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# Robust optimization applied to uncertain limit analysis and optimal plastic design

Jeremy BLEYER<sup>a</sup>, Vincent LECLERE<sup>b</sup>

a. Laboratoire Navier, Ecole des Ponts, Université Gustave Eiffel, CNRS (UMR 8205),  
jeremy.bleyer@enpc.fr

b. CERMICS, Ecole des Ponts, vincent.leclere@enpc.fr

## Résumé :

*Ce travail propose de recourir à la théorie de l'optimisation robuste dans le contexte de l'analyse limite et de l'optimisation de structures plastiques en présence d'incertitudes. L'analyse limite est une méthode directe efficace permettant de calculer la charge limite d'une structure compte-tenu de ses équations d'équilibre et de son critère de résistance. Dans le cas déterministe, la charge limite est obtenue en résolvant un problème d'optimisation convexe. Lorsque le chargement ou les propriétés de résistance sont aléatoires, les données de ce problème d'optimisation deviennent incertaines. L'optimisation robuste permet alors de calculer la charge limite la plus défavorable pour ces problèmes incertains. Nous introduisons la notion de critère de résistance robuste, par exemple, un critère de Mohr-Coulomb à cohésion et angle de frottement incertains. Enfin, nous appliquons également ces concepts à l'optimisation plastique de treillis.*

## Abstract :

*This contribution explores the use of robust optimization theory in the context of the limit analysis and plastic optimization of structures in presence of uncertainty. Limit analysis is an efficient tool for computing in a direct manner the ultimate load of a structure given its equilibrium and strength conditions. In the deterministic case, the ultimate load is obtained through the resolution of a convex optimization program. When loading or strength properties are random, the data of such an optimization problem become uncertain. Robust optimization theory allows to compute a worst-case ultimate load estimate of such uncertain problems. We introduce the notion of robust strength criteria e.g. the robust counterpart of a Mohr-Coulomb criterion with uncertain cohesion and friction angle. We also apply such concepts to the optimal plastic design of trusses.*

**robust optimization, limit analysis, optimal plastic design, truss layout optimization**

## 1 Introduction

Limit analysis (Hill, 1950; Salençon, 2013) is a powerful direct method enabling to estimate the collapse load of a structure consisting of a perfectly plastic material. The lower and upper bound approaches of

limit analysis are naturally formulated as convex optimization problems for which given data consist of a known material yield criterion, a known reference loading and a known geometry (Bleyer, 2019). However, in real applications, uncertainty may affect all these data, e.g. the amplitude or direction of some loading may not be known exactly, material strength properties can vary locally, etc. Furthermore, engineers are interested in designing a structure which would be robust to such uncertainties, meaning that the collapse load factor should always be safe for all possible realizations of the uncertain parameters. This requires to design the structure against a worst-case situation.

Classical limit analysis formulations can only take into account such uncertainties by either assuming a worst-case scenario for the parameters or by performing a stochastic analysis with random realizations of the unknown parameters. The first approach can be overly conservative and it is sometimes difficult to determine what would be a worst-case scenario in case of a complex loading mode for instance. The latter approach requires to postulate some probability distribution for the parameters and solving an extremely large amount of problems to determine the worst-case configuration, without ever reaching it.

As an alternative path, we propose, in the following, to rely on the developments of robust optimization theory (Ben-Tal et al., 2009; Bertsimas et al., 2011) to obtain a robust estimate of plastic limit loads in presence of uncertainty (Bleyer and Leclère, 2022).

