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HADAMARD STATES FOR LINEARIZED GRAVITY ON SPACETIMES WITH COMPACT CAUCHY SURFACES

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ABSTRACT. We consider the quantization of linearized Einstein equations. We prove the existence of Hadamard states in the harmonic gauge on any Einstein spacetime with compact Cauchy surfaces.

1. INTRODUCTION AND SUMMARY

Linearized gravity is an example of a linear *gauge theory*, for which the construction of states is significantly more difficult than for ordinary matter fields. While the structure of the classical linearized gravity needed for its algebraic quantization is now well understood [BFR, FH, BDM, HS], the rigorous construction of physical states, ie *Hadamard states*, remains an important open problem.

Let us now mention several works which are related to the present one.

The simplest example of a linear gauge theory is Maxwell equations, which were considered by in [F, FP, DS], Hadamard states being constructed in [FS]. For linearized Yang–Mills equations around the zero solution, Hadamard states were constructed in the BRST framework in [H]. The case of linearized Yang–Mills equations around a non zero solution was considered later in [GW2].

In these models, one can use spacetime deformation arguments which are not applicable to linearized gravity.

The case of linearized gravity on asymptotically flat spacetimes was studied in [BDM] with methods drawing from earlier works [AA, DMP], the quantization turns out however to be limited to a subspace of classical degrees of freedom due to divergences at null infinity.

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More recently in [GMW] the construction of Hadamard states for linearized gravity on *analytic spacetimes* was investigated using *Wick rotation*. This consists in applying Wick rotation in some Gaussian time coordinate with respect to a reference Cauchy surface Σ . The various d'Alembertian operators are transformed into elliptic Laplacians. In general these elliptic Laplacians are only defined in some strip in imaginary Gaussian time.

One can recover a quasi-free state for the Lorentzian theory from *Calderón projectors*, a well-known tool in elliptic boundary value problem. This method was first used in [GW2] to construct analytic Hadamard states for scalar fields on analytic spacetimes.

States obtained from Calderón projectors depend on less arbitrary choices than those constructed by pseudodifferential calculus. Therefore it is hoped that the crucial gauge invariance property will be automatically satisfied.

However there are still a number of difficulties to obtain gauge invariant Hadamard states from Calderón projectors.

Firstly the Wick rotated operators should be not only elliptic but also invertible. To define them properly one has to impose some boundary conditions on the boundary of the strip in which they are defined.

The Dirichlet boundary conditions used in [GMW] have the advantage of easily giving invertibility of the Wick rotated operators and a modified positivity property. However they are not gauge invariant. As a consequence in [GMW] the gauge invariance and positivity of the two-point functions are only obtained modulo the addition of some smooth corrections.

In this paper we reconsider the problem of existence of Hadamard states for linearized gravity by using a different strategy. We prove the following result:

Theorem 1.1. *Let (M, \mathbf{g}) a globally hyperbolic spacetime with $\dim M = 4$ and $\mathbf{Ric} = \Lambda \mathbf{g}$, $\Lambda \in \mathbb{R}$. Assume that (M, \mathbf{g}) has compact Cauchy surfaces. Then there exist Hadamard states for linearized gravity on (M, \mathbf{g}) .*

1.1. Description of the paper. We now briefly recall the algebraic quantization of linearized gravity, explain the difficulties encountered when trying to construct Hadamard states for linearized gravity, and describe the approach we use in this paper to overcome them.

1.1.1. Linearized Einstein equations. Let (M, \mathbf{g}) be a globally hyperbolic spacetime with $\dim M = 4$ solving the Einstein equations

$$\mathbf{Ric} = \Lambda \mathbf{g},$$

where $\Lambda \in \mathbb{R}$ is the cosmological constant. Let $V_k := \mathbb{C} \otimes_s^k T^*M$, $k = 1, 2$ be the complex bundle of symmetric $(0, k)$ -tensors on M . We consider the two differential operators

$$P = -\square_2 - I \circ d \circ \delta + 2 \mathbf{Riem}_{\mathbf{g}}, \quad K = I \circ d,$$

where

$$-\square_2 \text{ is the d'Alembertian, } (\square_2 u)_{ab} = \nabla^c \nabla_c u_{ab},$$

- I is the trace reversal $(Iu)_{ab} = u_{ab} - \frac{1}{2}\text{tr}_{\mathbf{g}}(u)\mathbf{g}_{ab}$,
- d the symmetric differential $(dw)_{ab} = \nabla_{(a}u_{b)}$,
- δ is the formal adjoint of d , $(\delta u)_a = -2\nabla^c u_{ca}$,
- $\mathbf{Riem}_{\mathbf{g}}$ is the Ricci operator $(\mathbf{Riem}_{\mathbf{g}}u)_{ab} = \mathbf{R}_a{}^{cd}u_{cd}$.

The *linearized Einstein equations* around \mathbf{g} are

$$Pu = 0, \quad (1.1)$$

where u is a (symmetric) $(0,2)$ -tensor. The identity $P \circ K = 0$ implies that $\text{Ker}_{\text{sc}} P$ is invariant under *linearized gauge transformations* given by $u \mapsto u + Kw$, where w is a $(0,1)$ -tensor. Therefore the natural 'on-shell' phase space is the quotient space:

$$\frac{\text{Ker}_{\text{sc}} P}{\text{Ran}_{\text{sc}} K}.$$

Here and below the subscripts sc resp. c refer to 'space compact', resp. 'compact' for example $\text{Ker}_{\text{sc}} P$ is the space of (smooth) space compact solutions of (1.1).

1.1.2. *The phase space for linearized gravity.* The operator P is not hyperbolic, hence does not have advanced/retarded propagators. To equip the phase space with a Hermitian structure, it is necessary to add a *subsidiary gauge condition*. We follow here the nice exposition in [HS]. In this paper we will use the *de Donder* or *harmonic gauge*:

$$K^*u = 0,$$

where $K^* = \delta$ is the adjoint of K for a Hermitian form $(\cdot|\cdot)_{I, V_2}$ involving I , see 2.5.1, for which P is formally selfadjoint. The quotient space $\frac{\text{Ker}_{\text{sc}} P}{\text{Ran}_{\text{sc}} K}$ is then isomorphic to

$$\frac{\text{Ker}_{\text{sc}} D_2 \cap \text{Ker}_{\text{sc}} K^*}{K \text{Ker}_{\text{sc}} D_1},$$

where

$$D_1 = K \circ K^* = -\square_1 - \Lambda,$$

$$D_2 = P + K \circ K^* = -\square_2 + 2\mathbf{Riem}_{\mathbf{g}}$$

are hyperbolic operators acting respectively on $(0,1)$ - and $(0,2)$ -tensors. Since D_2 is hyperbolic, it admits advanced/retarded propagators $G_{2\text{ret/adv}}$. One can then introduce the 'off shell' phase space:

$$\mathcal{V}_P = \frac{\text{Ker}_c K^*}{\text{Ran}_c P}$$

equipped with the Hermitian form

$$\overline{[u]} \cdot Q_P[u] := \overline{u} \cdot Q_{I,2}u,$$

where

$$\overline{u} \cdot Q_{I,2}u = i(u|IG_2u)_{V_2},$$

and $G_2 = G_{2\text{ret}} - G_{2\text{adv}}$ is the commutator function for D_2 .

The algebraic quantization of linearized gravity simply consists in constructing the CCR $*$ -algebra $\text{CCR}(\mathcal{V}_P, Q_P)$.

Note that other Hermitian spaces, isomorphic to (\mathcal{V}_P, Q_P) are useful. In this paper, after fixing a reference Cauchy surface Σ , we will rely on the Hermitian space of Cauchy data $(\frac{\text{Ker}_c K_\Sigma^\dagger}{\text{Ran}_c K_\Sigma}, q_{I,2})$, where $K_\Sigma, K_\Sigma^\dagger$ are Cauchy surface analogs of K, K^* , see 2.7.1.

1.1.3. *Hadamard states.* A quasi-free state on $\text{CCR}(\mathcal{V}_P, Q_P)$ is defined by a pair of Hermitian forms Λ_P^\pm on \mathcal{V}_P called *covariances* such that

$$\begin{aligned} i) \quad & \Lambda_P^\pm = \Lambda_P^{\pm*}, \quad \Lambda_P^\pm \geq 0, \\ ii) \quad & \Lambda_P^+ - \Lambda_P^- = Q_P. \end{aligned} \tag{1.2}$$

We will be interested in covariances obtained from a pair of sesquilinear forms on $C_0^\infty(M; V_2)$ by

$$\overline{[u]} \cdot \Lambda_P^\pm[u] := \overline{u} \cdot \Lambda_2^\pm u, \tag{1.3}$$

The forms Λ_2^\pm have to satisfy a number of conditions.

Firstly they should 'pass to quotient' i.e. (1.3) should be meaningful. This leads to the conditions:

$$\begin{aligned} (1) \quad & D_2^* \circ \Lambda_2^\pm = \Lambda_2^\pm \circ D_2 = 0, \\ (2) \quad & \Lambda_2^\pm = 0 \text{ on } \text{Ker}_c K^* \times \text{Ran}_c K. \end{aligned}$$

Condition (1) corresponds to the *field equations*, familiar from quantization of matter fields, while condition (2), specific to gauge fields, is the *gauge invariance*.

The next two conditions are

$$\begin{aligned} (3) \quad & \Lambda_2^\pm = \Lambda_2^{\pm*}, \quad \Lambda_2^\pm \geq 0, \text{ on } \text{Ker}_c K^*, \\ (4) \quad & \Lambda_2^+ - \Lambda_2^- = Q_{I,2}, \end{aligned}$$

and correspond to (1.2). Condition (3) is the *positivity*, while condition (4) corresponds to the *CCR*.

The last condition is the *Hadamard condition*, which singles out Hadamard states, considered as the physically meaningful states on $\text{CCR}(\mathcal{V}_P, Q_P)$. Denoting by $\lambda_2^\pm \in \mathcal{D}'(M \times M; L(V_2))$ the distributional kernels of Λ_2^\pm , one requires that

$$(5) \quad \text{WF}(\lambda_2^\pm)' \subset \mathcal{N}^\pm \times \mathcal{N}^\pm,$$

where \mathcal{N}^\pm are the two connected components of the characteristic manifold $\mathcal{N} = \{(x, \xi) \in T^*M \setminus o : \xi \cdot \mathbf{g}^{-1}(x)\xi = 0\}$.

Conditions (1), (4) and (5) are rather easy to satisfy. In fact the construction of Hadamard states for scalar fields via pseudodifferential calculus initiated in [J1, J2] and further developed in [GW1, GOW] can be adapted to the tensor case and produces a wealth of covariances satisfying (1), (4) and (5), see [GMW, Sect. 5].

Condition (3) (positivity) is much more delicate, because D_2 is selfadjoint only for a *non-positive* Hermitian form. This difficulty is at the origin of use of Krein spaces ('Hilbert spaces' with a non-positive scalar product) appearing in the Gupta-Bleuler approach in QED.

If true, condition (3) will in general only be satisfied on the subspace $\text{Ker}_c K^*$.

Condition (2) (gauge invariance) is also very difficult to impose, because it has to be satisfied exactly, not only modulo smoothing errors.

1.1.4. *The approach in this paper.* In this paper we circumvent the difficulties with conditions (2) and (3) by relying on *full gauge fixing*.

Working with the Cauchy surface phase space $\frac{\text{Ker}_c K_\Sigma^\dagger}{\text{Ran}_c K_\Sigma}$, we start with a pair $\lambda_{2\Sigma}^\pm$ of Cauchy surface covariances, see 2.9.1, whose associated Λ_2^\pm will satisfy (1), (4) and (5).

We next try to find a convenient supplementary space E of $\text{Ran}_c K_\Sigma$ in $\text{Ker}_c K_\Sigma^\dagger$. We can then identify the canonical phase space $\frac{\text{Ker}_c K_\Sigma^\dagger}{\text{Ran}_c K_\Sigma}$ with E using the associated projection $\pi : \text{Ker}_c K_\Sigma^\dagger \rightarrow E$.

The *modified covariances* $\pi^* \circ \lambda_{2\Sigma}^\pm \circ \pi$ will then automatically satisfy the gauge invariance condition.

The supplementary space E has however to be chosen appropriately. First $\lambda_{2\Sigma}^\pm$ should be *positive* on E if the positivity condition (3) is to be satisfied by the modified covariances. Second E has also to be adapted so that the projection π does not spoil the microlocal Hadamard condition (5).

We select the space E using a microlocal version of the *synchronous gauge condition*, see 4.1.2. The fact that E is supplementary to $\text{Ran}_c K_\Sigma$ is equivalent to the solvability of some *elliptic system* of equations on Cauchy data.

If the system is uniquely solvable, (the so called *regular case*, see Subsect. 4.4), then the existence of a Hadamard state for linearized gravity follows rather easily.

