represents an interesting benchmark due to its representation of the phenomenon of large vortex breaking into smaller and smaller structures, in further works other test cases will be considered, such as forced homogeneous isotropic turbulence (HIT) or test cases including boundary effects.

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