## 2 Robust limit analysis with strength uncertainties

### 2.1 Nominal and uncertain limit analysis problem

The *nominal* limit analysis problem amounts to computing the maximum load factor  $\lambda_N$  by solving the following convex maximization problem :

$$\begin{aligned} \lambda_N = \max_{\lambda, \boldsymbol{\sigma}} \quad & \lambda \\ \text{s.t.} \quad & \text{div } \boldsymbol{\sigma} + \lambda \mathbf{F} = 0 \quad \text{in } \Omega \\ & \boldsymbol{\sigma} \cdot \mathbf{n} = \lambda \mathbf{t} \quad \text{on } \partial\Omega_T \\ & \boldsymbol{\sigma} \in G \quad \text{in } \Omega \end{aligned} \quad (\text{N})$$

where  $\lambda$  is the load factor,  $\boldsymbol{\sigma}$  the Cauchy stress field in  $\Omega$ ,  $\mathbf{F}$  the body force,  $\mathbf{t}$  the contact force prescribed on some part  $\partial\Omega_T$  of the boundary and  $G$  is the material yield/strength criterion which we assume to be a convex set (possibly unbounded) containing 0. In the above, the first two constraints correspond to the local balance equation and traction boundary conditions, whereas the last one corresponds to the strength condition which must be satisfied at all point  $\mathbf{x} \in \Omega$ .

We now consider the case where the loading is certain but the material may possess uncertain properties such that the strength criterion is now written as  $G(\boldsymbol{\zeta})$  where  $\boldsymbol{\zeta} \in \mathcal{U} \subseteq \mathbb{R}^m$  is a vector of uncertain parameters and  $\mathcal{U}$  the corresponding *uncertainty set* in which the uncertainty must vary. Although it can be more general, we assume, to fix ideas, that  $\mathcal{U}$  is a convex ball of unit radius for some norm i.e.  $\mathcal{U} = \{\boldsymbol{\zeta} \text{ s.t. } \|\boldsymbol{\zeta}\| \leq 1\}$ . The maximum load factor now becomes uncertain i.e. it depends on the value of

the uncertainty realization :

$$\begin{aligned} \lambda^+(\zeta) &= \max_{\lambda, \sigma} \lambda \\ \text{s.t.} \quad & \text{div } \sigma + \lambda \mathbf{F} = 0 \\ & \sigma \cdot \mathbf{n} = \lambda \mathbf{t} \\ & \sigma \in G(\zeta) \end{aligned} \quad (1)$$

The main purpose of robust optimization is to provide worst-case solutions to a given optimization problem. Our proposed theory of robust limit analysis therefore aims at evaluating the worst-case limit load among all possible realizations. In the remaining of this section, we discuss various robust formulations.

## 2.2 Adjustable robust optimization

For a given loading and two different given realizations of the uncertainty, one expects that the stress field will be different depending on the uncertainty realizations. The most natural approach therefore consists in considering the stress field and the corresponding load factor to be recourse variables. Thus, we are faced with an *adjustable robust counterpart* (ARC) to problem (1) defined as follows :

$$\begin{aligned} \lambda_{\text{ARC}} &= \min_{\zeta \in \mathcal{U}} \lambda^+(\zeta) = \min_{\zeta \in \mathcal{U}} \max_{\sigma(\zeta), \lambda(\zeta)} \lambda(\zeta) \\ \text{s.t.} \quad & \text{div } \sigma(\zeta) + \lambda(\zeta) \mathbf{F} = 0 \\ & \sigma(\zeta) \cdot \mathbf{n} = \lambda(\zeta) \mathbf{t} \\ & \sigma(\zeta) \in G(\zeta) \end{aligned} \quad (\text{ARC})$$

i.e. we find the largest load factor such that, for each uncertainty realization there exists an optimal stress field in equilibrium, with the corresponding collapse load factor, satisfying the strength criterion.

In the following, we also make use of the following equivalent formulation of the ARC problem (Takeda et al., 2008; Marandi and Den Hertog, 2018) :

$$\begin{aligned} \lambda_{\text{ARC}} &= \max_{\bar{\lambda}} \bar{\lambda} \\ \text{s.t.} \quad & \forall \zeta \in \mathcal{U}, \exists \sigma, \lambda \text{ s.t. } \text{div } \sigma + \lambda \mathbf{F} = 0 \\ & \sigma \cdot \mathbf{n} = \lambda \mathbf{t} \\ & \sigma \in G(\zeta) \\ & \bar{\lambda} \leq \lambda \end{aligned} \quad (2)$$

where uncertainty of the objective function has been transferred to the constraints with the introduction of a static (non-adjustable) variable  $\bar{\lambda}$ .