If it is not uniquely solvable (the so called *singular case*, see Subsect. 4.5) and if the Cauchy surface Σ is *compact*, then by Fredholm theory it still has finite dimensional kernel and cokernel. We can further alter the modified covariances by some finite rank and smoothing operators to obtain a pair of Hadamard covariances.

1.2. **Notation.** We now collect various notations used throughout the paper.

1.2.1. *Isomorphisms of vector spaces.* If E, F are vector spaces and $A \in L(E, F)$ we write $A : E \xrightarrow{\sim} F$ if A is an isomorphism. If E, F are topological vector spaces, we use the same notation if A is a homeomorphism.

1.2.2. *Sesquilinear forms.* If E is a complex vector space, its antidual is denoted by E^* . A sesquilinear form A on E is an element of $L(E, E^*)$ and its action on elements of E is denoted by $\bar{u} \cdot Av$.

1.2.3. *Projections.* If $F \subset E$ are two vector spaces we say that $\pi : E \rightarrow F$ is a projection if $\pi^2 = \pi$ and $\text{Ran}\pi = F$.

1.2.4. *Operators on quotient spaces.* Let $F_i \subset E_i$, $i = 1, 2$ be vector spaces and let $A \in L(E_1, E_2)$. Then the induced map

$$[A] \in L(E_1/F_1, E_2/F_2),$$

is

- 1) well-defined if $AE_1 \subset E_2$ and $AF_1 \subset F_2$,
 - 2) injective iff $A^{-1}F_2 = F_1$,
 - 3) surjective iff $E_2 = AE_1 + F_2$.
- (1.4)

1.2.5. *Sesquilinear forms on quotients.* Let now $E \subset F$ be vector spaces and let $C \in L(E, E^*)$. We denote by $F^\circ \subset E^*$ the annihilator of F . Then the induced map

$$[C] \in L(E/F, (E/F)^*),$$

is

- 1) well-defined if $CE \subset F^\circ$, $F \subset \text{Ker } C$,
 - 2) non-degenerate iff $F = \text{Ker } C$.
- (1.5)

If C is hermitian or anti-hermitian then the condition $F \subset \text{Ker } C$ implies the other one $CE \subset F^\circ$ (and vice versa).

1.2.6. *Sections of vector bundles.* Let $V \xrightarrow{\pi} M$ be a finite rank complex vector bundle over a smooth manifold M .

- If $\Sigma \subset M$ is a smooth manifold we denote by $V|_\Sigma \xrightarrow{\pi} \Sigma$ the restriction of V to Σ .

- We denote by $C^\infty(M; V)$, resp. $C_0^\infty(M; V)$ the space of smooth, resp. compactly supported smooth sections of V .

- We denote by $\mathcal{D}'(M; V)$, resp. $\mathcal{E}'(M; V)$ the space of distributional, resp. compactly supported distributional sections of V .

We use the same notations if V is a finite dimensional vector space, i.e. we write simply V instead of the trivial vector bundle $M \times V$.

1.2.7. *Globally hyperbolic spacetimes.* We use the convention $(-, +, \dots, +)$ for the Lorentzian signature.

- If (M, \mathbf{g}) is a spacetime, we denote by $J_\pm(K)$ the future/past causal shadow of $K \subset M$.

- If M is a globally hyperbolic spacetime we denote by $C_{\text{sc}}^\infty(M; V)$ the space of space-compact sections, i.e. sections in $C^\infty(M; V)$ with compactly supported restriction to a Cauchy surface.

1.2.8. *Distributional kernels and wavefront sets.* -If $u \in \mathcal{D}'(M; V)$ we denote by $\text{WF}(u) \subset T^*M \setminus o$ its *wavefront set*, which is invariantly defined using local trivializations of V .

-If $V_i \xrightarrow{\pi} M_i$ are two vector bundles as above and $A : C_0^\infty(M_1; V_1) \rightarrow \mathcal{D}'(M_2; V_2)$ is linear continuous, then A admits a distributional kernel, still denoted by $A \in \mathcal{D}'(M_2 \times M_1; V_2 \boxtimes V_1)$.

- We denote by $\text{WF}(A)' \subset (T^*M_2 \times T^*M_1) \setminus o$ its *primed* wavefront set, defined by

$$\Gamma' = \{((x_2, \xi_2), (x_1, -\xi_1)) : ((x_2, \xi_2), (x_1, \xi_1)) \in \Gamma\} \text{ for } \Gamma \subset T^*M_2 \times T^*M_1.$$

2. LINEARIZED GRAVITY

In this section we review the quantization of linearized gravity, following [HS]. We also introduce the useful phase spaces of Cauchy data, following [GW2].

2.1. **Notation and background.** We start by fixing notation. Let (M, \mathbf{g}) be a 4-dimensional Lorentzian manifold.

2.1.1. *Convention for the Riemann tensor.* We use the same convention as in e.g. [R, FH, BDM] for the sign of the Riemann tensor i.e.

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)u_c = \mathbf{R}_{abc}{}^d u_d$$

on $(0, 1)$ -tensors. The Ricci tensor is $\mathbf{Ric}_{ab} = \mathbf{R}_{acb}{}^c = \mathbf{R}^c{}_{acb}$, and the scalar curvature $\mathbf{R} = \mathbf{g}^{ab} \mathbf{Ric}_{ab}$. The Einstein equations with cosmological constant Λ , i.e. $\mathbf{Ric} - \frac{1}{2} \mathbf{g} \mathbf{R} + \Lambda \mathbf{g} = 0$, are equivalent to

$$\mathbf{Ric} = \Lambda \mathbf{g}. \quad (2.1)$$

We will say that (M, \mathbf{g}) is Einstein if (2.1) is satisfied.

2.1.2. *Hermitian forms on tensors.* We denote by

$$V_k := \mathbb{C} \otimes_{\mathbb{S}}^k T^*M$$

the complex bundle of symmetric $(0, k)$ -tensors. We will only need the cases $k = 0, 1, 2$. V_k is equipped with the non-degenerate Hermitian form

$$(u|u)_{V_k} := k! \bar{u} \cdot (\mathbf{g}^{\otimes k})^{-1} u. \quad (2.2)$$

In abstract index notation,

$$(u|u)_{V_k} = k! \mathbf{g}^{a_1 b_1} \dots \mathbf{g}^{a_k b_k} \bar{u}_{a_1 \dots a_k} u_{b_1 \dots b_k}.$$

For example for $k = 2$ we have

$$(u|u)_{V_2} = 2 \text{tr}(u^* \mathbf{g}^{-1} u \mathbf{g}^{-1}). \quad (2.3)$$

The $k!$ normalization differs from the most common convention, it has however the advantage that various expressions involving adjoints look more symmetric.

For $U \subset M$ open, the Hermitian form (2.2) on fibers induces a Hermitian form

$$(u|v)_{V_k(U)} = \int_U (u(x)|v(x))_{V_k} d\text{vol}_{\mathbf{g}}, \quad u, v \in C_0^\infty(U; V_k). \quad (2.4)$$

The adjoint of $A : C^\infty(M; V_k) \rightarrow C^\infty(M; V_l)$ for those Hermitian forms will be denoted by A^* .

If $\Sigma \subset M$ is a Cauchy surface, we set

$$(u|v)_{V_k(\Sigma)} = \int_\Sigma (u(x)|v(x))_{V_k} d\text{vol}_{\mathbf{h}}, \quad u, v \in C_0^\infty(\Sigma; V_k),$$

where $d\text{vol}_{\mathbf{h}}$ is the induced density on Σ .

2.1.3. Decomposition of tensors. Let us assume that $M = I \times \Sigma$ where $I \subset \mathbb{R}$ is an open interval, Σ a smooth manifold with variables (t, \mathbf{x}) and

$$\mathbf{g} = -dt^2 + \mathbf{h}(t, \mathbf{x})d\mathbf{x}^2,$$

where $\mathbf{h} \in C^\infty(M, \otimes_s^2 T^*\Sigma)$ is a smooth t -dependent Riemannian metric on Σ . We set

$$V_{k\Sigma} = \mathbb{C} \otimes_s^k T^*\Sigma.$$

2.1.4. Decomposition of $(0, 1)$ -tensors. We identify

$$\begin{aligned} C^\infty(M; V_1) &\xrightarrow{\sim} C^\infty(I; C^\infty(\Sigma; V_{0\Sigma})) \oplus C^\infty(I; C^\infty(\Sigma; V_{1\Sigma})) \text{ by} \\ w &\mapsto (w_t, w_\Sigma), \\ w &=: w_t dt + w_\Sigma. \end{aligned} \quad (2.5)$$

The scalar product $(\cdot|\cdot)_{V_1}$ reads then

$$(w|w)_{V_1} = -|w_t|^2 + (w_\Sigma|w_\Sigma)_{V_{1\Sigma}} = -|w_t|^2 + (w_\Sigma|\mathbf{h}^{-1}w_\Sigma).$$

2.1.5. Decomposition of $(0, 2)$ -tensors. Similarly we identify

$$\begin{aligned} C^\infty(M; V_2) &\xrightarrow{\sim} C^\infty(I; C^\infty(\Sigma; V_{0\Sigma})) \oplus C^\infty(I; C^\infty(\Sigma; V_{2\Sigma})) \oplus C^\infty(I; C^\infty(\Sigma; V_{2\Sigma})) \text{ by} \\ u &\mapsto (u_{tt}, u_{t\Sigma}, u_{\Sigma\Sigma}), \\ u &=: u_{tt}dt \otimes dt + u_{t\Sigma} \otimes dt + dt \otimes u_{t\Sigma} + u_{\Sigma\Sigma}. \end{aligned} \quad (2.6)$$

The scalar product $(\cdot|\cdot)_{V_2}$ reads:

$$(u|u)_{V_2} = 2|u_{tt}|^2 - 4(u_{t\Sigma}|u_{t\Sigma})_{V_{1\Sigma}} + (u_{\Sigma\Sigma}|u_{\Sigma\Sigma})_{V_{2\Sigma}}. \quad (2.7)$$

2.2. The differential and its adjoint. Let

$$\begin{aligned} d : C^\infty(M; V_k) &\rightarrow C^\infty(M; V_{k+1}) \\ (du)_{a_1, \dots, a_{k+1}} &= \nabla_{(a_1} u_{a_2, \dots, a_{k+1})}, \end{aligned}$$

where $u_{(a_1 \dots a_k)}$ is the symmetrization of $u_{a_1 \dots a_k}$, and

$$\begin{aligned} \delta : C^\infty(M; V_k) &\rightarrow C^\infty(M; V_{k-1}) \\ (\delta u)_{a_1, \dots, a_{k-1}} &= -k \nabla^a u_{aa_1 \dots a_{k-1}}. \end{aligned}$$

With these conventions, we have $d^* = \delta$ w.r.t. the Hermitian form (2.4).

2.3. Operators on tensors.

2.3.1. *Trace reversal.* The operator of *trace reversal* I is given by

$$I := \mathbb{1} - \frac{1}{4}|\mathbf{g}\rangle(\mathbf{g}|,$$

where

$$(\mathbf{g}| : u_2 \mapsto (\mathbf{g}|u_2)_{V_2}, \quad |\mathbf{g}\rangle : u_0 \mapsto u_0\mathbf{g},$$

i.e. I is the orthogonal symmetry w.r.t. the line $\mathbb{C}\mathbf{g}$. Equivalently

$$(Iu)_{ab} = u_{ab} - \frac{1}{2}\mathrm{tr}_{\mathbf{g}}(u)\mathbf{g}_{ab}, \quad \mathrm{tr}_{\mathbf{g}}(u) := \mathbf{g}^{ab}u_{ab} = \frac{1}{2}(\mathbf{g}|u)_{V_2}.$$

It satisfies

$$I^2 = \mathbb{1}, \quad I = I^* \text{ on } C^\infty(M; V_2). \quad (2.8)$$

2.3.2. *Ricci operator.* The *Ricci operator* is

$$\mathbf{Riem}_{\mathbf{g}}(u)_{ab} := \mathbf{R}_a{}^c{}_b{}^d u_{cd} = \mathbf{R}^c{}_{ab}{}^d u_{cd}, \quad u \in C^\infty(M; V_2).$$

The fact that $\mathbf{Riem}_{\mathbf{g}}$ preserves symmetric $(0, 2)$ -tensors follows from the symmetries of the Riemann tensor.

Lemma 2.1. *The Ricci operator satisfies:*

- i) $\mathbf{Riem}_{\mathbf{g}}\mathbf{g} = -\mathbf{Ric}$,
- ii) $\mathbf{Riem}_{\mathbf{g}} \circ I = I \circ \mathbf{Riem}_{\mathbf{g}}$, if g is Einstein, (2.9)
- iii) $\mathbf{Riem}_{\mathbf{g}} = \mathbf{Riem}_{\mathbf{g}}^*$.

2.4. **Lichnerowicz operators.** Let $-\square_i$ be the rough d'Alembertian acting on sections of V_k :

$$-\square_i u_i = -\mathbf{g}^{ab}\nabla_{e^a, e^b}^2 u_i,$$

where $(e_a)_{0 \leq a \leq d}$ is a local frame. The *Lichnerowicz operators* $[L]$ acting on sections of V_k are defined by:

$$\begin{aligned} D_{0,L} &= -\square_0, \\ D_{1,L} &= -\square_1 + \mathbf{Ric} \circ \mathbf{g}^{-1}, \\ D_{2,L} &= -\square_2 + \mathbf{Ric} \circ \mathbf{g}^{-1} \circ \cdot + \cdot \circ \mathbf{g}^{-1} \circ \mathbf{Ric} + 2\mathbf{Riem}_{\mathbf{g}}. \end{aligned} \quad (2.10)$$

One has

$$D_{i,L} = D_{i,L}^*.$$

The proofs of the following facts can be found for example in [B].

Proposition 2.2. *If (M, \mathbf{g}) is Einstein then:*

$$\begin{aligned} D_{i+1,L} \circ d &= d \circ D_{i,L}, \quad \delta \circ D_{i+1,L} = D_{i,L} \circ \delta, \\ (\mathbf{g}| \circ D_{2,L} &= D_{0,L} \circ (\mathbf{g}|, \quad D_{2,L} \circ |\mathbf{g}\rangle = |\mathbf{g}\rangle \circ D_{0,L}. \end{aligned}$$

2.5. Linearized gravity as a gauge theory. In this subsection we follow [FH, HS]. Let (M, \mathbf{g}) be a globally hyperbolic spacetime of dimension 4. We assume that (M, \mathbf{g}) is Einstein. Let us introduce the differential operators

$$\begin{aligned} P &:= -\square_2 - I \circ d \circ \delta + 2 \mathbf{Riem}_{\mathbf{g}}, \\ K &:= I \circ d. \end{aligned} \quad (2.11)$$

$Pu = 0$ are the *linearized Einstein equations*. The condition $K^*u = 0$, where K^* is defined below, is the linearized *de Donder* or *harmonic gauge*.

2.5.1. *Physical Hermitian form.* We consider V_k , $k = 0, 1, 2$ as Hermitian bundles, where the Hermitian forms on fibers is now

$$(u|u)_{I, V_k} := (u|u)_{V_k}, \quad k = 0, 1, \quad (u|u)_{I, V_2} := (u|Iu)_{V_2}. \quad (2.12)$$

The corresponding Hermitian form on smooth sections of V_k , $k = 1, 2$ is

$$(u|u)_{I, V_k(U)} = \int_U (u(x)|u(x))_{I, V_k} d\text{vol}_{\mathbf{g}}, \quad u, v \in C_0^\infty(U; V_k). \quad (2.13)$$

We denote by A^* the corresponding formal adjoint of A for $(\cdot|\cdot)_{I, V_k(M)}$ to distinguish it from the formal adjoint A^* for $(\cdot|\cdot)_{V_k(M)}$. The two are related as follows:

$$\begin{aligned} A^* &= IA^*I \text{ if } A : C^\infty(M; V_2) \rightarrow C^\infty(M; V_2), \\ A^* &= A^*I \text{ if } A : C^\infty(M; V_k) \rightarrow C^\infty(M; V_2), \quad k = 0, 1 \\ A^* &= IA^* \text{ if } A : C^\infty(M; V_2) \rightarrow C^\infty(M; V_k), \quad k = 0, 1 \\ A^* &= A^* \text{ if } A : C^\infty(M; V_i) \rightarrow C^\infty(M; V_j) \quad i, j \neq 2. \end{aligned} \quad (2.14)$$

In particular,

$$K^* = K^* \circ I = \delta \circ I \circ I = \delta. \quad (2.15)$$

2.5.2. *Operators in linearized gravity.* Let us set:

$$D_k := D_{k,L} - 2\Lambda, \quad k = 0, 1, 2.$$

Then

$$\begin{aligned} K^*K &= D_1 = -\square_1 - \Lambda, \\ P + KK^* &= D_2 = -\square_2 + 2\mathbf{Riem}_{\mathbf{g}}. \end{aligned} \quad (2.16)$$

The operator D_0 is useful in connection with the *traceless gauge*. Note that

$$P = P^*, \quad D_2 = D_2^* = D_2^*, \quad [I, D_2] = 0.$$

The operators D_k are Green hyperbolic and hence admit unique *retarded/advanced* inverses $G_{k\text{ret}/\text{adv}}$. The *causal propagators* are

$$G_k := G_{k\text{ret}} - G_{k\text{adv}},$$

and satisfy $G_k^* = G_k^* = -G_k$.

2.6. Cauchy problem. Let $\Sigma \subset M$ a smooth space-like Cauchy surface. For $k = 0, 1, 2$ we set

$$\varrho_k u = \begin{pmatrix} u|_\Sigma \\ i^{-1} \nabla_\nu u|_\Sigma \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \quad u \in C_{\text{sc}}^\infty(M; V_k),$$

where ν is the future directed unit normal to Σ .

We denote by U_k the operator solving the Cauchy problem for D_k i.e.

$$\begin{cases} D_k U_k = 0, \\ \varrho_k U_k = \mathbb{1}. \end{cases} \quad (2.17)$$

2.6.1. Conserved charges. There exist a unique Hermitian form $q_k : C_0^\infty(\Sigma; V_k \otimes \mathbb{C}^2) \rightarrow C_0^\infty(\Sigma; V_k \otimes \mathbb{C}^2)^*$ called the *charge* of D_k , such that

$$(\phi_k | i G_k \phi_k)_{V_k(M)} = \overline{\varrho_k u_k} \cdot q_k \varrho_k u_k$$

for $\phi_k \in C_0^\infty(M; V_k)$ and $u_k = G_k \phi_k \in \text{Ker}_{\text{sc}} D_k$. One can compute q_k using the identity

$$\begin{aligned} & (u_k | D_k v_k)_{V_k(J_\pm(\Sigma))} - (D_k u_k | v_k)_{V_k(J_\pm(\Sigma))} \\ &= \pm i^{-1} \overline{\varrho_k u_k} \cdot q_k \varrho_k u_k, \quad u_k, v_k \in C_0^\infty(M; V_k). \end{aligned}$$

2.6.2. Operators on Cauchy data and physical charge. We follow here [GW2, Subsect. 2.4].

To the operator K we associate an operator K_Σ acting on Cauchy data by setting

$$K_\Sigma := \varrho_2 K U_1. \quad (2.18)$$

Similarly since $[I, D_2] = 0$ we can define

$$I_\Sigma := \varrho_2 I U_2 = I \otimes \mathbb{C}^2. \quad (2.19)$$

We obtain that $q_2 I_\Sigma = I_\Sigma^* q_2$ and as in 2.5.1 we define the Hermitian form

$$q_{I,2} := q_2 \circ I_\Sigma,$$

called the *physical charge* for D_2 .

We denote by K_Σ^\dagger the adjoint of K_Σ for the Hermitian forms $q_1, q_{I,2}$ i.e.

$$\overline{K_\Sigma^\dagger f_2} \cdot q_1 f_1 = \overline{f_2} \cdot q_{I,2} K_\Sigma f_1, \quad f_k \in C_0^\infty(\Sigma, V_k \otimes \mathbb{C}^2). \quad (2.20)$$

We have:

$$K_\Sigma^\dagger = \varrho_1 K^* U_2.$$

Lemma 2.3. *We have:*

- (1) $K \circ U_1 = U_2 \circ K_\Sigma$, $K^* \circ U_2 = U_1 \circ K_\Sigma^\dagger$;
- (2) $\varrho_2 \circ K = K_\Sigma \circ \varrho_1$ on $\text{Ker}_{\text{sc}} D_1$, $\varrho_1 \circ K^* = K_\Sigma^\dagger \circ \varrho_2$ on $\text{Ker}_{\text{sc}} D_2$;
- (3) $K_\Sigma^\dagger \circ K_\Sigma = 0$.

2.7. Phase spaces.

Proposition 2.4. *The maps*

$$\begin{aligned} [G_2] &: \frac{\text{Ker}_c K^*}{\text{Ran}_c P} \longrightarrow \frac{\text{Ker}_{\text{sc}} P}{\text{Ran}_{\text{sc}} K}, \\ [Id] &: \frac{\text{Ker}_{\text{sc}} D_2 \cap \text{Ker}_{\text{sc}} K^*}{K \text{Ker}_{\text{sc}} D_1} \longrightarrow \frac{\text{Ker}_{\text{sc}} P}{\text{Ran}_{\text{sc}} K}, \end{aligned}$$

are well defined and bijective.

Let us define the Hermitian forms Q_k on $C_0^\infty(M; V_k)$:

$$\overline{u}_k \cdot Q_k u_k := i(u_k | G_k u)_{V_k(M)},$$

and the *physical charge*

$$Q_{I,2} := Q_2 \circ I = I^* \circ Q_2.$$

Definition 2.5. *The physical phase space is the Hermitian space (\mathcal{V}_P, Q_P) , where:*

$$\mathcal{V}_P = \frac{\text{Ker}_c K^*}{\text{Ran}_c P}, \quad [\overline{u}] \cdot Q_P [u] = \overline{u} \cdot Q_{I,2} u, \quad [u] \in \frac{\text{Ker}_c K^*}{\text{Ran}_c P}.$$

Q_P is a well-defined Hermitian form on \mathcal{V}_P .

2.7.1. *Phase space of Cauchy data.* The following results are proved in [GW2].

Proposition 2.6. *The induced map*

$$[\varrho_2] : \frac{\text{Ker}_{\text{sc}} D_2 \cap \text{Ker}_{\text{sc}} K^*}{K \text{Ker}_{\text{sc}} D_1} \longrightarrow \frac{\text{Ker}_c K_\Sigma^\dagger}{\text{Ran}_c K_\Sigma}$$

is well defined and bijective.

Proposition 2.7. *The map*

$$[\varrho_2 G_2] : \left(\frac{\text{Ker}_c K^*}{\text{Ran}_c P}, Q_P \right) \longrightarrow \left(\frac{\text{Ker}_c K_\Sigma^\dagger}{\text{Ran}_c K_\Sigma}, q_{I,2} \right)$$

is an isomorphism of Hermitian spaces.

2.8. Quantization. The algebraic quantization of linear gauge theories is discussed in detail in [GW2, Sect. 3]. The algebraic framework reduces the quantization problem to showing the existence of physically relevant quantum states on the CCR $*$ -algebra $\text{CCR}(\mathcal{V}_P, Q_P)$ associated to the Hermitian space (\mathcal{V}_P, Q_P) defined in Subsect. 2.7. The notions of CCR $*$ -algebras, quasi-free states and covariances are explained for example in [G, Chap. 4].

2.8.1. *Covariances.* A quasi-free state on $\text{CCR}(\mathcal{V}_P, Q_P)$ is determined by a pair Λ_P^\pm of *covariances*, i.e. of Hermitian forms on \mathcal{V}_P such that

- i) $\Lambda_P^\pm = \Lambda_P^{\pm*}, \Lambda_P^\pm \geq 0,$
- ii) $\Lambda_P^+ - \Lambda_P^- = Q_P.$

We will consider quasi-free states ω on $\text{CCR}(\mathcal{V}_P, Q_P)$ with covariances obtained from a pair of continuous Hermitian forms Λ_2^\pm on $C_0^\infty(M; V_2)$ (called the *spacetime covariances* of ω) by:

$$[\bar{u}] \cdot \Lambda_P^\pm[u] = \bar{u} \cdot \Lambda_2^\pm u, [u] \in \frac{\text{Ker}_c K^*}{\text{Ran}_c P}. \quad (2.21)$$

Lemma 2.8. *Suppose that $\Lambda_2^\pm \in L(C_0^\infty(M; V_2), C_0^\infty(M; V_2)^*)$ are such that:*

- i) $D_2^* \circ \Lambda_2^\pm = \Lambda_2^\pm \circ D_2 = 0,$
- ii) $\Lambda_2^+ - \Lambda_2^- = Q_{I,2}$ on $\text{Ker } K_c^*,$
- iii) $\Lambda_2^\pm = 0$ on $\text{Ker}_c K^* \times \text{Ran}_c K,$
- v) $\Lambda_2^\pm = \Lambda_2^{\pm*}, \Lambda_2^\pm \geq 0$ on $\text{Ker}_c K^*.$

Then Λ_2^\pm are the covariances of a quasi-free state on $\text{CCR}(\mathcal{V}_P, Q_P)$.