## 2.3 Static robust optimization

Unfortunately, adaptive recourse problem are numerically challenging. We follow here a conservative static robust counterpart (RC) in which we look for a stress field  $\sigma$  and a load factor  $\lambda$ , independent of the exact realization of the uncertainty, which satisfy the strength condition  $G(\zeta)$  for all  $\zeta \in \mathcal{U}$ . The corresponding problem can be formulated as follows :

$$\begin{aligned} \lambda_{\text{RC}} &= \max_{\lambda, \sigma} \lambda \\ \text{s.t.} \quad & \text{div } \sigma + \lambda \mathbf{F} = 0 \\ & \sigma \cdot \mathbf{n} = \lambda \mathbf{t} \\ & \sigma \in G(\zeta) \quad \forall \zeta \in \mathcal{U} \end{aligned} \quad (3)$$

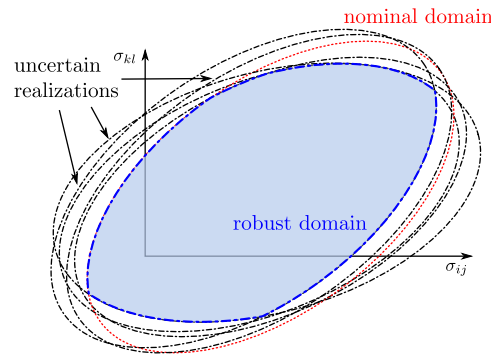


FIGURE 1 – Robust strength domain  $G_{RC}$  (in blue) obtained as the intersection of various uncertain realizations  $G(\zeta)$  (in black) of a nominal domain (in red).

What makes problem (3) a *robust optimization* problem is the condition  $\forall \zeta \in \mathcal{U}$  in the last constraint. This implies that the constraint  $\sigma \in G(\zeta)$  must be fulfilled for any possible value of  $\zeta \in \mathcal{U}$ . It is therefore an infinite-dimensional constraint. One of the main goals of robust optimization theory is to make such a problem tractable using standard convex optimization algorithms.

For instance, the robust constraint can be reformulated as :

$$\sigma \in G(\zeta) \quad \forall \zeta \in \mathcal{U} \quad \Leftrightarrow \quad \sigma \in G_{RC} \quad (4)$$

when introducing :

$$G_{RC} = \bigcap_{\zeta \in \mathcal{U}} G(\zeta) \quad (5)$$

the robust counterpart to the uncertain strength criterion. In order for a stress field to be admissible with respect to any possible realization of the uncertain strength criterion  $G(\zeta)$ , it has to belong to the intersection of all such domains (see Figure 1).

Now, problem (3) writes as :

$$\begin{aligned} \lambda_{RC} = \max_{\lambda, \sigma} \quad & \lambda \\ \text{s.t.} \quad & \text{div } \sigma + \lambda \mathbf{F} = 0 \\ & \sigma \cdot \mathbf{n} = \lambda \mathbf{t} \\ & \sigma \in G_{RC} \end{aligned} \quad (\text{RC})$$

which is now independent of the uncertainty realization. As a result, problem (RC) is a classical limit analysis problem with a different strength criterion given by (5). This makes problem (RC) very appealing provided that a simple expression for  $G_{RC}$  can be found. It is however very hard to determine a simple expression for the infinite-dimensional set intersection appearing in (5). Exact or approximate reformulation of strength criteria robust counterparts are discussed in section 3.

## 2.4 Affinely adjustable robust optimization

Unfortunately, if (RC) problem are numerically tractable, the obtained approximation might be unreasonably conservative (Bertsimas and Sim, 2004). A middle ground is the *affinely adjustable robust counterpart* (AARC), which consists in looking for adjustable variables  $\sigma(\zeta)$  and  $\lambda(\zeta)$  that are affine

functions of the uncertain variable, the so-called *affine decision rule* (Ben-Tal et al., 2004) :

$$\boldsymbol{\sigma}(\boldsymbol{\zeta}) = \boldsymbol{\sigma}_0 + \sum_{j=1}^m \boldsymbol{\sigma}_j \zeta_j \quad (6a)$$

$$\lambda(\boldsymbol{\zeta}) = \lambda_0 + \sum_{j=1}^m \lambda_j \zeta_j \quad (6b)$$

where the  $\boldsymbol{\sigma}_i$  (resp.  $\lambda_i$ ) represent  $1 + m$  different stress fields (load factor variables) which are now static optimization variables. Inserting the affine decision rules (6a)-(6b) into (ARC), the corresponding AARC reads :

$$\begin{aligned} \lambda_{\text{AARC}} = \max_{\boldsymbol{\sigma}_i, \lambda_i} \min_{\boldsymbol{\zeta} \in \mathcal{U}} & \lambda_0 + \sum_{j=1}^m \lambda_j \zeta_j \\ \text{s.t.} & \operatorname{div} \left( \boldsymbol{\sigma}_0 + \sum_{j=1}^m \boldsymbol{\sigma}_j \zeta_j \right) + \left( \lambda_0 + \sum_{j=1}^m \lambda_j \zeta_j \right) \mathbf{F} = 0 \\ & \left( \boldsymbol{\sigma}_0 + \sum_{j=1}^m \boldsymbol{\sigma}_j \zeta_j \right) \cdot \mathbf{n} = \left( \lambda_0 + \sum_{j=1}^m \lambda_j \zeta_j \right) \mathbf{t} \\ & \left( \boldsymbol{\sigma}_0 + \sum_{j=1}^m \boldsymbol{\sigma}_j \zeta_j \right) \in G(\boldsymbol{\zeta}) \end{aligned} \quad (7)$$

which can also be reformulated as follows :

$$\begin{aligned} \lambda_{\text{AARC}} = \max_{\bar{\lambda}, \boldsymbol{\sigma}_i, \lambda_i} & \bar{\lambda} \\ \text{s.t.} & \operatorname{div}(\boldsymbol{\sigma}_j) + \lambda_j \mathbf{F} = 0 \quad \forall j = 0, \dots, m \\ & \boldsymbol{\sigma}_j \cdot \mathbf{n} = \lambda_j \mathbf{t} \quad \forall j = 0, \dots, m \\ & \left( \boldsymbol{\sigma}_0 + \sum_{j=1}^m \boldsymbol{\sigma}_j \zeta_j \right) \in G(\boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \mathcal{U} \\ & \bar{\lambda} \leq \lambda_0 + \sum_{j=1}^m \lambda_j \zeta_j \quad \forall \boldsymbol{\zeta} \in \mathcal{U} \end{aligned} \quad (\text{AARC})$$

in which we removed the uncertainty from the objective function and replaced the minimization over  $\boldsymbol{\zeta}$  with robust constraints. Note that equality constraints depending on  $\boldsymbol{\zeta}$  have been re-expressed by identifying the corresponding terms of the expansion in terms of  $\zeta_i$  since  $\mathcal{U}$  is full dimensional.

## 2.5 Comparison between the different approaches

Summarizing, (RC) is the most conservative formulation yielding the smallest limit load. (AARC) is more flexible since it considers additional static variables  $\boldsymbol{\sigma}_j, \lambda_j$  for  $j = 1, \dots, m$  and reduces to (RC) if we fix all  $\boldsymbol{\sigma}_j = 0$ . As mentioned, (ARC) is less conservative than (AARC) since we allow for more general decision rules but is generally untractable. Finally, all of these formulations guard against all possible realizations of the uncertainty such that we have the following ordering :

$$\lambda_{\text{RC}} \leq \lambda_{\text{AARC}} \leq \lambda_{\text{ARC}} \leq \lambda^+(\boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \mathcal{U} \quad (8)$$