2.9. Hadamard condition. The general consensus is that the *Hadamard condition* singles out the physically meaningful states. We use the following definition of Hadamard states [SV]. We recall that

$$\mathcal{N} = \{(x, \xi) \in T^*M \setminus \circ : \xi \cdot \mathbf{g}^{-1}(x)\xi = 0\}$$

is the characteristic set of the wave operator on (M, \mathbf{g}) , and

$$\mathcal{N}^\pm = \mathcal{N} \cap \{(x, \xi) \in T^*M \setminus \circ : \pm v \cdot \xi > 0 \ \forall v \in T_x M \text{ future-directed time-like}\}$$

are its two connected components, corresponding to the upper/lower energy shells.

To formulate the Hadamard condition, we need to identify the Hermitian forms Λ_2^\pm with distributional kernels $\lambda_2^\pm(\cdot, \cdot) \in \mathcal{D}'(M \times M; L(V_2))$, called *two-point functions*.

This identification is defined by the formal identity

$$\bar{u} \cdot \Lambda_2^\pm v =: \int_{M \times M} (u(x) | \lambda_2^\pm(x, y) v(y))_{V_2} d\text{vol}_g(x) d\text{vol}_g(y), \quad u, v \in C_0^\infty(M; V_2).$$

One can of course use other Hermitian forms on the fibers of V_2 to do this identification, like for example $(\cdot | \cdot)_{I, V_2}$ or a Hilbertian scalar product as will be done in Sect. 3. This change amounts to compose $\lambda_2^\pm(x, y)$ by smooth linear operators acting on the fibers of V_2 over x and y and does not change the Hadamard condition (2.23) below.

Definition 2.9. *A quasi-free state ω on $\text{CCR}(\mathcal{V}_P, Q_P)$ given by covariances Λ_2^\pm as in Lemma 2.8 is Hadamard if in addition to (2.22) it satisfies:*

$$\text{WF}(\lambda_2^\pm)' \subset \mathcal{N}^\pm \times \mathcal{N}^\pm. \quad (2.23)$$

2.9.1. *Hadamard condition on a Cauchy surface.* One can equivalently consider Hermitian forms $\lambda_{2\Sigma}^\pm$ on the space of Cauchy data $C_0^\infty(\Sigma; V_2 \otimes \mathbb{C}^2)$ called *Cauchy surface covariances*. Namely assume that we have a pair of Hermitian forms

$$\lambda_{2\Sigma}^\pm \in L(C_0^\infty(\Sigma; V_2 \otimes \mathbb{C}^2), C_0^\infty(\Sigma; V_2 \otimes \mathbb{C}^2)^*)$$

and set

$$\Lambda_2^\pm = (\varrho_2 G_2)^* \lambda_{2\Sigma}^\pm (\varrho_2 G_2). \quad (2.24)$$

The conditions on $\lambda_{2\Sigma}^\pm$ corresponding to (2.22) are

$$\begin{aligned} i) \quad & \lambda_{2\Sigma}^+ - \lambda_{2\Sigma}^- = q_{I,2} \text{ on } \text{Ker}_c K_\Sigma^\dagger, \\ ii) \quad & \lambda_{2\Sigma}^\pm = 0 \text{ on } \text{Ker}_c K_\Sigma^\dagger \times \text{Ran}_c K_\Sigma, \\ iii) \quad & \lambda_{2\Sigma}^\pm = \lambda_{2\Sigma}^{\pm*}, \lambda_{2\Sigma}^\pm \geq 0 \text{ on } \text{Ker}_c K_\Sigma^\dagger. \end{aligned} \quad (2.25)$$

Since $q_{I,2}$ is non-degenerate, we can set

$$\lambda_{2\Sigma}^\pm =: \pm q_{I,2} \circ c_2^\pm. \quad (2.26)$$

Proposition 2.10. *Suppose $c_2^\pm : C_0^\infty(\Sigma; V_2 \otimes \mathbb{C}^2) \rightarrow C^\infty(\Sigma; V_2 \otimes \mathbb{C}^2)$ is a pair of operators such that:*

$$\begin{aligned} i) \quad & c_2^+ + c_2^- = \mathbb{1}, \\ ii) \quad & c_2^\pm : \text{Ran}_c K_\Sigma \rightarrow \text{Ran} K_\Sigma, \\ iii) \quad & q_{I,2} \circ c_2^\pm = c_2^{\pm*} \circ q_{I,2}, \pm q_{I,2} \circ c_2^\pm \geq 0, \text{ on } \text{Ker}_c K_\Sigma^\dagger. \end{aligned} \quad (2.27)$$

Then Λ_2^\pm given by (2.24) and (2.26) are the covariances of a quasi-free state on $\text{CCR}(\mathcal{V}_P, Q_P)$. Furthermore if for some neighborhood \mathcal{U} of Σ in M we have:

$$iv) \quad \text{WF}(U_2 \circ c_2^\pm)' \subset (\mathcal{N}^\pm \cup \mathcal{F}) \times T^*\Sigma$$

over $\mathcal{U} \times \Sigma$, where $\mathcal{F} \subset T^*M$ is a conic set with $\mathcal{F} \cap \mathcal{N} = \emptyset$, then the associated state is Hadamard.

The proof of (2.27) is analogous to the one found in [GW2, Sect. 3.4]. The proof of the statement on the Hadamard condition can be found in [G, Sect. 11.1].

Remark 2.11. *If $[c_2^\pm, I_\Sigma] = 0$ then we can replace the first condition in (2.27) iii) by the simpler*

$$q_2 \circ c_2^\pm = c_2^{\pm*} \circ q_2.$$

Conversely if c_2^\pm satisfy the conditions in Prop. 2.10 then setting

$$\hat{c}_2^\pm = \frac{1}{2}(c_2^\pm + I_\Sigma \circ c_2^\pm \circ I_\Sigma),$$

we obtain that \hat{c}_2^\pm satisfy also the conditions in Prop. 2.10 and $[\hat{c}_2^\pm, I_\Sigma] = 0$. The only point deserving some attention is the microlocal condition iv), which follows from the fact that $\text{WF}(I)'$ and $\text{WF}(I_\Sigma)'$ are included in the diagonal of $T^*M \times T^*M$ and $T^*\Sigma \times T^*\Sigma$ respectively.

3. HADAMARD COVARIANCES

In [GMW, Sect. 5] we constructed Hadamard covariances $\lambda_{i\Sigma}^\pm$ for the operators D_i , $i = 1, 2$ appearing in linearized gravity. Some properties of these covariances were deduced from Wick rotation, ie $\lambda_{i\Sigma}^\pm$ were obtained from Calderón projectors associated to elliptic operators \tilde{D}_i obtained from D_i by a Wick rotation in a Gaussian time coordinate associated to some Cauchy surface Σ .

This procedure requires analyticity of the metric \mathbf{g} (or at least partial analyticity in Gaussian time). The essential property of the covariances constructed in this way is a positivity property with respect to an Euclidean charge \tilde{q} defined in (3.32).

In absence of analyticity, one can replace the Wick rotated metric $\tilde{\mathbf{g}}$ by an almost analytic extension of \mathbf{g} and obtain the same conclusions.

Another possibility is to prove the positivity property directly, which is what we will do in this section.

3.1. The framework. In order to keep the exposition relatively short, we will adopt the framework in [GMW, Sect. 5] to which we refer the reader for notation and proofs.

3.1.1. Spacetime and Hermitian bundle. We set $M = I_t \times \Sigma_x$, where $I \subset \mathbb{R}$ is an interval with $0 \in I$ and (Σ, \mathbf{h}_0) a d -dimensional Riemannian manifold of bounded geometry.

Note that in later sections Σ will be assumed to be compact, so all the assumptions below related to bounded geometry are automatically satisfied.

We set $\Sigma_t = \{t\} \times \Sigma$ and identify Σ_0 with Σ . The dual variables to (t, x) are denoted by (τ, k) .

We fix a t -dependent Riemannian metric on Σ ,

$$\mathbf{h} : I \ni t \mapsto \mathbf{h}(t) \in C_b^\infty(I; BT_2^0(\Sigma, \mathbf{h}_0)).$$

We assume that $\mathbf{h}(0) = \mathbf{h}_0$ and for ease of notation often denote $\mathbf{h}(t)$ by \mathbf{h}_t .

We equip M with the Lorentzian metric

$$\mathbf{g} := -dt^2 + \mathbf{h}_t dx^2. \quad (3.28)$$

We fix a finite rank complex vector bundle $V \xrightarrow{\pi} \Sigma$ of bounded geometry over (Σ, \mathbf{h}_0) . We still denote by V the vector bundle over $M: I \times V \xrightarrow{\pi} M$ which is a vector bundle with the same fibers as V .

We denote by $\text{Diff}(M; V)$, resp. $\text{Diff}(\Sigma; V)$ the space of differential operators on M resp. Σ acting on sections of V .

We assume that $V \xrightarrow{\pi} M$ is equipped with a non-degenerate fiberwise *Hermitian* structure $(\cdot|\cdot)_V$, which is *independent* of t .

We fix a reference fiberwise *Hilbertian* structure $(\cdot|\cdot)_{\tilde{V}}$ on the fibers of V which is also independent of t .

We write \tilde{V} instead of V to emphasize that V is tacitly equipped with the Hilbertian structure $(\cdot|\cdot)_{\tilde{V}}$.

We will use this Hilbertian structure to identify sesquilinear forms on the fibers of V with linear operators.

If $x \in M$ and $u, v \in V_x$ we have

$$(u|v)_V = (u|\tau_x v)_{\tilde{V}}, \quad \tau_x \in L(V_x),$$

and we denote by $\tau \in C^\infty(M; L(V))$ the corresponding section, which is independent on t .

Note that $\tau = \tau^*$ and without loss of generality we can assume that

$$\tau^* \tau = \mathbb{1}, \text{ i.e. } \tau \text{ is unitary for } (\cdot|\cdot)_{\tilde{V}}.$$

If $a \in L(V_x)$ for $x \in M$ we denote by a^* , resp. a^\star , the adjoints of a for $(\cdot|\cdot)_{\tilde{V}}$, resp. $(\cdot|\cdot)_V$. Then,

$$a^\star = \tau_x^{-1} a^* \tau_x \tag{3.29}$$

for some $\tau_x \in L(V_x)$.

For $u, v \in C_{\text{sc}}^\infty(M; V)$ we set

$$\begin{aligned} (u|v)_{\tilde{V}(\Sigma_t)} &:= \int_{\Sigma_t} (u|v)_{\tilde{V}} |\mathbf{h}_0|^{\frac{1}{2}} dx, \\ (u|v)_{V(\Sigma_t)} &:= \int_{\Sigma_t} (u|v)_V |\mathbf{h}_0|^{\frac{1}{2}} dx = (u|\tau v)_{\tilde{V}(\Sigma_t)}, \\ (u|v)_{\tilde{V}(M)} &:= \int_M (u(t)|v(t))_{\tilde{V}} |\mathbf{h}_0|^{\frac{1}{2}} dt dx, \\ (u|v)_{V(M)} &:= \int_M (u(t)|v(t))_V |\mathbf{h}_0|^{\frac{1}{2}} dt dx = (u|\tau v)_{\tilde{V}(M)}. \end{aligned}$$

If $\Omega \subset M$ is some open set, we also denote

$$\begin{aligned} (u|v)_{\tilde{V}(\Omega)} &:= \int_\Omega (u(t)|v(t))_{\tilde{V}} |\mathbf{h}_0|^{\frac{1}{2}} dt dx, \\ (u|v)_{V(\Omega)} &:= \int_\Omega (u(t)|v(t))_V |\mathbf{h}_0|^{\frac{1}{2}} dt dx = (u|\tau v)_{\tilde{V}(\Omega)}. \end{aligned}$$

We denote by $L^2(\Sigma; \tilde{V})$ the L^2 space obtained from the Hilbertian scalar product $(\cdot|\cdot)_{\tilde{V}(\Sigma)}$.

3.1.2. Adjoints. If $a \in C_b^\infty(I; \text{Diff}(\Sigma; V))$, resp. $A \in \text{Diff}(M; V)$, we denote by a^* resp. A^* its formal adjoint for $(\cdot|\cdot)_{\tilde{V}(\Sigma_t)}$ resp. $(\cdot|\cdot)_{\tilde{V}(M)}$. We set $\text{Re} a = \frac{1}{2}(a + a^*)$.

We denote by a^\star resp. A^\star its formal adjoint for $(\cdot|\cdot)_{V(\Sigma_t)}$ resp. $(\cdot|\cdot)_{V(M)}$. As above we have:

$$a^\star = \tau^{-1} a^* \tau, \quad A^\star = \tau^{-1} A^* \tau.$$

3.1.3. Hyperbolic operator. We fix a t -dependent differential operator $a = a(t, \mathbf{x}, D_{\mathbf{x}})$ belonging to $C_b^\infty(I; \text{Diff}^2(\Sigma; V))$ and denote by $\sigma_{\text{pr}}(a) \in C^\infty(T^*\Sigma; L(V))$ its principal symbol.

We assume the following properties:

- (H1) $a(t) = a^\star(t)$, $t \in I$,
- (H2) $\sigma_{\text{pr}}(a)(t)(\mathbf{x}, \mathbf{k}) = \mathbf{k} \cdot \mathbf{h}_t^{-1}(\mathbf{x}) \mathbf{k} \mathbb{1}_V$, $t \in I$.

We set

$$D := \partial_t^2 + a(t) \text{ acting on } C_0^\infty(M; V), \quad (3.30)$$

which is a hyperbolic operator with scalar principal part. Note that $D = D^*$, but of course $D \neq D^*$ in general.

The Cauchy problem

$$\begin{cases} Du = 0 \text{ in } M \\ \varrho u = f \in C_0^\infty(\Sigma; V \otimes \mathbb{C}^2) \end{cases} \quad (3.31)$$

is well-posed, where

$$\varrho u = \begin{pmatrix} u(0) \\ i^{-1} \partial_t u(0) \end{pmatrix}.$$

We denote by $u = Uf$ the unique solution of (3.31). We set:

$$\tilde{q} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad q := \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix}. \quad (3.32)$$

The Cauchy evolution for D is pseudo-unitary for q .

3.2. Hadamard projectors. In [GMW, Sect. 5] we constructed projectors c^\pm , acting on $C_0^\infty(\Sigma; \tilde{V} \otimes \mathbb{C}^2)$, called *Hadamard projectors* such that

- 1) $c^+ + c^- = \mathbb{1}$,
 - 2) $qc^\pm = c^{\pm*}q$,
 - 3) $\text{WF}(U \circ c^\pm)' \subset (\mathcal{N}^\pm \cup \mathcal{F}) \times T^*\Sigma$ for $\mathcal{F} = \{k = 0\} \subset T^*M$.
- (3.33)

We define Cauchy surface covariances by

$$\bar{f} \cdot \lambda_\Sigma^\pm f = \pm (f | qc^\pm f)_{\tilde{V}(\Sigma) \otimes \mathbb{C}^2},$$

and the associated spacetime covariances by

$$\Lambda^\pm = (\varrho \circ G)^* \lambda_\Sigma^\pm (\varrho \circ G),$$

where G is the causal propagator for D . The charge Q is

$$\bar{u} \cdot Qu = i(u | Gu)_{V(M)}, \quad u \in C_0^\infty(M; V).$$

We have then

- i)* $D^* \Lambda^\pm = \Lambda^\pm D = 0, \Lambda^+ - \Lambda^- = Q$,
- ii)* $\Lambda^\pm = \Lambda^{\pm*}$,
- iii)* $\text{WF}(\lambda^\pm)' \subset \mathcal{N}^\pm \times \mathcal{N}^\pm$.

In other words, except for the positivity, $\Lambda^\pm, \lambda_\Sigma^\pm$ are Hadamard covariances for D .

The following lemma states the positivity of c^\pm for the Euclidean charge \tilde{q} defined in (3.32).

Lemma 3.1. *We can construct the Hadamard projectors c^\pm so that in addition to (3.33) one has*

$$\pm (c^\pm f | \tilde{q} c^\pm f)_{\tilde{V}(\Sigma) \otimes \mathbb{C}^2} \geq 0, \quad \forall f \in C_0^\infty(\Sigma; \tilde{V} \otimes \mathbb{C}^2). \quad (3.34)$$

Moreover

$$\pi_1 c^\pm f = b^\pm \pi_0 c^\pm f \text{ for } \pi_i f = f_i, \quad f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \quad (3.35)$$

where $b^+ = b$, $b^- = -b^*$ and b is an elliptic first order pseudodifferential operator such that $b : H^s(\Sigma; \tilde{V}) \rightarrow H^{s-1}(\Sigma; \tilde{V})$ is an isomorphism for any $s \in \mathbb{R}$.

Proof. We use the notation in [GMW, 5.2.7-5.2.10]. The projectors c^\pm are given by

$$c^\pm = T \pi^\pm T^{-1}, \quad \pi^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.36)$$

where T is defined in [GMW, equ. (5.13)].

A concrete expression for c^\pm is:

$$c^\pm = \begin{pmatrix} \mp(b^+ - b^-)^{-1} b^\mp & \pm(b^+ - b^-)^{-1} \\ \mp b^\pm (b^+ - b^-)^{-1} b^\mp & \pm b^\pm (b^+ - b^-)^{-1} \end{pmatrix}, \quad (3.37)$$

where $b^+ = b$, $b^- = -b^*$ and b is an elliptic first order pseudodifferential operator on Σ .

To prove that $\pm c^{\pm*} \tilde{q} c^\pm \geq 0$, we need to compute $T^* \tilde{q} T$. For the operator b constructed in [GMW, Prop. 5.2] and c defined in [GMW, 5.2.9] we have

$$T = S \circ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad S = -i \begin{pmatrix} 1 & -1 \\ b & \tau^* b^* \tau \end{pmatrix} (b + \tau^* b^* \tau)^{-1}.$$

We compute T^* and using that $\tilde{q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we obtain that

$$T^* \tilde{q} T = (c(b + \tau^* b^* \tau)^{-1})^* \begin{pmatrix} b + b^* & -b^* + \tau^* b^* \tau \\ -b + \tau^* b \tau & -\tau^* (b + b^*) \tau \end{pmatrix} (c(b + \tau^* b^* \tau)^{-1}).$$

Therefore using (3.36) we see that $\pm c^{\pm*} \tilde{q} c^\pm \geq 0$ iff $b + b^* \geq 0$. The construction of b is given in [GMW, Prop. 5.2]. Concretely we have:

$$b = \epsilon + b_0(1 - \chi_R(a_{\text{ref}})),$$

$R \gg 1$ being a large parameter. The first order elliptic pseudodifferential operator ϵ satisfies $\text{Re} \epsilon \geq 1$. Therefore

$$b + b^* = (2\text{Re} \epsilon)^{\frac{1}{2}} (1 - \tilde{s}_{-1}) (2\text{Re} \epsilon)^{\frac{1}{2}},$$

where

$$\tilde{s}_{-1} = -(2\text{Re} \epsilon)^{-\frac{1}{2}} (b_0(1 - \chi_R(a_{\text{ref}})) + (1 - \chi_R(a_{\text{ref}})) b_0^*) (2\text{Re} \epsilon)^{-\frac{1}{2}}.$$

As in the proof of [GMW, Prop. 5.2] the norm of \tilde{s}_{-1} in $B(L^2(\Sigma; \tilde{V}))$ tends to 0 when $R \rightarrow +\infty$, so choosing $R \gg 1$ we obtain that $b + b^* \geq c > 0$. This completes the proof of the first statement of the lemma.

The second statement follows from (3.37), except for the fact that b can be chosen invertible. The principal symbol of b is $(k \cdot \mathbf{h}_0(x)k)^{\frac{1}{2}} \mathbb{1}$, so b is elliptic. We have seen above that we can choose $R \gg 1$ so that $b + b^* \geq c > 0$. The sesquilinear form associated to b with domain $H^{\frac{1}{2}}(\Sigma; \tilde{V})$ is closed and coercive, so $b = H^{\frac{1}{2}}(\Sigma; \tilde{V}) \rightarrow H^{-\frac{1}{2}}(\Sigma; \tilde{V})$ is an isomorphism. This extends to any $s \in \mathbb{R}$ by the usual argument. \square

3.3. Application to D_k . Let us now recall how to apply the previous constructions to the operators D_k , $k = 1, 2$.

If (M, \mathbf{g}) is globally hyperbolic and Σ is a smooth spacelike Cauchy surface, then if (M, \mathbf{g}) is of *bounded geometry* near Σ , see [GMW, Def. 3.2] for the precise definition, then using Gaussian normal coordinates to Σ one can isometrically map a neighborhood of Σ in M to $I \times \Sigma$, equipped with a metric as in (3.28). The bounded geometry assumption is automatically satisfied if Σ is compact, as we will assume in later sections.

3.3.1. Reduced setting. Conjugating D_k by an isomorphism corresponding to parallel transport along ∂_t one can then reduce oneself to the situation in Subsect. 3.1, see [GMW, Subsect. 4.4]. For $k = 2$ this isomorphism maps the background metric \mathbf{g} to $\mathbf{g}_0 = -dt^2 + \mathbf{h}_0$.

In this reduced setting, several operators take simpler forms: for example we have

$$D_k = \partial_t^2 + a_k(t, \mathbf{x}, \partial_{\mathbf{x}}),$$

$$Iu_2 = u_2 - \frac{1}{4}(\mathbf{g}_0|u_2)v_2,$$

and the expression of d can be found in [GMW, Prop. 4.11]. The operators τ_k relating the Hermitian and Hilbertian structures on V_k are

$$\tau_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where we use the decompositions of $(0, k)$ -tensors recalled in 2.1.3.

3.3.2. Gauge invariance modulo smooth errors. To complete this subsection, we write an easy lemma. We state and prove it only in the case when Σ is compact, but the result extends easily to the bounded geometry framework.

Lemma 3.2. *Assume that Σ is compact. Let c_k^\pm for $k = 1, 2$ be Hadamard projectors for D_k . Then*

$$c_2^\pm K_\Sigma = K_\Sigma c_1^\pm \pm r_{-\infty},$$

where $r_{-\infty} = c_2^+ K_\Sigma c_1^- - c_2^- K_\Sigma c_1^+ \in \Psi^{-\infty}(\Sigma)$ is smoothing.

Proof. Let $f_1 \in \mathcal{E}'(\Sigma; V_1 \otimes \mathbb{C}^2)$, $u_1 = U_1 c_1^+ f_1$, $u_2 = Ku_1$ and $f_2 = \varrho_2 u_2 = K_\Sigma c_1^+ f_1$. We have $\text{WF}u_1 \subset \mathcal{N}^+$ hence $\text{WF}u_2 \subset \mathcal{N}^+$ since K is a differential operator. On the other hand $\text{WF}U_2 c_2^\pm f_2 \subset \mathcal{N}^\pm$ hence $\text{WF}U_2 c_2^- f_2 \subset \mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$ since $c_2^- = 1 - c_2^+$. Therefore $c_2^- K_\Sigma c_1^+$ is smoothing \square

3.3.3. *Additional symmetry.* The operator D_2 has the additional symmetry $[I, D_2] = 0$, which can be carried over to b_2 or the Hadamard projectors c_2^\pm .

In fact if b_2 is the operator entering in the construction of c_2^\pm , we see that $I \circ b_2 \circ I$ also satisfies the conditions in [GMW, Prop. 5.2], which characterize b uniquely modulo a smoothing error. Therefore we can replace b_2 by $\frac{1}{2}(b_2 + I \circ b_2 \circ I)$ modulo a smoothing error and assume that $[I, b_2] = 0$.

This replacement does not invalidate the properties of c_2^\pm summarized in Subsect. 3.2. The projectors c_2^\pm have now the additional property

$$[I_\Sigma, c_2^\pm] = 0.$$

Therefore c_2^\pm are also selfadjoint for the physical Hermitian form $q_{I,2} = q_2 \circ I_\Sigma$ ie

$$(f_2 | q_2 I_\Sigma c_2^\pm f_2)_{\tilde{V}_2 \otimes \mathbb{C}^2} = (c_2^\pm f_2 | q_2 I_\Sigma f_2)_{\tilde{V}_2 \otimes \mathbb{C}^2}. \quad (3.38)$$

4. GAUGE FIXING

4.1. **Introduction.** To motivate the constructions in this section we start with some comments on conditions (2.22) and (2.23) in Subsect. 2.8 on covariances generating a quasi-free state on $\text{CCR}(\mathcal{V}_P, Q_P)$.

For a quantum field theory associated to D_2 , one usually assumes condition (2.22) *i*) and $\lambda_2^+ - \lambda_2^- = iG_2$, ie (2.22) *ii*) extended to the larger space $C_0^\infty(M; V_2)$. Together with (2.23), these two conditions fix λ_2^\pm uniquely, modulo smooth kernels.

The positivity condition (2.22) *v*) is in general not satisfied on arbitrary test fields, but only on $\text{Ker}_c K^*$. This comes from the fact that the fiber scalar product $(\cdot | \cdot)_{V_2}$ for which D_2 is formally selfadjoint is non positive.

The condition (2.22) *iv*) that λ_2^\pm map $\text{Ran}_c K$ into $\text{Ran} K$ is the crucial *gauge invariance condition*, which implies that Λ^\pm are well defined on the physical phase space $\frac{\text{Ker}_c K^*}{\text{Ran}_c P}$. This condition is the most difficult to impose.

A way out of this difficulty is to try to eliminate the remaining gauge freedom.