### 3 Robust strength conditions

#### 3.1 Uncertain strength conditions and a tractable approximation

Tractability of robust formulations such as (AARC) is essentially driven by how the uncertain strength criterion  $G$  depends on  $\zeta$ . Unfortunately, we are not aware of any general results. However, in most applications, such uncertain constraints can be written in the following form :

$$g(\boldsymbol{\sigma} + \boldsymbol{\Sigma}\zeta) \leq 1 - \mathbf{b}^T \zeta, \quad \forall \zeta \in \mathcal{U} \quad (9)$$

with  $\boldsymbol{\sigma} \in \mathbb{R}^d$ ,  $\boldsymbol{\Sigma} \in \mathbb{R}^{d \times m}$ ,  $d$  being the dimension of the stress space,  $\mathbf{b} \in \mathbb{R}^m$  and  $g$  is a convex homogeneous function.

Exact reformulations of such a constraint are possible only if  $G$  or  $\mathcal{U}$  is polyhedral. In the general case, one can benefit from the following safe approximation due to Bertsimas and Sim (2006) : the robust constraint (9) can be safely approximated as follows

$$g(\boldsymbol{\sigma}) + \|\mathbf{s}\|_* \leq 1 \quad (10)$$

where  $\|\cdot\|_*$  is the dual norm of  $\|\cdot\|$  and where for  $j = 1, \dots, m$  :

$$s_j = \max\{g(\boldsymbol{\Sigma}_j) + b_j, g(-\boldsymbol{\Sigma}_j) - b_j\} \quad (11)$$

with  $\boldsymbol{\Sigma}_j$  denoting the  $j$ -th column of  $\boldsymbol{\Sigma}$ .

#### 3.2 Example on a robust Mohr-Coulomb criterion

Let us consider the case of a Mohr-Coulomb strength criterion where the cohesion  $c$  and the friction angle  $\phi$  are uncertain. A negative correlation is often encountered between both parameters, i.e. soils with low cohesion tend to exhibit higher friction angles than with higher cohesion. We denote by  $\rho$  the correlation coefficient between  $c$  and  $\phi$ , with typical values ranging from  $-0.5$  to  $-0.9$  (Wang and Akeju, 2016). Let us therefore consider that  $\mathbf{k} = (c, \phi)$  is given by :

$$\mathbf{k}(\zeta) = \mathbf{k}_0 + \mathbf{K}\zeta, \quad \text{for } \zeta \in \mathcal{U} \quad (12)$$

where  $\mathbf{k}_0$  corresponds to the nominal values and where the "correlation" matrix  $\mathbf{K}$  is such that :

$$\mathbf{K}^T \mathbf{K} = \begin{bmatrix} \Delta c^2 & \rho \Delta c \Delta \phi \\ \rho \Delta c \Delta \phi & \Delta \phi^2 \end{bmatrix} \quad \text{i.e. } \mathbf{K} = \begin{bmatrix} \Delta c & \rho \Delta \phi \\ 0 & \Delta \phi \sqrt{1 - \rho^2} \end{bmatrix} \quad (13)$$

where  $\Delta c, \Delta \phi$  are the parameters typical variations and are assumed to be positive. Note that if such variations were taken as the standard deviations of the corresponding parameters,  $\mathbf{K}^T \mathbf{K}$  would be the corresponding covariance matrix.

The robust counterpart of the Coulomb criterion therefore reads :

$$\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3) \sin \phi(\zeta) - 2c(\zeta) \cos \phi(\zeta) \leq 0 \quad \forall \zeta \in \mathcal{U} \quad (14)$$

where  $\sigma_1$  (resp.  $\sigma_3$ ) is the maximum (resp. minimum) principal stress.

Assuming that the variations  $\Delta c$ ,  $\Delta\phi$  are small, linearization around  $\mathbf{k}_0$  results in :

$$\begin{aligned} \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)(\sin\phi_0 + \cos(\phi_0)K_{22}\zeta_2) \\ - 2(c_0 + K_{11}\zeta_1 + K_{12}\zeta_2)\cos\phi_0 \\ + 2c_0\sin\phi_0K_{22}\zeta_2 \leq 0 \quad \forall \zeta \in \mathcal{U} \end{aligned} \quad (15)$$

with  $K_{ij}$  being the components of  $\mathbf{K}$  defined in (13).