Working for example with the phase space $\frac{\text{Ker}_{\text{sc}} D_2 \cap \text{Ker}_{\text{sc}} K^*}{K \text{Ker}_{\text{sc}} D_1}$ this amounts to impose more gauge fixing conditions in addition to the harmonic gauge condition $K^* u_2 = 0$ in order to eliminate the remaining gauge freedom corresponding to $K \text{Ker}_{\text{sc}} D_1$.

The additional gauge fixing conditions should hence uniquely specify a supplementary space to $K \text{Ker}_{\text{sc}} D_1$ inside $\text{Ker}_{\text{sc}} D_2 \cap \text{Ker}_{\text{sc}} K^*$. These additional gauge fixing conditions should moreover be chosen so that the positivity condition on the covariances is satisfied on this supplementary space.

4.1.1. *TT gauge condition.* A first way to reduce the gauge freedom is to impose the *traceless condition* by requiring that $(\mathbf{g} | u_2)_{V_2} = 0$. Together with in $K^* u_2 = 0$, this is called the *transverse-traceless* (TT) *gauge*.

This is always possible if $\Lambda \neq 0$ see eg [FH, Thm. 2.7]. Setting

$$K_0 : C^\infty(M; V_0) \ni u_0 \mapsto u_0 \mathbf{g} \in C^\infty(M; V_2),$$

we have $K_0^* u_2 = -(\mathbf{g}|u_2)_{V_2}$ and one can show that

$$[Id] : \frac{\text{Ker}_{\text{sc}} D_2 \cap \text{Ker}_{\text{sc}} K^* \cap \text{Ker}_{\text{sc}} K_0^*}{K \text{Ker}_{\text{sc}} D_1 \cap \text{Ker}_{\text{sc}} K_0^*} \longrightarrow \frac{\text{Ker}_{\text{sc}} P}{\text{Ran}_{\text{sc}} K} \quad (4.1)$$

is again an isomorphism. The key fact is that

$$K_0 D_0 = D_2 K_0, \quad (4.2)$$

by Prop. 2.2, where $D_0 = -\square_0 - 2\Lambda$ was introduced in Subsect. 2.5.

This does not fully eliminate the gauge freedom, ie we still have a quotient space in (4.1).

4.1.2. Synchronous gauge condition. A possibility is to impose the *synchronous gauge* condition. After fixing a Cauchy surface Σ and introducing Gaussian normal coordinates (t, \mathbf{x}) to Σ , one requires that $u_{t\Sigma} = 0$ near Σ . It is shown in [FH, Thm. 2.8] that for any $u_2 \in C^\infty(M; V_2)$ such that $Pu_2 = 0$, there exists $u_1 \in C^\infty(M; V_1)$, such that $(u_2 - Ku_1)_{t\Sigma} = 0$ near Σ . However K^*Ku_1 does not necessarily vanish, ie the harmonic gauge condition $K^*u_2 = 0$ is destroyed by this gauge transformation.

One can weaken the synchronous gauge condition by requiring only that

$$u_{2t\Sigma}|_{\Sigma} = 0, \nabla_\nu u_{2t\Sigma}|_{\Sigma} = 0. \quad (4.3)$$

Note that unlike the harmonic and traceless gauge conditions, the above condition does not 'propagate' to the whole spacetime M , because in general $u_{2t\Sigma}$ does not solve a hyperbolic equation, even if $D_2 u_2 = 0$.

One can ask if is possible to impose (4.3), together with the TT gauge condition $K^*u_2 = (g|u_2)_{V_2} = 0$. One can call this the *TT-synchronous* gauge condition.

Given $u_2 \in \text{Ker}_{\text{sc}} D_2 \cap \text{Ker}_{\text{sc}} K^*$, we need to find $u_1 \in C^\infty(M; V_1)$ such that

$$\left\{ \begin{array}{l} D_1 u_1 = 0, \\ 1) \quad (\mathbf{g}|Ku_1)_{V_2}|_{\Sigma} = (\mathbf{g}|u_2)_{V_2}|_{\Sigma}, \\ 2) \quad (\mathbf{g}|\nabla_\nu Ku_1)_{V_2}|_{\Sigma} = (\mathbf{g}|\nabla_\nu u_2)_{V_2}|_{\Sigma}, \\ 3) \quad (Ku_1)_{t\Sigma}|_{\Sigma} = u_{2t\Sigma}|_{\Sigma}, \\ 4) \quad (\nabla_\nu Ku_1)_{t\Sigma}|_{\Sigma} = \nabla_\nu u_{2t\Sigma}|_{\Sigma}. \end{array} \right. \quad (4.4)$$

If $\tilde{u}_2 = u_2 - Ku_1$, then $K^*u_2 = 0$, the initial conditions (1) and (2) ensure that $(\mathbf{g}|\tilde{u}_2)_{V_2} = 0$, using (4.2), while (3) and (4) ensure that \tilde{u}_2 satisfies (4.3).

The system (4.4) can be rewritten as an elliptic system of equations in terms of the Cauchy data $f_1 = \varrho_1 u_1$.

If Σ is compact, this elliptic system is Fredholm. Except for a finite dimensional subspace in $\text{Ker} D_2$ (4.4) has a unique solution, modulo a finite dimensional subspace in $\text{Ker} D_1$.

4.2. Microlocal TT-synchronous gauge. Even if we ignore the problem with the possible non invertibility of (4.4), the TT-synchronous gauge condition is not convenient for the construction of gauge invariant Hadamard states for linearized gravity. In particular states constructed using the TT-synchronous gauge fixing will in general not be Hadamard states.

It turns out that it is much better to adapt it to the Hadamard projectors c_2^\pm for D_2 . Let us now define this modified gauge condition.

In the rest of the section we assume that the Cauchy surface Σ is compact.

As before we use Gaussian normal coordinates to Σ to isometrically identify a neighborhood of Σ in M with $I_t \times \Sigma_x$ for some interval $I \ni 0$, equipped with the metric $-dt^2 + h_t(x)dx^2$. Under this identification Σ is identified with $\{t = 0\}$.

We denote by $l : C^\infty(\Sigma; V_2) \rightarrow C^\infty(\Sigma; V_1)$ the map:

$$lu_2 := \begin{pmatrix} \frac{1}{2}(\mathbf{g}_0|u)_{V_2}|_\Sigma \\ 2(u_{2t\Sigma})|_\Sigma \end{pmatrix},$$

where in the rhs we identify as usual $(v_t, v_\Sigma) \in C^\infty(\Sigma; \mathbb{C} \oplus T^*\Sigma)$ with $v_t dt + v_\Sigma \in C^\infty(\Sigma; V_1)$.

If c_2^\pm are the Hadamard projectors whose construction is recalled in Sect. 3 we set for $f_2 \in C^\infty(\Sigma; V_2 \otimes \mathbb{C}^2)$

$$R_\Sigma f_2 := \begin{pmatrix} l\pi_0 c_2^+ f_2 \\ l\pi_0 c_2^- f_2 \end{pmatrix} \in C^\infty(\Sigma; V_1 \otimes \mathbb{C}^2). \quad (4.5)$$

Here $\pi_0 : V_2 \otimes \mathbb{C}^2 \rightarrow V_2$ denotes the projection on the first component.

Definition 4.1. *The microlocal TT-synchronous gauge condition is defined (in terms of Cauchy data on Σ) by:*

$$\begin{aligned} (1) \quad & K_\Sigma^\dagger f_2 = 0, \\ (2) \quad & R_\Sigma f_2 = 0. \end{aligned} \quad (4.6)$$

4.3. Properties of R_Σ and $R_\Sigma K_\Sigma$. Imposing (4.6) by a gauge transformation is equivalent to find $f_1 \in C^\infty(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2)$ solving

$$R_\Sigma K_\Sigma f_1 = R_\Sigma f_2, \quad (4.7)$$

for some given $f_2 \in C^\infty(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \cap \text{Ker } K_\Sigma^\dagger$, so that $f_2 - K_\Sigma f_1$ satisfies (4.6). In this subsection we study the operator $R_\Sigma K_\Sigma$ appearing in (4.7) and we prove an important positivity property of the microlocal TT-synchronous gauge condition.

4.3.1. Some equivalent norms. The convenient Sobolev spaces for Cauchy data are:

$$\mathcal{H}^s(\Sigma; \tilde{V}_k \otimes \mathbb{C}^2) = H^s(\Sigma; \tilde{V}_k) \oplus H^{s-1}(\Sigma; \tilde{V}_k), \quad s \in \mathbb{R}$$

equipped with the norm

$$\|f\|_s^2 = \|f_0\|_{H^s(\Sigma; \tilde{V}_k)}^2 + \|f_1\|_{H^{s-1}(\Sigma; \tilde{V}_k)}^2, \quad f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

Lemma 4.2. *The map:*

$$\begin{aligned} & \mathcal{H}^s(\Sigma; \tilde{V}_k \otimes \mathbb{C}^2) \rightarrow H^s(\Sigma; \tilde{V}_k \otimes \mathbb{C}^2) \\ L_k : f & \mapsto L_k f = \begin{pmatrix} \pi_0 c_k^+ f \\ \pi_0 c_k^- f \end{pmatrix} =: \begin{pmatrix} v^+ \\ v^- \end{pmatrix} = v \end{aligned} \quad (4.8)$$

is an isomorphism.

We will denote by $\pi^\pm : C^\infty(\Sigma; \tilde{V}_k \otimes \mathbb{C}^2) \rightarrow C^\infty(\Sigma; \tilde{V}_k)$ the maps $v = \begin{pmatrix} v^+ \\ v^- \end{pmatrix} \mapsto v^\pm$.

It is often more convenient to use $\begin{pmatrix} v^+ \\ v^- \end{pmatrix}$ instead of f .

Proof. From the expression (3.37) of c^\pm , we deduce that c^\pm are bounded on \mathcal{H}^s and since $c^+ + c^- = \mathbb{1}$, an equivalent norm on \mathcal{H}^s is given by

$$\|f\|_{s, \text{mod}}^2 = \|c^+ f\|_s^2 + \|c^- f\|_s^2.$$

Using the second statement in Lemma 3.1, we obtain that an equivalent norm is

$$(\|v^+\|_{H^s(\Sigma; \tilde{V}_i)}^2 + \|v^-\|_{H^s(\Sigma; \tilde{V}_i)}^2)^{\frac{1}{2}}, \quad v^\pm = \pi_0 c^\pm f,$$

which proves the lemma. \square

4.3.2. Properties of $R_\Sigma K_\Sigma$.

Proposition 4.3. (1) $R_\Sigma K_\Sigma : \mathcal{H}^s(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2) \rightarrow H^{s-1}(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2)$ is Fredholm of index 0 for any $s \in \mathbb{R}$;

(2) $\pi^\pm R_\Sigma c_2^\mp = 0$;

(3) $\pi^\pm R_\Sigma K_\Sigma c_2^\mp$ is smoothing.

Proof. From Lemma 3.2 we obtain that

$$R_\Sigma K_\Sigma = \begin{pmatrix} l\pi_0 K_\Sigma c_1^+ & 0 \\ 0 & l\pi_0 K_\Sigma c_1^- \end{pmatrix} + R_{-\infty}, \quad (4.9)$$

where $R_{-\infty}$ is a smoothing operator.

We recall that K_Σ decomposes as $K_\Sigma = I_\Sigma \circ T_\Sigma$, corresponding to $K = I \circ d$. Since $I\mathbf{g}_0 = -\mathbf{g}_0$ and $I = I^*$, we obtain that

$$l\pi_0 K_\Sigma f_1 = \begin{pmatrix} -\frac{1}{4}(\mathbf{g}_0 | \pi_0 T_\Sigma f_1)_{V_2} \\ 2(\pi_0 T_\Sigma f_1)_{t\Sigma} \end{pmatrix}.$$

We recall that we work in the reduced setting, explained in 3.3.1.

If w is the solution of $D_1 w = 0$ with $\rho w = f_1$, then from [GMW, Prop. 4.11] we have:

$$\begin{aligned} -\frac{1}{4}(\mathbf{g}_0 | \pi_0 T_\Sigma f_1)_{V_2} &= (dw)_{tt} - \frac{1}{2}(\mathbf{h}_0 | (dw)_{\Sigma\Sigma})_{V_2\Sigma} \\ &= \partial_t w_t + \delta_\Sigma w_\Sigma + \frac{1}{2}\text{tr}(\mathbf{r}_0)w_t, \\ 2(\pi_0 T_\Sigma f_1)_{t\Sigma} &= 2(dw)_{t\Sigma} = \partial_t w_\Sigma - \frac{1}{2}\text{tr}(\mathbf{r}_0)w_\Sigma - \mathbf{r}_0 w_\Sigma + d_\Sigma w_t, \end{aligned} \quad (4.10)$$

where $\mathbf{r}_0 = \frac{1}{2}\partial_t \mathbf{h}_0 \mathbf{h}_0^{-1}$ and d_Σ, δ_Σ are the symmetric differential and co-differential on (Σ, \mathbf{h}_0) .