One can see that (15) is of the form (9). Using the approximation (11), one obtains the following robust counterpart :

$$\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi_0 - 2c_0\cos\phi_0 + \|\mathbf{s}\|_* \leq 0 \quad (16)$$

where :

$$\mathbf{s} = \begin{pmatrix} 2\Delta c \cos\phi_0 \\ ((\sigma_1 + \sigma_3)\cos(\phi_0) + 2c_0\sin\phi_0)\sqrt{1-\rho^2}\Delta\phi - 2c_0\cos\phi_0\rho\Delta\phi \end{pmatrix} \quad (17)$$

Let us now investigate the simple case of no cross-correlation  $\rho = 0$  with  $\mathcal{U} = \{(\zeta_1, \zeta_2) \text{ s.t. } \|\zeta\|_\infty \leq 1\}$ . The previous expression reduces to :

$$\mathbf{s} = \begin{pmatrix} 2\Delta c \cos\phi_0 \\ ((\sigma_1 + \sigma_3)\cos(\phi_0) + 2c_0\sin\phi_0)\Delta\phi \end{pmatrix} \quad (18)$$

$$\|\mathbf{s}\|_* = \|\mathbf{s}\|_1 = 2\Delta c \cos\phi_0 + |(\sigma_1 + \sigma_3)\cos(\phi_0) + 2c_0\sin\phi_0|\Delta\phi \quad (19)$$

so that the robust Mohr-Coulomb criterion (16) reduces to :

$$\begin{aligned} \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi_0 \\ + |(\sigma_1 + \sigma_3)\cos(\phi_0) + 2c_0\sin\phi_0|\Delta\phi \leq 2(c_0 - \Delta c)\cos\phi_0 \end{aligned} \quad (20)$$

which can be further expressed as follows :

$$\begin{cases} \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)(\sin\phi_0 + \cos(\phi_0)\Delta\phi) \leq 2c_{\min}\cos\phi_0 - 2c_0\sin\phi_0\Delta\phi \\ \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)(\sin\phi_0 - \cos(\phi_0)\Delta\phi) \leq 2c_{\min}\cos\phi_0 + 2c_0\sin\phi_0\Delta\phi \end{cases} \quad (21)$$

where  $c_{\min} = c_0 - \Delta c$  is the worst-case cohesion. Introducing  $\phi_{\min} = \phi_0 - \Delta\phi$  the worst-case friction angle and  $\phi_{\max} = \phi_0 + \Delta\phi$  the best-case friction angle and using the fact that  $\sin(\phi_{\max/\min}) \approx \sin\phi_0 \pm \cos(\phi_0)\Delta\phi$  and  $\cos(\phi_{\max/\min}) \approx \cos\phi_0 \mp \sin(\phi_0)\Delta\phi$ , the previous criterion is, in fact, a first-order approximation (in terms of  $\Delta c$ ,  $\Delta\phi$ ) to the following multi-surface criterion :

$$\begin{cases} \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi_{\max} \leq 2c_{\min}\cos(\phi_{\max}) \\ \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\phi_{\min} \leq 2c_{\min}\cos(\phi_{\min}) \end{cases} \quad (22)$$

i.e. the obtained robust counterpart, for this specific case, (approximately) corresponds to the intersection of two Coulomb criteria with the worst-case cohesion and either the best or the worst-case friction



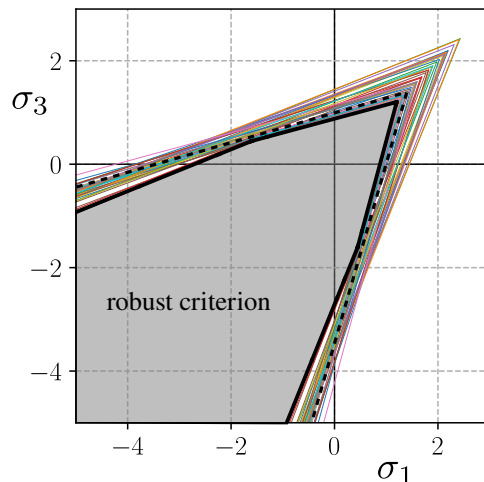


FIGURE 2 – Robust and uncertain Mohr-Coulomb criterion :  $c_0 = 1$  MPa,  $\phi_0 = 30^\circ$ ,  $\Delta c = 150$  kPa,  $\Delta\phi = 5^\circ$ . Black dashed lines denote the nominal surface, thin coloured lines denote random realizations of the uncertain criterion.

angle. An illustration of such a result is given in Figure 2. The yield surface corresponding to random realizations of  $c(\zeta)$  and  $\phi(\zeta)$  are also represented. One can indeed see that the obtained robust strength criterion forms a tight lower bound to the various realizations and is made of two sets of lines approximately characterized by the minimum and maximum friction angle  $\phi_{\min}$  and  $\phi_{\max}$ .

## 4 Application to truss plastic design

In this section we consider the optimization of plastic trusses using a ground-structure approach (Gilbert and Tyas, 2003; He et al., 2019). A set of potential trusses connecting given nodes is considered and one aims at finding the minimal volume truss able to carry given loads with some strength condition for each truss. More precisely, one looks for the value of the cross-section  $\mathbf{a}$  for each truss. This problem can be formulated as follows :

$$\begin{aligned} \min_{\mathbf{a}, \mathbf{N}} \quad & \ell^T \mathbf{a} \\ \text{s.t.} \quad & \mathbf{H} \mathbf{N} = \mathbf{F} \\ & |\mathbf{N}| \leq \sigma_0 \mathbf{a} \end{aligned} \quad (23)$$

where  $\ell$  is the vector of truss lengths,  $V = \ell^T \mathbf{a}$  being the total volume to be minimized,  $\mathbf{N}$  is the vector of normal forces in each truss,  $\sigma_0$  is the strength of the constitutive material,  $\mathbf{H}$  is the equilibrium matrix and  $\mathbf{F}$  the vector of given loads.

We consider here the case of loading uncertainties where the load vector is given as follows :

$$\mathbf{F}(\zeta) = \mathbf{F}_0 + \sum_{j=1}^m \mathbf{F}_j \zeta_j, \quad \forall \zeta \in \mathcal{U} \quad (24)$$

where  $\mathbf{F}_0$  is the nominal load and  $\mathbf{F}_j$  are additional loading modes of uncertain amplitude.