By Lemma 3.1, we know that if $f_1 = c_1^\pm f_1$, then $\partial_t w|_\Sigma = ib^\pm w|_\Sigma$. Therefore

$$l\pi_0 K_\Sigma c_1^\pm f_1 = (ib^\pm + B)\pi_0 c_1^\pm f_1 \quad (4.11)$$

where

$$Bv = \begin{pmatrix} \frac{1}{2}\text{tr}(\mathbf{r}_0) & \delta_\Sigma \\ d_\Sigma & -\frac{1}{2}\text{tr}(\mathbf{r}_0) - \frac{1}{2}\mathbf{r}_0 \end{pmatrix} v, \quad v = \begin{pmatrix} v_t \\ v_\Sigma \end{pmatrix} \in H^s(\Sigma; \tilde{V}_1).$$

Summarizing we obtain

$$R_\Sigma K_\Sigma = \begin{pmatrix} (ib^+ + B)\pi_0 c_1^+ & 0 \\ 0 & (ib^- + B)\pi_0 c_1^- \end{pmatrix} + R_{-\infty}, \quad (4.12)$$

By Lemma 3.1, b^\pm are elliptic and invertible and $\pm \text{Re} b^\pm \geq C(-\Delta_{\tilde{h}_0} + 1)^{\frac{1}{2}}$. The operator B belongs to $\Psi^1(\Sigma; \tilde{V}_1)$ and is formally selfadjoint. Therefore the maps $(ib^\pm + B) : H^s(\Sigma; \tilde{V}_1) \rightarrow H^{s-1}(\Sigma; \tilde{V}_1)$ are boundedly invertible, which implies (1) since $R_{-\infty}$ is smoothing. (2) is obvious and (3) follows from (4.12). \square

4.3.3. Positivity property.

Lemma 4.4. *We have*

$$\pm q_{I,2} \circ c_2^\pm \geq 0 \text{ on } \text{Ker } R_\Sigma.$$

Proof. Using (3.38) and the fact that c_2^\pm are projections we obtain that

$$\pm (f_2 | q_{I,2} c_2^\pm f_2)_{\tilde{V}_2 \otimes \mathbb{C}^2} = \pm (c_2^\pm f_2 | q_{I,2} c_2^\pm f_2)_{\tilde{V}_2 \otimes \mathbb{C}^2},$$

and if $R_\Sigma f_2 = 0$ we obtain that

$$\pm (c_2^\pm f_2 | q_{I,2} c_2^\pm f_2)_{\tilde{V}_2 \otimes \mathbb{C}^2} = \pm (c_2^\pm f_2 | \tilde{q}_2 c_2^\pm f_2)_{\tilde{V}_2 \otimes \mathbb{C}^2}.$$

This is positive by Lemma 3.1. \square

4.4. Gauge fixing in the regular case. In order to lighten notation, we often write in this subsection $\bar{f} \cdot qf$ for $(f | qf)_{\tilde{V} \otimes \mathbb{C}^2}$, \mathcal{H}^s for $\mathcal{H}^s(\Sigma; \tilde{V} \otimes \mathbb{C}^2)$, H^s for $H^s(\Sigma; \tilde{V} \otimes \mathbb{C}^2)$.

We assume in this subsection that $R_\Sigma K_\Sigma : \mathcal{H}^s(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2) \rightarrow H^{s-1}(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2)$ is *invertible*. We set

$$T = 1 - K_\Sigma (R_\Sigma K_\Sigma)^{-1} R_\Sigma. \quad (4.13)$$

Proposition 4.5. (1) $T : \mathcal{H}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \rightarrow \text{Ker } R_\Sigma$ is a bounded projection on $\mathcal{H}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2)$,

(2) $T \circ K_\Sigma = 0$,

(3) T preserves $\text{Ker } K_\Sigma^\dagger$.

(4) $c_2^\pm T c_2^\mp$ is smoothing.

Proof. From (3.37) we obtain that $R_\Sigma : \mathcal{H}^s(\Sigma; V_2 \otimes \mathbb{C}^2) \rightarrow H^s(\Sigma; V_1 \otimes \mathbb{C}^2)$ is bounded. Using the ellipticity of $R_\Sigma K_\Sigma$ we obtain that T is bounded. We have $R_\Sigma \circ T = 0$ and $T = \mathbb{1}$ on $\text{Ker } R_\Sigma$ which implies (1). (2) and (3) are clear.

To prove (4) it suffices to show that $c_2^\pm K_\Sigma (R_\Sigma K_\Sigma)^{-1} R_\Sigma c_2^\mp$ is smoothing, or using Lemma 3.2 that $c_1^\pm (R_\Sigma K_\Sigma)^{-1} R_\Sigma c_2^\mp$ is smoothing. This follows from Prop. 4.3 (2) and (3). \square

Remark 4.6. *Property (4) in Prop. 4.5 is the key property of the microlocal TT-synchronous gauge condition. It will be used in the next subsection for the construction of Hadamard states. It is not satisfied by the usual TT-synchronous gauge condition.*

4.5. Gauge fixing in the singular case. Assume now that $R_\Sigma K_\Sigma$ is *not invertible*.

4.5.1. *Notation.* The charges q_2 and hence $q_{I,2}$ are well defined on \mathcal{H}^s for $s \geq \frac{1}{2}$, which we will assume in the sequel. The orthogonal of a subspace $E \subset \mathcal{H}^s$ for $q_{I,2}$ will be denoted by $E^{q_{I,2}}$.

If $A : C^\infty(\Sigma; \tilde{V}_k \otimes \mathbb{C}^2) \rightarrow C^\infty(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2)$ we define $A^\dagger : \mathcal{D}'(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2) \rightarrow \mathcal{D}'(\Sigma; \tilde{V}_k \otimes \mathbb{C}^2)$ by

$$(u|Af)_{L^2(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2)} := \overline{A^\dagger u} \cdot q_{I,k} f,$$

where we recall that $q_{I,2} = q_2 \circ I$, $q_{I,1} = q_1$. Denoting by A^* the usual adjoint obtained from the Hilbertian structure of $\tilde{V}_k \otimes \mathbb{C}^2$ we have $A^* = q_{I,k} A^\dagger$.

4.5.2. *The space $\mathcal{H}_{\text{reg}}^s$.* We set:

$$\mathcal{H}_{\text{reg}}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) := \{f_2 \in \mathcal{H}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) : R_\Sigma f_2 \in \text{Ran } R_\Sigma K_\Sigma\},$$

and we will sometimes write $\mathcal{H}_{\text{reg}}^s$ for $\mathcal{H}_{\text{reg}}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2)$. The gauge fixing equation (4.7) can be solved iff $f_2 \in \mathcal{H}_{\text{reg}}^s$. Clearly

$$\text{Ker } R_\Sigma \cap \mathcal{H}^s \subset \mathcal{H}_{\text{reg}}^s, \quad K_\Sigma \mathcal{H}^{s+1} \subset \mathcal{H}_{\text{reg}}^s.$$

By ellipticity $\text{Ker}(R_\Sigma K_\Sigma)^*$ is a finite dimensional space in $C^\infty(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2)$, and there exist $u_i \in \text{Ker}(R_\Sigma K_\Sigma)^*$, $1 \leq i \leq n$ such that $u \in \text{Ran } R_\Sigma K_\Sigma$ iff $(u_i|u)_{\tilde{V}_1 \otimes \mathbb{C}^2} = 0$, $1 \leq i \leq n$.

A routine computation gives that

$$R_\Sigma^\dagger = \begin{pmatrix} (b^+ - b^-)^{-1} & (b^+ - b^-)^{-1} \\ b^+(b^+ - b^-)^{-1} & -b^-(b^+ - b^-)^{-1} \end{pmatrix} \times \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix},$$

for $J : H^s(\Sigma; \tilde{V}_1) \rightarrow H^s(\Sigma; \tilde{V}_2)$ defined by

$$(Jv)_{tt} = \frac{1}{2}v_t, \quad (Jv)_{t\Sigma} = v_\Sigma, \quad (Jv)_{\Sigma\Sigma} = -\frac{1}{2}v_t \mathbf{h}_0.$$

In particular we see that $R_\Sigma^\dagger : C^\infty(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2) \rightarrow C^\infty(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2)$, hence $v_i = R_\Sigma^\dagger u_i \in C^\infty(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2)$. Since $(R_\Sigma K_\Sigma)^* u_i = q_1 K_\Sigma^\dagger R_\Sigma^\dagger u_i = 0$ we get

$$K_\Sigma^\dagger v_i = 0.$$

Without loss of generality, we can assume that the v_i are linearly independent and we set $\mathcal{V} = \text{Vect}\{v_i : 1 \leq i \leq n\}$. We have

$$\mathcal{H}_{\text{reg}}^s = \{f_2 \in \mathcal{H}^s : \bar{v}_i \cdot q_{I,2} f_2 = 0, 1 \leq i \leq n\} = \mathcal{V}^{q_{I,2}},$$

and $\mathcal{H}_{\text{reg}}^s$ is of codimension n in \mathcal{H}^s . Let us pick a supplementary space \mathcal{V}_1 in \mathcal{V} of $\mathcal{V} \cap \text{Ran} K_\Sigma$. We can assume that $v_1, \dots, v_p \in \mathcal{V}_1$ and $v_{p+1}, \dots, v_n \in \mathcal{V} \cap \text{Ran} K_\Sigma$. Since $\text{Ker} K_\Sigma^\dagger = \text{Ran} K_\Sigma^{q_{I,2}}$, we obtain that

$$\text{Ker} K_\Sigma^\dagger \cap \mathcal{H}_{\text{reg}}^s = \text{Ker} K_\Sigma^\dagger \cap \mathcal{V}_1^{q_{I,2}},$$

and $\text{Ker} K_\Sigma^\dagger \cap \mathcal{H}_{\text{reg}}^s$ is of codimension p in $\text{Ker} K_\Sigma^\dagger$. Therefore we can find $w_1, \dots, w_p \in \text{Ker} K_\Sigma^\dagger \cap C^\infty(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2)$ such that $A = [\bar{w}_i \cdot q_{I,2} v_j]_{1 \leq i, j \leq p}$ is invertible.

Since $\mathcal{H}_{\text{reg}}^s$ is of codimension n in \mathcal{H}^s , we can complete the w_i for $1 \leq i \leq p$ by $w_i \in C^\infty$ for $p+1 \leq i \leq n$ so that $Q = [\bar{w}_i \cdot q_{I,2} v_j]_{1 \leq i, j \leq n}$ is invertible. Using that $w_i \in \text{Ker} K_\Sigma^\dagger$ and $v_j \in \text{Ran} K_\Sigma$ for $i \leq p, j \geq p+1$ we have $Q = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$, where $A \in M_p(\mathbb{C})$, $C \in M_{n-p}(\mathbb{C})$ are invertible. Replacing w_i by $\sum_k \bar{T}_{ik} w_k$ for $T = \begin{pmatrix} A^{-1} & 0 \\ -C^{-1} B A^{-1} & C^{-1} \end{pmatrix}$ we obtain

$$\begin{aligned} \bar{w}_i \cdot q_{I,2} v_j &= \delta_{ij}, \\ w_i &\in \text{Ker} K_\Sigma^\dagger \text{ for } i \leq p, \\ v_j &\in \text{Ker} K_\Sigma^\dagger \text{ for } j \leq n, \\ v_j &\in \text{Ran} K_\Sigma \text{ for } j \geq p+1. \end{aligned} \tag{4.14}$$

We set now

$$\pi f_2 = \sum_{j=1}^n \bar{v}_j \cdot q_{I,2} f_2 w_j. \tag{4.15}$$

Lemma 4.7. (1) $1 - \pi : \mathcal{H}^s \rightarrow \mathcal{H}_{\text{reg}}^s$ is a projection;

(2) $\pi \circ K_\Sigma = 0$;

(3) $1 - \pi$ preserves $\text{Ker} K_\Sigma^\dagger$;

(4) π^\dagger preserves $\text{Ran} K_\Sigma$.