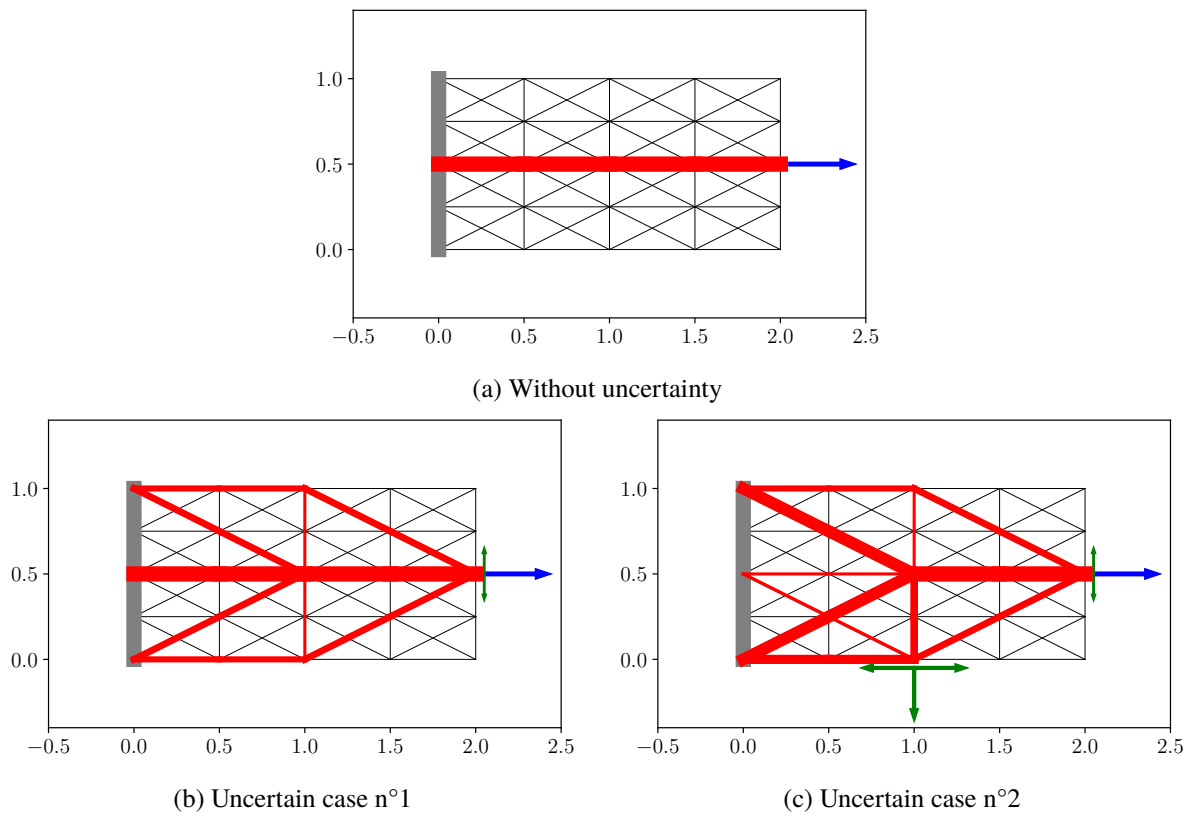


FIGURE 3 – Optimal truss as a function of the considered loading. Red lines represent the optimized cross-sections. In blue : main force ; in green : uncertain perturbations.

Applying the previous robust optimization concepts with an affine decision rule for  $\mathbf{N}$  yields the following AARC formulation :

$$\begin{aligned}
 \min_{\mathbf{a}, \mathbf{N}_j} \quad & \ell^T \mathbf{a} \\
 \text{s.t.} \quad & \mathbf{H} \mathbf{N}_j = \mathbf{F}_j \quad \forall j = 0, \dots, m \\
 & \left| \mathbf{N}_0 + \sum_{j=1}^m \mathbf{N}_j \zeta_j \right| \leq \sigma_0 \mathbf{a} \quad \forall \zeta \in \mathcal{U}
 \end{aligned} \tag{25}$$

where the last constraint can be equivalently formulated as follows :

$$\left| \mathbf{N}_0 + \sum_{j=1}^m \mathbf{N}_j \zeta_j \right| \leq \sigma_0 \mathbf{a} \quad \forall \zeta \in \mathcal{U} \tag{26}$$

$$\Leftrightarrow \sup_{\zeta \in \mathcal{U}} \left\{ \pm \mathbf{N}_0 \pm \sum_{j=1}^m \mathbf{N}_j \zeta_j \right\} \leq \sigma_0 \mathbf{a} \tag{27}$$

$$\Leftrightarrow \pm \mathbf{N}_0 + \|\overline{\mathbf{N}}\|_* \leq \sigma_0 \mathbf{a} \tag{28}$$

where  $\overline{\mathbf{N}} = [\mathbf{N}_1, \dots, \mathbf{N}_m]$  and the norm is taken with respect to the column axis.

In Figure 3a, one can see that the optimized truss for a pure horizontal concentrated loading corresponds

to a single bar in traction. Such a structure is obviously not robust at all to slight perturbations. In Figures 3b-3c, additional uncertain loadings have been considered (in green) which result in a much more robust structure compared to the nominal case.

## 5 Conclusions

Concepts of robust optimization theory have been applied to the problem of limit analysis and optimal design of plastic structures. Simple decision rules must generally be used for optimization variables depending on the realization of the uncertainty. In some instances, robust strength constraints can be safely approximated to obtain a tractable formulation. In future works, such robust formulations will be used on 2D/3D structural problems and compared against Monte-Carlo simulations with respect to the worst-case limit loads.

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