Proof. The fact that π is a projection is easy. We have $\text{Ker} \pi = \mathcal{H}_{\text{reg}}^s$ which proves (1). We have $\pi \circ K_\Sigma = 0$ since $v_i \in \text{Ker} K_\Sigma^\dagger$ and

$$\pi^\dagger f_2 = \sum_{j=1}^n \bar{w}_j \cdot q_{I,2} f_2 v_j.$$

By (4.14) this implies that π^\dagger preserves $\text{Ran}K_\Sigma$ which implies that π preserves $\text{Ker}K_\Sigma^\dagger$. \square

Let us now construct the analog of the projection T in the singular case. We fix two projections

$$\begin{aligned}\pi_1 &: \mathcal{H}^s(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2) \rightarrow \text{Ker}R_\Sigma K_\Sigma, \\ \pi_2 &: \mathcal{H}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \rightarrow \text{Ran}K_\Sigma \cap \text{Ker}R_\Sigma.\end{aligned}$$

The projections π_i are finite rank with smooth distributional kernels. Let us define the map

$$(R_\Sigma K_\Sigma)^{-1} R_\Sigma : \mathcal{H}_{\text{reg}}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \rightarrow \mathcal{H}^{s+1}(\Sigma; \tilde{V}_1 \otimes \mathbb{C}^2) \cap \text{Ker}\pi_1,$$

such that $f_1 = (R_\Sigma K_\Sigma)^{-1} R_\Sigma f_2$ for $f_2 \in \mathcal{H}_{\text{reg}}^s$ is the unique solution of

$$\begin{cases} R_\Sigma K_\Sigma f_1 = R_\Sigma f_2, \\ \pi_1 f_1 = 0. \end{cases} \quad (4.16)$$

We set

$$T_{\text{reg}} = \mathbb{1} - K_\Sigma (R_\Sigma K_\Sigma)^{-1} R_\Sigma : \mathcal{H}_{\text{reg}}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \rightarrow \mathcal{H}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2). \quad (4.17)$$

Lemma 4.8. (1)

$$T_{\text{reg}} : \mathcal{H}_{\text{reg}}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \rightarrow \text{Ker}R_\Sigma$$

is a projection;

$$(2) T_{\text{reg}} \text{ preserves } \mathcal{H}_{\text{reg}}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \cap \text{Ker}K_\Sigma^\dagger;$$

$$(3) T_{\text{reg}} K_\Sigma = K_\Sigma \pi_1.$$

Proof. We have $R_\Sigma T_{\text{reg}} = 0$ hence $\text{Ran}T_{\text{reg}} \subset \text{Ker}R_\Sigma$ and $T_{\text{reg}} = \mathbb{1}$ on $\text{Ker}R_\Sigma$ which implies (1). (2) follows from $K_\Sigma^\dagger K_\Sigma = 0$. If $f_2 = K_\Sigma g_1$, the unique solution of (4.16) is $f_1 = (1 - \pi_1)g_1$ which implies (3). \square

We set now

$$T = (1 - \pi_2) \circ T_{\text{reg}} \circ (1 - \pi) : \mathcal{H}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2) \rightarrow \mathcal{H}^s(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2). \quad (4.18)$$

Proposition 4.9. (1) $T : \mathcal{H}^s \rightarrow \text{Ker}R_\Sigma \cap \text{Ker}\pi_2$ is a projection;

$$(2) TK_\Sigma = 0;$$

$$(3) T \text{ preserves } \text{Ker}K_\Sigma^\dagger;$$

$$(4) c_2^\pm T c_2^\mp \text{ is smoothing.}$$

Proof. From Lemma 4.8 (1) and $\text{Ran}\pi_2 \subset \text{Ker}R_\Sigma$ we get that $R_\Sigma T = 0$, so $\text{Ran}T \subset \text{Ker}R_\Sigma \cap \text{Ker}\pi_2$. Since $\text{Ker}R_\Sigma \subset \mathcal{H}_{\text{reg}}^s$ we have $\pi = 0$ on $\text{Ker}R_\Sigma$, hence $T_{\text{reg}}(1 - \pi) = \mathbb{1}$ on $\text{Ker}R_\Sigma$ hence $T = \mathbb{1}$ on $\text{Ker}R_\Sigma \cap \text{Ker}\pi_2$. This proves (1).

By Lemma 4.7 (2) we have $(1 - \pi)K_\Sigma = K_\Sigma$ hence $TK_\Sigma = (1 - \pi_2)K_\Sigma \pi_1 = 0$ by Lemma 4.8 (3) and the definition of π_1, π_2 . This proves (2).

(3) follows from Lemma 4.8 (2) and Lemma 4.7 (3). To prove (4) we argue as in the proof of Prop. 4.5 (3) using Prop. 4.3, and additionally the fact that π, π_2 have smooth distributional kernels. \square

5. HADAMARD STATES

In this section we construct a Hadamard state by modifying the Hadamard projectors c_2^\pm using the projection T . As in the previous section we assume that Σ is compact.

We start with the simpler regular case.

5.1. **Regular case.** Let us set

$$\lambda_{2\Sigma}^\pm := \pm T^* \circ q_{I,2} c_2^\pm \circ T. \quad (5.1)$$

and as in 2.9.1:

$$\Lambda_2^\pm := (\rho_2 G_2)^* \lambda_{2\Sigma}^\pm (\rho_2 G_2).$$

Theorem 5.1. *The pair $\lambda_{2\Sigma}^\pm$ are the Cauchy surface covariances of a gauge invariant Hadamard state for P , i.e.*

- (1) $\text{WF}(\Lambda_2^\pm)' \subset \mathcal{N}^\pm \otimes \mathcal{N}^\pm$.
- (2) $\lambda_{2\Sigma}^+ - \lambda_{2\Sigma}^- = q_{I,2}$ on $\text{Ker } K_\Sigma^\dagger$,
- (3) $\lambda_{2\Sigma}^\pm = 0$ on $\text{Ker}_c K_\Sigma^\dagger \times \text{Ran } K_\Sigma^\dagger$
- (4) $\lambda_{2\Sigma}^\pm = \lambda_{2\Sigma}^{\pm*}$, $\lambda_{2\Sigma}^\pm \geq 0$ on $\text{Ker}_c K_\Sigma^\dagger$.

Therefore Λ_2^\pm are the covariances of a quasi-free Hadamard state on $\text{CCR}(\mathcal{V}_P, Q_P)$.

Proof. Let us first prove (1). We apply [G, Prop 11.1.1]. Let $U_2(t, s)$ be the Cauchy evolution for D_2 . Writing $T^* q_{I,2} =: q_{I,2} T^\dagger$, we need to show that $\text{WF}(U_2(\cdot, 0) T^\dagger c_2^\pm T)' \subset (\mathcal{N}^\pm \cup \mathcal{F}) \times T^* \Sigma$ for some conic set $\mathcal{F} \subset T^* M$ with $\mathcal{F} \cap \mathcal{N} = \emptyset$.

By Prop. 4.5 $c_2^\pm T c_2^\mp$ is smoothing, hence $c_2^\pm T^\dagger c_2^\mp$ is smoothing, since $c_2^\pm = (c_2^\pm)^\dagger$. Therefore $U_2(\cdot, 0) T^\dagger c_2^\pm T = U_2(\cdot, 0) c_2^\pm T^\dagger T$ modulo a smooth kernel. So the result follows from the Hadamard property of c_2^\pm .

Let us now prove (2). We have $\lambda_2^+ - \lambda_2^- = T^* q_{I,2} T$. T preserves $\text{Ker } K_\Sigma^\dagger$ by Prop. 4.5 (3) and $T f_2 = f_2 \text{ mod } \text{Ran } K_\Sigma^\dagger$ hence $\lambda_2^+ - \lambda_2^- = q_{I,2}$ on $\text{Ker } K_\Sigma^\dagger$.

(3) follows from Prop. 4.5 (2).

Since $\text{Ran } T \subset \text{Ker } R_\Sigma$ and $\pm q_2 c_2^\pm \geq 0$ on $\text{Ker } R_\Sigma$ by Lemma 4.4, we obtain (4). \square

5.2. **Singular case.** We now consider the singular case. We need an additional modification of c_2^\pm since $T^* q_{I,2} T \neq q_{I,2}$ on $\text{Ker } K_\Sigma^\dagger$ because of the projection π . This modification is inspired by a construction in [FS, Subsect. 4.4]. Let

$$\tilde{\pi} f_2 = \sum_{j=1}^p \bar{v}_j \cdot q_{I,2} f_2 w_j,$$

and note using (4.14) that $\tilde{\pi} = \pi$ on $\text{Ker } K_\Sigma^\dagger$.

Consider the hermitian form

$$\begin{aligned} \nu &:= q_{I,2} - (1 - \tilde{\pi})^* q_{I,2} (1 - \tilde{\pi}) \\ &= q_{I,2} \tilde{\pi}^\dagger + q_{I,2} \tilde{\pi} - q_2 \tilde{\pi}^\dagger \tilde{\pi}. \end{aligned} \quad (5.2)$$

acting on $C^\infty(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2)$. It has a smooth distributional kernel and a finite rank. Identifying it with a selfadjoint operator using the Hilbertian scalar product $(\cdot|\cdot)_{\tilde{V}_2 \otimes \mathbb{C}^2}$, we can find $u_i \in C^\infty(\Sigma; \tilde{V}_2 \otimes \mathbb{C}^2)$, $1 \leq i \leq q$ linearly independent such that

$$\nu = \sum_{i=1}^q \alpha_i |u_i\rangle \langle u_i|, \quad \alpha_i \neq 0.$$

We set

$$\overline{f_2} \cdot \lambda_{2\Sigma}^\pm f_2 := \pm \overline{Tf_2} \cdot q_{I,2} c_2^\pm Tf_2 \pm (f_2 | \mathbb{1}_{\mathbb{R}^\pm}(\nu) \nu f_2)_{\tilde{V}_2 \otimes \mathbb{C}^2}, \quad (5.3)$$

$$\Lambda_2^\pm := (\rho_2 G_2)^* \lambda_{2\Sigma}^\pm (\rho_2 G_2).$$

Theorem 5.2. *The pair $\lambda_{2\Sigma}^\pm$ are the Cauchy surface covariances of a gauge invariant Hadamard state for P , i.e.*

- (1) $\text{WF}(\Lambda_2^\pm)' \subset \mathcal{N}^\pm \otimes \mathcal{N}^\pm$.
- (2) $\lambda_{2\Sigma}^+ - \lambda_{2\Sigma}^- = q_{I,2}$ on $\text{Ker } K_\Sigma^\dagger$,
- (3) $\lambda_{2\Sigma}^\pm = 0$ on $\text{Ker}_c K_\Sigma^\dagger \times \text{Ran } K_\Sigma$
- (4) $\lambda_{2\Sigma}^\pm = \lambda_{2\Sigma}^{\pm*}$, $\lambda_{2\Sigma}^\pm \geq 0$ on $\text{Ker}_c K_\Sigma^\dagger$.

Therefore Λ_2^\pm are the covariances of a quasi-free Hadamard state on $\text{CCR}(\mathcal{V}_P, \mathcal{Q}_P)$.

Proof. The proof of (1) is identical to Thm. 5.1. Note that the additional term in (5.3) produces a smooth additional term in the two-point functions Λ_2^\pm .

Let us prove (2). We have using (5.2):

$$\lambda_{2\Sigma}^+ - \lambda_{2\Sigma}^- = T^* q_{I,2} T + q_{I,2} - (1 - \tilde{\pi})^* q_{I,2} (1 - \tilde{\pi}).$$

Let us compute the first term in the r.h.s. on $\text{Ker } K_\Sigma^\dagger$. Recall that $T = (1 - \pi_2) T_{\text{reg}} (1 - \pi)$. We have $T_{\text{reg}} (1 - \pi) f_2 = (1 - \pi) f_2$ modulo $\text{Ran } K_\Sigma$ by (4.17). Since $\text{Ran } \pi_2 \subset \text{Ran } K_\Sigma$ we obtain that $T f_2 = (1 - \pi) f_2$ modulo $\text{Ran } K_\Sigma$. By Prop. 4.9 T preserves $\text{Ker } K_\Sigma^\dagger$ hence:

$$\overline{Tf_2} \cdot q_{I,2} Tf_2 = \overline{(1 - \pi)f_2} \cdot q_{I,2} (1 - \pi)f_2 = \overline{(1 - \tilde{\pi})f_2} \cdot q_{I,2} (1 - \tilde{\pi})f_2,$$

since $\pi = \tilde{\pi}$ on $\text{Ker } K_\Sigma^\dagger$. Using the definition of ν in (5.2) we obtain that $\lambda_{2\Sigma}^+ - \lambda_{2\Sigma}^- = q_{I,2}$ on $\text{Ker } K_\Sigma^\dagger$.

We now prove (3). By Prop. 4.9 (2) $TK_\Sigma = 0$. Since $v_j, w_j \in \text{Ker } K_\Sigma^\dagger$ for $1 \leq j \leq p$, see (4.14), we obtain that $\tilde{\pi} K_\Sigma = \tilde{\pi}^\dagger K_\Sigma = 0$. Therefore $\nu = 0$ on $\text{Ran } K_\Sigma$ hence $\lambda_{2\Sigma}^\pm = 0$ on $\text{Ran } K_\Sigma$. This proves (3).

It remains to prove (4). The first term in the rhs of (5.3) is positive by the same argument as in the proof of Thm. 5.2, since $\text{Ran } T \subset \text{Ker } R_\Sigma$. The second term is also clearly positive. This completes the proof of the theorem. \square

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